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18 July 2014

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Abstract. We define a bargaining set for finite economies using Aubin's veto mechanism and show its coincidence with the set of Walrasian allocations. Then, we rewrite our notion in terms of replicated economies showing that, in contrast with Anderson, Trockel and Zhou's (1997) non-convergence result, this Edgeworth bargaining set shrinks to the set of Walrasian allocations.

JEL Classification: D51, D11, D00.

Keywords: Bargaining sets, coalitions, core, veto mechanism.

* We are especially grateful to Carlos Hervés-Beloso for his helpful comments and suggestions. We are also indebted to Joao Correia-da-Silva, Marta Faias, and the participants in the XXII and XXIII European Workshop on General Equilibrium Theory (EWGET 2013 and 2014) and in the V Workshop on Economic Analysis held in Naples in January, 2014.

** This work is partially supported by Research Grants ECO2012-38860-C02-01 and 02 (Ministerio de Economía y Competitividad).

1 Introduction

The core of an economy is defined as the set of allocations which cannot be blocked by any coalition. Thus, the veto mechanism that defines the core implicitly assumes that individuals are not forward-looking. However, one may ask whether an objection or veto is credible or, on the contrary, not consistent enough so other agents in the economy may react to it and propose an alternative or counter-objection.

The first outcome of this two-step conception of the veto mechanism was the work by Aumann and Maschler (1964), who introduced the concept of bargaining set, containing the core of a cooperative game. This original concept of bargaining set was later adapted to atomless economies by Mas-Colell (1989). The main idea is to inject a sense of credibility and stability to the veto mechanism, hence permitting the implementation of some allocations which otherwise would be formally blocked, although in a non-credible way. Thus, only objections without counter-objections are considered as credible or justified, and consequently, blocking an allocation becomes more difficult.

In the case of pure exchange economies with a finite number of traders the set of Walrasian allocations is a strict subset of the core which is also strictly contained in the bargaining set. Under conditions of generality similar to those required in Aumann's (1964) core-Walras equivalence theorem, Mas-Colell (1989) showed that the bargaining set and the competitive allocations coincide for continuum economies. These equivalence results provide foundations for the Walrasian market equilibrium and, at the same time, bring up the question of whether there are analogies in economies with a large, but finite number of agents. A classical contribution in this direction is the one by Debreu and Scarf (1963), who stated a first formalization of Edgeworth's (1881) conjecture, showing that the core and the set of Walrasian allocations become arbitrarily close whenever a finite economy is replicated a sufficiently large number of times. However, in contrast with the Debreu-Scarf core convergence theorem, the work by Anderson, Trockel and Zhou (1997), ATZ from now on, proved that the bargaining set does not shrink to the set of Walrasian allocations in a sequence of replicated economies as the core does.[1]

[1]The replica sequence in the example stated by ATZ satisfies the hypotheses of the Debreu-Scarf theorem (1963); preferences are smooth and the economy is regular.

Therefore, unlike the core, the Mas-Colell bargaining set does not lead to a convergence result in large finite economies. Roughly speaking, this is basically due to the fact that the notion of a justified objection is very stringent. Thus, given the difficulties in finding such credible blocking, the bargaining set may become very large. The example stated by ATZ highlights this point: they define a sequence of replica economies in which there is a unique Walrasian equilibrium but the bargaining set eventually occupies the full measure of the set of all individually rational and Pareto optimal allocations having the equal treatment property. Nevertheless, as ATZ pointed out, the argument supporting their non-convergence example depends crucially on the use of a replica structure to enlarge the economy. Consequently, they leave open the possibility that other ways of enlarging the set of agents, and in turn, strengthening the blocking power of coalitions in the economy, might lead to other results.

Instead of starting from Aumann's core-Walras equivalence, in this paper we build upon Debreu-Scarf's core convergence and the Edgeworth equilibrium notion that Aubin (1979) turned into his veto mechanism, where agents can participate in coalitions with a part of their endowments, showing that the core resulting from this blocking system equals the set of Walrasian allocations. The veto mechanism *à la Aubin* actually represents a way of enlarging the set of coalitions. Furthermore, the Aubin core-Walras equivalence leads us to consider the Aubin veto to define objections and counter-objections. Thus, we define a concept of bargaining set for finite economies that involves not only more possible objections but also counter-objections. Note that enlarging the number of coalitions in this way may be a double-edged sword. Having more coalitions implies more possibilities to object but, at the same time, produces more ways of counter-objecting. That is, objecting becomes easier but having a justified objection becomes harder. This highlights the fact that the overall effect of enlarging the number of coalitions is not straightforward.

It could appear that this notion is nothing but Mas-Colell's for the particular case of a n-types continuum economy, but it is not. There are actually conceptual differences between both concepts with important implications regarding the nature of justified objections.

Our first result states that the set of Walrasian allocations coincides with this Aubin bargaining set, providing a finite approach to the characterization obtained by Mas-Colell (1989) of competitive allocations. Our Walras-bargaining equivalence allows us to deduce that the bargaining set we have defined is also consistent in the sense of Dutta *et al.* (1989) as happens with the Mas-Colell bargaining set for atomless economies. Furthermore, we also provide a discrete approach to the characterization of justified objections stated by Mas-Colell (1989) by means of a notion of Walrasian objections which reflects the main differences between Mas-Colell's bargaining set and ours. The fact that any Walrasian objection is justified and vice-versa for finite economies, allows us to refine our Walras-bargaining equivalence and its proof in terms of Walrasian objections.

Our result (and also Mas-Colell's) implicitly requires the formation of all coalitions. In other words, the bargaining set concept requires checking the whole set of possible coalitions in order to test whether any group of agents can improve upon an allocation by using their own resources, both in the objection and counter-objection processes. It is usually argued that the costs arising from forming a coalition are not at all negligible; incompatibilities among different agents may appear and a large amount of information and communication might be needed to really get together a coalition. This idea leads us to study the possibility of restricting the formation of coalitions by assuming that not all the parameters, which specify the degree of participation of agents when they become members of a coalition, are admissible. Then, we analyze the consequences that this condition has with regard to the bargaining set solution. We show that both for objections and counter-objections, the participation rates of the agents can be restricted to those arbitrarily small without changing the bargaining set. However, we show that this does not hold if we consider parameters close enough to complete participation. We also prove that the participation rates in the counter-objection system can be restricted to rational numbers, which leads us to an analysis of the convergence properties of the bargaining set when the economy is enlarged via replicas which constitutes a central point in this paper.

The Aubin bargaining set concept can be rewritten in terms of replicated economies by just considering rational numbers as participation rates, resulting in what we refer to as Edgeworth bargaining set. This is so because it works by taking into account the whole replica structure, and not only what happens at each step as the economy is replicated. Actually, going back to the work by ATZ, we show that it cannot be used to prove non-convergence for the Edgeworth bargaining set and, at the same time, this analysis allows us to obtain an alternative and simple proof of the result by ATZ. Furthermore, we provide an example that shows the impossibility of obtaining an exact convergence result for the Edgeworth bargaining set. The example points out why it is not possible to get to that convergence result and how this could be fixed. Indeed, considering a continuity property of the equilibrium correspondence, we obtain a generic convergence result for the Edgeworth bargaining set. Next, considering a notion of leader in the objection process we show that the corresponding Edgeworth bargaining set shrinks and converges to the set of Walrasian allocations, providing an exact convergence result.

The Walras-bargaining equivalence and the convergence properties of the Edgeworth bargaining set we obtain can be summarized in the following tables.

Atomless economies	core-Walras equivalence. (Aumann, 1964)	Mas-Colell's bargaining set-Walras equivalence. (Mas-Colell, 1989)
Finite economies	Aubin core-Walras equivalence. (Aubin, 1979)	(Aubin) bargaining set -Walras equivalence. This paper: Theorem 3.1.

Table 1: Equivalence results for Walrasian equilibria.

Bargaining set	Non-Convergence	Convergence
Mas-Collel's (1989)	Anderson, Trockel and Zhou (1997) Alternative, simple non-convergence proof. This paper (Section 5.2)	
Geanakoplos' (1978)		Anderson (1998)
Edgeworth bargainig set. This paper (Section 5.1)	Non-convergence example. This paper (Section 5.3)	A generic convergence result. This paper: Theorem 5.1
Edgeworth bargainig set with leader. This paper (Section 5.4)		This paper: Theorem 5.2

Table 2: Convergence properties for bargaining sets of economies.

Finally, we try to make the best use of our results by recasting in terms of the bargaining set some characterizations of the Walrasian allocations already present throughout the literature. First, we focus on a result by Hervés-Beloso, Moreno-García and Yannelis (2005) that characterizes Walrasian allocations as those that are not blocked by the coalition formed by all the agents in a collection of perturbed economies. Then, we revisit the approach followed by Hervés-Beloso and Moreno-García (2009), who showed that Walrasian equilibria can be identified by using a non-cooperative two-player game. Both equivalence theorems constitute now additional characterizations of the bargaining set for finite economies.

The rest of the work is structured as follows. In Section 2 we collect notations and preliminaries. In Section 3, a Walras-bargaining equivalence and a characterization of justified objections via Walrasian objections are provided. Section 4 elaborates on the possibility of restricting the coalitions that are allowed to form and still get the bargaining set. In Section 5, we introduce the notion of Edgeworth bargaining set and analyze convergence properties. In Section 6, specific equivalence theorems for Walrasian equilibrium are presented as further characterizations of the bargaining sets. In order to facilitate the reading of the paper, the proofs of the results are contained in a final Appendix.

2 Preliminaries

Let $\mathcal{E}$ be an exchange economy with a finite set of agents $N = \{1, \ldots, n\}$, who trade a finite number ℓ of commodities. Each consumer i has a preference relation $\succsim_i$ on the set of consumption bundles $\mathbb{R}^\ell_+$, with the properties of continuity, convexity[2] and strict monotonicity. This implies that preferences are represented by utility functions $U_i, i \in N$. Let $\omega_i \in \mathbb{R}^\ell_{++}$ denote the endowments of consumer i. So the economy is $\mathcal{E} = (\mathbb{R}^\ell_+, \succsim_i, \omega_i, i \in N)$.

An allocation x is a consumption bundle $x_i \in \mathbb{R}^\ell_+$ for each agent $i \in N$. The allocation x is feasible in the economy $\mathcal{E}$ if $\sum_{i=1}^n x_i \leq \sum_{i=1}^n \omega_i$. A price system is an element of the $(\ell - 1)$-dimensional simplex of $\mathbb{R}^\ell_+$. A Walrasian equilibrium for the economy $\mathcal{E}$ is a pair (p, x), where p is a price system and x is a feasible allocation such that, for every agent i, the bundle x_i maximizes the utility function U_i in the budget set $B_i(p) = \{y \in \mathbb{R}^\ell_+ \text{ such that } p \cdot y \leq p \cdot \omega_i\}$. We denote by $W(\mathcal{E})$ the set of Walrasian allocations for the economy $\mathcal{E}$.

A coalition is a non-empty set of consumers. An allocation y is said to be attainable or feasible for the coalition S if $\sum_{i \in S} y_i \leq \sum_{i \in S} \omega_i$. Let $x \in \mathbb{R}^{\ell n}_+$ be a feasible allocation in the economy $\mathcal{E}$. The coalition S blocks x if there exists

[2]The convexity of preferences we require is the following: If a consumption bundle z is strictly preferred to $\hat{z}$ so is the convex combination $\lambda z + (1 - \lambda)\hat{z}$ for any $\lambda \in (0, 1)$. This convexity property is weaker than strict convexity and it holds, for instance, when the utility functions are concave.

an allocation y which is attainable for S, such that $y_i \succsim_i x_i$ for every $i \in S$ and $y_j \succ_j x_j$ for some member j in S. A feasible allocation is efficient if it is not blocked by the grand coalition, formed by all the agents. The core of the economy $\mathcal{E}$, denoted by $C(\mathcal{E})$, is the set of feasible allocations which are not blocked by any coalition of agents.

It is known that, under the hypotheses above, the economy $\mathcal{E}$ has Walrasian equilibrium and that any Walrasian allocation belongs to the core (in particular, it is efficient). Moreover, the blocking power of coalitions in finite economies is not able to eliminate every non-Walrasian allocation. Therefore, in order to characterize the Walrasian equilibria in terms of the core, we have to enlarge the set of coalitions or, alternatively, increase somehow their veto power. This line of arguments has been carried out in different ways. For instance, Aubin (1979) extended the notion of ordinary veto by allowing members to participate with a portion of their endowments when joining a coalition. We refer to this veto system as *Aubin veto* or *veto in the sense of Aubin*. An allocation x is blocked in the sense of Aubin by the coalition S via the allocation y if there exist coefficients $\alpha_i \in (0,1]$, for each $i \in S$, such that (i) $\sum_{i\in S} \alpha_i y_i \leq \sum_{i \in S} \alpha_i \omega_i$, and (ii) $y_i \succsim_i x_i$, for every $i \in S$ and $y_j \succ_j x_j$ for some $j \in S$. The Aubin core of the economy $\mathcal{E}$, denoted by $C_A(\mathcal{E})$, is the set of all feasible allocations which cannot be blocked in the sense of Aubin. Under the standard assumptions stated above, Aubin (1979) showed that $C_A(\mathcal{E}) = W(\mathcal{E})$.

As with the core, the Aubin core does not assess the "credibility" of the objections; any attainable allocation which is blocked by a coalition is dismissed. The argument that objections might be met with counter-objections leads to bargaining set notions. Since the original bargaining set notion was introduced by Aumann and Maschler (1964) for cooperative games, several versions have been defined and studied. More specifically, Mas-Colell (1989) defined the bargaining set for atomless economies.[3] The idea of the definition is that this set contains all the feasible allocations of the economy that are not blocked in a credible, justified way. Recently, the original bargaining set was extended by Yang, Liu and Liu (2011) to Aubin bargaining sets for games which they refer to as convex cooperative fuzzy games. Shortly after, Liu and Liu (2012) gave a modification of the previous extension and obtained both existence and equivalence results

[3]Mas-Colell (1989) not only adapted the original concept of bargaining set to atomless economies but also proved, under conditions of generality similar to the Aumann's (1964) core equivalence theorem, that the bargaining set and the set of competitive allocations coincide.

with other cooperative solutions. However, they remarked that finding a proper definition of the Aubin bargaining set is not an easy task.

In the next section, we provide a concept of bargaining set by means of the Aubin veto instead of the usual blocking mechanism. Thus, we extend and adapt the notions of bargaining sets recently provided by Yang, Liu and Liu (2011) and Liu and Liu (2012) for (transferable utility) cooperative games to finite exchange economies. In addition, we will use the fact that, regarding Walrasian equilibria, a finite economy $\mathcal{E}$ with n consumers is equivalent to a continuum economy $\mathcal{E}_c$ with n-types of agents as we specify below.

Consider a continuum economy where the set of agents is represented by the unit real interval $[0,1]$ endowed with the Lebesgue measure μ (as in Aumann, 1964). There are only a finite number of types of consumers. Thus, $I = [0,1] = \bigcup_{i=1}^{m} I_i$, with $\mu(I_i) = n_i/n$ (i.e., $\mu(I_i)$ is a rational number).[4] Every $t \in I_i$ has the same endowments ω_i and preference $\succsim_i$, that is, all the consumers in I_i are of the same type i. Note that we can write $I_i = \bigcup_{j=1}^{n_i} I_{ij}$ with $\mu(I_{ij}) = 1/n$ for every i, j. Consider now a finite economy with n agents and n_i consumers of each type i. Note that a feasible allocation $x = (x_1, \dots, x_n)$, with $x_i = (x_{ij}, j = 1, \dots, n_i)$, in the finite economy defines a feasible allocation f_x in the continuum economy which is given by $f_x(t) = x_{ij}$ for every $t \in I_{ij}$. Reciprocally, a feasible allocation f in the continuum economy defines a feasible allocation x^f in the finite economy which is given by $x_{ij}^f = \frac{1}{\mu(I_{ij})} \int_{I_{ij}} f(t) d\mu(t)$. Moreover, x (respectively f) is an equal-treatment allocation if and only if f_x (respectively x^f) also is.

Under continuity and convexity of preferences, if (x, p) is a Walrasian equilibrium in the n-agent economy, then (f_x, p) is a competitive allocation in the n-types continuum economy. Conversely, if (f, p) is a competitive equilibrium in the continuum economy then (x^f, p) is a Walrasian equilibrium in the finite economy. (See, for instance, García-Cutrín and Hervés-Beloso,1993).

Consider now the economy $\mathcal{E}$ that we have defined at the beginning of this section. Let $\mathcal{E}_c$ be the associated continuum economy, where the set of agents is $I = [0,1] = \bigcup_{i=1}^{n} I_i$, where $I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right)$ if $i \neq n$; $I_n = \left[\frac{n-1}{n}, 1\right]$; and all the agents in the subinterval I_i are of the same type i. In this particular case, $x = (x_1, \dots, x_n)$ is a Walrasian allocation in the finite economy $\mathcal{E}$ if and only if the step

[4] Without loss of generality one can take $I_i = [a_i, a_{i+1})$, for every $i \in \{1, \dots, m-1\}$; with $a_1 = 0$, $a_{i+1} - a_i = n_i/n$ and $I_m = [a_m, 1]$. Equivalently, we can also take $I = [0, n]$ and $I_i = [n_i, n_i + n_{i+1})$, for every $i \in \{1, \dots, m-1\}$; with $n_1 = 0$ and $I_m = [n_m, n]$.

function f_x (defined by $f_x(t) = x_i$ for every $t \in I_i$) is a competitive allocation in the continuum economy $\mathcal{E}_c$. In short, the initial finite economy $\mathcal{E}$ and the associated continuum economy $\mathcal{E}_c$ are equivalent regarding market equilibrium.

3 A Walras-bargaining equivalence for finite economies

In economies with a continuum of agents that trade a finite number of commodities, the competitive equilibrium is not only characterized by the core (Aumann, 1964), but also by the bargaining set (Mas-Colell, 1989). The Mas-Colell bargaining set is well defined for finite economies and, in this case, it can be larger than the core (see example in Section 6 in Mas-Colell, 1989).

To specify the notion of the Mas-Colell bargaining set for the finite economy $\mathcal{E}$, let x be a feasible allocation that is blocked by a coalition S via the allocation y. Thus, the objection (S, y) to x has a counter-objection if there exists a coalition T and an attainable allocation z for T such that $z_i \succ_i y_i$ for every $i \in T \cap S$ and $z_i \succ_i x_i$ for every $i \in T \setminus S$, where $T \setminus S$ is the set of agents which are in T but not in S.

An objection which cannot be counter-objected is said to be justified. Thus, the Mas-Colell bargaining set of an economy contains all the feasible allocations which, if they are objected (or blocked), could also be counter-objected. Let $B_{MC}(\mathcal{E})$ denote the Mas-Colell bargaining set for the economy $\mathcal{E}$ with n consumers.

3.1 A bargaining set notion for finite economies

In this section we provide a definition of bargaining set for finite economies using Aubin's veto mechanism that will allow us to prove that the set of Walrasian allocations and the bargaining set coincide.

An *Aubin objection* to x in the economy $\mathcal{E}$ is a pair (S, y), where S is a coalition that blocks x via y in the sense of Aubin. Note that the coalition S can be also defined by the parameters which specify the participation of its members.

An *Aubin counter-objection* to the objection (S, y) is a pair (T, z), where T is a coalition and z is an allocation defined on T, for which there exist $\lambda_i \in (0, 1]$

for each $i \in T$, such that:

(i) $\sum_{i \in T} \lambda_i z_i \leq \sum_{i \in T} \lambda_i \omega_i$,

(ii) $z_i \succ_i y_i$ for every $i \in T \cap S$ and

(iii) $z_i \succ_i x_i$ for every $i \in T \setminus S$.

Remark. Consider that the parameters defining the participations rates of each member in a blocking coalition S are rational numbers. Then, there are natural numbers $a_i, i \in S$ and $r \geq \max\{a_i, i \in S\}$, such that $\lambda_i = a_i/r$ for every $i \in S$. That is, we can say that the blocking coalition is formed by a_i agents of type i. Therefore, when the participation rates are rational numbers, the veto mechanism in the sense of Aubin is the standard veto system in sequence of replicated economies.

From now on in this section and in the related proofs, every time we are in a finite economy framework and write block, objection, counter-objection, or any other concept related with a veto system, we refer to those notions in the sense of Aubin unless stated otherwise.

Definition 3.1 *A feasible allocation belongs to the (Aubin) bargaining set of the finite economy if it has no justified objection. A justified objection is an objection that has no counter-objection.*

We denote by $B(\mathcal{E})$ the bargaining set of the economy $\mathcal{E}$ as we have defined above. Note that $W(\mathcal{E}) = C_A(\mathcal{E}) \subseteq B(\mathcal{E})$.

Our notion of bargaining set differs from the one by Mas-Colell. To clarify this point, let us highlight the main differences between the sets $B_{MC}(\mathcal{E})$ and $B(\mathcal{E})$. In our definition agents can join a coalition for objection or counter-objection process, with a part of their initial endowments. That is, regarding the bargaining system, agents can cooperate with different participation levels and the attainable bundles depend on these degrees of involvement. Furthermore, whenever an agent i is assigned the commodity bundle y_i within a coalition involved in an objection, if she also joins a coalition for a counter-objection, then she necessarily needs to be assigned a bundle that improves her upon y_i, independently of the rate of participation of agent i in the coalition.[5] This fact embodies one of the main conceptual differences between the Mas-Colell bargaining set and the bargaining set using the veto mechanism in the sense of Aubin.

To be precise, considering the notion of the Mas-Colell bargaining set, if a coalition with a justified objection includes only part of some type of agents then it is not possible for these agents to strictly improve at the objection.[6] This is not the case with our notion of justified objections. In particular, if we have a justified objection (S, y) to the allocation x in a finite economy $\mathcal{E}$, with rates of participation $\lambda_i, i \in S$, then the pair $(\tilde{S}, \tilde{y})$ given by any coalition $\tilde{S}$ in the associated continuum economy $\mathcal{E}_c$, such that the set of members in $\tilde{S}$ of type i (denoted by $\tilde{S}_i$) has measure λ_i, and $\tilde{y}(t) = y_i$ for every $t \in \tilde{S}_i$, is an objection to the step allocation f_x in $\mathcal{E}_c$, although it is not necessarily a justified objection. Basically, this contrast is due to the somehow leadership condition that a type obtains whenever any agent of such a type takes part in an objection, independently of the degree of participation.

3.2 A Walras-bargaining equivalence result

The bargaining set we consider constitutes indeed an adequate way of "enlarging" the economy, allowing us to characterize Walrasian allocations in finite economies as allocations with no justified objections. To this end, we show a preliminary result that we will use in the proof of our Walras-bargaining equivalence for economies with a finite number of consumers.

Lemma 3.1 *Let x be an allocation in $\mathcal{E}$. If (S, g) is a justified objection (in the sense of Mas-Colell) to f_x in the associated n-types continuum economy $\mathcal{E}_c$, then $(\bar{S}, \bar{g})$ is a justified objection to x in the finite $\mathcal{E}$, where $\bar{S} = \{i \in N \mid \mu(S \bigcap I_i) > 0\}$ and $\bar{g}_i = \frac{1}{\mu(S_i)} \int_{S_i} g(t) d\mu(t)$, for every $i \in \bar{S}$.*

Note that, in particular, we can conclude that if (S, g) is a justified objection (in the sense of Mas-Colell) to f_x in $\mathcal{E}_c$, then so is $(S, \hat{g})$, where $\hat{g}(t) = \bar{g}_i$ for

[5]This remark provides a different way to overcome the weakness (pointed out by Liu and Liu, 2012) of the related fuzzy bargaining set introduced by Yang, Liu and Liu (2011) for (transferable utility) cooperative games.

[6]For more details, see Remark 5 in Mas-Colell (1989). See also the related Lemma 3.4 in Anderson, Trockel and Zhou (1997).

every $t \in S_i = S \cap I_i$ and every $i \in \bar{S}$.[7] We remark that, in the proof of this Lemma, we just use the corresponding notions of justified objections in $\mathcal{E}$ and $\mathcal{E}_c$, respectively, and we do not use the characterization of justified objections that Mas-Colell (1989) showed and which can be applied to the associated n-types continuum economy.

Theorem 3.1 *The bargaining set of the finite economy $\mathcal{E}$ coincides with the set of Walrasian allocations.*

Enlarging the set of coalitions has a double effect. On the one hand, objecting is easier and allows for more justified objections which, in turn, would make the bargaining set smaller. On the other hand, counter-objecting is also easier, which would eliminate more objections, making it more difficult for the equivalence to hold. There is still another effect that comes from the aforementioned fact that if a type participates in both an objection and counter-objection, then an improvement is required in the counter-objection with respect the objection for such a type. The aggregate effect is therefore not clear, which makes our equivalence result not trivial.

Let us remember that Dutta *et al.* (1989) introduced the concept of consistency regarding the bargaining set, going one step further and trying to assess not only the credibility of the objections but also of the counter-objections involved in the process. They establish a notion of consistent bargaining set meaning that each objection in a "chain" of objections is tested (credible) in precisely the same way as its predecessor. However, the authors recognize that in a context of an exchange economy with a continuum of agents, the equivalence result by Mas-Colell (1989) implies that his bargaining set is consistent. Since we provide an equivalence result , there is also consistency in our bargaining set.

3.3 Justified objections as Walrasian objections

We remark that Theorem 3.1 states that any non Walrasian allocation has a justified objection. We finish this section by characterizing justified objections

[7] We stress that when preferences are not strictly convex we cannot ensure that every justified objection in the n-types continuum economy has the equal-treatment property. However, the Lemma 3.1 ensures that given a justified objection in $\mathcal{E}_c$, there is also an equal-treatment justified objection.

as Walrasian objections. This characterization is a discrete approach to the one stated by Mas-Colell (1989) for continuum economies. The concept of Walrasian objection requires the introduction of a price system p, and is based on a self selection property: members that participate in a coalition in a Walrasian objection against an allocation are those who would rather trade at the price vector p than get the consumption bundle they receive by such an allocation. The following notion of Walrasian objection differs from the one by Mas-Colell (1989) and reflects the differences between $B_{MC}(\mathcal{E})$ and $B(\mathcal{E})$.

Definition 3.2 *Let x be an allocation in the finite economy $\mathcal{E}$. An (Aubin) objection (S, y) to x is said to be Walrasian if there exists a price system p such that (i) $p \cdot v \geq p \cdot \omega_i$ if $v \succsim_i y_i$, $i \in S$ and (ii) $p \cdot v \geq p \cdot \omega_i$ if $v \succsim_i x_i$, $i \notin S$.*

We remark that, under the assumptions of monotonicity and strict positivity of the endowments, we know that $p \gg 0$, and therefore conditions *(i)* and *(ii)* above can be written as follows: $v \succ_i y_i$ implies $p \cdot v > p \cdot \omega_i$, for $i \in S$ and $v \succ_i x_i$ implies $p \cdot v > p \cdot \omega_i$ for $i \notin S$.

Observe that the notion of Walrasian objection in the finite economy $\mathcal{E}$ does not depend explicitly on the rates of participation of the members in the coalition that objects an allocation. To be precise, in order to check whether the objection (S, y) is Walrasian, no importance is attached to the degree of participation of the individuals joining the coalition S that make the allocation y attainable *à la Aubin*; what does become important is the set of consumers who are involved in the objection.

Proposition 3.1 *Let x be a feasible allocation in the finite economy $\mathcal{E}$. Then, any objection to the allocation x is justified if and only if it is a Walrasian objection.*

The fact that any Walrasian objection is a justified objection in finite economies allows us to refine our Walras-bargaining equivalence and its proof in terms of Walrasian objections. To see this, let x be a feasible allocation in $\mathcal{E}$. Note that we can now guarantee that if x is not a Walrasian allocation, then it has a Walrasian objection. Moreover, applying Proposition 3.1, Lemma 3.1 states that if (S, g) is a Walrasian objection (in the sense of Mas-Colell) to f_x in the associated n-types continuum economy $\mathcal{E}_c$, then $(\bar{S}, \bar{g})$ is a Walrasian objection to x in the finite $\mathcal{E}$, where $\bar{S} = \{i \in \{1, \ldots, n\} \mid \mu(S_i) = \mu(S \bigcap I_i) > 0\}$ and $\bar{g}_i = \frac{1}{\mu(S_i)} \int_{S_i} g(t) d\mu(t)$, for every $i \in \bar{S}$.

Let x be a feasible allocation in $\mathcal{E}$ and (S, y) an objection to x, being α_i the participation of each $i \in S$. Denote by $\mathcal{E}_S(\alpha)$ the continuum economy formed only by consumers of types in S and such that the measure of the set of agents of type i is α_i. From Proposition 3.1, we can deduce that when $S = N$, the objection (S, y) is justified if and only if y is a competitive allocation in the restricted continuum economy $\mathcal{E}_N(\alpha)$. However, note that in general an objection given by a coalition S and a competitive allocation of $\mathcal{E}_S(\alpha)$ is not necessarily a justified (or Walrasian) objection. Being a Walrasian objection is much more demanding. We also remark that the fact that (S, y) is a justified objection to x and $y_i \succ_i x_i$ does not imply $\alpha_i = 1$. This is in contrast to Mas-Colell's notion for which if a coalition with a justified objection includes only part of some type of agents then it is not possible for these agents to strictly improve with the objection.

In short, we stress that, since justified and Walrasian objections coincide, one can conclude that such a characterization points out that the concept of Walrasian objection in the finite framework is also more than a technical tool to refine the Walras-bargaining equivalence.

4 Restricting coalition formation

Both Mas Colell's (1989) result and our Walras-bargaining equivalence implicitly require the formation of all coalitions in the objection and counter-objecting mechanism. That is, checking whether a given allocation belongs to the bargaining set seems to require contemplating the whole set of possible coalitions in order to test whether any group of agents, by using their own resources, can improve upon an allocation either in the objection or counter-objection process. This will be a complicated task, even when the economy is small, provided that agents can participate in a coalition with a part of their endowments. Indeed, the Aubin veto system in a finite economy is equivalent to the blocking scheme in the associated continuum economy, with a finite number of types, conducted by equal-treatment allocations.

We also remark that the formation of coalitions may imply some theoretical difficulties. It is usually argued that the costs, which arise from forming a coalition, are not at all negligible. Incompatibilities among different agents may appear and a large amount of information and communication might be needed to really form a coalition. Thus, sometimes, it will not suffice to merely say that several agents constitute a coalition since it may result in high formation costs, commitments and constraints, which make it difficult to assume that the veto mechanism underlying cooperative solutions, like the core or the bargaining set, works freely and spontaneously.

In this section, the difficulty in arguing that coalition formation is costless leads us to consider a restricted veto mechanism in the procedure leading to the bargaining set. Thus, we assume that not all the parameters, which specify the degree of participation of agents when they become members of a coalition, are admissible. Next we will study the consequences that this assumption has with regard to the bargaining set solution.

To this end, we consider that a coalition S is defined by the rates of participation of its members, which is given by a vector $\lambda_S = (\lambda_i, i \in S) \in (0,1]^{|S|}$, where $|S|$ denotes the cardinality of S.

Consider that for each coalition S the participation rates are restricted to a subset $\Lambda_S \subset [0,1]^{|S|}$. Let us denote by $B_\Lambda(\mathcal{E})$ (respectively $B^\Lambda(\mathcal{E})$) the bargaining set where a coalition S can object (respectively counter-object) only with participation rates in Λ_S. When the set of coalitions is restricted in the objection (respectively counter-objection) process, it becomes harder to block an allocation (respectively to counter-object an objection) and thus we have $B^\Lambda(\mathcal{E}) \subseteq B(\mathcal{E}) \subseteq B_\Lambda(\mathcal{E})$. In addition, if $\Lambda, \widehat{\Lambda}$ are such that $\Lambda_S \subseteq \widehat{\Lambda}_S$ for every coalition S, then $B^\Lambda(\mathcal{E}) \subseteq B^{\widehat{\Lambda}}(\mathcal{E})$ but $B_{\widehat{\Lambda}}(\mathcal{E}) \subseteq B_\Lambda(\mathcal{E})$. Therefore, restricting the set of coalitions which are able to object enlarges the bargaining set, whereas restricting the coalition formation in the counter-objection mechanism diminishes the bargaining set. This is so because when not all the coalitions can take part in the bargaining mechanism, on the one hand, blocking is harder but on the other hand, it is easier for an admissible objection to become credible or justified.

In the case of continuum economies, following Schmeidler (1972), we can interpret the measure of a coalition as the amount of (or cost of) information and communication needed in order to form such a coalition. Consequently, it may be meaningful to consider those coalitions whose size converges to zero; that is, the coalitions with small formation cost. We apply this argument to economies with a finite number of agents where the veto system in the sense of Aubin is considered. To this effect, given $\delta \in (0,1]$, let δ-$B(\mathcal{E})$ denote the bargaining set of the economy $\mathcal{E}$ where the participation rate of any agent in any coalition, both in the objection and counter-objection procedure, is restricted to be less or equal than δ.

The next result is related to the remark on the core of atomless economies stated by Schmeilder (1972), showing that in order to obtain the core of a continuum economy, it is enough to consider the blocking power of arbitrarily small coalitions.

Lemma 4.1 *All the δ-bargaining sets are equal and coincide with the bargaining set in the finite economy $\mathcal{E}$. That is, δ-$B(\mathcal{E}) = B(\mathcal{E})$, for every $\delta \in (0,1]$.*

The above result is in contrast to the work by Schjødt and Sloth (1994) who showed that, in continuum economies, when one restricts the coalitions participating in objections and counter-objections to those whose size is arbitrarily small, then the Mas-Colell bargaining set becomes strictly larger than the original one. In other words, in atomless economies and contrary to the core, the formation of only arbitrarily small coalitions in the bargaining process does not allow the characterization of the competitive allocations. This is due to the fact that limiting the size of coalitions in continuum economies prevents obtaining justified objections. This is not the case in economies with a finite number of agents when one restricts the participation rates of members forming a coalition to those arbitrarily small. Thus, the previous lemma marks a further contrast between Mas-Collel's bargaining set for continuum economies and our finite approach.

Symmetrically to Schmeidler's (1972) and Grodal's (1972)[8] core characterizations for atomless economies, Vind (1972) showed that in order to block any non-competitive allocation it is enough to consider the veto power of arbitrarily large coalitions. This result allows us to show that in order to obtain the Aubin core the formation of only one coalition is sufficient, namely, the big coalition, which is formed by all the agents in the economy; moreover, for every consumer the endowment participation rate can be chosen to be arbitrarily close to one, i.e., the parameters defining the degree of joining in the big coalition can be

[8]Grodal extended Schmeidler's result by showing that, given $\delta \in (0,1)$, the blocking coalitions can be restricted to those with measure less than δ that are also union of at most $\ell + 1$ subcoalitions with diameter less than δ.

restricted to those close to the total participation (see Hervés-Beloso and Moreno-García, 2001 and Hervés-Beloso, Moreno-García and Yannelis, 2005). The next example shows that this restriction on coalition formation cannot be adapted to the bargaining set solution we address.

Example 1. Let $\mathcal{E}$ be an economy with two consumers who trade two commodities, a and b. Both agents have the same preference relation represented by the utility function $U(a,b) = ab$, and both are initially endowed with one unit of each commodity. Let us consider the feasible allocation x which assigns the bundle $x_1 = (2,2)$ to the individual 1 and the bundle $x_2 = (0,0)$ to individual 2. The allocation x does not belong to the bargaining set (it does not belong to the core and it is not a Walrasian allocation). In fact, x is blocked in the sense of Aubin by $S = \{2\}$ with any participation rate $\lambda \in (0,1]$. Moreover, every objection $(\{2\},(1,1))$, with any $\lambda \in (0,1]$, has no counter-objection *à la Aubin* and, therefore, is justified.

Note that there exists y such that the coalition $\{1,2\}$ objects x in the sense of Aubin via $y = (y_1, y_2)$, with strictly positive weights. That is, there exists $(\lambda_1, \lambda_2) \in (0,1]^2$ such that $\lambda_1 y_1 + \lambda_2 y_2 \leq (\lambda_1 + \lambda_2)(1,1)$. In addition, $U(y_1) \geq 4$ and $U(y_2) \geq 0$, with at least one strict inequality. This implies that $U(y_2) < U(\omega_2) = 1$. Therefore, any objection where the participation parameters are restricted to be strictly positive for every consumer is counter-objected by individual 2.

We conclude that in contrast to the Aubin core, we cannot restrict the coalition formation to the grand coalition with parameters close enough to the total participation. Next we state a similar example showing that we cannot state such a restriction in the counter-objecting mechanism either.

Example 2. Let $\mathcal{E}$ be an economy with three consumers who trade two commodities, a and b. All the agents have the same preference relation represented by the utility function $U(a,b) = ab$, and are initially endowed with one unit of each commodity. Let us consider the feasible allocation x which assigns the bundle $x_1 = (3,3)$ to individual 1 and the bundle $x_2 = x_3 = (0,0)$ to individuals 2 and 3. The allocation x is blocked in the sense of Aubin by $S = \{2\}$ with any participation rate $\lambda \in (0,1]$. Note also that $(\{3\},(1,1))$ is a counter-objection to the objection $(\{2\},(1,1))$. However, there is no counter-objection to $(\{2\},(1,1))$ if all the participation rates are required to be, for instance, larger than $1/2$.[9] To

[9]The same remains true if the parameters are required to be larger than any number in

see this, assume that $\{1,2,3\}$ counter-objects, with weights $\lambda_i, i = 1,2,3$. Given the preference relations, we can conclude that $3\lambda_1 + \lambda_2 < \lambda_1 + \lambda_2 + \lambda_3$. We obtain a contradiction with the fact that $\lambda_1, \lambda_3 \in (1/2, 1]$.

To finish this section, we consider a quite different restriction for the participation rates of the agents in coalitions. As the following lemma states, it turns out that the bargaining set is entirely characterized when the participation rates of agents in coalitions involved in counter-objections are rational numbers.

Lemma 4.2 *Let $B^Q(\mathcal{E})$ denote the bargaining set of the economy $\mathcal{E}$ where only rational numbers are allowed as participation rates in the counter-objection process. Then, $B^Q(\mathcal{E}) = B(\mathcal{E})$.*

The restriction in the previous lemma is equivalent to the veto mechanism in the sequence of replicated economies with equal treatment allocations. Then, we conclude that an Aubin objection (S, y) to x is justified if and only if the allocation (feasible or not) which assigns y_i to agents of type $i \in S$ and x_i to agents of type $i \in N \setminus S$ is not objected in any replicated economy.

We remark that, taking into account the observations on restricting coalition formation in the previous section, Lemma 4.2 can be obtained as an immediate consequence of our bargaining-Walras equivalence. However, in the Appendix we provide a proof which does not use the equality $W(\mathcal{E}) = B(\mathcal{E})$.

5 Convergence

Since models with a continuum of agents are thought of as idealizations of large economies, one might expect the Mas-Colell bargaining set to become approximately competitive in sequences of economies as the number of agents increases. However, ATZ showed that the bargaining set does not shrink to the set of Walrasian allocations by replicating the economy. They state a replica sequence of economies where the Mas-Colell bargaining set does not converge no matter how nice the preferences may be.[10] Thus, the work by ATZ gives insights into

$(1/2, 1)$.

[10]They provide a non-convergence result for Zhou's (1994) bargaining set, which requires additional restrictions on counter-objections. These restrictions make justified objections easier to form and thus make this bargaining set smaller than Mas-Colell's.

the discrepancy between the behavior of the Mas-Colell bargaining set in the continuum and its behavior in sequences of large finite economies.

In this section we provide a different notion of bargaining set that makes significant use of the replica structure and allows us to obtain convergence results, reinforcing Edgeworth's conjecture and Debreu-Scarf's (1963) result in the light of bargaining set concepts.

5.1 Edgeworth bargaining set

We emphasize that the Aubin veto mechanism becomes the blocking system in replicated economies as long as the participation rates are fractions (rational numbers) and equal-treatment allocations are considered in the replicas. Thus, in what follows, we rewrite and analyze our bargaining set concept for replicated economies in the spirit of Edgeworth's conjecture.

Consider the finite economy $\mathcal{E} = (\mathbb{R}^{\ell}_{+}, \succsim_i, \omega_i, i \in N)$. For each positive integer r, the r-fold replica economy $r\mathcal{E}$ of $\mathcal{E}$ is a new economy with rn agents indexed by ij $j = 1, \ldots, r$, such that each consumer ij has a preference relation $\succsim_{ij}=\succsim_i$ and endowments $\omega_{ij} = \omega_i$. That is, $r\mathcal{E}$ is a pure exchange economy with r agents of type i for every $i \in N$. Given a feasible allocation x in $\mathcal{E}$ let rx denote the corresponding equal treatment allocation in $r\mathcal{E}$, which is given by $rx_{ij} = x_i$ for every $j \in \{1, \ldots, r\}$ and $i \in N$.

The allocation rx is objected in $r\mathcal{E}$ if there exists a collection S of types and r_i agents of each type $i \in S$ which are able to attain an equal treatment allocation which improves rx; to be precise, if there exist commodity bundles $y_i, i \in S$ such that $\sum_{i \in S} r_i y_i \leq \sum_{i \in S} r_i \omega_i$ and $y_i \succsim_i x_i$ for every $i \in S$, with strict preference for some $i_0 \in S$.

Let (S, y) be an objection to rx in the economy $r\mathcal{E}$. The pair (T, z) is a counter-objection to the objection (S, y) if there exist natural numbers $n_i, i \in T$, such that $\sum_{i \in T} n_i z_i \leq \sum_{i \in T} n_i \omega_i$ and $z_i \succ_i y_i$ for every $i \in T \cap S$ and $z_i \succ_i x_i$ for ever $i \in T \setminus S$.

An objection to rx in the economy $r\mathcal{E}$ is justified if it is not counter-objected in any replicated economy. We say that the feasible allocation $x = (x_1, \ldots, x_n)$ belongs to the bargaining set of $r\mathcal{E}$ and write $x \in B(r\mathcal{E})$ if the allocation rx has no justified objection in $r\mathcal{E}$.

We refer to this bargaining set of a replicated economy as the *Edgeworth bargaining set*. We remark that, according to the notion above, if rx has a justified objection in $r\mathcal{E}$, then the same objection is also justified in $\hat{r}\mathcal{E}$ for any $\hat{r} \geq r$. Thus, as it happens with the core, this Edgeworth bargaining set shrinks under replication, i.e., for any natural number r we have that $B((r+1)\mathcal{E}) \subseteq B(r\mathcal{E})$.

5.2 ATZ's counterexample revisited

Let us now analyze the same example considered by ATZ under this notion of Edgeworth bargaining sets. There are two consumers and two commodities denoted by a and b. The endowments are $\omega_1 = (3,1)$ and $\omega_2 = (1,3)$. Both consumers have the same utility function $U(a,b) = \sqrt{ab}$. Let $\mathcal{H}$ denote the set of individually rational, Pareto optimal and equal-treatment allocations in the sequence of replicated economies. Given $\alpha \in [0,4]$, let $h(\alpha)$ be the allocation that gives (α,α) to agents of type 1 and $(4-\alpha, 4-\alpha)$ to agents of type 2. Then, $\mathcal{H} = \left\{h(\alpha), \text{ with } \alpha \in \left[\sqrt{3}, 4-\sqrt{3}\right]\right\}$. ATZ showed that the measure of the set of allocations in $\mathcal{H}$ which are not in the Mas-Colell and Zhou bargaining sets tends to zero as the economy is replicated. Therefore, they provide a non-convergence example for the Mas-Colell bargaining set in sequences of replicated economies.

Consider r_1 agents of type 1 and r_2 of type 2. Let a be numeraire and let p denote de price of b. Let $\tau = r_1/r_2$. Some calculations show that the Walrasian equilibrium for this restricted replicated economy $\mathcal{E}(\tau)$ is given by the price $p(\tau) = \frac{3\tau+1}{\tau+3}$, and the allocation which assigns $x_1(\tau) = \left(\frac{3\tau+5}{\tau+3}, \frac{3\tau+5}{3\tau+1}\right)$ and $x_2(\tau) = \left(\frac{5\tau+3}{\tau+3}, \frac{5\tau+3}{3\tau+1}\right)$ to agents of type 1 and 2, respectively.

For each $\tau \in \mathbb{R}_+$, let $V_i(\tau) = \left(U(x_i(\tau))\right)^2$, for $i = 1,2$. The function V_1 is decreasing and convex whereas V_2 is increasing and concave. For each $\alpha \in (\sqrt{3}, 4-\sqrt{3})$, there exist τ_α and τ^α such that $V_1(\tau_\alpha) = \alpha^2$ and $V_2(\tau^\alpha) = (4-\alpha)^2$. Note that $\alpha = 2$ defines the Walrasian allocation and $V_1(1) = V_2(1) = 4$. However, for any $\alpha \neq 2$, we have $\tau^\alpha < \tau_\alpha$. To see this, note that since $h(\alpha)$ is not a Walrasian allocation, there is a Walrasian objection in the sense of Mas-Colell in the associated continuum economy. That is, there are $\beta_1, \beta_2 \in (0,1]$ and an allocation (y_1, y_2) such that $\beta_1 y_1 + \beta_2 y_2 \leq \beta_1\omega_1 + \beta_2\omega_2$, $U(y_1) \geq \alpha$ and $U(y_2) \geq (4-\alpha)$ with one strict inequality. Since $h(\alpha)$ is efficient, $\beta_1 \neq \beta_2$. Assume

$\beta_1 < \beta_2$ and let $\tau^* = \beta_1/\beta_2$. Then $\tau^* = \tau_\alpha$ and $V_2(\tau^*) > (4-\alpha)^2 = V_2(\tau^\alpha)$.[11] Since V_2 is an increasing function, $\tau^\alpha < \tau_\alpha = \tau^*$. Since V_1 is decreasing, the case $\beta_1 > \beta_2$ is analogous.

Let $\alpha \in (\sqrt{3}, 2) \cup (2, 4-\sqrt{3})$. Then, $V_1(\tau) > \alpha^2$ and $V_2(\tau) > (4-\alpha)^2$, for any $\tau \in (\tau^\alpha, \tau_\alpha)$. For each rational number $\tau \in (\tau^\alpha, \tau_\alpha)$, let $r_1(\tau), r_2(\tau)$ be natural numbers such that $\tau = r_1(\tau)/r_2(\tau)$. We can conclude that the coalition formed by $r_1(\tau)$ consumers of type 1 and $r_2(\tau)$ of type 2 with the allocation $x(\tau)$ is a Walrasian objection to $rh(\alpha)$ for any replicated economy $r\mathcal{E}$ with $r \geq \max\{r_1(\tau), r_2(\tau)\}$. Our Proposition 4.1 allows us to conclude that the objection we have obtained is justified. Therefore, the argument by ATZ does not lead to a non-convergence result for the notion of the Edgeworth bargaining set we have proposed.

An alternative proof for the non-convergence of the Mas-Colell bargaining set. The previous argument leads to a different way to prove that the Mas-Colell bargaining set does not converge when we replicate the economy. To show this, consider the allocation $\hat{x}$ given by $\hat{x}_1 = (4,4) - x_2(\sqrt{2})$ and $\hat{x}_2 = x_2(\sqrt{2})$. Note that $\hat{x}$ is not Walrasian. We find a unique positive number $\hat{\tau}$ such that $(U(\hat{x}_1))^2 = V_1(\hat{\tau})$.[12] Consider the two types associated economy where agents of type 1 are represented by the interval [0,1] and agents of type 2 by (1,2]. Since V_1 is decreasing and $\hat{x}$ is individually rational, the set of all potential justified objections (in the sense of Mas-Colell) is given by the interval $[\sqrt{2}, \hat{\tau}]$ (see figure below). Any coalition $S \subset [0,2]$ such that $\mu(S \cap [0,1]) = 1$ and $\mu(S \cap (1,2]) = 1/\sqrt{2}$ blocks $f_{\hat{x}}$ (the step function given by $\hat{x}$) via the allocation that assigns $x_1(\sqrt{2})$ to agents in $S \cap [0,1]$ and $x_2(\sqrt{2})$ to agents in $S \cap (1,2]$. Furthermore, these objections are the unique Walrasian objections (in the sense of Mas-Colell) to $f_{\hat{x}}$.[13] This implies that the only coalitions able to make a justified objection are those with measure $1 + 1/\sqrt{2}$. In other words, although every $\tau \in [\sqrt{2}, \hat{\tau}]$ defines an objection to $f_{\hat{x}}$, the unique which is (Mas-Colell) justified is given by $\tau = \sqrt{2}$. Thus we conclude that it is not possible to find a justified objection (in the sense of Mas-Colell) in any replicated economy, that is, $r\hat{x}$ belongs to the Mas-Colell bargaining set of $r\mathcal{E}$ for every r, which proves

[11] See Remark 5 in Mas-Colell (1989).

[12] Equivalently, $\frac{9\hat{\tau}^2+30\hat{\tau}+25}{3\hat{\tau}^2+10\hat{\tau}+3} = \frac{62\sqrt{2}-5}{10\sqrt{2}+9}$, and some calculations show that $\hat{\tau} \simeq 1.6634$.

[13] This is so because if a coalition with a Mas-Colell justified objection includes only part of some type of agents, then it is not possible for these agents to strictly improve with the objection.

the non-convergence.[14]

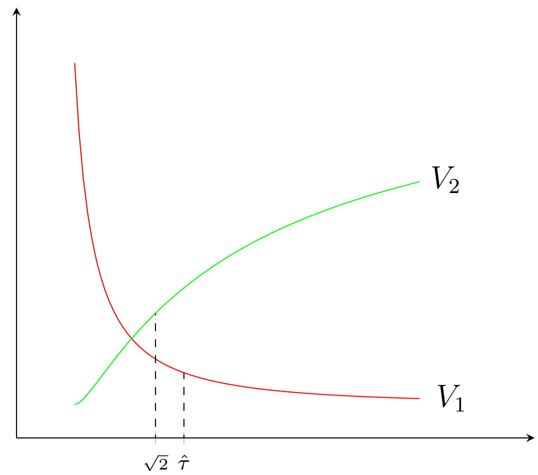


Fig 1: $(U(\hat{x}_1))^2 = V_1(\hat{\tau})$ and $(U(\hat{x}_2))^2 = V_2(\sqrt{2})$.

5.3 A counterexample and a generic convergence result

Next, we first state a counterexample showing that we cannot obtain an exact convergence result for the Edgeworth bargaining set. However, the example itself also indicates the nature of the problem with achieving such a convergence and the type of condition that will be needed to obtain it. As we will see, such a condition is a continuity property of the correspondence between economies and prices. This condition may be expected to hold in a wide range of situations, so the example is essentially the exception rather than the rule.

Counterexample. Let $\mathcal{E}$ be an exchange economy with two commodities and two agents, endowed with $\omega_1 = (\omega_1^x, \omega_1^y) = (2, 1)$ and $\omega_2 = (\omega_2^x, \omega_2^y) = (1, 2)$ respectively, who have the same utility function U, defined as follows:

$$U(x,y) = \begin{cases} \frac{1}{2^{1/4}}\sqrt{x} + \sqrt{y} & \text{if } \ x > \sqrt{2}\, y, \text{ and} \\ \\ \sqrt{x} + (2 - 2^{1/4})\sqrt{y} & \text{if } \ x \leq \sqrt{2}\, y. \end{cases}$$

Let x be the numeraire good and let p denote the price of y. The demand function for each agent i is

[14]Note that any $\tau \in [\sqrt{2}, \hat{\tau}]$ defines an (Aubin) justified objection to $\hat{x}$ via the Walrasian allocation $x(\tau)$ of the economy $\mathcal{E}(\tau)$. Then, any rational number $\tau \in \left[\sqrt{2}, \hat{\tau}\right]$ leads to a justified objection for some replicated economy. This implies that $\hat{x}$ does not belong to the Edgeworth bargaining set for any large enough replicated economy.

$$d_i(p) = \begin{cases} \left(\frac{p\ \left(\omega_i^x + p\omega_i^y\right)}{p+\sqrt{2}}, \ \frac{\sqrt{2}\ \left(\omega_i^x + p\omega_i^y\right)}{p^2+\sqrt{2}\ p} \right) & \text{if} \ \ p > \overline{p}, \\ \left(\frac{\sqrt{2}\ \left(\omega_i^x + p\omega_i^y\right)}{p+\sqrt{2}}, \ \frac{\omega_i^x + p\omega_i^y}{p+\sqrt{2}} \right) & \text{if } p \in \left[\underline{p}, \overline{p}\right], \text{ and} \\ \left(\frac{p\ \left(\omega_i^x + p\omega_i^y\right)}{p+\left(2-2^{1/4}\right)^2}, \ \frac{\left(2-2^{1/4}\right)^2\ \left(\omega_i^x + p\omega_i^y\right)}{p^2+p\left(2-2^{1/4}\right)^2} \right) & \text{if} \ \ p < \underline{p}, \end{cases}$$

where $\underline{p} = 2^{1/4}(2 - 2^{1/4})$ and $\overline{p} = \sqrt{2}$.

The Walrasian equilibrium price for this economy is $p^* = 2 - 2^{1/4}$, and the resulting Walrasian allocation assigns the bundle $d_1(p^*) = \left(\frac{4-2^{1/4}}{3-2^{1/4}}, \frac{4-2^{1/4}}{3-2^{1/4}}\right)$, $d_2(p^*) = \left(\frac{5-2^{5/4}}{3-2^{1/4}}, \frac{5-2^{5/4}}{3-2^{1/4}}\right)$ to agent 1 and 2, respectively.

Now consider there are r_1 agents of type 1 and r_2 of type 2 and let $\tau = r_1/r_2$. Some calculations show that the Walrasian equilibrium prices for this restricted replicated economy, $\mathcal{E}(\tau)$, are

$$p(\tau) = \begin{cases} 2^{1/4}\sqrt{\frac{2\tau+1}{\tau+2}} & \text{if} \ \ \tau > \tau^*, \\ \left[\underline{p}, \overline{p}\right] & \text{if} \ \ \tau = \tau^*, \text{ and} \\ (2-2^{1/4})\sqrt{\frac{2\tau+1}{\tau+2}} & \text{if} \ \ \tau < \tau^*, \end{cases}$$

where $\tau^* = 1 + \frac{3}{2}\sqrt{2}$.

Note that there is a continuum of Walrasian equilibria for the restricted economy $\mathcal{E}(\tau^*)$ and a unique Walrasian equilibrium for any other economy $\mathcal{E}(\tau)$ with $\tau \neq \tau^*$. For each $\tau \in \mathbb{R}_+$, the utility levels which can be attained for each type of consumers at a Walrasian allocation of the economy $\mathcal{E}(\tau)$ are given by the mappings $V_i(\tau) = U(d_i(p(\tau))),\ i = 1, 2$, whose graphical representations are shown in the following figure, where $\alpha_i = \min\{V_i(\tau^*)\}$ and $\beta_i = \max\{V_i(\tau^*)\}$:

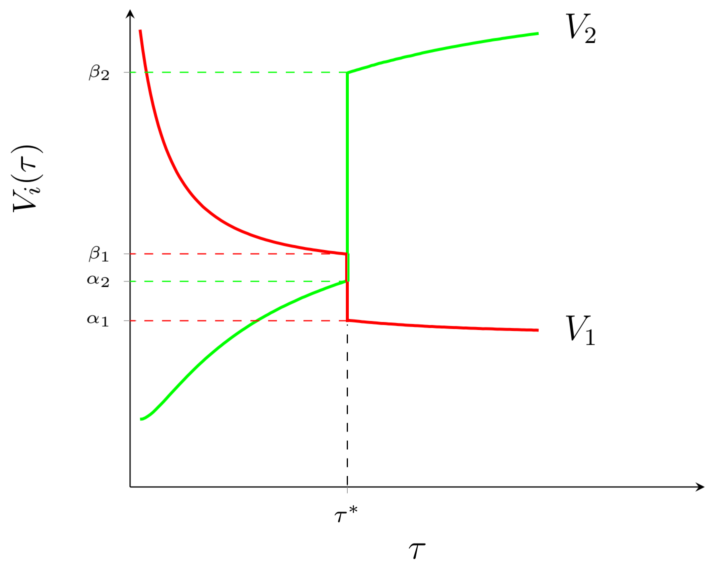
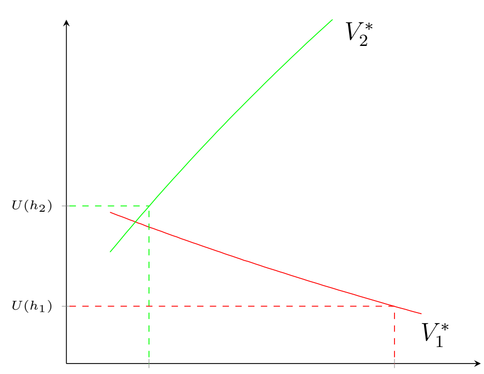


Fig. 2: V_1 and V_2 are not lower semicontinuous at τ^*.

Now consider a feasible allocation $h = (h_1, h_2)$ such that $U(h_i) \in (\alpha_i, \beta_i)$.[15] This allocation is individually rational and therefore, in order to block h in a replicated economy, both types need to be present. In addition, there is no justified objection for h whenever $\tau > \tau^*$ or $\tau < \tau^*$. It is possible, though, to find justified objections given by a set of Walrasian allocations in the economy $\mathcal{E}(\tau^*)$, which has a continuum of Walrasian equilibria. Let p_i be the Walrasian equilibrium price for $\mathcal{E}(\tau^*)$ such that $U(d_i(p_i)) = U(h_i)$. As illustrated in the figure below, any price in $[p_2, p_1] \subset [\underline{p}, \overline{p}]$ leads to a justified objection. However, since τ^* is an irrational number, such set of justified objections cannot be attained in any replicated economy, which proves the non-convergence.



[15] For instance, we can take $h_1 = \left(\frac{11^2}{5^2\left(3-2^{1/4}\right)^2}, \frac{11^2}{5^2\left(3-2^{1/4}\right)^2} \right)$ and $h_2 = (3,3) - h_1$.

Fig. 3: We get this grafic by "zooming in" on the figure 2 when $\tau = \tau^*$.

$V_i^*(p) = U(d_i(p))$, with $p \in p(\tau^*)$.

The example shows the impossibility of obtaining a convergence result if we allow for discontinuities of the equilibrium correspondence. Nevertheless, we are able to show a generic convergence result. Indeed, next we will show that under a continuity property of the equilibrium price correspondence, the Walrasian allocations of a finite economy are characterized as allocations that belong to the Edgeworth bargaining set of every replicated economy. Before presenting this generic equivalence result, let us state some previous lemmas.

Lemma 5.1 *Let x be a non-Walrasian feasible allocation in the economy $\mathcal{E}$. Then, for each i, there exists a sequence of rational numbers $r_i^k \in (0,1]$ converging to 1 and there is a sequence of allocations $(x^k, k \in \mathbb{N})$ which converges to x such that: (i) $\sum_{i=1}^n r_i^k x_i^k \leq \sum_{i=1}^n r_i^k \omega_i$, (ii) $x_i^k \succ_i x_i$ for every i, and (iii) $x_i^k \succ_i x_i^{k+1}$ for every k and every i.*

This lemma shows that if we have a non-Walrasian allocation x in the finite economy $\mathcal{E}$, then there is a sequence of (Aubin) objections converging to x where rational rates of participation are arbitrarily close to 1 for every consumer. In particular, we have a sequence of objections to x in the replicated economies in which every objection is given by a coalition involving all the types of agents and an equal-treatment allocation.

To state our next lemma, let us consider the rational parameters $r_i^k \in (0,1], i \in N$ obtained in Lemma 5.1 and state the following notation. Let $r^k = \sum_{i=1}^n r_i^k$ and $\hat{r}_i^k = r_i^k / r^k, i = 1, \ldots, n$. Let also $\bar{r}_i^k = \sum_{h=0}^{i-1} \hat{r}_h^k$, with $\hat{r}_0^k = 0$. Finally, let $\mathcal{E}_c^k$ be the continuum economy with n types of agents, where consumers in the subinterval I_i^k are of type i (i.e, have endowments ω_i and preferences $\succsim_i$), being $I_i^k = \left[\bar{r}_{i-1}^k, \bar{r}_i^k\right)$ for every $i = 1, \ldots, n-1$ and $I_n^k = \left[\bar{r}_{n-1}^k, 1\right]$. Note that the allocation x^k defines a feasible allocation f^k in the continuum economy $\mathcal{E}_c^k$ given by the step function $f^k(t) = x_i^k$ for every $t \in I_i^k$.

Lemma 5.2 *Assume that x is not a Walrasian allocation but belongs to the Edgeworth bargaining set of every replicated economy. Then, for every k, there* *is a justified objection in the sense of Mas-Colell*[16] *to the allocation f^k in the continuum economy $\mathcal{E}_c^k$.*

Let $\Delta = \left\{p \in \mathbb{R}_+^\ell | \sum_{h=1}^{\ell} p_h = 1\right\}$ and let d_i (from Δ into $\mathbb{R}_+^\ell$) denote the demand correspondence for consumer i, characterized by preferences $\succsim_i$ and endowments $\omega_i \in \mathbb{R}_{++}^\ell$, in the finite economy $\mathcal{E}$. The excess demand correspondence for consumer i is given by $Z_i(p) = d_i(p) - \omega_i$ for each $p \in \Delta$. Let Π be the mapping that associates each economy with its Walrasian equilibrium prices. Thus, $p \in \Pi(\mathcal{E})$ if and only if $0 \in \sum_{i=1}^{n} Z_i(p) = Z(p)$.

Note that when determining the market-clearing prices of an economy, it is sufficient to consider only the excess demand mappings. Let $\mathcal{Z}$ denote the set of excess demand correspondences from Δ to $\mathbb{R}^\ell$ endowed with a metric topology.[17] Consider the excess demands $Z_1, \ldots, Z_n$ of the n consumers in $\mathcal{E}$ and the associated n-types continuum economy $\mathcal{E}_c$. Then, to examine $\Pi(\mathcal{E})$ or equivalently $\Pi(\mathcal{E}_c)$, it suffices to describe $\mathcal{E}_c$ by the measure η on $\mathcal{Z}$ defined by $\eta(F) = \sum_{i \in T_F} \mu(I_i)$, where F is any Borel subset of $\mathcal{Z}$ and $T_F = \{i \in N | Z_i \in F\}$.

Given a general continuum economy, where the set of agents is represented by the interval $I = [0,1]$, the measure which describes it is given by $\upsilon(F) = \mu\left(\{t \in I | Z_t \in F\}\right)$ for each Borel set $F \subset \mathcal{Z}$, being Z_t the excess demand correspondence of the agent $t \in I$.

Now, for each k let us consider the justified objection (S^k, g^k) to f^k obtained in the proof of the Lemma 5.2. In order to define a sequence of auxiliary continuum economies restricted to the coalitions S^k where the set of consumers is the interval $[0,1]$ for every k, we state the following notation. Let $\gamma_i^k = \mu(S^k \cap I_i^k)$, $T^k = \left\{i \in N | \gamma_i^k > 0\right\}$, t^k denotes the cardinality of T^k and $m_k = \max\left\{i | i \in T^k\right\}$. Let $\hat{\gamma}_0^k = 0$ and $\hat{\gamma}_i^k = \gamma_i^k / \mu(S^k)$ for every $i \in \{1, \ldots n\}$. Note that $\hat{\gamma}_i^k = 0$ for every i which does not belong to T^k. For each $i \in T^k$, let $\hat{I}_i^k = \left[\bar{\gamma}_{i-1}^k, \bar{\gamma}_i^k\right)$ if $i \neq m_k$ and $\hat{I}_i^k = \left[\bar{\gamma}_{m_k-1}^k, 1\right]$ if $i = m_k$, where $\bar{\gamma}_i^k = \sum_{h=0}^{i-1} \hat{\gamma}_h^k$. Finally, let $\hat{\mathcal{E}}_c^k$ be the continuum economy with t^k types of agents, where for each $i \in T^k$, consumers in the subinterval $\hat{I}_i^k$ are of type i.

Lemma 5.3 *Let ν^k be the measure describing the auxiliary continuum economy*

[16]We emphasize that a justified objection in the sense of Mas-Colell defines an Aubin objection in the economy $\mathcal{E}$ which is justified.

[17]We do not specify a topology here. Later on we will restrict ourselves to some subsets of $\mathcal{Z}$ with a particular topology to obtain useful results.

$\hat{\mathcal{E}}_c^k$, defined by the justified objection to f^k, in which the measure of agents of type i is $\hat{\gamma}_i^k$. There exists a subsequence of measures which converges weakly to a measure ν describing the limit economy $\hat{\mathcal{E}}_c$.

Next, under a continuity assumption regarding the equilibrium prices mapping, we state a convergence result for the Edgeworth bargaining set we have defined.

Theorem 5.1 *Assume that the equilibrium price correspondence is continuous at the measure ν describing the economy $\hat{\mathcal{E}}_c$. Then, an allocation is Walrasian in the finite economy $\mathcal{E}$ if and only if it belongs to the Edgeworth bargaining set of every replicated economy. That is,*

$$W(\mathcal{E}) = \bigcap_{r \in \mathbb{N}} B(r\mathcal{E}).$$

The assumptions on endowments and preferences in our finite economy $\mathcal{E}$ allow us to ensure that the excess demands $Z_i, i \in N$, obey the desirability condition that if a sequence of prices p_n converges to p, a boundary point of Δ, then $\|Z_i(p_n)\|$ goes to ∞. Assume, in addition, that the excess demand mappings are limited to be continuously differentiable functions from the interior of Δ. If we metrize the space $\mathcal{Z}$ by requiring uniform convergence of the functions and their first derivatives on compact sets, the set of economies (described by measures, as above) on which the equilibrium price correspondence Π is continuous is open and dense in the topology of weak convergence. (See Dierker, 1973, or Hildenbrand, 1974). Moreover, if we drop the requirement that the functions be continuously differentiable, requiring only continuity, we still have that the set of economies on which Π is continuous is a dense subset. In fact, it is a residual set, that is, the countable intersection of open dense sets. Thus, in this framework, we can say that the convergence of the Edgeworth bargaining sets to the Walrasian allocations is generic.

5.4 An exact convergence result: Edgeworth bargaining set with leader

Neither the concept of bargaining set by Mas-Colell (1989) nor our Edgeworth bargaining set imposes any restriction on the members that may belong to an objecting or counter-objecting coalition. However, the definition of bargaining set for cooperative games introduced by Aumann and Maschler (1964) and Davis and Maschler (1963), requires that the original objection has to be proposed by an agent that acts as a "leader", meaning that this agent cannot belong to any counter-objecting coalition. In addition, Geanakoplos (1978) gave an alternative definition of leader, modifying the one by Aumann-Davis-Maschler in such a way that the "leader" could be not just one agent, but a group of agents. Thus, the Aumann-Davis-Maschler concept of leader would be a particular case of Geanakoplos'.

It is important to remark that the designation of a leader makes a profound difference in the resulting bargaining sets, especially when the economy is enlarged with the aim of studying convergence properties. Indeed, the bargaining sets convergence results that have already been obtained in the related literature depend crucially on the presence of a leader or a group of leaders (see Geanakoplos, 1978, Shapley and Shubik, 1984 and Anderson, 1998). In this section, we provide a notion of bargaining set which involves the concept of a leader that is understood as a type of agents. This solution allows us to show that when we replicate the economy, the bargaining set shrinks and converges to the set of Walrasian allocations, in a similar way as the Debreu-Scarf's convergence theorem for the core, without any additional continuity property of the equilibrium correspondence as it has been required for the previous generic convergence result. Thus, in what follows, we incorporate the presence of a leader to the Edgeworth bargaining set concept and then we obtain an exact convergence result.

Consider an objection (S, y) to the allocation rx in $r\mathcal{E}$. That is, there are $r_i \leq r$ agents of each type $i \in S$ such that $\sum_{i \in S} r_i y_i \leq \sum_{i \in S} r_i \omega_i$ and $y_i \succsim_i x_i$ for every $i \in S$, with strict preference for some $j \in S$. We remark that without loss of generality we assume $r_h = r$ for some $h \in S$.

The objection (S, y) to rx in the economy $r\mathcal{E}$ is L-counter-objected if for every $i \in S$, with $r_i = r$, there exists a counter-objection (T, z), with $i \notin T$, in some replicated economy $\hat{r}\mathcal{E}$ with $\hat{r} \geq r$. In other words, an objection (S, y) to rx in the economy $r\mathcal{E}$ is L-justified if there exists $i \in S$, with $r_i = r$, such that any counter-objection (T, z) in $\hat{r}\mathcal{E}$ with $\hat{r} \geq r$ requires that i belongs to T.

We say that the feasible allocation x belongs to the leader bargaining set of $r\mathcal{E}$ and we write $x \in B_L(r\mathcal{E})$ if the allocation rx has no L-justified objection. We must remember that, for every r, the set of Walrasian allocations of the economy $\mathcal{E}$ is contained in the core of $r\mathcal{E}$ which is contained in $B_L(r\mathcal{E})$.

We stress that in our definition, a leader consists in a group of individuals of the same type. Furthermore, every type that participates with all its agents in an objection can be designated as a leader. Consequently, in our notion a leader becomes a type. Moreover, according to our leader bargaining set, for any natural number r, there is $\hat{r} \geq r$ such that $B_L(\hat{r}\mathcal{E}) \subseteq B_L(r\mathcal{E})$. To see this, note that obviously we have $B_L(2r\mathcal{E}) \subseteq B_L(r\mathcal{E})$.

Theorem 5.2 *The allocation x is Walrasian in the economy $\mathcal{E}$ if and only if x belongs to the leader bargaining set of every replicated economy. That is,*

$$\bigcap_{r \in \mathbb{N}} B_L(r\mathcal{E}) = W(\mathcal{E}).$$

This convergence result depends crucially on the consideration of "leaders" (understood as types) when an objection is proposed in the sequence of replicated economies. The underlying argument is that when an objection is proposed by a leader, any counter-objecting coalition must exclude this leader. It is also the presence of a leader (either as an individual or as a group) in the objection process which allows for the convergence results that have already been obtained in the literature. Geanakoplos (1978) considered a modified notion[18] of the Davis-Machler definition and showed that his bargaining set becomes asymptotically competitive as the number of agents grows. Shapley and Shubik (1984) showed that the Aumann-Davis-Maschler bargaining set converges in replica sequences of TU exchange economies with smooth preferences. Anderson (1998) extended both Geanakoplos and Shapley and Shubik results to sequences of NTU exchange economies, weakening some assumptions such as smoothness of preferences.

[18]Geneakoplos (1978) modified the Davis-Maschler definition by considering that the "leader" was a group of agents containing a fixed (but small) fraction of the number of agents in the economy; thus, as the number of agents grows along the sequence of economies, the number of individuals in the "leader" grows proportionately. However, this modified notion does not require the individuals in the group to be of the same type as our notion does.

Roughly speaking, the aforementioned convergence results show that different notions of bargaining set involving the presence of a leader can approximately be decentralized by prices for large economies. Therefore, these works point out that the Geneakoplos bargaining set and the Aumann and Maschler bargaining set have better convergence properties than Mas-Colell's.

Our convergence theorem adds to this line of research, showing that it makes a fundamental difference for the asymptotic analysis of the Edgeworth bargaining sets whether one requires that there be a group of leaders or not. The notion of the bargaining set with leader we state differs from those which have been considered in the related literature and, in turn, neither our convergence result can be deduced from the previous ones nor vice-versa. Moreover, we show that the intersection of the bargaining sets of the sequence of the replicated economies coincides with the set of Walrasian allocations, providing an extension of the Debreu-Scarf core-convergence to bargaining sets which is not the case of the already obtained asymptotic theorems that show a convergence in measure (Anderson, 1998).

6 Final additional characterizations

Given our equivalence results, any characterization of Walrasian equilibrium for finite economies turns immediately into an additional characterization of the bargaining set. In this section we pick up two different ways of identifying Walrasian allocations and recast them in terms of bargaining sets as corollaries.

First, let us consider a feasible allocation $x = (x_1, \ldots, x_n)$ in the economy $\mathcal{E}$. Following Hervés-Beloso, Moreno-García and Yannelis (2005), we define a family of economies denoted by $\mathcal{E}(a, x)$, $a = (a_1, \ldots, a_n) \in [0, 1]^n$, which coincide with $\mathcal{E}$ except for the endowments that, for each agent $i \in N$, are defined by $\omega_i(a, x) = a_i x_i + (1 - a_i)\omega_i$. An allocation (feasible or not) is said to be dominated in the economy $\mathcal{E}$ if it is blocked by the grand coalition N.

In the aforementioned work it was proved that, under the assumptions we have considered, an allocation x is Walrasian in the economy $\mathcal{E}$ if and only if it is not dominated in any perturbed economy $\mathcal{E}(a, x)$. This characterization allows us to write the next corollary as an immediate consequence of the Walras-bargaining equivalence we have obtained in Theorem 3.1.

Corollary 6.1 *An allocation x belongs to the bargaining set of $\mathcal{E}$ (equivalently, to the leader bargaining set of every replicated economy $r\mathcal{E}$) if and only if it is not dominated in any economy $\mathcal{E}(a, x)$.*

An alternative way of stating the above result is: *The allocation x has a justified objection (equivalently, a Walrasian objection) in the economy $\mathcal{E}$ if and only if x is blocked by the grand coalition in some economy $\mathcal{E}(a, x)$.*

The essence of the second characterization of Walrasian equilibrium that we recast for bargaining sets differs substantially from the previous ones. It follows a non-cooperative game theoretical approach and provides insights into the mechanism through which the bargaining process is conducted.

Given the finite economy $\mathcal{E} = (\mathbb{R}^{\ell}_{+}, \succsim_i, \omega_i, i \in N)$, let us define an associated game $\mathcal{G}$ as follows. There are two players. The strategy sets for the players are given by:

$$S_1 = \{\ x = (x_1, ..., x_n) \in \mathbb{R}^{\ell n}_{+} \text{ such that } x_i \neq 0 \text{ and } \textstyle\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} \omega_i\}.$$

$$S_2 = \{(a, y) \in [\alpha, 1]^n \times \mathbb{R}^{\ell n}_{+} \text{ such that } \textstyle\sum_{i=1}^{n} a_i y_i \leq \sum_{i=1}^{n} a_i \omega_i\},$$

where α is a real number such that $0 < \alpha < 1$.

Given a strategy profile $s = (x, (a, y)) \in S_1 \times S_2$, the payoff functions Π_1 and Π_2, for player 1 and 2, respectively, are defined as $\Pi_1(x, (a, y)) = \min_i\{U_i(x_i) - U_i(y_i)\}$ and $\Pi_2(x, (a, y)) = \min_i\{a_i\ (U_i(y_i) - U_i(x_i))\}$.

Note that if $\Pi_2(x, (a, y)) > 0$, then the allocation x is blocked via y by the big coalition being a_i the participation rate of each consumer i. Actually, player 2 gets a positive payoff if and only if the big coalition objects in the sense of Aubin the allocation proposed by player 1.

As an immediate consequence of our bargaining-Walras equivalence and Theorem 4.1 in Hervés-Beloso and Moreno-García (2009), we obtain the following corollary.

Corollary 6.2 *x belongs to the bargaining set of the economy $\mathcal{E}$, if and only if $(x, (\mathbf{b}, x))$ with $\mathbf{b}_i = b$, for every $i = 1, \dots, n$, (for instance $(x, (\mathbf{1}, x))$) is a Nash equilibrium for the game $\mathcal{G}$.*

To finish, we remark that the spirit of the bargaining set notions we have considered seems to indicate that additional and finer characterizations for such cooperative concepts could be obtained through non-cooperative solutions of different games, in which a player represents the objection system whereas another one is in charge of the counter-objection mechanism. This is part of our further research.

Appendix

Proof of Lemma 3.1. Let us assume that f_x is objected by (S, g) meaning that: $\int_S g(t)d\mu(t) \leq \int_S \omega(t)d\mu(t)$, $g \succsim_t f_x$ for every $t \in S$ and $\mu(\{t \in S | g \succ_t f_x\}) > 0$. Let $S_i = S \cap I_i$ and $\bar{S} = \{i \in N | \mu(S_i) > 0\}$. Since S blocks f_x via g, we have that there exists a type $k \in N$ and a set $A \subset S_k = S \cap I_k$, with $\mu(A) > 0$, such that $g(t) \succ_k f_x$, for every $t \in A$.

Let $\bar{g}$ be the allocation given by $\bar{g}_i = \frac{1}{\mu(S_i)} \int_{S_i} g(t)d\mu(t)$, for every $i \in \bar{S}$. Then, by convexity of the preferences, we have $\bar{g}_i \succsim_i x_i = f_x(t)$ for every $t \in S_i = S \cap I_i$ and $i \in \bar{S}$; and $\bar{g}_k \succ_k x_k = f_x(t)$ for every $t \in S_k$.[19] Thus, $(\bar{S}, \bar{g})$ is an objection *à la Aubin* to the allocation x in the economy $\mathcal{E}$, since we have that: *(i)* $\sum_{i \in \bar{S}} \mu(S_i)\bar{g}_i \leq \sum_{i \in \bar{S}} \mu(S_i)\omega_i$, *(ii)* $\bar{g}_i \succsim_i x_i$ for every $i \in \bar{S}$ and *(iii)* there exists $k \in \bar{S}$ such that $\bar{g}_k \succ_k x_k$.

Assume that the objection $(\bar{S}, \bar{g})$ has a counter-objection $(\bar{T}, z)$, that is, there exists $\{\lambda_i\}_{i \in \bar{T}}$ with $\lambda_i \in (0, 1]$ for every $i \in \bar{T}$, such that: *(i)* $\sum_{i \in \bar{T}} \lambda_i z_i \leq \sum_{i \in \bar{T}} \lambda_i \omega_i$, *(ii)* $z_i \succ_i \bar{g}_i$ for every $i \in \bar{T} \cap \bar{S}$ and *(iii)* $z_i \succ_i x_i$ for every $i \in \bar{T} \setminus \bar{S}$.

If $\bar{T} \cap \bar{S} = \emptyset$ then, in the associated continuum economy $\mathcal{E}_c$, any coalition $T = \bigcup_{i \in \bar{T}} T_i \subset I$ with $\mu(T_i) = \lambda_i$, counter-objects the objection (S, g) via the allocation f_z given by $f_z(t) = z_i$ for every $t \in T_i$. Otherwise (i.e., $\bar{T} \cap \bar{S} \neq \emptyset$), from the previous condition *(ii)* we can deduce that for every $i \in \bar{T} \cap \bar{S}$, there exists $A_i \subset S_i$ with $\mu(A_i) > 0$, such that $z_i \succ_i g(t)$ for every $t \in A_i$. This is again a consequence of the convexity property of preferences. Let $a = \min\{\mu(A_i), i \in \bar{T} \cap \bar{S}\}$ and take M large enough such that $\alpha_i = \frac{\lambda_i}{M} \leq a$ for every $i \in \bar{T}$.

Consider a coalition $T \subset I$ in the continuum economy $\mathcal{E}_c$ with $T = \cup_{i \in \bar{T}} T_i$, such that $T_i \subset A_i$, if $i \in \bar{T} \cap \bar{S}$; $T_i \subset I_i$, if $i \in \bar{T} \setminus \bar{S}$ and $\mu(T_i) = \alpha_i$, for every $i \in \bar{T}$. Then, defining the step function h as $h(t) = z_i$ if $t \in T_i$, we have that: *(i)*

[19] See the Lemma in García-Cutrín and Hervés-Beloso (1993) for further details.

$\int_T h(t)d\mu(t) = \sum_{i\in\bar{T}} \alpha_i z_i \le \sum_{i\in\bar{T}} \alpha_i\omega_i = \int_T \omega(t)d\mu(t)$, *(ii)* $h(t) \succ_i g(t)$ for every $t \in T_i$ with $i \in \bar{T}\cap\bar{S}$; and *(iii)* $h(t) \succ_i x_i = f_x(t)$ for every $t \in T_i$ with $i \in \bar{T}\setminus\bar{S}$.

Note that *(ii)* and *(ii)* mean $h(t) \succ_t g(t)$ for every $t \in T\cap S$ and $h(t) \succ_t f_x(t)$ for every $t \in T\setminus S$, respectively. In other words, we have constructed a counter-objection (T, h) for the objection (S, g), which concludes the proof.

Q.E.D.

Proof of Theorem 3.1. Since the Aubin core coincides with the set of Walrasian allocations for the economy $\mathcal{E}$ (see Aubin, 1979), we have that any Walrasian allocation has no objection in the sense of Aubin and therefore belongs to the bargaining set of $\mathcal{E}$.

Let us show that $B(\mathcal{E}) \subseteq W(\mathcal{E})$. Consider an allocation $x \in B(\mathcal{E})$ and the step function[20] f_x which is a feasible allocation in the associated n-types continuum economy $\mathcal{E}_c$. It suffices to show that f_x belongs to the Mas-Colell bargaining set of $\mathcal{E}_c$.[21] Let us assume that f_x is blocked by the coalition S via the allocation g in $\mathcal{E}_c$ and that (S, g) is a justified objection to f_x in the sense of Mas-Colell. By Lemma 3.1 we can ensure that $(\bar{S}, \bar{g})$ is a justified objection to x in $\mathcal{E}$, where $\bar{g}_i = \frac{1}{\mu(S_i)}\int_{S_i} g(t)d\mu(t)$, for every $i \in \bar{S} = \{i \in N \mid \mu(S\cap I_i) > 0\}$. This is in contradiction to the fact that $x \in B(\mathcal{E})$ and concludes the proof.

Q.E.D.

Proof of Proposition 3.1. Let (S, y) be an objection *à la Aubin* to x. Assume (T, z) is a counter-objection in the sense of Aubin to (S, y). Then, there exist coefficients$\lambda_i \in (0, 1]$ for each $i \in T$, such that: $\sum_{i\in T}\lambda_i z_i \le \sum_{i\in T}\lambda_i\omega_i$; $z_i \succ_i y_i$ for every $i \in T\cap S$ and $z_i \succ_i x_i$ for every $i \in T\setminus S$. Since (S, y) is a Walrasian objection at prices p we have that $p\cdot z_i > p\cdot\omega_i$, for every $i \in T\cap S$ and $p\cdot z_i > p\cdot\omega_i$, for every $i \in T\setminus S$. This implies $p\cdot\sum_{i\in T}\lambda_i z_i > p\cdot\sum_{i\in T}\lambda_i\omega_i$, which contradicts that z is attainable by T with weights $\lambda_i, i \in T$. Thus, we conclude that (S, y) is a justified objection.

To show the converse, let (S, y) be a justified objection to x and let $a = (a_1, \ldots, a_n)$ be an allocation (not necessarily feasible) such that $a_i = y_i$ if $i \in S$ and $a_i = x_i$ if $i \notin S$. For every consumer i define $\Gamma_i = \{z \in \mathbb{R}^\ell | z+\omega_i \succsim_i a_i\}\bigcup\{0\}$

[20] For every $t \in [0, 1]$, $f_x(t) = x_i$ if $t \in I_i$

[21] This is so because the Mas-Colell bargaining set of $\mathcal{E}_c$ equals the set of competitive allocations (Mas-Colell, 1989), which is also equivalent to the core (Aumann, 1964), and f_x is competitive in $\mathcal{E}_c$ if and only if x is Walrasian in $\mathcal{E}$.

and let Γ be the convex hull of the union of the sets $\Gamma_i, i \in N$.

Let us show that $\Gamma \bigcap(-\mathbb{R}^\ell_{++})$ is empty. Assume that $\delta \in \Gamma \bigcap(-\mathbb{R}^\ell_{++})$. Then, there is $\lambda = (\lambda_i, i \in N) \in [0,1]^n$, with $\sum_{i=1}^n \lambda_i = 1$, such that $\delta = \sum_{i=1}^n \lambda_i z_i \in \Gamma$. This implies that the coalition $T = \{j \in N \mid \lambda_j > 0\}$ counter-objects (S, y) via the allocation $\hat{z}$ where $\hat{z}_i = z_i + \omega_i - \delta$ for each $i \in T$. Indeed, $\sum_{j \in T} \lambda_j \hat{z}_j = \sum_{j \in T} \lambda_j \omega_j$. Moreover, since $z_i \in \Gamma_i$ for every $i \in T$ and $\delta \ll 0$, by monotonicity of preferences, $\hat{z}_i \succ_i y_i$ for every $i \in T \cap S$ and $\hat{z}_i \succ_i x_i$ for every $i \in T \setminus S$. This is a contradiction.

Thus, $\Gamma \bigcap(-\mathbb{R}^\ell_{++}) = \emptyset$, which implies that 0 is a frontier point of Γ. Therefore, there exists a hyperplane that supports Γ at 0. That is, there exists a price system p such that $p \cdot z \geq 0$ for every $z \in \Gamma$. This means that $p \cdot v \geq p \cdot \omega_i$, if $v \succsim_i a_i$. Therefore, we conclude that (S, y) is a Walrasian objection.

Q.E.D.

Proof of Lemma 4.1. Let an allocation y be attainable for a coalition S with participation rates $\lambda_i, i \in S$. That is, $\sum_{i \in S} \lambda_i y_i \leq \sum_{i \in S} \lambda_i \omega_i$. It suffices to note that there exists $(\alpha_i, i \in S)$, with $\alpha_i \leq \delta$ for every $i \in S$ such that $\sum_{i \in S} \alpha_i y_i \leq \sum_{i \in S} \alpha_i \omega_i$. To see this, let M be large enough so that $\alpha_i = \lambda_i / M \leq \delta$, for every $i \in S$. Thus, the same allocation y is also attainable for the same coalition S with participation rates arbitrarily small. The same reasoning holds for the case of both objections and counter-objections.

Q.E.D.

Proof of Lemma 4.2. Let x be a feasible allocation and (S, y) an objection to x. Let (T, z) be a counter-objection to (S, y). This means that there exist coefficients $\alpha_i,\ i \in T$, such that (i) $\sum_{i \in T} \alpha_i z_i = \sum_{i \in T} \alpha_i \omega_i$ and (ii) $z_i \succ_i y_i$ for every $i \in T \cap S$, and $z_i \succ_i x_i$ for every $i \in T \setminus S$.

For every natural $k \in \mathbb{N}$, we define a_i^k, $i \in T$, as the smallest integer greater than or equal to $k\alpha_i$. Let us denote $z_i^k = \dfrac{k\alpha_i}{a_i^k}(z_i - \omega_i) + \omega_i$. Since $\lim\limits_{k \to \infty} z_i^k = z_i$ for every $i \in T$, by continuity of preferences, we have that $z_i^k \succ_i y_i$ for every $i \in T \cap S$ and $z_i^k \succ_i x_i$ for every $i \in T \setminus S$, for all k large enough.

By construction, we have $\sum_{i \in T} a_i^k (z_i^k - \omega_i) = 0$. Denoting $q_i^k = \frac{a_i^k}{\sum_{i \in T} a_i^k}$ we obtain (i) $\sum_{i \in T} q_i^k z_i^k = \sum_{i \in T} q_i^k \omega_i$ and (ii) $z_i^k \succ_i y_i$ for every $i \in T \cap S$, and $z_i^k \succ_i x_i$ for every $i \in T \setminus S$, for all k large enough.

Q.E.D.

Proof of Lemma 5.1 Observe that if a sequence of allocations x^k converges to x and $x_i^k \succ_i x_i$, for every i and k, then, under continuity of preferences, condition (iii) holds by taking a subsequence if necessary.

Let x be a feasible allocation. If x is not Pareto optimal, then, for every i, there exists y_i such that $\sum_{i=1}^n y_i \leq \sum_{i=1}^n \omega_i$ and $y_i \succ_i x_i$. The sequence given by $x_i^k = \frac{1}{k} y_i + (1 - \frac{1}{k}) x_i$ fulfills the requirements of the Lemma with $r_i^k = 1$ for all i and k.

Now let x be a non-Walrasian feasible allocation which is efficient. Then, by the assumptions on endowments and preferences, there exist rational numbers $a_i \in (0,1]$ (with $a_j < 1$ for some j; otherwise x would be non Pareto optimal) and bundles y_i for all $i = 1, \dots, n$, such that $\sum_{i=1}^n a_i(y_i - \omega_i) = -\delta$, with $\delta \in \mathbb{R}_{++}^{\ell}$ and $y_i \succ_i x_i$, for every i (see Hervés-Beloso and Moreno-García, 2001, for details).

Let $a = \sum_{i=1}^n a_i$. Given $\varepsilon \in (0,1]$, let $y_i^{\varepsilon} = \varepsilon y_i + (1-\varepsilon) x_i$. By convexity of preferences, $y_i^{\varepsilon} \succ_i x_i$ for every i. Consider the bundle $x_i^{\varepsilon} = x_i + \frac{\varepsilon \delta}{a^{\varepsilon}}$, where $a^{\varepsilon} = (1-\varepsilon)(n-a)$. By monotonicity of preferences, $x_i^{\varepsilon} \succ_i x_i$ for every i.

Take a sequence of rational numbers ε_k converging to zero and, for each k and i, let $a_i^k = (1-\varepsilon_k)(1-a_i)$, $r_i^k = a_i + a_i^k \in (0,1]$, and define the commodity bundle $x_i^k = \frac{a_i}{r_i^k} y_i^{\varepsilon_k} + \frac{a_i^k}{r_i^k} x_i^{\varepsilon_k}$. Therefore, by construction, the sequences r_i^k and x_i^k ($i = 1, \dots, n$ and $k \in \mathbb{N}$) verify the required properties.[22]

Q.E.D.

Proof of Lemma 5.2 Let q^k be a natural number such that $r_i^k = b_i^k / q^k$, with $b_i^k \in \mathbb{N}$ for each $i = 1, \dots, n$. Since $x \in \bigcap_{r \in \mathbb{N}} B(r\mathcal{E})$, we have that the allocation x^k cannot be a Walrasian allocation for the economy formed by b_i^k agents of type i; otherwise, the coalition formed by b_i^k members of each type i joint with x^k would define a justified objection in the q^k-replicated economy.[23] Then, f^k cannot be

[22]Note that by construction the next equalities hold:

$$\begin{aligned}
\sum_{i=1}^n r_i^k \left(x_i^k - \omega_i\right) &= \sum_{i=1}^n \left(a_i y_i^{\varepsilon_k} + a_i^k x_i^{\varepsilon_k}\right) - \sum_{i=1}^n a_i \omega_i - \sum_{i=1}^n a_i^k \omega_i = \\
&= \varepsilon_k \sum_{i=1}^n a_i (y_i - \omega_i) + \sum_{i=1}^n (1-\varepsilon_k) x_i - \sum_{i=1}^n (1-\varepsilon_k) a_i \omega_i - \sum_{i=1}^n a_i^k \omega_i + \\
&\quad \tfrac{\varepsilon_k \delta}{(n-a)} \sum_{i=1}^n (1-a_i) = \sum_{i=1}^n (1-\varepsilon_k) x_i - \sum_{i=1}^n (1-\varepsilon_k) a_i \omega_i - \sum_{i=1}^n a_i^k \omega_i
\end{aligned}$$

[23]We remark that with our notion of justified objection in the replicated economies, any objecting coalition involving all types joint with a Walrasian allocation for such a coalition defines a justified objection. This is not the case for the corresponding Mas-Colell's notion

a competitive allocation in the continuum economy $\mathcal{E}_c^k$. By Mas-Colell's (1989) equivalence result, f^k is blocked by a Walrasian objection in the economy $\mathcal{E}_c^k$. That is, there is a coalition S^k blocking f^k via g^k that is a competitive allocation at equilibrium price p^k for the economy restricted to the coalition S^k. Thus, by convexity of preferences, we can consider without loss of generality that g^k is an equal-treatment allocation. In addition, $p^k \cdot y > p^k \cdot \omega_i$ if $y \succ_i x_i^k$, for every i such that $\mu(S^k \cap I_i) = 0$.

Q.E.D.

Proof of Lemma 5.3 Since the number of types of consumers we deal with is finite, without loss of generality we can consider, taking a subsequence if necessary, that $T^k = T$ for every k. Note that $\hat{\gamma}_i^k \in (0,1]$ for every $i \in T$ and $\sum_{i \in T} \hat{\gamma}_i^k = 1$ for every k. Therefore, there exists a subsequence of $(\hat{\gamma}_i^k, i \in T)$ that converges to $(\gamma_i, i \in T)$ and $\sum_{i \in T} \gamma_i = 1$. We use the same notation for such a subsequence and write γ_i^k converges to γ_i for every $i \in T$. Let $\hat{\mathcal{E}}_c$ be the continuum economy with a finite number of types where the set of agents of type i is represented by a subinterval of $[0,1]$ whose measure is γ_i.

Let $(Z_i, i \in T)$ be the excess demand correspondences of the types that are actually present in every economy $\hat{\mathcal{E}}_c^k$. The measure ν^k that describes $\hat{\mathcal{E}^k}_c$ is given by $\nu^k(F) = \mu\left(\{t \in I \text{ such that } Z_t \in F\}\right)$ for each subset F of $\mathcal{Z}$. Let us define a function τ which assigns to each $F \subset \mathcal{Z}$ the subset of types $\tau(F) = \{i \in T | Z_i \in F\}$. Then, $\nu^F = \sum_{i \in \tau(F)} \hat{\gamma}_i^k$. We deduce that

$$\lim_{k \to \infty} \nu^k(F) = \lim_{k \to \infty} \sum_{i \in \tau(F)} \hat{\gamma}_i^k = \sum_{i \in \tau(F)} \gamma_i = \nu(F),$$

where ν is the measure describing the economy $\hat{\mathcal{E}}_c$. Therefore, we can conclude that ν^k converges weakly to ν.

Q.E.D.

Proof of Theorem 5.1 Since $W(\mathcal{E})$ is included in the core of every replicated economy $r\mathcal{E}$, it is immediate that $W(\mathcal{E}) \subseteq \bigcap_{r \in \mathbb{N}} B(r\mathcal{E})$.

To show the converse, assume that x is not a Walrasian allocation but x belongs to the Edgeworth bargaining set of every replicated economy. By the previous lemmas, for each natural number k, there is a subset T of types and competitive equilibrium (p^k, g^k) in the continuum economy $\hat{\mathcal{E}}_c^k$ such that:

which requires that if a set of agents of type i becomes strictly better off in a justified objection, then all the agents of type i have to be members of the objecting coalition.

(i) $g_i^k \succsim_i x_i^k$ for every $i \in T$, with $g_j^k \succ_j x_j^k$ for some $j \in T$, and $g_i^k \in d_i(p^k)$ for every $i \in T$, and

(ii) $x_i^k \succsim_i d_i(p^k)$ for every $i \in N \setminus T$.[24]

Let us consider the following sets of types that do not belong to T: $A_k = \left\{i \notin T | x_i \succsim_i d_i(p^k)\right\}$, $B_k = \left\{i \notin T | x_i \prec_i d_i(p^k)\right\}$. Since the number of types is finite, without loss of generality we can consider, taking a subsequence if it is necessary, that $A_k = A$ and $B_k = B$ for every k.

We recall that the economy $\hat{\mathcal{E}}_c^k$, which is described by the measure ν^k on $\mathcal{Z}$, is formed by agents of type $i \in T$ and each type i is represented by the subinterval $\hat{I}_i^k$ with measure $\hat{\gamma}_i^k$. Moreover, the sequence of measures $\left(\nu^k\right)_{k \in \mathbb{N}}$ converges weakly to ν. Let us choose a sequence of numbers $\delta_k \in (0,1)$ converging to 1 and let $\varepsilon^k = 1 - \delta_k$, which converges to zero. For each $i \in B$ take $\varepsilon_i^k > 0$ such that $\varepsilon^k = \sum_{i \in B} \varepsilon_i^k$. Let $T_1 = T \cup B$ and for each $i \in T_1$ define $\widetilde{\gamma}_i^k \in (0,1)$ as follows:

$$\widetilde{\gamma}_i^k = \begin{cases} \delta_k \hat{\gamma}_i^k & \text{if } i \in T \\ \varepsilon_i^k & \text{if } i \in B \end{cases}$$

Note that $\sum_{i \in T_1} \widetilde{\gamma}_i^k = 1$.

For each k, define the continuum economy $\widetilde{\mathcal{E}}_c^k$ formed by agents of types in T_1 and such that each agent of type i is represented by a subinterval with measure $\widetilde{\gamma}_i^k$. Let $\widetilde{\nu}^k$ denote the measures on $\mathcal{Z}$ describing the economy $\widetilde{\mathcal{E}}_c^k$. Note that $\lim_{k\to\infty} \widetilde{\gamma}_i^k = \lim_{k\to\infty} \hat{\gamma}_i^k = \gamma_i$ for every $i \in T$ and $\widetilde{\gamma}_i^k$ goes to zero as k increases for every $i \in B$. Then, the economy $\widetilde{\mathcal{E}}_c^k$ differs from $\hat{\mathcal{E}}_c^k$ only in at most a finite set of types of agents whose measure goes to zero when k increases. Therefore, the sequence of measures $\left(\widetilde{\nu}^k\right)_{k \in \mathbb{N}}$ also converges weakly to ν.

Now, for each k and for each $i \in T_1 = T \cup B$, take a sequence of positive rational numbers r_i^{km} converging to $\widetilde{\gamma}_i^k$ when m increases and such that $\sum_{i \in T_1} r_i^{km} = 1$ for every m. In this way, for each k, let us define a sequence of continuum economies $\mathcal{E}_c^{km}$ formed by agents of types in T_1 and such that each agent of type i is represented by a subinterval with rational measure r_i^{km}. Let us take the diagonal sequence of economies $\left(\mathcal{E}_c^{kk}\right)_{k \in \mathbb{N}}$ and let ν^{kk} be the measure on $\mathcal{Z}$ that describes $\mathcal{E}_c^{kk}$. Note that $\lim_{k\to\infty} r_i^{kk} = \lim_{k\to\infty} \hat{\gamma}_i^k$ for every $i \in T$ and $\lim_{k\to\infty} r_i^{kk} = 0$ for

[24]Note that, given a price vector p, all the bundles in $d_i(p)$ are indifferent; thus, when we write $z \succsim_i d_i(p)$ it means $z \succsim_i d$ for every $d \in d_i(p)$.

every $i \in B$. Then, the sequence of measures $\left(\nu^{kk}\right)_{k\in\mathbb{N}}$ converges weakly to ν as well.

Therefore, by the continuity of the equilibrium price correspondence at ν and for k large enough, any competitive equilibrium price of the economy $\hat{\mathcal{E}}_c^k$ is arbitrarily close to an equilibrium price of the economy $\mathcal{E}_c^{kk}$, while both of them lie within a neighborhood of the set of equilibrium price of the limit economy $\hat{\mathcal{E}}_c$ described by the measure ν.

Then, by the continuity of the equilibrium mapping at ν and the continuity of preferences, we deduce that for every k large enough there is an equilibrium price $\widetilde{p}_1^k$ for the economy $\mathcal{E}_c^{kk}$ such that $d_i(\widetilde{p}_1^k) \succ_i x_i$ for every $i \in T_1$. If $x_i \succsim_i d_i(\widetilde{p}_1^k)$ for every $i \in A$, we have found a Walrasian objection to x in a replicated economy, which is in contradiction to the fact that x belongs to the Edgeworth bargaining set of every replicated economy. Otherwise, let $\widetilde{A}_k = \left\{i \notin T_1 | x_i \succsim_i d_i(\widetilde{p}_1^k)\right\}$, $\widetilde{B}_k = \left\{i \notin T_1 | x_i \prec_i d_i(\widetilde{p}_1^k)\right\}$. As before, without loss of generality, taking a subsequence if it is necessary, we can consider $\widetilde{A}_k = \widetilde{A}$ and $\widetilde{B}_k = \widetilde{B}$ for every k. Let $T_2 = T_1 \cup \widetilde{B}$ and repeat the analogous argument. In this way, after a finite number h of iterations, we have either *(i)* $T_h = N = \{1, \dots, n\}$ or *(ii)* $N \setminus T_h \neq \emptyset$ but $\left\{i \notin T_h | x_i \prec_i d_i(\widetilde{p}_h^k)\right\} = \emptyset$. If *(i)* occurs we find a justified objection to x in a replicated economy which involves all the types of agents. If *(ii)* is the case, there is also a justified objection to x in a replicated economy but involving only a strict subset of types. In any situation we obtain a contradiction.

Q.E.D.

Proof of Theorem 5.2. Since $W(\mathcal{E}) \subset C(r\mathcal{E}) \subset B(r\mathcal{E})$, it is immediate that $W(\mathcal{E}) \subseteq \bigcap_{r\in\mathbb{N}} B_L(r\mathcal{E})$.

To show the converse, consider $x \in \bigcap_{r\in\mathbb{N}} B_L(r\mathcal{E})$ and assume that x is not a Walrasian allocation in the economy $\mathcal{E}$. Let us consider the corresponding step function f_x in the associated continuum economy $\mathcal{E}_c$. We have that f_x does not belong to $B_{MC}(\mathcal{E}_c)$. Then, there exists a justified objection to f_x following Mas-Colell's definition in $\mathcal{E}_c$. By convexity of preferences, Remark 5 in Mas-Colell (1989) allows us to ensure that there is a justified objection to x that is given by (S, y) and parameters $\alpha_i, i \in S$, such that $\sum_{i\in S} \alpha_i y_i \leq \sum_{i\in S} \alpha_i \omega_i,\ y_i \succsim_i x_i$ for every $i \in S$ and $y_j \succ_j x_j$ for some $j \in S$. Moreover, $\alpha_j = 1$ and $y_i \sim_i x_i$ for every i such that $\alpha_i < 1$.

If $S = \{j\}$ the pair $(\{j\}, y_j)$ is an objection in every replicated economy. Then, for every $r\mathcal{E}$ there is a collection T of types which excludes j and an allocation z such that (T, z) counter-objects $(\{j\}, y_j)$. Then we can find a counter-objection in $\mathcal{E}_c$ to the justified objection, which is a contradiction.

Now consider that S contains not only the type j. By continuity of preferences, we can take ε such that $(1-\varepsilon)y_j \succ_j x_j$. Let $\alpha = \sum_{\substack{i \in S \\ i \neq j}} \alpha_i$ and define the allocation $\tilde{y}$ as follows:

$$\tilde{y}_i = \begin{cases} (1-\varepsilon)y_i & \text{if } i = j \\ y_i + \frac{\varepsilon y_j}{\alpha} & \text{if } i \neq j \end{cases}$$

By construction, $\sum_{i \in S} \alpha_i \tilde{y}_i \leq \sum_{i \in S} \alpha_i \omega_i$. Since preferences are monotone $\tilde{y}_i \succ_i x_i$ for every $i \in S$. Actually, $\tilde{y}_i \succ_i y_i \succsim_i x_i$, for every $i \neq j$.

As in the proof of Lemma 5.2, for every natural $k \in \mathbb{N}$, let α_i^k, $i \in S$ be the smallest integer greater than or equal to $k\alpha_i$. Let us denote $y_i^k = \frac{k\alpha_i}{\alpha_i^k}(\tilde{y}_i - \omega_i) + \omega_i$. Note that y_i^k converges to $\tilde{y}_i$ for every $i \in S$ and then, by continuity of preferences, we have that $y_i^k \succ_i x_i$ for every $i \in S$ and for all k large enough. In addition, $y_i^k \succ_i y_i \succsim_i x_i$ for every $i \neq j$ and for all k large enough. We remark that $y_j^k = (1-\varepsilon)y_j$ and $\alpha_j^k = 1$ for every k.

Then, the coalition with α_i^k agents of type $i \neq j$ with $i \in S$, and k agents of type j, blocks x via y^k in the replicated economy $k\mathcal{E}$. Therefore, there exists a counter-objection (T, z) to the objection (S, y^k) in some replicated economy $r\mathcal{E}$ with $r \geq k$, such that $j \notin T$. Thus, for every $i \in T$, there exists a natural number $\beta_i \leq r$, such that $\sum_{i \in T} \beta_i z_i \leq \sum_{i \in T} \beta_i \omega_i$, $z_i \succ_i y_i^k \succ_i y_i$ for every $i \in T \cap S$ and $z_i \succ_i x_i$ for every $i \in T \setminus S$. This is a contradiction with the fact that the objection (S, y) defines a justified objection to f_x in the associated continuum economy.

Q.E.D.

References

Anderson, R.M. (1998): Convergence of the Aumann-Davis-Maschler and Geanakoplos Bargaining Sets. Economic Theory 11, 1–37.

Anderson, R. M., Trockel, W., Zhou, L. (1997): Nonconvergence of the Mas-Colell and Zhou bargaining sets. Econometrica 65 (5), 1227–1239.

Aubin, J.P. (1979): Mathematical methods of game economic theory. North-

Holland, Amsterdam, New York, Oxford.

Aumann, R.J. (1964): Markets with a continuum of traders. Econometrica 32, 39–50.

Aumann, R., Maschler, M. (1964): The bargaining set for cooperative games, in: M. Dresher, L. S. Shapley and A. W. Tucker, eds., Advances in game theory (Princeton University Press, Princeton, NJ), 443–447.

Davis, M., Maschler, M. (1963): Existence of stable payoff configurations for cooperative games. Bulletin of the American Mathematical Society 69, 106–108.

Debreu, G., Scarf, H. (1963): A limit theorem on the core of an economy. International Economic Review 4 (3), 235–246.

Dierker, E. (1973): Topological methods in Walrasian Economics. Berlin, Heidelberg, and New York Springer-Verlag.

Dutta, B., Ray, D., Sengupta, K., Vohra, R. (1989): A consistent bargaining set. Journal of Economic Theory 49, 93–112.

Edgeworth, F.Y. (1881): Mathematical Psychics. Paul Kegan, London.

García-Cutrín, J., Hervés-Beloso, C. (1993): A discrete approach to continuum economies. Economic Theory 3, 577-584.

Geanakoplos, J. (1978): The bargaining set and nonstandard analysis. Chapter 3 of Ph.D. Dissertation, Department of Economics, Harvard University, Cambridge, MA.

Grodal, B. (1972): A second remark on the core of an atomless economy. Econometrica 40, 581–583.

Hervés-Beloso, C., Moreno-García, E. (2001): "The veto mechanism revisited"; in Approximation, Optimization and Mathematical Economics; p. 147-159. Marc Lassonde (Ed.). Heidelberg, New York. Physica-Verlag.

Hervés-Beloso, C., Moreno-García, E., Yannelis, N.C. (2005): Characterization and incentive compatibility of Walrasian expectations equilibrium in infinite dimensional commodity spaces. Economic Theory, 26, 361-381.

Hervés-Beloso, C., Moreno-García, E. (2009): Walrasian analysis via two-player games. Games and Economic Behavior 65, 220–233.

Hildenbrand, W. (1974): Core and equlibria in a large economy. Princeton, N.J.: Princeton University Press.

Liu, J., Liu, X. (2012): Fuzzy extensions of bargaining sets and their existence in cooperative fuzzy games. Fuzzy sets and systems 188, 88–101.

Mas-Colell, A. (1989): An equivalence theorem for a bargaining set. Journal of Mathematical Economics 18, 129–139.

Schjødt, U., Sloth, B. (1994): Bargaining sets with small coalitions. International Journal of Game Theory 23, 49–55.

Schmeidler, D. (1972): A remark on the core of an atomless economy. Econometrica 40, 579–580.

Shapley, L.S., Shubik, M. (1984): Convergence of the bargaining set for differentiable market games, Appendix B. In: Martin Shubik, A Game Theoretic Approach to Political Economy, volume II of Game Theory in the Social Sciences. Cambridge, MA: M.I.T. Press.

Vind, K. (1972): A third remark on the core of an atomless economy. Econometrica 40, 585–586.

Yang, W., Liu, J., Liu, X. (2011): Aubin cores and bargaining sets for convex cooperative fuzzy games. International Journal of Game Theory 40, 467–479.

Zhou, L. (1994): A new bargaining set of an n-person game and endogenous coalition formation. Games and Economic Behavior 6, 512-526.