Menu Auctions and Influence Games with Private Information

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MENU AUCTIONS AND INFLUENCE GAMES WITH PRIVATE INFORMATION[1]

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We study games in which multiple principals influence the choice of a privately-informed agent by offering action-contingent payments. We characterize the equilibrium allocation set as the maximizers of an endogenous aggregate virtual-surplus program. The aggregate maximand for every equilibrium includes an information-rent margin which captures the confluence of the principals' rent-extraction motives. We illustrate the economic implications of this novel margin in two applications: a public goods game in which players incentivize a common public good supplier, and a lobbying game between conflicting interest groups who offer contributions to influence a common political decision-maker.

KEYWORDS: Menu auctions, influence games, common agency, screening contracts, public goods games, lobbying games.

[1] An earlier version of this paper was circulated as "Public Contracting in Delegated Agency Games" (April 2013). The present papers studies a less general set of influence games, but includes a large variety of applications of the results. We thank numerous seminar participants for their thoughtful comments on this project. We are especially indebted to John Birge, Philippe Jehiel, Michel LeBreton, Stephano Lovo, David Rahman and Aggey Semenov.

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1. INTRODUCTION

Economists have long been interested in strategic settings in which several interested parties (perhaps with opposed interests) attempt to influence a common agent to do their bidding. In the almost three decades that have passed since the seminal strategic analysis by Bernheim and Whinston (1986), the *truthful* equilibrium of their complete-information model of menu auctions and influence games has become a work horse in a wide range of settings. Applications include international trade (e.g., Grossman and Helpman (1994, 1995, 2001), Dixit, Grossman and Helpman (1997), Goldberg and Maggi (1997)), combinatorial auction design (e.g., Milgrom (2007)), industrial organization (e.g., Bernheim and Whinston (1989), Inderst and Wey (2007)) and political economy and public finance (e.g., Aidt (1998), Laussel and LeBreton (1998, 2001), Persson and Tabellini (2002), Bellettini and Ottaviano (2005)).

The menu auctions game of Bernheim and Whinston (1986) owes its success, in part, to the simplicity and robustness of its equilibrium characterization, even in what may at first glance appear to be very complicated strategic settings. To review, the basic game consists of n principals and a single common agent. The agent chooses some action, $q \in \mathcal{Q}$, that has payoff consequences for each of the principals. Prior to taking an action, however, the principals may each offer the agent enforceable payment schedules – menus of promised payment-action pairs (possibly subject to constraints such as nonnegativity). After receiving a menu offer from each principal, the agent chooses an action to maximize utility. Bernheim and Whinston (1986) show that there are a large number of equilibria to this influence game, but there is always an equilibrium in which the agent chooses an action which maximizes the collective surplus of the principals and the agent. Such a surplus-maximizing equilibrium can be supported with "truthful" menus in which each principal offers a transfer schedule whose marginal transfer is equal to the principal's marginal benefit of action. Furthermore, this truthful equilibrium allocation is also the only one that is immune to a reasonable class of renegotiations – formally, the equilibrium is a Coaltional-Proof Nash Equilibrium. As a result, Bernheim and Whinston argue that the surplus-maximizing outcome (relative to the set of principals and the agent) is a reasonable equilibrium to use for predicting behavior in general menu auction games with complete information.

The novel contribution of this paper is to reconsider influence games under the assumption that the agent has private information.[1] When the common agent possesses private information,[2] two technical difficulties are introduced. First, if principals are allowed to choose discontinuous payment schedules, then we must entertain the possibility that the equilibrium payment functions are discontinuous. In this case, each individual principal's optimization problem is itself discontinuous. As such, standard control-theory techniques which assume continuity (and typically piecewise differentiability) cannot be applied to the problem. Fortunately, we are able to modify results from the mathematics literature on non-smooth optimal control to address this problem. A second difficulty introduced from asymmetric information is that an individual principal may only choose to influence a strict subset of types in equilibrium. The sets of types on which each principal is *active* must be determined in order to construct equilibrium tariffs, but the equilibrium tariffs, in turn, determine the regions of activity. In short, the equilibrium activity sets must be jointly determined at the fixed point of the principals' best-response correspondences.[3] Fortunately, we are able to address this difficulty and obtain closed-form solutions by

[1]Although there are a few antecedents to this approach – Laffont and Tirole (1991), Martimort (1996, 2007), LeBreton and Salanié (2003) – these papers rely on highly-stylized settings and (sometimes implicit) equilibrium refinements for their results. Moreover, these earlier models ignored the endogeneity of the principals' activity sets by either assuming symmetry or assumptions which guaranteed full coverage. Our paper provides the first general analysis of this class of influence games and explicitly characterizes the principals' activity sets.

[2]The present paper is related to an older literature which examines common agency with a privately informed agent under the assumption that each principal can only contract over a distinct subset of the agent's actions. This has been referred to as the *private contracts* assumption, to distinguish it from the present setting of *public* contracting. For example, Martimort and Stole (2009) assume that the principals are firms selling their output to a common consumer. Principal 1, however, can only condition its contract on the consumer's purchases of its good, q_1; the consumer's purchases of the rival's product, q_2, is not contractible by principal 1. Thus, $(q_1, q_2) \in \mathcal{Q}_1 \times \mathcal{Q}_2$ is the agent's action, but principal i is only allowed to offer a payment schedule $t_i : \mathcal{Q}_i \to \mathbb{R}$.

[3]In previous work, Martimort and Stole (2012), we allowed transfer payments to be negative and assumed that the agent must accept all or none of the offered contracts. Formally, this is known as the case of *intrinsic* common agency. This is the appropriate setting if the principals have some control of the agent's choice as in the case of public regulation by different government agencies. When common agency is intrinsic, the activity sets of the principals always coincide, so equilibrium analysis of these games avoids the difficulties in the present paper. Nonetheless, intrinsic common agency with public contracts provides an interesting comparison for the influence games in the current paper, which we discuss in a subsequent section of this paper.

assuming that each principal has linear preferences over $\mathcal{Q}$.

As in Bernheim and Whinston's (1986) complete information game, we find an infinite number of equilibria. Rather than propose an equilibrium refinement at the outset, we instead construct conditions for the set of *all equilibria*. Our main theoretical contribution, Theorem 1, demonstrates that all equilibria exhibit the same confluence of information-rent terms and that the θ-type agent's equilibrium choice $\overline{q}(\theta)$ is identical across all equilibria in which there is full separation in the neighborhood of θ. One particular equilibrium allocation – what we call the *maximal* equilibrium – always exists, is easy to compute, exhibits maximal separation, and is implemented with continuous contracts. The limit of the expectation of the marginal tariff in the maximal equilibrium as the game becomes one of complete information also equals the marginal tariff in the truthful equilibrium of Bernheim and Whinston (1986). Additionally, for any equilibrium allocation, and for any subset of types for which there is full separation in the equilibrium allocation, we demonstrate that the equilibrium allocation must coincide with the maximal allocation. These properties motivate us to focus on the maximal equilibrium allocation when performing comparative static exercises in the applications of Section V, though we remain somewhat agnostic about imposing this refinement. To this end, we also characterize a particularly interesting discontinuous equilibrium in one of our applications and illustrate how one can construct such equilibria in general.

Two settings provide motivating applications which we study. First, we consider games in which the preferences of all principals reflect a general consensus that more activity by the agent is desirable, holding the amount of transfers fixed. In this case, we say that preferences are *congruent*. As an application, suppose that the n principals all value some public good, q, produced by the agent, but the principals value the good with possibly different intensities (relative to their marginal utility of money). The agent has private information about the cost of providing the good, q, and so each principal will individually consider the tradeoff of greater provision against the reduction of the agent's information rent. In a world with one principal, the setting is analogous to the government regulation of a monopolist with unknown marginal cost, Baron and Myerson (1982). With multiple principals, however, we will see that there are additional effects that may generate an allocation that looks considerably different from either the first-best allocation (which is also the truthful equilibrium of Bernheim and Whinston's complete information game) or the Baron-Myerson allocation.

Second, we also consider games in which some principals prefer higher q, and other principals prefer a lower q. Thus, we can think of there being two "interest groups" with opposed preferences, although the principals within each group have congruent preferences and act noncooperatively. The most natural application is one where the agent is a politician with some a privately-known ideal policy point, and the principals are lobbyists offering political contributions as functions of the politician's policy choice, q. As in the public good example, each principal will consider the tradeoff between influencing the agent to choose a more preferred policy against the ability to reduce the information rents paid to the agent (e.g., total campaign contributions). Unlike the setting in which all principals are congruent in their preferences, we will find that the informational-rent distortions of the principals are combined into an interesting and novel marginal distortion.

In both the public goods game and the lobbying game, we also characterize interesting comparative statics on principal preferences and the equilibrium influence the principals exert on the agent. Among other results, applying Jensen's inequality to the equilibrium information-rent confluence function implies that dispersed preferences within an interest group (small and large stakeholders rather than homogenous stakeholders) leads to more influence. We also show that an increase in one principal's preferences leads to more influence, but with some crowding out of the contributions of liked-minded principals. Lastly, we demonstrate that a mean-preserving spread of principal preferences in the political lobbying game that does not affect the first-best policy choice, nonetheless leads to a mean-preserving spread in the distribution of equilibrium policies.

The basic influence game with incomplete information is presented in Section 2. In Section 3 we analyze the best-response correspondence of a typical principal and present our key building block for our discontinuous control program in Lemma 1. Necessary conditions characterizing the role of information rents in any equilibrium are presented in Section 4, along with sufficient conditions for the maximal equilibrium. We turn to applications in Section 5 and analyze the maximal equilibria in both the public goods and lobbying games. We also further explore discontinuous equilibria in the setting of lobbying games, illustrating in a concrete example the similarities and differences between the maximal and discontinuous equilibria. Section 6 concludes.

2. A MODEL OF INFLUENCE WITH INCOMPLETE INFORMATION

Our economic influence game is a setting in which each of n principals simultaneously offers a common agent a nonnegative payment schedule that rewards the agent for the choice of action, $q \in \mathcal{Q} \subset \mathbb{R}$. Formally, we allow each principal i to present to the agent any upper-semicontinuous function, $t_i : \mathcal{Q} \to \mathbb{R}_+$, as its contract offer.[4] Following such offers, the agent chooses an action to maximize utility given the promised payments.

Each principal has preferences that are linear in both the agent's choice of q and in monetary transfers. Given a contract t_i and a choice of q by the agent, we denote principal i's payoff as simply $s_i q - t_i(q)$. We will indicate that a principal has a positive preference for q, $s_i > 0$, by the index set $i \in \mathcal{A}$; similarly, a principal with negative preferences for q, $s_i < 0$, will be indicated by $i \in \mathcal{B}$. The set of all principals is $\mathcal{N} = \mathcal{A} \cup \mathcal{B}$.

The agent has heterogeneous preferences, indexed by $\theta \in \Theta = [\underline{\theta}, \overline{\theta}]$, chosen by nature at the start of the game according to a commonly-known distribution function, F, with an associated positive, atomless density function f. The agent's preferences over action q and monetary transfers t, conditional on θ, is

$$S_0(q) - \theta q + t,$$

where S_0 is a continuously differentiable, strictly concave function. We make two additional assumptions on the distribution of types to avoid technical complications in the arguments which follow. First, in order to guarantee that the solutions to a relaxed program are monotone, we make the familiar assumption that the distribution function, F, and its complement, $1 - F$, are log concave. Second, in order to obtain interior principal activity sets, we assume that each principal's marginal preference, s_i, is not too large relative to the heterogeneity of agent preferences. Specifically, we require that there exists an interior type $\hat{\theta}_i$ such that $s_i = F(\hat{\theta}_i)/f(\hat{\theta}_i)$ for $i \in \mathcal{A}$ and $|s_i| = (1 - F(\hat{\theta}_i)/f(\hat{\theta}_i)$ for $i \in \mathcal{B}$. The implication of this assumption (yet to be shown) is that each principal will actively influence a proper subset of agent types.[5]

[4] Requiring the schedules be nonnegative is without loss of generality if the agent has the option to reject any subset of the offered schedules (i.e., if common agency is *delegated*). When the agent must either accept of reject the entire set of the n offers, then common agency is *intrinsic*. The set of equilibria for this simpler setting is explored in Martimort and Stole (2012).

[5] It is straightforward to extend the analysis to the case in which principals might be active for every agent

The timing of the influence game is simple. First, nature chooses the agent's type. Second, each principal i chooses a transfer function, $t_i \in \mathcal{T}$, where $\mathcal{T}$ is the set of nonnegative, upper-semicontinuous functions on $\mathcal{Q}$. We will denote $T_{-i}(q) \equiv \sum_{j \neq i} t_j(q)$ and $T(q) \equiv \sum_{i \in \mathcal{N}} t_i(q)$ as the associated aggregate transfers of the principals from this stage. Third, the agent chooses an optimal action given the aggregate transfers offered in the second stage, $q_0(\theta \mid T)$. Finally, payments are made by the principals in accord with their contractual obligations.

Our solution concept is pure-strategy Perfect Bayesian equilibria. We say that the strategy profile $\{\overline{q}_0, \overline{t}_1, \ldots, \overline{t}_n\}$ is an *equilibrium* of the influence game if for all $\theta \in \Theta$

$$(1) \qquad \overline{q}_0(\theta \mid T) \in \arg\max_{q \in \mathcal{Q}} \; S_0(q) - \theta q + \overline{T}(q),$$

and for all $i \in \mathcal{N}$

$$(2) \qquad \overline{t}_i \in \arg\max_{t_i \in \mathcal{T}} \int_{\Theta} \left(s_i \overline{q}_0(\theta \mid \overline{T}_{-i} + t_i) - t_i(\overline{q}_0(\theta \mid \overline{T}_{-i} + t_i)) \right) f(\theta) d\theta.$$

For any aggregate equilibrium transfer function, $\overline{T}$, we will refer to the *equilibrium allocation* pair, $(\overline{q}, \overline{U})$, as defined by $\overline{q}(\theta) \equiv \overline{q}_0(\theta \mid \overline{T})$ and $\overline{U}(\theta) = S_0(\overline{q}(\theta)) - \theta \overline{q}(\theta) + \overline{T}(\overline{q}(\theta))$ for all $\theta \in \Theta$.

3. PRELIMINARIES

We begin with a consideration of principal i's best response under the belief that the other principals will offer the aggregate influence schedule $\overline{T}_{-i}$. From principal i's vantage point, it is as if he is designing a contract for an agent with residual preferences given by

$$S_0(q) - \theta q + \overline{T}_{-i}(q).$$

Absent an agreement with principal i, the agent can always secure the following indirect utility with the remaining $n-1$ principals:

$$\overline{U}_{-i}(\theta) \equiv \max_{q \in \mathcal{Q}} \; S_0(q) - \theta q + \overline{T}_{-i}(q).$$

Note that if the agent is offered a nonnegative schedule by principal i, it necessarily follows that the agent's indirect utility of contracting with principal i weakly exceeds $\overline{U}_{-i}$.

type. Our main characterization theorem, for example, extends to this setting with additional notation and treatment of additional special cases.

Similarly, if the agent's indirect utility exceeds $\overline{U}_{-i}$, then the agent must choose an action for which principal i has offered a positive payment. Hence, we can replace the requirement that $t_i \geq 0$ with the requirement that $U \geq \overline{U}_{-i}$, where U is the agent's utility when contracting with all principals.

Framed in this manner, we can think of principal i as choosing an allocation (q, U) that is individually rational and incentive compatible for the agent relative to some outside option, $\overline{U}_{-i}$. As is well known in the literature, these requirements can be expressed formally as

(3) $\qquad U(\theta) \geq \overline{U}_{-i}(\theta), \quad \text{for all } \theta \in \Theta, \quad \text{(individual rationality)}$

(4) $\qquad U(\theta) \ \text{ convex}, \quad -q(\theta) \in \partial U(\theta), \quad \text{(incentive compatibility).}$

Here, we have chosen to state the incentive compatibility requirement in (4) using the subdifferential of U to allow for the possibility that U may have points of non differentiability.

Given the characterization of implementability, principal i's problem of choosing an optimal t_i can be reformulated as choosing an allocation (q, U) ($q : \Theta \to \mathcal{Q}$, Lebesgue measurable, and $U : \Theta \to \mathbb{R}$, absolutely continuous) to solve the following program:

$$\text{Program } \mathcal{P}_i: \quad \max_{(q,U)} \int_{\Theta} \left\{ s_i q + S_0(q(\theta)) - \theta q(\theta) + \overline{T}_{-i}(q(\theta)) - U(\theta) \right\} f(\theta) d\theta,$$

subject to (3)-(4). If $\overline{T}_{-i}$ is known to be piecewise differentiable and the integrand is known to be concave, we could apply standard optimal control results to obtain a characterization of the the optimal contract. Assuming that $\overline{T}_{-i}$ is continuous and almost everywhere differentiable, however, imposes an equilibrium refinement that we should make explicit.[6]

[6]Note that in the above description of principal i's program, we have implicitly allowed the principal to resolve the agent's indifference in her favor if the agent's best-response set is multi-valued. Because incentive compatibility requires that the agent's indirect utility function is convex, however, and because a convex function has at most a countable number of kinks, the set of types who do not have a unique optimal choice is necessarily of measure zero. Thus, we may arbitrarily break ties in cases of agent-indifference without any measurable impact on the best-responses of the players. We thank Thomas Mariotti for raising this concern.

To provide a general solution to principal i's program that requires only that $\overline{T}_{-i}$ be upper-semicontinuous, we utilize necessary and sufficient conditions for control programs with type-dependent participation constraints and possibly discontinuous objective functions that we have developed elsewhere (Martimort and Stole (2014)). Intuitively, one can show that the solution to the program in which the objective function is replaced with its concavification is also a solution to the original program. The concavification, while continuous, is possibly nondifferentiable at points, and so tools from nonsmooth optimal control are employed. These tools, fortunately, allow us to state necessary and sufficient conditions using a distribution of Lagrange multipliers that is reminiscent of Jullien (2000).[7]

Before presenting the solution to Principal i's program, we introduce one remaining piece of notation – the virtual marginal valuation of principal i:

$$(5) \qquad \beta_i(\theta) \equiv \begin{cases} \max\left\{s_i - \frac{F(\theta)}{f(\theta)}, 0\right\} & i \in \mathcal{A}, \\ \\ \min\left\{s_i + \frac{1-F(\theta)}{f(\theta)}, 0\right\} & i \in \mathcal{B}. \end{cases}$$

Recall that our assumption on the distribution of types required that principal preferences are not too strong relative to the agent heterogeneity. Formally, for each principal i, there exists a $\hat{\theta}_i \in (\underline{\theta}, \overline{\theta})$ such that $\hat{\theta}_i$ is the largest (resp., smallest) type such that $\beta_i(\theta) = 0$ for $i \in \mathcal{B}$ (resp., $i \in \mathcal{A}$).

We can now present the main building block of our analysis.

LEMMA 1 *Given the aggregate transfer function, $\overline{T}_{-i}$, and the agent's corresponding outside option, $\overline{U}_{-i}(\theta)$, the allocation $(\overline{q}, \overline{U})$ is a solution to Principal i's program if and only if it satisfies (3)-(4), and for almost every $\theta \in \Theta$*

$$(6) \qquad \beta_i(\theta) = 0 \iff \overline{U}(\theta) = \overline{U}_{-i}(\theta),$$

[7] Jullien (2000) provides necessary and sufficient conditions for control problems with pure type-dependent state constraints under the assumption that the objective function is continuous and piecewise differentiable. Martimort and Stole (2014) demonstrate that a slight variation of Jullien's conditions can be applied to discontinuous models as well. It is worth noting that the simplicity of these conditions is a consequence of the assumption that the objective function is linear in the state variable. Because the preferences of the players are quasi linear in money, this assumption is satisfied in the present setting.

$$\text{(7)} \qquad \overline{q}(\theta) \in \arg\max_{q\in\mathcal{Q}}\ S_0(q) + (\beta_i(\theta)-\theta)q + \overline{T}_{-i}(q).$$

In words, the Lemma informs us that for any type for which $\beta_i(\theta)=0$, principal i finds it optimal not to influence the agent's choice. Indeed, the optimal transfer t_i which implements $(\overline{q},\overline{U})$ above will have the property that $t_i(\overline{q}(\theta))=0$ for all θ such that $\beta_i(\theta)=0$. For these types, we say that principal i is *inactive*. For any θ for which principal i is active, $\beta_i(\theta)>0$, we have $\overline{U}(\theta)>\overline{U}_{-i}(\theta)$ and principal i offers a marginal payment given by $t_i'(\overline{q}(\theta))=\beta_i(\theta)$, wherever t_i is differentiable, and (7) gives the solution to the agent's optimal choice program.

4. EQUILIBRIA

Our influence game is an aggregate game because – after reducing the agent's choice to the function $\overline{q}_0(\theta|T)$ – principal i's preferences over strategy profiles can be reduced to preferences over t_i and the aggregate T. Although the influence game has infinite-dimensional strategies and incomplete information, it also has the convenient property that it is quasi-linear in strategies (i.e., payoffs are linear in transfer functions), and so following Martimort and Stole (2012), we can apply the *aggregate concurrence principle*, in tandem with Lemma 1, to deduce an immediate necessary condition for the set of equilibria. In the present context, this is done simply noting that $\overline{q}$ must solve (7) for each principal i. Hence, $\overline{q}$ must also maximize the sum of the objectives from these individual programs:

$$\text{(8)} \qquad \begin{aligned} \overline{q}(\theta) \ \in\ & \arg\max_{q\in\mathcal{Q}} \sum_{i\in\mathcal{N}} S_0(q) + (\beta_i(\theta)-\theta)q + \overline{T}_{-i}(q) \\ & = \arg\max_{q\in\mathcal{Q}}\ S_0(q) + (\beta(\theta)-\theta)\,q + (n-1)\left(S_0(q)-\theta q+\overline{T}(q)\right), \end{aligned}$$

where $\overline{T}$ implements $\overline{q}$ and $\beta(\theta)\equiv\sum_{i\in\mathcal{N}}\beta_i(\theta)$ is the aggregate virtual preferences of the principals.

Because $\overline{T}$ appears in the objective in (8) and it must also implement $\overline{q}$, this necessary condition contains a fixed point: For a given $\overline{T}$, there exists a $\overline{q}$, which in turn must be a solution to the program in (8). As we will demonstrate, there are generally an infinite number of solutions to this program. One solution, however, stands out for special consideration. Define the allocation $\overline{q}^{\mathcal{Q}}$ as the unique solution to

$$\overline{q}^{\mathcal{Q}}(\theta) = \arg\max_{q\in\mathcal{Q}} \; S_0(q) + (\beta(\theta) - \theta)\, q, \tag{9}$$

and denote $\overline{T}^{\mathcal{Q}}$ as the aggregate transfer function which implements $\overline{q}^{\mathcal{Q}}$. Implementability, in turn, implies that $\overline{q}^{\mathcal{Q}}$ also satisfies

$$\overline{q}^{\mathcal{Q}}(\theta) \in \arg\max_{q\in\mathcal{Q}} \; S_0(q) - \theta q + \overline{T}^{\mathcal{Q}}(q). \tag{10}$$

As a consequence, $\overline{q}^{\mathcal{Q}}$ maximizes any objective which is a linear combination of the maximands in programs (9) and (10), and therefore $\overline{q}^{\mathcal{Q}}$ is a solution to (8). Indeed, as we show below in our main characterization result, $\overline{q}^{\mathcal{Q}}$ is an equilibrium allocation. It has the additional properties that it is continuous, strictly decreasing over any interval in which some principal is active, and it is implemented by an almost everywhere differentiable aggregate transfer function. That said, we emphasize that there are an infinity of solutions to (8) that are equilibria but do not satisfy (9).[8] Fortunately, all equilibria have a similar structure, which we now characterize in one of this paper's main theoretical contributions.

THEOREM 1 *If $\overline{q}$ is an equilibrium allocation, then*

$$\overline{q}(\theta) \in \arg\max_{q\in\overline{q}(\Theta)} \; S_0(q) + (\beta(\theta) - \theta)q, \quad \textit{for all } \theta \in \Theta. \tag{11}$$

Moreover, the allocation $\overline{q}^{\mathcal{Q}}$ satisfying

$$\overline{q}^{\mathcal{Q}}(\theta) \in \arg\max_{q\in\mathcal{Q}} \; S_0(q) + (\beta(\theta) - \theta)\, q, \tag{12}$$

is an equilibrium allocation.

The difference between the two conditions is subtle, but significant. The sufficient condition in (12) is stronger than the necessary condition (11) since it requires optimality over the whole set of possible actions, $\mathcal{Q}$; on the contrary, the necessary condition in (11) requires optimality relative to the (typically) smaller range of equilibrium allocations, $\overline{q}(\Theta)$.

[8] Note that although any solution that satisfies (9) and (10) must also satisfy (8), there exist allocations which satisfy (8) and (10), but fail to satisfy (9).

It is for this reason that we denote the solution to (12) with a $\mathcal{Q}$ as superscript to indicate the optimization is taken over the whole of $\mathcal{Q}$ rather than the equilibrium range, $\overline{q}(\Theta) \subseteq \mathcal{Q}$, and accordingly we will refer to such an equilibrium as *maximal*. We will demonstrate below that there are equilibrium allocations which satisfy (11) but do not satisfy (12), and so the latter condition implicitly refines the equilibrium set. When, for instance, S_0 is strictly concave, this restriction implies that $\overline{q}$ is in fact continuous. Instead, condition (11) is a priori compatible with the existence of discontinuities in the output profile. In fact, the equilibrium set is shown to contain an infinite number of discontinuous equilibria.

Although the domains of optimization differ in (11) and (12), it is worth emphasizing that the fundamental character of both conditions is similar. Theorem 1 makes it clear that in any equilibrium – and in particular in the continuous *maximal* equilibrium – there is an information-rent distortion generated by the presence of n principals each trading off the cost of bilateral inefficiency against the gain of surplus extraction. The manner in which these information-rent margins combine to generate departures from efficiency is the same across all equilibria and involves the comparison of the principals' collective margins under full information,

$$\sum_{i \in \mathcal{N}} s_i$$

versus their collective virtual margins under incomplete information,

$$\sum_{i \in \mathcal{A}} \max\left\{ s_i - \frac{F(\theta)}{f(\theta)}, 0 \right\} + \sum_{i \in \mathcal{B}} \min\left\{ s_i + \frac{1 - F(\theta)}{f(\theta)}, 0 \right\}.$$

This universal comparison is possibly the most compelling reason to refine the set of equilibria and focus on $\overline{q}^{\mathcal{Q}}$, although we remain largely agnostic. It is also worth noting that although any equilibrium allocation $\overline{q}$ may differ from $\overline{q}^{\mathcal{Q}}$, if for some θ it is the case that $\overline{q}(\theta)$ is fully separating over the neighborhood of θ, then $\overline{q}$ and $\overline{q}^{\mathcal{Q}}$ must coincide. Formally,

COROLLARY 1 *For any equilibrium allocation, $\overline{q}$, and for any $\theta \in \Theta$,*

$$\overline{q}(\theta) \in \operatorname{int} \overline{q}(\Theta) \implies \overline{q}(\theta) = \overline{q}^{\mathcal{Q}}(\theta).$$

Hence, any differences between two equilibrium allocations arise at the boundaries of the equilibrium range.

Given the simplicity in computing maximal equilibrium allocations for applications, we provide here a complete characterization of the corresponding maximal equilibrium tariffs. The nature of these tariffs is established in the constructive proof to (12) in Theorem 1.

COROLLARY 2 *The maximal equilibrium allocation*

$$\overline{q}^{\mathcal{Q}}(\theta) \equiv \arg\max_{q \in \mathcal{Q}} \; S_0(q) + (\beta(\theta) - \theta)q,$$

is supported by the continuous equilibrium tariffs which satisfy, for each i,

(a) $\overline{t}_i^{\mathcal{Q}}(q) = 0$ *if either* $q \notin \overline{q}^{\mathcal{Q}}(\Theta)$, *or* $q = \overline{q}^{\mathcal{Q}}(\theta)$ *and* $\beta_i(\theta) = 0$, *and*

(b) $\frac{d}{dq}\overline{t}_i^{\mathcal{Q}}(q) = \beta_i(\overline{\vartheta}^{\mathcal{Q}}(q))$, *where* $\theta = \overline{\vartheta}^{\mathcal{Q}}(q)$ *is the inverse of* $q = \overline{q}^{\mathcal{Q}}(\theta)$.[9]

A typical feature of non-maximal equilibrium allocations are the presence of discontinuities in $\overline{q}$ (or equivalently, gaps in the equilibrium range $\overline{q}(\Theta)$) and the bunching of types. For equilibrium allocations that are discontinuous, Theorem 1 imposes additional structure for the points at which there must be bunching.

COROLLARY 3 *If $\overline{q}$ is an equilibrium allocation with a discontinuity at θ_0, then*

(13) $$S_0(q_1) + (\beta(\theta_0) - \theta_0)q_1 = S_0(q_2) + (\beta(\theta_0) - \theta_0)q_2,$$

where $q_1 = \lim_{\theta \to \theta_0^+} \overline{q}(\theta)$ *and* $q_2 = \lim_{\theta \to \theta_0^-} \overline{q}(\theta) > q_1$. *Moreover, there exist* $\theta_1 > \theta_2$ *such that* $q_1 = \overline{q}^{\mathcal{Q}}(\theta_1)$ *and* $q_2 = \overline{q}^{\mathcal{Q}}(\theta_2)$, *and*

$$\overline{q}(\theta) = \begin{cases} q_1 = \overline{q}^{\mathcal{Q}}(\theta_1), & \text{for } \theta \in (\theta_0, \theta_1] \\ \\ q_2 = \overline{q}^{\mathcal{Q}}(\theta_2), & \text{for } \theta \in [\theta_2, \theta_0). \end{cases}$$

The above characterization bears strong similarities with the literature on mechanism design without transfers in monopolistic screening environments.[10] In that literature,

[9]This inverse is *a priori* a correspondence which is single-valued at any point where $\overline{t}_i^{\mathcal{Q}}(q)$ is differentiable.

[10]See Holmström (1984), Melumad and Shibano (1991), Alonso and Matoushek (2008), and Martimort and Semenov (2006), among others.

much effort has been devoted to characterize possible actions implementable in a context with asymmetric information when the preferences of the principal and the agent differ. The basic lesson there is that any implementable action may either be flat over some range and not responsive to the agent's private information or correspond to the latter's ideal point. In light of this literature, everything happens as if, over a range $\overline{Q}$, the equilibrium output $\overline{q}$ were chosen by a surrogate principal who aggregates the behavior of all principals and maximizes their aggregate virtual surplus as defined in (11). Of course, the objective of this surrogate principal differs from what would be optimal had principals merged; the difference being related to the fact that in a non-cooperative context, each principal introduces output distortions for rent extraction reasons.

We complete our equilibrium analysis in this section by demonstrating that one can take any equilibrium allocation and add an additional discontinuity to create a new equilibrium allocation, provided the discontinuity is not too large.

PROPOSITION 1 *Let $\overline{q}$ be any equilibrium allocation and let $(\hat{\theta}, \hat{q})$ be any point on this allocation around which there in an open neighborhood such that $\overline{q}$ is continuous, strictly decreasing, and two or more principals are active. Then there exists an equilibrium allocation, $\tilde{q}$, whose range corresponds to $\overline{q}$, except for the an open neighborhood (q_1, q_2) containing $\hat{q}$:*

$$\tilde{q}(\theta) \in \arg\max_{q \in \mathcal{Q} \setminus (q_1, q_2)} S_0(q) + (\beta(\theta) - \theta)q. \tag{14}$$

In addition, every agent type weakly prefers the original equilibrium allocation, $\overline{U}(\theta) \geq \tilde{U}(\theta)$, with strict preference for some positive measure of types.

The Proposition makes clear that an arbitrary number of discontinuities may be introduced into any equilibrium allocation. We also show in the constructive proof to this proposition that if types are uniformly distributed, the equilibrium tariffs which implement $\tilde{q}$ are simply $\tilde{t}_i(q) = 0$ for $q \in (q_1, q_2)$ and $\tilde{t}_i(q) = \bar{t}_i(q)$ otherwise.[11] In Section 5.2 we will explicitly construct such a discontinuous equilibrium in the context of a lobbying game and demonstrate that even large gaps can be introduced.

[11]More generally, when types are not uniformly distributed, the constructed tariff for each active principal i must have a change in constant at either q_1 or q_2, but the aggregate tariff still satisfies $\tilde{T}(q) = 0$ for $q \in (q_1, q_2)$ and $\tilde{T}(q) = \overline{T}(q)$ otherwise.

5. APPLICATIONS

We have chosen two classes of games to illustrate the equilibrium characterization in Theorem 1 that we believe are of particular interest as applications. In the first section, our focus is on settings in which all principals exhibit congruent preferences (e.g., $\mathcal{A} = \mathcal{N}$ and $\mathcal{B} = \emptyset$), and our particular application is the provision of a public good by citizen-principals employing a privately informed agent-supplier. In the subsequent section, our focus shifts to environments in which principals disagree about the preferred direction of action, and we apply our framework to games in which principal-lobbyists attempt to influence the policy choice of a privately informed agent-legislator. Our focus in both games is primarily on the maximal equilibrium allocations, but as a comparison we also characterize non-maximal equilibria in the context of the lobbying game.

5.1. *Congruent Preferences – Private Provision of Public Goods*

In the public-good game we consider, there are n principal-citizens and a privately-informed supplier of a public good. Each principal values the public good, but the principals may differ in the intensities of their preferences. Formally, we order the n principals such that $s_1 \geq ... \geq s_n > 0$ and will denote a configuration of principal preferences by the vector $\boldsymbol{s} \equiv (s_1, \dots, s_n)$. Each principal offers the common agent a contribution schedule, t_i, which promises a payment $t_i(q)$ to the agent for q units of public good. We take the domain of public goods to be $\mathcal{Q} = [0, q_{\max}]$, with $q_{\max}$ larger than the first-best level of public good, and for simplicity we assume that $S_0(0) = S_0'(0) = 0$.

5.1.1. *Properties of the maximal equilibrium allocation*

Specializing (12) from Theorem 1 to the public goods setting, the maximal equilibrium allocation satisfies

$$\overline{q}^{\mathcal{Q}}(\theta) = \arg\max_{q \in \mathcal{Q}} \; S_0(q) - \theta q + \left(\sum_{i \in N} \max\left\{ s_i - \frac{F(\theta)}{f(\theta)}, 0 \right\} \right) q. \tag{15}$$

There are two natural benchmarks for comparison. Under complete information, the first-best, full-information allocation satisfies

$$\overline{q}^{fb}(\theta) = \arg\max_{q \in \mathcal{Q}} \; S_0(q) - \theta q + \left(\sum_{i \in N} s_i \right) q,$$

and so we have the immediate (and familiar) result that the presence of incomplete information results in a downward distortion in activity except at the most efficient type, $\underline{\theta}$. In the present setting, however, there are two distinct reasons for this downward distortion.

A second benchmark allows us to decompose this further. Suppose that there is a single principal who has linear preferences for the public good given by $\sum_i s_i$; alternatively, one can think of a cooperative formed with all principals designing their compensation schedule to maximize their collective surplus. In this case, the optimal allocation coincides with familiar allocation of Baron and Myerson (1982) in which the government regulates a monopolist with unknown marginal cost. Formally, the solution is

$$\overline{q}^{bm}(\theta) = \arg\max_{q \in \mathcal{Q}} \; S_0(q) - \theta q + \max\left\{ \left(\sum_{i \in N} s_i \right) - \frac{F(\theta)}{f(\theta)}, 0 \right\} q.$$

Close inspection reveals that

$$\overline{q}^{fb}(\theta) \geq \overline{q}^{bm}(\theta) \geq \overline{q}^{\mathcal{Q}}(\theta)$$

for all θ with strict inequality for a positive measure of types. The difference between the first-best allocation and the Baron-Myerson solution is well understood as the outcome of a tradeoff between surplus extraction and inefficient output. The additional downward distortion between the Baron-Myerson allocation and the maximal allocation in the non-cooperative game can be understood as a tragedy of the commons in which each individual principal "over harvests" the agent's information rent. This is clearest to see when the principals have symmetric preferences, $s_i = s$ for all i, and we further assume that the principals are active for all types. In this case,

$$\overline{q}^{bm}(\theta) = \arg\max_{q \in \mathcal{Q}} \; S_0(q) + nsq - \theta q - \frac{F(\theta)}{f(\theta)} q,$$

but in the noncooperative setting

$$\overline{q}^{\mathcal{Q}}(\theta) = \arg\max_{q \in \mathcal{Q}} \; S_0(q) + nsq - \theta q - n\frac{F(\theta)}{f(\theta)} q.$$

Evidently, the information-rent term is magnified by a factor of n as each principal attempts to extract a margin of rents from the agent. The noncooperative public-goods game induces an n-fold marginalization that is in the same spirit as the problem of double marginalizations that arise in vertical sales relationships.

It is worth noting that this over harvesting of the agent's information rent disappears as one takes the limit of the maximal allocation as information becomes complete. Unlike the classic public-goods game which does not include a common agent and in which each principal contributes a fixed level of output to the public good, under complete information the presence of a common agent and the use of nonlinear prices allows the first-best allocation to arise in equilibrium. The noncooperative nature of the game introduces additional distortions *only because of the presence of incomplete information.* This is one of the novel insights of our approach.

5.1.2. *Comparison with* Intrinsic *Common-Agency Games*

In this paper, we require that each principal offers nonnegative transfers to the agent, which is formally equivalent to allowing the agent to accept only a subset of contract offers. Elsewhere in Martimort and Stole (2012), we considered the simpler setting in which the principals could offer negative transfers but the agent was restricted to either accept or reject to entire set of offers. In this case of *intrinsic* common agency in which rejection must be uniform across all principals, every principal is active for the same set of agent types. Moreover, if the principals' preferences are congruent, then the analogue of the maximal equilibrium in the case of intrinsic common agency is

$$\overline{q}^I(\theta) = \arg\max_{q\in\mathcal{Q}} S_0(q) - \theta q + \max\left\{\sum_{i\in\mathcal{N}}\left(s_i - \frac{F(\theta)}{f(\theta)}\right), 0\right\} q. \tag{16}$$

Comparing (16) to (15), we see that the fact that only a subset of principals may be active for a given type results in a higher (more efficient) level of activity compared to the game in which the agent must either accept or reject the entire set of contract offers. In the case of intrinsic common agency, if the agent is active, then the information term $F(\theta)/f(\theta)$ necessarily has a weight of n attached to it, while in the case of our influence game in which transfers are nonnegative, there may be a lower coefficient attached to the rent term for some less-efficient types.

5.1.3. *Comparative statics on principals' preferences*

The fact that the cooperative allocation is weakly higher than the noncooperative allocation (and strict for some types) is actually a special case of a more general phenomena and follows from an application of Jensen's inequality to the characterization in (15), which is convex in the principals' preference configuration.

PROPOSITION 2 *In the public goods game, consider two configurations of principal preferences, $\boldsymbol{s} = (s_1, \ldots, s_n)$ and $\tilde{\boldsymbol{s}} = (\tilde{s}_1, \ldots, \tilde{s}_n)$, where $\tilde{\boldsymbol{s}}$ is a mean-preserving spread [12] of $\boldsymbol{s}$. The associated maximal allocations in each game have the property that for all θ*

$$\overline{q}^{\mathcal{Q}}_{\tilde{\boldsymbol{s}}}(\theta) \geq \overline{q}^{\mathcal{Q}}_{\boldsymbol{s}}(\theta),$$

with a strict inequality for some positive measure.

As an illustration, consider the case of full principal integration; this is equivalent to one principal having preferences $\tilde{s}_1 = \sum_i s_i$ and the other $(n-1)$ principals having preferences $\tilde{s}_j = 0$ for $j \neq 1$. It follows that $\tilde{\boldsymbol{s}}$ is more disperse than $\boldsymbol{s}$ and therefore, from Proposition 2, the Baron-Myerson outcome with a collective of principals generates a higher allocation than in the noncooperative setting.

As another application, consider the case in which all of the principals are symmetric, $s_i = S/n$, and the total marginal benefit is fixed independent of n. As one increases n, the mean benefit is unchanged, but the dispersion of preferences increases.[13] It follows that an increase in n reduces public good provision. Indeed, using again (15) for this case, we have

$$\overline{q}^{\mathcal{Q}}(\theta) = \underset{q \in \mathcal{Q}}{\arg\max} \; S_0(q) - \theta q + n \left(\max \left\{ \frac{S}{n} - \frac{F(\theta)}{f(\theta)}, 0 \right\} \right) q,$$

which for $n \to \infty$ converges to

$$\overline{q}^{\mathcal{Q}}(\theta) = \underset{q \in \mathcal{Q}}{\arg\max} \; S_0(q) - \theta q.$$

Thus, in the limit no public good is provided. Note that this asymptotic inefficiency result arises from a very different source than the asymptotic inefficiency result in the public goods game of Mailath and Postlewaite (1990). In our setting, inefficiency arises because

[12] To be clear, given two configurations $\boldsymbol{s}$ and $\tilde{\boldsymbol{s}}$ with the same mean (i.e., $\sum_i s_i = \sum_i \tilde{s}_i$), we define the associated discrete distributions on the combined domain $\cup_i s_i \ \cup_j \tilde{s}_j$. If the distribution for $\boldsymbol{s}$ second-order stochastically dominates the distribution for $\tilde{\boldsymbol{s}}$, then we say that $\tilde{\boldsymbol{s}}$ is a mean-preserving spread of $\boldsymbol{s}$.

[13] To be precise, one should imagine a population of $m > n$ principals exists for which n principals have the preferences $s_i = S/n$ and $m - n$ principals have preferences $s_i = 0$. Now, as n increases toward m, the profile of principals' preferences increase in dispersion.

each principal attempts to extract the agent's marginal rent, ignoring the externality this has on the others; in Mailath and Postewaite's setting, each principal has private information about their willingness to pay (the agent's preferences are known), and the probability that any individual principal is pivotal goes to zero as the number of players increases.

More generally, Proposition 2 makes clear that the equilibrium allocation is not invariant with respect to redistributions of the principals' preferences, keeping the aggregate $\sum_i s_i$ constant. Thus, a unit tax on principal 1's use of the public good that is exactly offset by a unit subsidy on principal 2's use could have a real impact on the equilibrium allocation of public goods if this changed the set of active principals.[14] The conduit for how mean-preserving variations in the principals' preferences can have real impacts in the final allocation is reminiscent of findings in the public finance literature on voluntary contribution games (see, e.g., Bergstrom, Blume and Varian (1986), et al.). This literature, which has focused on complete information games in which players' strategies are scalar contributions (as opposed to schedules of contributions), demonstrates that neutrality arises in simple public goods games precisely when the set of contributors is unaffected by a variation in preferences or incomes; when the set of contributors is affected, however, the level of public good provision is typically altered. Similarly, we find in our richer incomplete-information setting with a privately-informed agent that the key source of non-neutrality is that an underlying variation can impact the set of principals who are actively influencing some type.

In the context of interest groups, the finding in Proposition 2 formalizes the ideas of Olson (1965) and Stigler (1974) that a group is more likely to be influential if the group's preferences are heterogeneous (e.g., a combination of small and large stakeholders, rather than a group of equal stakeholders). This idea has also been formalized in a simple setting of binary actions and preferences by LeBreton and Salanié (2003). The present paper shows that this result remains prominent in a richer setting.

Another political effect noted by Olson (1965) is that an increase in the stake of one interest group member raises that person's contribution, possibly lowers the contribution of others, but on net raises the total contribution (i.e., crowding out may arise, but it

[14]This is not the case in models of intrinsic common agency, as shown in Martimort and Stole (2012), because in such games all principals are active on the same type set and the allocation is unchanged by mean-preserving variations in the principals' preferences.

is never complete). We can find a similar result in the case of public goods where the increase in stake is modeled by an increase in s_i, and we can ask what happens to the maximal equilibrium allocation (and the marginal transfers of all principals) in this case.

PROPOSITION 3 *In the public goods game, consider two principal preference configurations,* $\boldsymbol{s}$ *and* $\tilde{\boldsymbol{s}}$*, in which* $\tilde{s}_i = s_i + \Delta_i$*,* $\Delta_i > 0$*, but* $\tilde{s}_j = s_j$ *for* $j \neq i$*. Then the associated maximal equilibrium allocations satisfy*

$$\overline{q}^{\mathcal{Q}}_{\tilde{\boldsymbol{s}}}(\theta) \geq \overline{q}^{\mathcal{Q}}_{\boldsymbol{s}}(\theta),$$

with strict inequality for some positive measure of types.

Furthermore, both the marginal aggregate payment function and the marginal payment function of principal i *weakly increase over the set of equilibrium choices (and strictly so for a subset of outputs), while the marginal payment functions of the other principals,* $j \neq i$*, weakly decrease over the set of equilibrium choices (and strictly so for a subset of outputs). Crowd out is less than perfect.*

This result follows directly from an application of (15): Because $\overline{q}^{\mathcal{Q}}(\theta)$ is weakly increasing in s_i (and strictly increasing in s_i for some positive measure of types), it follows that the maximal equilibrium allocation must weakly increase (strictly over the same measure of types). Hence, the aggregate marginal contribution schedule, $\overline{T}^{\mathcal{Q}\prime}(\overline{q})$, cannot decrease for any $q \in \overline{q}^{\mathcal{Q}}(\Theta)$ and must strictly increase for at least some range of q that are chosen in equilibrium by the agent. Next consider the marginal payments made by principals $j \neq i$ (whose stakes have remained constant). From Corollary 2, the marginal transfer of principal j is given by

$$\overline{t}^{\mathcal{Q}\prime}_j(q) = \beta_i(\overline{\vartheta}(q)) = \max\left\{s_j - \frac{F(\overline{\vartheta}(q))}{f(\overline{\vartheta}(q))}, 0\right\},$$

where $\theta = \overline{\vartheta}(q)$ is the inverse function of $q = \overline{q}^{\mathcal{Q}}(\theta)$ and is uniquely defined at every point of differentiability of $\overline{t}^{\mathcal{Q}}_j(q)$. For any region of types for which $\overline{q}^{\mathcal{Q}}$ is decreasing and strictly higher, it follows that $\overline{\vartheta}(q)$ is also decreasing and strictly higher. From the marginal payment equation, $\overline{t}^{\mathcal{Q}\prime}_j(q)$ must be lower following the change in principal i's preferences for these q. Of course, we know that $\overline{T}^{\mathcal{Q}\prime}(q)$ is strictly higher for this q, so it follows that $\overline{t}^{\mathcal{Q}\prime}_i(q)$ must be increase more than the reduction of $\sum_j \overline{t}^{\mathcal{Q}\prime}_j(q)$. Hence, crowd out occurs, but it is less than perfect.

5.1.4. *A worked example*

We conclude our study of the public goods game with a worked example to illustrate the properties of the maximal equilibrium. To this end, we assume that $n = 2$, $S_0(q) = -\frac{1}{2}q^2$, $\mathcal{Q} = [0, q_{\max}]$ with $q_{\max}$ sufficiently large[15], and that θ is distributed uniformly on $[0, \overline{\theta}]$. As benchmarks, the efficient output is $q^{fb}(\theta) = \max\{s_1 + s_2 - \theta, 0\}$, and the cooperative Baron-Myerson allocation is $q^{bm}(\theta) = \max\{s_1 + s_2 - 2\theta, 0\}$. We use (15) to obtain a closed-form solution for the maximal allocation:

$$\overline{q}^{\mathcal{Q}}(\theta) = \max\left\{\sum_{i=1}^{2} \max\left\{s_i - \frac{F(\theta)}{f(\theta)}, 0\right\} - \theta, 0\right\}.$$

In our setting with $n = 2$, we can alternatively characterize $\overline{q}^{\mathcal{Q}}$ as the pointwise maximum of

$$\overline{q}^{1}(\theta) \equiv \max\left\{s_1 - \theta - \frac{F(\theta)}{f(\theta)}, 0\right\},$$

$$\overline{q}^{2}(\theta) \equiv \max\left\{s_2 - \theta - \frac{F(\theta)}{f(\theta)}, 0\right\},$$

and

$$q^{\{1,2\}}(\theta) \equiv \max\left\{s_1 + s_2 - \theta - 2\frac{F(\theta)}{f(\theta)}, 0\right\},$$

where we can think of $\overline{q}^{i}$ as the allocation that principal i would implement in isolation, and $\overline{q}^{\{1,2\}}$ as the allocation that arises whenever principals 1 and 2 are both actively influencing the agent with double marginalizations. Thus,

$$\overline{q}^{fb}(\theta) \geq \overline{q}^{bm}(\theta) \geq \max\{\overline{q}^{1}, \overline{q}^{2}, \overline{q}^{\{1,2\}}\} = \overline{q}^{\mathcal{Q}}(\theta).$$

An illustration makes clear these orderings.

[15] It suffices that $q_{\max} \geq -\underline{\theta} + \sum_{i \in N} s_i$ to prevent bunching at $q_{\max}$.

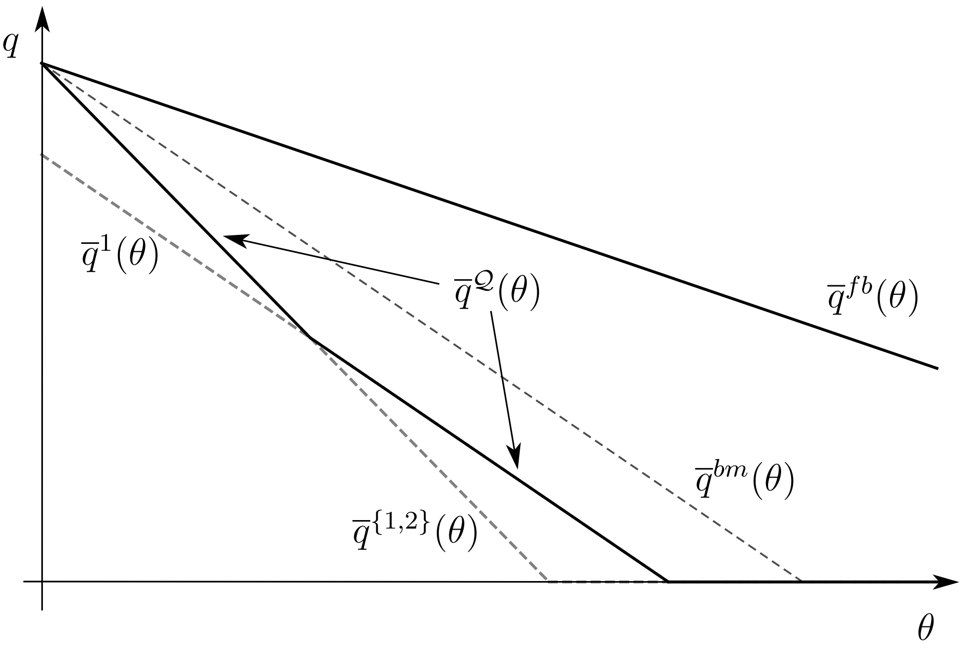

Figure 1: Voluntary provision of a public good. Figure is drawn for the case of θ uniformly distributed on $[0, \frac{5}{2}]$, $s_1 = \frac{7}{2} > s_2 = \frac{3}{4}$, and $S_0(q) = -\frac{1}{2}q^2$.

This example illustrates how distortions in delegated common agency games with congruent principals manifest themselves in two dimensions. First, because each active principal contributes less than his marginal valuation, inefficient provision arises at the intensive margin arises. The equilibrium output is lower than the cooperative solution and features the same two-fold distortion that is present in intrinsic common-agency games. A second distortion, novel to delegated agency games, emerges from limited participation by the weaker principal; output is also distorted at the extensive margin. In this example, there exists a non-empty interval of types, $s_2 \leq \frac{F(\theta)}{f(\theta)} \leq s_1$, such that only principal 1 is active under asymmetric information while both principals would be active as a cooperative or if information was complete.

The fact that output is inefficiently low in the noncooperative setting relative to the cooperative Baron-Myerson outcome may suggest that the familiar free-riding problem in classic public-goods contributions games is also present in public-goods games with more complex strategy spaces. While this is true in a sense, we again emphasize that the source of this multi-principal problem is incomplete information. If information were complete (which is tantamount to eliminating the inverse-hazard terms from the equation), the maximal equilibrium leads to full efficiency: each principal offers the marginal tariff $t_i'(q) = s_i$. This is the same efficient equilibrium outcome that arises in Bernheim and Whinston's (1986) "truthful equilibrium." Thus, free riding need not arise in complete-information public-goods games if the principals have the ability to offer nonlinear tariffs to a common agent rather than making direct, one-dimensional contributions to the public good. When incomplete information is present, however, each principal has a private incentive to distort the agent's output choice to extract additional information rent. Because each principal ignores the negative externality that this imposes on others, from a collective viewpoint, the principals inefficiently extract too much rent. The public goods free-riding problem present in our setting more closely fits the narrative of a "tragedy of the commons" in which each principal over harvests the common resource – the agent's information rent.

Another interpretation of the limited participation that may arise under asymmetric information is that some form of exclusive contracting emerges endogenously even if exclusivity clauses cannot be enforced at the outset. This is so even if both principals would otherwise have contracted with the agent under complete information. This finding is reminiscent of an important insight developed by Bernheim and Whinston (1998) in their study of vertical relationships between manufacturers and retailers. They showed that exclusive dealing in marketing practices arises when the agency costs of a common representation are too large compared with those under exclusive dealing. There is, however, an important difference between their result and ours. They assume that the possibility of exclusive representation arises ex ante, i.e., before the realization of uncertainty. Although their general contracting model is thus consistent with hidden actions or hidden information, it cannot account with the possibility of exclusivity arising for some realization of shocks and not for others. In this regard, our model, where contracting takes place once the agent is already informed, generates richer patterns of behaviors.

5.2. *Conflicting Preferences – Lobbying for Influence*

We next turn to settings in which the principals can be divided into two "interest" groups – $\mathcal{A}$ and $\mathcal{B}$ – with opposed objectives. In short, in games with conflicting preferences, it is as if there are two influence groups, each composed of principals who are contributing to the public good of their own group. Thus, previous results in the context of public goods that illustrated how changes in preferences affect influence continue to hold provided the comparative static is taken over one group in isolation. In the present section, we will see that there is an additional role of conflict between groups that generates non-nested sets of influence, and that an increase in dispersion across all principals can easily generate an increased dispersion of policies relative to the first best.

5.2.1. *A simple $n = 2$ model of conflict*

We begin with the simplest setting of two principals with conflicting preferences $s_1 > 0 > s_2$. As a motivation, principal 1 prefers a higher tax rate, q, whereas principal 2 prefers a lower tax rate. The decision-maker (agent) has some ideal policy he would like to pursue in the absence of any influence by lobbying groups. For simplicity, we model these preferences by taking $S_0(q) = -\frac{q^2}{2}$ where $q \in \mathcal{Q} = [-q_{max}, q_{max}]$ with q_{max} being large enough to ensure interior solutions to (9) allowing us to focus on first-order conditions. We will assume also that the agent's ideal point $q_0(\theta) = -\theta$ is symmetrically distributed over $[-\delta, \delta]$ with $\delta < 1$.[16] Choosing this bliss point gives a status quo payoff $U_0(\theta) = \frac{\theta^2}{2}$ to the agent.

5.2.2. *The maximal allocation*

Applying the general methodology developed in Theorem 1, we obtain:

PROPOSITION 4 *The maximal equilibrium allocation of the lobbying game with $s_1 > 0 > s_2$ is*

$$\bar{q}^{\mathcal{Q}}(\theta) = \max\left\{s_1 - \frac{F(\theta)}{f(\theta)}, 0\right\} + \min\left\{s_2 + \frac{1 - F(\theta)}{f(\theta)}, 0\right\} - \theta, \tag{17}$$

and the equilibrium marginal tariffs are given by

$$\bar{t}_j^{\mathcal{Q}\prime}(q) = \beta_i(\bar{\vartheta}(q)) = \begin{cases} \max\left\{s_j - \frac{F(\bar{\vartheta}(q))}{f(\bar{\vartheta}(q))}, 0\right\} & i \in \mathcal{A}, \\ \\ \max\left\{s_j - \frac{F(\bar{\vartheta}(q))}{f(\bar{\vartheta}(q))}, 0\right\} & i \in \mathcal{B}, \end{cases} \tag{18}$$

where $\theta = \bar{\vartheta}(q)$ is the inverse function of $q = \bar{q}^{\mathcal{Q}}(\theta)$.

If θ is uniformly distributed, the activity sets of the principals are

$$\bar{\Theta}_1 = [-\delta, \min\{s_1 - \delta, \delta\}) \quad \text{and} \quad \bar{\Theta}_2 = (\max\{\delta + s_2, -\delta\}, \delta]. \tag{19}$$

[16] Since principals are symmetrically biased in opposite directions, they would just agree on letting the agent choose his status quo policy had they cooperated.

If type heterogeneity is small relative to the strength of the principals' preferences,

$$\delta < \frac{s_1 + |s_2|}{2},$$

then the principals commonly influence a positive measure of intermediate-type agents; otherwise, each principal has a separate domain of influence.

The lobbying model shows that decision-makers with mild preferences receive contributions from both interest groups; unchallenged influence arises in our model endogenously for the decision-makers who are the most "ideologically" oriented.[17,18] This is, of course, a much richer pattern of influence and contributions than what is predicted by complete information lobbying games as in Grossman and Helpman (1994) or Dixit and al. (1997). In those complete information models, group i enjoys exclusive influence on policy only when other potential interest groups are just indifferent between that policy induced by group i and other policies that they may induce with positive contributions. Moreover, the absence of heterogeneity in the decision-maker's preferences in those models makes it impossible to generate different patterns of contributions and thus it remains a puzzle in that literature as to why some groups target some legislators and not others.

[17]Martimort and Semenov (2008) derive further results on the patterns of contributions in a lobbying game with a different objective function for the agent.

[18]Such a finding is loosely consistent with empirical work by Kroszner and Stratmann (1998) who document situations in which political action committees (PACs) representing rival constituents in the financial services industry contribute similar amounts to the same legislators, providing that they are not on the House Banking Committee. For members of the House Banking Committee, however, rival PACs tend not to match each others' contributions. Indeed, commercial-bank PAC contributions are negatively correlated with contributions from securities and insurance PACs, though this correlation is statistically insignificant.

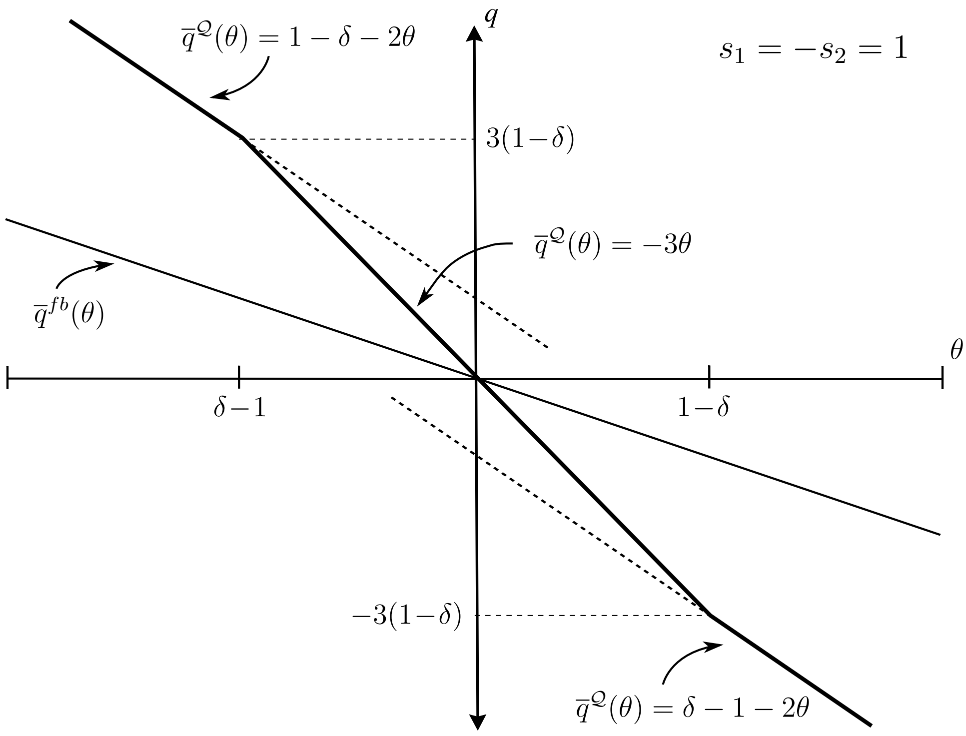

Figure 2: Maximal equilibrium allocation in a symmetric lobbying game, $s_1 = -s_2 = 1$, with $\delta \in (\frac{1}{2}, 1)$. Common influence arises for types $\theta \in (\delta - 1, 1 - \delta)$.

5.2.3. *A competitive nonlinear pricing reinterpretation*

Interestingly, the lobbying model can be transposed *mutatis mutandis* to an industrial organization setting to study how a consumer having private information on his most preferred bundle mixes between two goods marketed by two competing sellers. Suppose that this consumer wants to acquire one unit of a homogenous good and is located at a point $\theta \in [0, 1]$ on a unit line, with one seller's product being located at each extreme. The consumer has a valuation v for the good and incurs a quadratic loss of $-\frac{1}{2}(q - \theta)^2$ when consuming something that differs from his ideal of $q_0(\theta) = \theta$ from principal 1 and $1 - q_0(\theta)$ from principal 2. Up to some normalizations, the consumer and the sellers' profits are similar to those of the lobbying model above when the sellers' marginal costs are constant. Our previous results can be reinterpreted as giving conditions under which a share of the market is always covered by both sellers. When type-heterogeneity is sufficiently high, mixed bundling arises and global exclusivity cannot be an equilibrium. Hoernig and Valletti (2011) have independently derived a similar insight but, at the outset, restricted their analysis to smooth tariffs. As we will see below when studying discontinuous equilibria in the (similar) lobbying game, this restriction may indeed be justified because such smooth equilibrium may have attractive welfare properties among a much larger class of equilibria allowing for discontinuities. Nevertheless, there are discontinuous equilibria worth consideration. The approach in the present paper can also be applied to the more general issue of firms offering discounts to their customers based on their consumption mix, as in the recent debate over Intel's use of market-share discounts with electronics manufacturers who are also (potential) customers of AMD.[19]

5.2.4. *Discontinuous equilibria in the lobbying game.*

In the lobbying context, we establish the existence of discontinuous equilibria by applying the arguments used in the proof of Proposition 1 to construct arbitrary discontinuous allocations and then verifying that they are supported by equilibrium transfer functions. To this end, we assume a uniform and symmetric distribution around zero, and we introduce a single discontinuity to the maximal (continuous) allocation at $\theta = 0$. The following proposition provides an exact upper bound on the size of the equilibrium discontinuity; this bound makes clear that such discontinuity gaps may be significant.

PROPOSITION 5 *Suppose that* $s_1 = -s_2 = 1 < 2\delta$, $S_0(q) = -\frac{q^2}{2}$ *and that* θ *is uniformly distributed on* $\Theta = [-\delta, \delta]$. *For any* $q_0 \in (0, (1-\delta)\sqrt{3}]$, *there exists an equilibrium with a discontinuity at* $\theta_0 = 0$ *and such that* $\overline{q}(0^-) = -\overline{q}(0^+) = q_0$. *Both the agent's rent and the principals' expected payoffs in such discontinuous equilibria are lower than at the maximal equilibrium.*

In the proof of Proposition 5, we provide a construction of the tariffs supporting the discontinuous allocation and show that indeed the tariffs comprise an equilibrium to the common-agency game. The tariffs have a very natural structure. If $\bar{t}_i^Q$ is principal i's equilibrium tariff in the maximal equilibrium, and if the hypotheses of Proposition 5 are sat-

[19] Calzolari and Denicolo (2013) study a market-discount game and characterize one differentiable equilibrium in which the firms coordinate on extracting the customer's preference for variety; a general analysis of the larger set of equilibria is not undertaken. As in our analysis, the presence of two firms trying to extract the information rent of the consumer leads to greater distortions in consumption.

isfied, then the modified tariffs

$$\bar{t}_i(q) = \begin{cases} 0 & \text{for } q \in (-q_0, q_0), \\ \\ \bar{t}_i^{Q}(q) & \text{otherwise,} \end{cases}$$

support the discontinuous equilibrium.

To sustain those equilibria, principals design their contracts with "non-serious" out-of-equilibrium offers. For instance, principal 2 stipulates zero payments for outputs within the discontinuity gap $[\overline{q}(\theta_0^+), \overline{q}(\theta_0^-)]$ which are such that principal 1 is just indifferent to inducing the agent with type θ_0 to produce any output within that range. This construction makes it possible to sustain the discontinuity in the agent's choice.[20] Importantly, we demonstrate in the Appendix that a discontinuity can only be sustained if the equilibrium schedules lie below the maximal ones on the discontinuity gap. On the range of equilibrium outputs corresponding to those discontinuous equilibria, principals offer schedules which have the same margin as the maximal equilibria. So doing ensures that the agent still chooses the maximal output on any connected set in that range.

Consider our previous example of a symmetric lobbying game. One such discontinuous equilibrium which has a natural appeal exhibits extreme polarization: both lobbyists offer sufficiently strong incentives for their own cause such that no politician chooses an action in the middle of the policy space.

[20] By the same token, such construction could be replicated to sustain equilibria with multiple discontinuities.

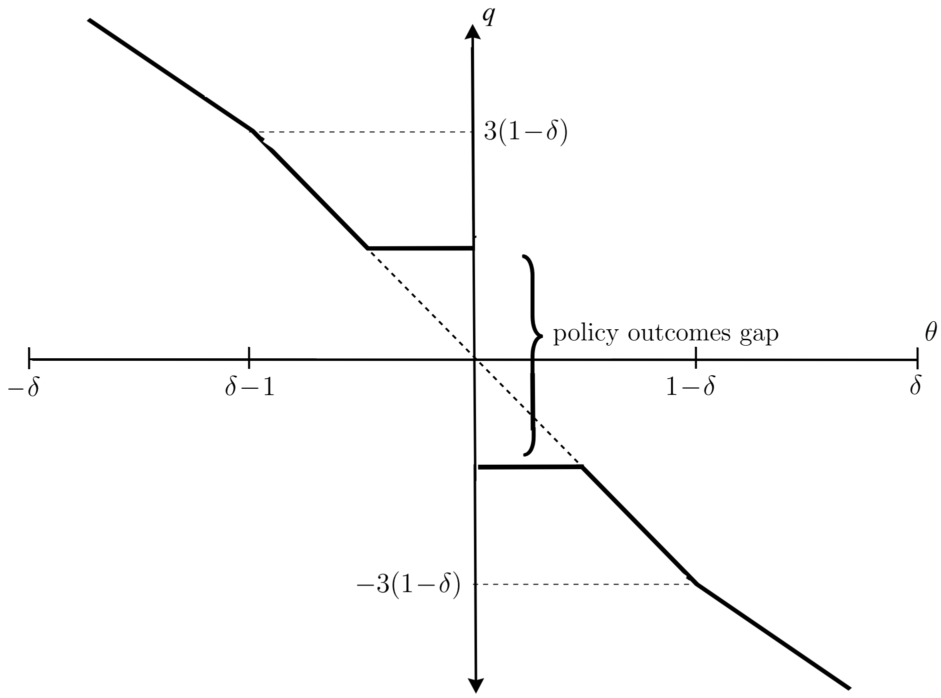

Figure 3: Discontinuous, "polarized" equilibrium in a symmetric lobbying game with $s_1 = -s_2 = 1$, $\delta \in (\frac{1}{2}, 1)$. Moderate policies, $q \in (-(1-\delta)\sqrt{3}, (1-\delta)\sqrt{3})$, are not chosen.

The comparison of the players' payoffs across equilibria in the lobbying context shows that the maximal equilibrium Pareto dominates, making it of focal interest. Not only the agent but also principals lose from coordinating on a discontinuous equilibrium. From Proposition 1, this result is clear for the agent since aggregate payments in those discontinuous equilibria are lower than at the maximal one. To explain the principals' preferences, observe that not paying the agent for policy choice within the discontinuity gap has two effects. First, it increases polarization since types nearby the discontinuity now pool at the boundaries of that discontinuity gap. This corresponds to more extreme policies than under the maximal equilibrium. Because principals have opposite preferences, this reallocation effect has no impact on their aggregate gross surplus. Second, those types who pool on decisions on each side of the policy gap end up being paid excessively compared with the maximal equilibrium. This is costly for the principals. That said, the polarization that arises in the discontinuous equilibrium does not seem inappropriate as a model of political lobbying. The fact that the equilibrium is inefficient relative to the maximal allocation does not persuade us to reject its relevance *a priori*. Indeed, experimental work by Kirchsteiger and Prat (2001) is suggestive that in complete information settings, the truthful (and efficient) equilibrium is not typically played and that instead a polarized, "natural"equilibrium is more focal for players, even though it is not as efficient. Our discontinuous equilibrium allocation has the flavor of the natural equilibrium in the complete information game.

5.2.5. *The impact of preference dispersion in lobbying games.*

The final issue we wish to address is the effect of a mean-preserving spread in principal preferences on the distribution of policies in our original framework of an arbitrary number of n principals (rather than $n = 2$). We maintain our assumption that $S_0(q) = -\frac{1}{2}q^2$, though the results below will generalize to any symmetric benefit function for the agent. For any configuration of principal preferences, we again order the preferences from highest to lowest, $s_1 \geq \cdots \geq s_k > 0 > s_{k+1} \geq \cdots \geq s_n$, where $\mathcal{A} = \{1, \ldots, k\}$ and $\mathcal{B} = \{k+1, \ldots, n\}$. With a slight abuse of notation, we will denote $\boldsymbol{s} = (\boldsymbol{s}_A, \boldsymbol{s}_B)$ to highlight the two separate vector components. The argument establishing our previous result in Proposition 2 directly extends to questions of preferences changes within one of the groups, $\mathcal{A}$ or $\mathcal{B}$, while holding the preferences of the other group fixed.

PROPOSITION 6 *Consider two configurations of principal preferences, $\boldsymbol{s} = (\boldsymbol{s}_A, \boldsymbol{s}_B)$ and $\tilde{\boldsymbol{s}} = (\tilde{\boldsymbol{s}}_A, \tilde{\boldsymbol{s}}_B)$. If $\tilde{\boldsymbol{s}}_A$ is a mean-preserving spread of $\boldsymbol{s}_A$ and $\tilde{\boldsymbol{s}}_B = \boldsymbol{s}_B$, then the associated maximal allocations in each game have the property that for all θ*

$$\overline{q}^{\mathcal{Q}}_{\tilde{\boldsymbol{s}}}(\theta) \geq \overline{q}^{\mathcal{Q}}_{\boldsymbol{s}}(\theta),$$

with a strict inequality for some positive measure. Similarly, if $\tilde{\boldsymbol{s}}_A = \boldsymbol{s}_A$ and $\tilde{\boldsymbol{s}}_B$ is a mean-preserving spread of $\boldsymbol{s}_B$, then

$$\overline{q}^{\mathcal{Q}}_{\tilde{\boldsymbol{s}}}(\theta) \leq \overline{q}^{\mathcal{Q}}_{\boldsymbol{s}}(\theta),$$

with a strict inequality for some positive measure.

We may immediately conclude that if the stakes of the players in group $\mathcal{A}$, for example, become more disperse, the equilibrium influence of group $\mathcal{A}$ on the distribution of policy increases to the detriment of group $\mathcal{B}$. More generally, more heterogeneous groups (holding mean preferences constant) have more influence.

Our result in Proposition 3 also has an immediate generalization that provides insight for the lobbying game. Suppose that some principal $i \in \mathcal{A}$ has an increased stake, but all remaining principals continue with the same stakes as before. Then it follows that $\overline{q}^{\mathcal{Q}}$ must weakly increase pointwise (in accord with the positive objectives of group $\mathcal{A}$), and this happens in spite of crowd out of contributions from other principals $j \neq i, j \in \mathcal{A}$ and in spite of reduced marginal contributions by principals $j \in \mathcal{B}$ in equilibrium.

PROPOSITION 7 *Consider two principal preference configurations, $\boldsymbol{s}$ and $\tilde{\boldsymbol{s}}$, in which $\tilde{s}_i = s_i + \Delta,\ \Delta > 0$, but $\tilde{s}_j = s_j$ for $j \neq i$. Then if $i \in \mathcal{A}$ (resp., $i \in \mathcal{B}$), the associated maximal equilibrium allocations satisfy*

$$\overline{q}^{\mathcal{Q}}_{\tilde{\boldsymbol{s}}}(\theta) \geq \ (resp., \leq)\ \overline{q}^{\mathcal{Q}}_{\boldsymbol{s}}(\theta),$$

with strict inequality for some positive measure of types.

Furthermore, if $i \in \mathcal{A}$ (resp., $i \in \mathcal{B}$) both the marginal aggregate payment function and the marginal payment function of principal i weakly increase (resp., decrease) over the set of equilibrium choices (and strictly so for a subset of outputs), while the marginal payment functions of the other principals, $j \neq i$, weakly decrease (resp., increase) over the set of equilibrium choices (and strictly so for a subset of outputs). Crowd out is less than perfect.

We want to conclude our analysis of dispersion by considering the effects of an increase in preference heterogeneity across all principals (not just within influence groups). In order to generate crisp predictions, we restrict our attention to situations in which the opposing interest groups are symmetric. Specifically, we assume that S_0 is a symmetric loss function around $q = 0$, that the density of types, f, is symmetric on $\Theta = [-\delta, \delta]$, and that the principals' preference configuration is *symmetric between interest groups*, $\boldsymbol{s}_A = (s_1, \ldots, s_k)$ and $\boldsymbol{s}_B = (-s_1, \ldots, -s_k)$. We have reordered the preferences of $\mathcal{B}$ ranging from largest in absolute value to smallest in absolute value, so that we can speak of the ith pair of principals to mean the pair in which $i \in \mathcal{A}$ has preference s_i and $i \in \mathcal{B}$ has preference $-s_i$. Observe, however, that we allow for arbitrary heterogeneity within groups. As a benchmark, note that under this symmetric specification of the influence game, aggregate principal preferences are zero and the first best policy outcome is $\overline{q}^{fb}(\theta) = -\theta$.

We consider a special form of a mean-preserving spread which preserves the original symmetry between interest groups so that $\tilde{\boldsymbol{s}}_A = -\tilde{\boldsymbol{s}}_B$. In particular, we say that $\tilde{\boldsymbol{s}}$ is a *pairwise mean-preserving spread* if there exists a vector of positive increments, $\Delta = (\Delta_1, \ldots, \Delta_k)$

such that $\tilde{\mathbf{s}}_A = \mathbf{s}_A + \Delta$ and $\tilde{\mathbf{s}}_B = \mathbf{s}_B - \Delta$.[21] For a given configuration of preferences, $\mathbf{s}$, we can compute the maximal allocation $\overline{q}^Q_{\mathbf{s}}$, and the implied distribution of policy choices:

$$G(q \mid \mathbf{s}) \equiv \text{Prob}_\theta \left[\overline{q}^Q_{\mathbf{s}}(\theta) \leq q \right],$$

The following result is derived from (9).

PROPOSITION 8 *Consider two configurations of principal preferences, $\mathbf{s}$ and $\tilde{\mathbf{s}}$, which are symmetric between interest groups. If $\tilde{\mathbf{s}}$ is a pairwise mean-preserving spread of $\mathbf{s}$, then $G(q \mid \tilde{\mathbf{s}})$ is a mean-preserving spread of $G(q \mid \mathbf{s})$.*

In general, a symmetric mean-preserving spread applied to a preference configuration that is itself symmetric between interest groups results in an increase in the dispersion of policy outcomes. As a reference point, notice that the distribution of first-best policies depends only upon the aggregate preferences of the principals and, therefore, is invariant to mean-preserving spreads. In our particular setting in which $\overline{q}^{fb}(\theta) = -\theta$, the first-best distribution is simply $G^{fb}(q) = 1 - F(-q)$.

To illustrate this phenomena, consider a simple stylized setting for $n = 2$, symmetric preferences $s_1 = -s_2 = s$ and θ uniformly distributed on $[-\delta, \delta]$ as before. For any preference parameter s, we can determine $\overline{q}^Q_{\mathbf{s}}$ and, in tandem with the original distribution over types θ, construct the implied equilibrium distribution of policies, $G(q|s)$, and its associated density, $g(q \mid s)$. Below, we plot the density $g(q|s)$ for various values of s and also plot the first-best distribution of policies, which coincides with the uniform distribution of θ.

[21] Technically, we have assumed a special case of a symmetric mean-preserving spread in that we require that the spread can be decomposed as k separate pairwise spreads. This makes the proof of Proposition 8 straightforward. We conjecture that a more general result is available for any symmetric, mean-preserving spread.

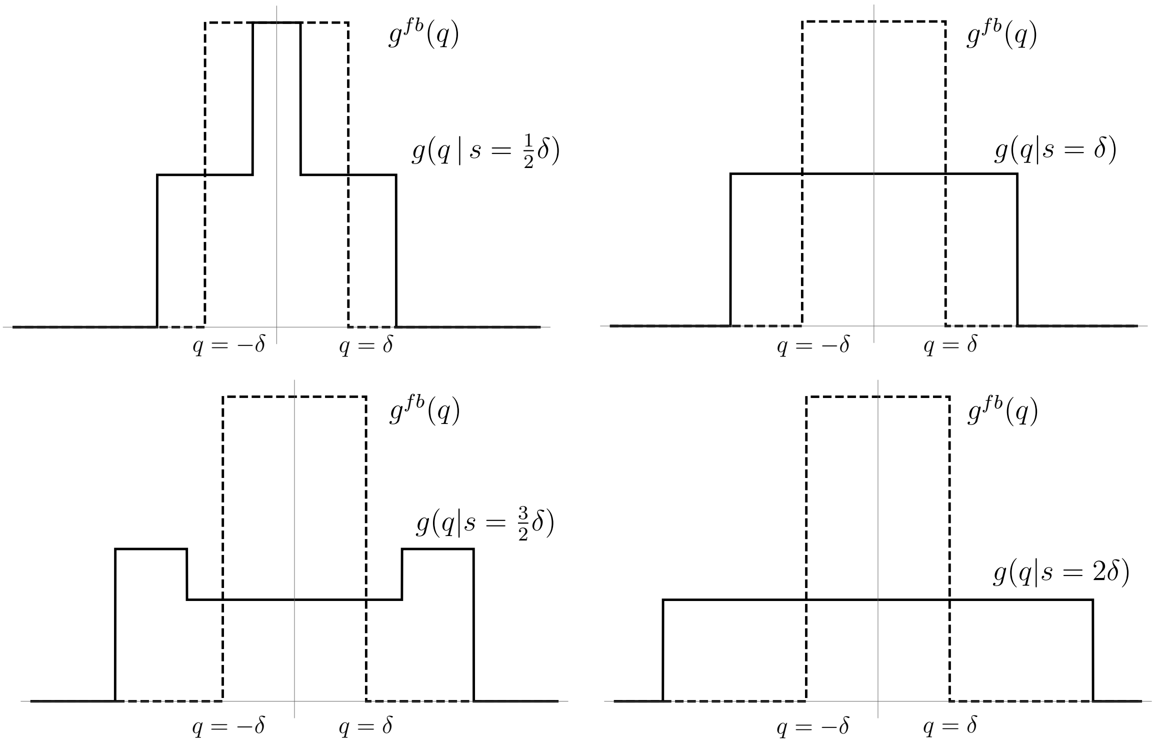

Figure 4: Equilibrium probability distributions of policies. $g^{fb}(q)$ is the probability density of the first-best policies; $g(q \mid s)$ is the equilibrium density of (maximal) equilibrium policies, where s varies from least-dispersed preferences, $s = \frac{1}{2}\delta$ (exclusive spheres of influence), to most dispersed preferences at $s = 2\delta$ (both principals actively influence all types).

In accord with the proposition, the greater the dispersion in preferences, the more disperse the distribution of policies.[22]

6. CONCLUSION

We have taken a large class of influence games with a privately-informed agent and shown a common feature of all equilibria is a confluence of the principals' marginal virtual valuations. If one is prepared to focus on maximal equilibria, their properties can easily be computed and comparative statics on underlying preferences yield a rich set of predictive relationships. One goal for this paper was to illustrate the simplicity of using

[22] McCarty, et al. (2006) have documented a strong correlation between increased campaign contributions and political polarization. In particular, contributors with extreme conservative or liberal preferences gave a disproportionately large share of the increased soft money allocated to congressional elections over the period 1982-2002. In the context of the lobbying model, such a finding can be generated with either an increase in the dispersion of preferences among contributors or, equivalently, with an increase in the value of a dollar to legislators relative to the value of the contributors.

the influence game with incomplete information, especially if one is prepared to impose the refinement of maximal equilibria. To this end, we have focused on two work horses – public goods games and lobbying games – but other applications such as nonlinear pricing are equally natural in this framework.

Of course, discontinuous equilibria also exist, but they are also straightforward to compute (once an assumption is made about where the discontinuities arise). The polarization in the discontinuous equilibrium of the lobbying game – though inefficient relative to the maximal equilibrium from the view of every player – has its own appeal and may naturally arise for similar reasons as the "natural" equilibrium in the experiments of Kirchsteiger and Prat (2001). For now, we remain agnostic about the selection of equilibria.

The class of influence games in this paper was chosen to be as large as possible while still allowing for powerful characterizations. We should note, however, that a limitation of our framework is our assumption that each principal has linear preferences over q. This can be relaxed under some stronger assumptions on the type distribution (as we noted in an earlier version of this paper, Martimort and Stole (2013)), but with considerable technical difficulty. The restriction that q is one dimensional, we conjecture, is a much less important assumption if one is willing to focus on continuous equilibria (a variation of maximal equilibria) and focus attention on characterizing the information-rent distortions. We leave the multi-dimensional generalization of the present setting to future work.

APPENDIX

PROOF OF LEMMA 1:

The proof of Lemma 1 proceeds in three steps. First, using a result in Martimort and Stole (2014), we provide a set of conditions that are necessary and sufficient for the solution to principal i's relaxed program (ignoring the convexity constraint on U). Second, we demonstrate the adjoint equations in these conditions can be further simplified given that the principal's preferences are linear in q. Third, we show that the solution to the relaxed and simplified program is a solution to the original program.

STEP 1: THE RELAXED PROGRAM. Consider the relaxed program, $(\mathcal{P}_i^r)$, that ignores the convexity constraint in (4):

$$(\mathcal{P}_i^r) : \max_{(U,q)} \int_\Theta \left(s_i q(\theta) + S_0(q(\theta)) + \overline{T}_{-i}(q(\theta)) - \theta q(\theta) - U(\theta)\right) f(\theta) d\theta$$

$$\text{subject to } U(\theta) \geq \overline{U}_{-i}(\theta) \text{ and } -q(\theta) \in \partial U(\theta) \text{ for all } \theta.$$

We rewrite this program using a change of variables in order to get it into a more useful format for applying a result from non-smooth control. Specifically, define the net utility that principal i's contract provides to the agent: $\Delta_i(\theta) = U(\theta) - \overline{U}_{-i}(\theta)$. It follows that, a.e., $\overline{q}_{-i}(\theta) - q(\theta) \in \partial\Delta_i(\theta)$ for $\overline{q}_{-i}(\theta) \in \arg\max_{q\in\mathcal{Q}} S_0(q) - \theta q + \overline{T}_{-i}(q)$.

We use Δ_i as the state variable and $q(\theta) - \overline{q}_{-i}(\theta)$ as the control variable in our new optimal control problem. Because $\overline{q}_{-i}(\theta)$ is data to this given program, $q(\theta)$ is effectively the control variable of principal i. Now we can state principal i's *relaxed* program in net payoffs as

$$\max_{(\Delta_i,q)} \int_\Theta \left(s_i(q(\theta)-\overline{q}_{-i}(\theta)) + S_0(q(\theta)) + \overline{T}_{-i}(q(\theta)) - \theta q(\theta) - \overline{U}_{-i}(\theta) - \Delta_i(\theta)\right) f(\theta) d\theta$$

$$\text{subject to } q(\theta) - \overline{q}_{-i}(\theta) \in -\partial\Delta_i(\theta),\ \Delta_i(\theta) \geq 0.$$

We apply Theorem 1 from Martimort and Stole (2014) and conclude that for any transfer $\overline{T}_{-i}$ offered by rival principals, the rent-output profile $(\overline{U}, \overline{q})$ is a solution to $(\mathcal{P}_i^r)$ if and only if $(\overline{U}, \overline{q})$ satisfies $\overline{U}(\theta) \geq \overline{U}_{-i}(\theta)$ and $-\overline{q}(\theta) \in \partial U(\theta)$ for all θ, and there exists a probability measure μ_i defined over the Borel subsets of Θ with an associated adjoint function, $\overline{M}_i : \Theta \to [0,1]$, defined by $\overline{M}_i(\underline{\theta}) = 0$ and for $\theta > \underline{\theta}$,

$$\overline{M}_i(\theta) \equiv \int_{[\underline{\theta},\theta)} \mu_i(d\theta),$$

such that the following two conditions are satisfied:

$$\text{(20)} \qquad supp\,\{\mu_i\} \subseteq \left\{\theta \,|\, \overline{U}(\theta) = \overline{U}_{-i}(\theta)\right\},$$

$$\overline{q}(\theta) \in \arg\max_{q \in \mathcal{Q}} \; s_i q + S_0(q) + \overline{T}_{-i}(q) + \left(\frac{\overline{M}_i(\theta) - F(\theta)}{f(\theta)} - \theta \right) q, \; a.e. \tag{21}$$

STEP 2: CHARACTERIZATION OF ADJOINT, $\overline{M}_i$. We prove the following simplifying lemma.

LEMMA 2 *In the linear common-agency game, if $(\overline{q}, \overline{U})$ is an equilibrium allocation, then for each principal i, $(\overline{q}, \overline{U})$ satisfies conditions (20) and (21) using the adjoint function*

$$\overline{M}_i(\theta) = \begin{cases} \max\{F(\theta) - s_i f(\theta), 0\}, & i \in \mathcal{A}, \; \forall \theta \in (\underline{\theta}, \overline{\theta}], \\ \\ \min\{F(\theta) - s_i f(\theta), 1\}, & i \in \mathcal{B}, \; \forall \theta \in (\underline{\theta}, \overline{\theta}], \end{cases} \tag{22}$$

such that $\overline{M}_i(\underline{\theta}) = 0$.

Proof of Lemma 2: We present the proof for $i \in \mathcal{A}$; the case for $i \in \mathcal{B}$ proceeds accordingly.

1. Define $\hat{\theta}_i$ as the unique solution to $s_i f(\hat{\theta}_i) = F(\hat{\theta}_i)$. Two properties are immediately implied for the region $(\hat{\theta}_i, \overline{\theta}]$. First, the monotone hazard rate property implies that

$$(\hat{\theta}_i, \overline{\theta}] = \{\theta \mid F(\theta) - s_i f(\theta) > 0\}.$$

   Moreover, the slope of $F(\theta) - s_i f(\theta)$ is positive if $f(\theta) > s_i f'(\theta)$; because the monotone hazard rate condition also requires $f'(\theta)/f(\theta) \leq f(\theta)/F(\theta)$, it follows that $F - s_i f$ is increasing if $F(\theta)/f(\theta) > s_i$. We conclude a second property of $(\hat{\theta}_i, \overline{\theta}]$ is that $F(\theta) - s_i f(\theta)$ is strictly increasing on this interval.

2. We next show that the set of types for whom principal i is active (i.e., $U(\theta) > \overline{U}_{-i}(\theta)$) is a lower interval, $\overline{\Theta}_i = [\underline{\theta}, \theta_0)$ where $\theta_0 \leq \hat{\theta}_i$.

   Suppose that on $[\theta_0, \theta_1] \subseteq \text{int}\Theta$ we have $\overline{U}(\theta) = \overline{U}_{-i}(\theta)$, but for $\varepsilon > 0$ sufficiently small we have $\overline{U}(\theta) > \overline{U}_{-i}(\theta)$ on the adjacent neighborhoods, $\theta \in (\theta_0 - \varepsilon, \theta_0) \cup (\theta_1, \theta_1 + \varepsilon)$. Because $\overline{U}(\theta) > \overline{U}_{-i}(\theta)$ on $(\theta_1, \theta_1 + \varepsilon)$ and those rent functions are continuous, convex with $\overline{q}(\theta_1) \in \partial\overline{U}(\theta_1)$ and $\overline{q}_{-i}(\theta_1) \in \partial\overline{U}_{-i}(\theta_1)$, it must be that $\overline{q}(\theta) < \overline{q}_{-i}(\theta)$ on this region for ε sufficiently small. For this inequality to be satisfied, (21) requires that $\overline{M}_i(\theta) < F(\theta) - s_i f(\theta)$ for all $\theta \in (\theta_1, \theta_1 + \varepsilon)$. Because the participation constraint is slack on $(\theta_1, \theta_1 + \varepsilon)$, $\overline{M}_i(\theta)$ is constant equal to $\overline{M}_i(\theta_1)$, and we have also $\overline{M}_i(\theta_1) < F(\theta) - s_i f(\theta)$. Because $\overline{M}_i(\theta) \geq 0$, it follows that $F(\theta) - s_i f(\theta) > 0$ on this interval which implies $\theta_1 \geq \hat{\theta}_i$. Because $F - s_i f$ is increasing for all $\theta > \hat{\theta}_i$, we can also conclude that

$$(23) \qquad s_i f(\theta) + \overline{M}_i(\theta_1) - F(\theta) \leq s_i f(\theta_1) + \overline{M}_i(\theta_1) - F(\theta_1) < 0 \quad \forall \theta \in (\theta_1, \overline{\theta}].$$

Suppose now that the participation constraint is binding on a second interval $[\theta_2, \theta_3]$ (possibly reduced to a point) with ϵ small enough so that $\theta_1 + \epsilon < \theta_2 - \epsilon$. On the interval (θ_1, θ_2), the fact that the participation constraint remains slack implies that $\overline{M}_i(\theta) = \overline{M}_i(\theta_1)$ on that interval. Because the participation constraint binds at θ_2, it must be that $\overline{q}(\theta) > \overline{q}_{-i}(\theta)$ on $(\theta_2 - \epsilon, \theta_2)$ which, using (21), would mean $s_i f(\theta) + \overline{M}_i(\theta_1) - F(\theta) > 0$ on that interval. A contradiction with (23). Thus, there is at most one region of binding participation, $[\theta_0, \theta_1]$.

Suppose now that the participation constraint is binding on $[\theta_0, \theta_1]$, $\theta_1 < \overline{\theta}$ and the participation constraint is slack in the right-neighborhood of θ_1. Because $\overline{U}$ and $\overline{U}_{-i}$ are convex functions and $\overline{U}_{-i}$ is a lower envelope of $\overline{U}$ on $[\theta_0, \theta_1]$, it follows that there is a neighborhood, $(\theta_1, \theta_1 + \varepsilon)$ such that $\overline{q}(\theta) < \overline{q}_{-i}(\theta)$ for all $\theta \in (\theta_1, \theta_1 + \varepsilon)$ for $\varepsilon > 0$ sufficiently small.

Suppose that principal i uses t_i to implement the conjectured equilibrium allocation. Then there exists a variation of this transfer, $\tilde{t}_i$ that creates a strict improvement. Define

$$\tilde{t}_i(q) = \begin{cases} \max\{t_i(q) - \eta, 0\} & \text{if } q \leq q_{-i}(\theta_1) \\ t_i(q) & \text{otherwise.} \end{cases}$$

We take $\eta > 0$ sufficiently small such that the allocation becomes $\tilde{q}(\theta) = \overline{q}_{-i}(\theta)$ for all $\theta \in (\theta_1, \theta_1 + \varepsilon)$ and $\tilde{q}(\theta) = \overline{q}(\theta)$ otherwise. The principal first gains from increasing quantity over $\theta \in (\theta_1, \theta_1 + \varepsilon)$ and not paying anything for that but he also gains from reducing payments by η for all $q \leq q_{-i}(\theta_1)$.

Therefore, we conclude that the participation constraint $\overline{U}(\theta) \geq \overline{U}_{-i}(\theta)$ is binding on an interval $[\theta_0, \overline{\theta}]$.

3. Because the activity set is of the form $[\underline{\theta}, \theta_0)$, (21) implies $\overline{M}_i(\theta) = 0$ on that interval.

4. We now establish that $\theta_0 \leq \hat{\theta}_i$.

   Because the activity set is of the form $[\underline{\theta}, \theta_0)$, $\overline{U}(\theta) \geq \overline{U}_{-i}(\theta)$ and thus necessarily $\overline{q}(\theta) > \overline{q}_{-i}(\theta)$ for $\theta \in (\theta_0 - \varepsilon, \theta_0)$ for ε small enough. Moreover, the structure of the activity set implies that $\overline{M}_i(\theta) = 0$ on that interval.

   Note also that over such interval, it is almost surely true that $\overline{q}_{-i}(\theta)$ is the unique maximizer of $S_0(q) + \overline{T}_{-i}(q) - \theta q$; this is because the convexity of $\overline{U}_{-i}$ implies that almost everywhere the best-response correspondence $\overline{q}_{-i}(\theta) = \partial \overline{U}_{-i}(\theta)$ is single-valued.

Using (21) and from the previous item $\overline{M}_i(\theta) = 0$ on $(\theta_0 - \varepsilon, \theta_0)$, we thus get for such θ :

$$\left(s_i - \frac{F(\theta)}{f(\theta)}\right)(\overline{q}(\theta) - \overline{q}_{-i}(\theta)) \geq$$

$$S_0(\overline{q}_{-i}(\theta)) + \overline{T}_{-i}(\overline{q}_{-i}(\theta)) - \theta\overline{q}_{-i}(\theta) - \left(S_0(q) + \overline{T}_{-i}(q) - \theta q\right) \geq 0$$

where the last inequality follows from the definition of $\overline{q}_{-i}(\theta)$. Thus

$$\left(s_i - \frac{F(\theta)}{f(\theta)}\right)(\overline{q}(\theta) - \overline{q}_{-i}(\theta)) \geq 0$$

where the last inequality follows from the definition of $\overline{q}_{-i}(\theta)$. Furthermore, the last inequality is thus strict almost everywhere on $(\theta_0 - \varepsilon, \theta_0)$. Thus, we deduce that $0 > F(\theta) - s_i f(\theta)$ almost everywhere on that interval and therefore by continuity everywhere. Thus, we conclude that $0 \geq F(\theta_0) - s_i f(\theta_0)$ and thus $\theta_0 \leq \hat{\theta}_i$.

5. We have now established that $\hat{\theta}_i$ lies in the inactive region $\left[\theta_0, \overline{\theta}\right]$. On $\left[\hat{\theta}_i, \overline{\theta}\right]$, we may as well choose $\overline{M}_i(\theta) = F(\theta) - s_i f(\theta)$. This choice of the adjoint function indeed satisfies conditions (20) (from item 1. $\overline{M}_i(\theta)$ is increasing over $\left[\hat{\theta}_i, \overline{\theta}\right]$) and (21).

   We next characterize $\overline{M}_i$ over the (possibly empty interior) interval $[\theta_0, \hat{\theta}_i)$ where $\theta_0 < \hat{\theta}_i$. An implication of (21) is that there exists an adjoint $\overline{M}_i(\theta)$ such that:

$$s_i + \frac{\overline{M}_i(\theta) - F(\theta)}{f(\theta)} \in \partial\overline{co}\{S_0(\overline{q}_{-i}(\theta)) + \overline{T}_{-i}(\overline{q}_{-i}(\theta)) - \theta\overline{q}_{-i}(\theta)\}.$$

   Because $\overline{M}_i \geq 0$ and $\theta < \hat{\theta}_i$, over the interval $[\theta_0, \hat{\theta}_i)$, we have

$$s_i + \frac{\overline{M}_i(\theta) - F(\theta)}{f(\theta)} > 0.$$

   Moreover, by definition of $\overline{q}_{-i}(\theta)$, we have also:

$$0 \in \partial\overline{co}\{S_0(\overline{q}_{-i}(\theta)) + \overline{T}_{-i}(\overline{q}_{-i}(\theta)) - \theta\overline{q}_{-i}(\theta)\}.$$

   Because $\partial\overline{co}\{S_0(\overline{q}_{-i}(\theta)) + \overline{T}_{-i}(\overline{q}_{-i}(\theta)) - \theta\overline{q}_{-i}(\theta)\}$ is an interval and thus convex, $s_i - \frac{F(\theta)}{f(\theta)}$ which is a convex combination of $s_i + \frac{\overline{M}_i(\theta) - F(\theta)}{f(\theta)}$ and 0 also satisfies:

$$s_i - \frac{F(\theta)}{f(\theta)} \in \partial\overline{co}\{S_0(\overline{q}_{-i}(\theta)) + \overline{T}_{-i}(\overline{q}_{-i}(\theta)) - \theta\overline{q}_{-i}(\theta)\}.$$

   In other words, there would be no loss of generality in taking $\overline{M}_i(\theta) = 0$ over the interval $[\theta_0, \hat{\theta}_i)$. We have now completely characterized the adjoint as in Lemma 2. (Incidentally, because $\partial\overline{co}\{S_0(\overline{q}_{-i}(\theta)) + \overline{T}_{-i}(\overline{q}_{-i}(\theta)) - \theta\overline{q}_{-i}(\theta)\}$ is almost everywhere single-valued, it follows that $s_i + \frac{\overline{M}_i(\theta)-F(\theta)}{f(\theta)} = 0$, which implies that $\overline{M}_i(\theta)$ has to be a negative number, a contradiction to the existence of $\theta_0 < \hat{\theta}_i$.) □

STEP 3: THE SOLUTION TO THE RELAXED PROGRAM IS CONVEX. What remains is to demonstrate that the solution $\overline{q}$ to the relaxed program is weakly decreasing (equivalently, that $\overline{U}$ is convex). Given Lemma 2, we can replace (20) and (21) with the conditions (6) and (7). The latter requires that

$$\overline{q}(\theta) \in \arg\max_{q\in\mathcal{Q}}\ S_0(q) + (\beta_i(\theta) - \theta)q + \overline{T}_{-i}(q).$$

Given that $\beta_i(\theta) - \theta$ is strictly decreasing in θ, it follows that $\overline{q}$ is weakly decreasing in θ for any upper semi-continuous $\overline{T}_{-i}$. Hence, the solution to the relaxed program is a solution to the original program. □

PROOF OF THEOREM 1:

NECESSITY. Lemma 1 must hold for any equilibrium allocation. Adding up (7) across all n principals, we obtain the condition, for almost every θ, the allocation satisfies

$$\overline{q}(\theta) \in \arg\max_{q\in\mathcal{Q}}\ S_0(q) + (\beta(\theta) - \theta)q + (n-1)(S_0(q) - \theta q + \overline{T}(q)), \tag{24}$$

where $\overline{T}$ implements $(\overline{q}, \overline{U})$. Simple revealed preference arguments show that $\overline{q}(\theta)$ is necessarily non-decreasing since $\beta(\theta) - \theta$ is itself non-increasing.

Define the value function of this program by

$$V(\theta) \equiv \max_{q\in\mathcal{Q}}\ S_0(q) + (\beta(\theta) - \theta)q + (n-1)(S_0(q) - \theta q + \overline{T}(q)).$$

From the fact that the maximand above is absolutely continuous in θ, upper semi-continuous in q and $\mathcal{Q}$ is compact, it follows that $V(\theta)$ is absolutely continuous. Moreover, given that $(\overline{q}, \overline{U})$ is an incentive-compatible allocation which solves this program,

$$V(\theta) = S_0(\overline{q}(\theta)) + (\beta(\theta) - \theta)\overline{q}(\theta) + (n-1)\bar{U}(\theta).$$

Because V is absolutely continuous, it is almost everywhere differentiable and for any pair (θ, θ'),

$$V(\theta) - V(\theta') = \int_{\theta'}^{\theta} (\beta'(x) - n)\overline{q}(x)dx.$$

Because $\overline{U}$ is implementable, it is absolutely continuous and therefore for any pair (θ, θ') we have

$$\overline{U}(\theta) - \overline{U}(\theta') = -\int_{\theta'}^{\theta} \overline{q}(x)dx.$$

Note that

$$S_0(\overline{q}(\theta)) + (\beta(\theta) - \theta)\overline{q}(\theta) - \left[S_0(\overline{q}(\theta')) + (\beta(\theta') - \theta')\overline{q}(\theta')\right]$$
$$= V(\theta) - V(\theta') - (n-1)\left[\overline{U}(\theta) - \overline{U}(\theta')\right]$$

or more simply

$$S_0(\overline{q}(\theta)) + (\beta(\theta) - \theta)\overline{q}(\theta) - \left[S_0(\overline{q}(\theta')) + (\beta(\theta') - \theta')\overline{q}(\theta')\right] = \int_{\theta'}^{\theta} (\beta'(x) - 1)\overline{q}(x)dx. \tag{25}$$

Using the relationship

$$((\beta(\theta) - \theta) - (\beta(\theta') - \theta'))\overline{q}(\theta') = \int_{\theta'}^{\theta} (\beta'(x) - 1)\overline{q}(\theta')dx,$$

and the fact that β and $\overline{q}$ are both weakly non-increasing, we obtain:

$$S_0(\overline{q}(\theta)) + (\beta(\theta) - \theta)\overline{q}(\theta) - \left[S_0(\overline{q}(\theta')) + (\beta(\theta) - \theta)\overline{q}(\theta')\right]$$
$$= \int_{\theta'}^{\theta} (\beta'(x) - 1)(\overline{q}(x) - \overline{q}(\theta'))dx \geq 0.$$

Because any $q' \in \overline{q}(\Theta)$ can be identified with some $\theta' \in \Theta$, the inequality implies $\overline{q}(\theta)$ satisfies (11) pointwise in θ.

SUFFICIENCY. Suppose that $\overline{q}^{\mathcal{Q}}$ satisfies (12). Because $\beta(\theta) - \theta$ is decreasing, $\overline{q}^{\mathcal{Q}}$ is non-increasing. Define the inverse of $\overline{q}^{\mathcal{Q}}$ as the correspondence

$$\overline{\vartheta}^{\mathcal{Q}}(q) \equiv \left[\min\{\theta | q = \overline{q}^{\mathcal{Q}}(\theta)\}, \max\{\theta | q = \overline{q}^{\mathcal{Q}}(\theta)\}\right].$$

Because $\overline{q}^{\mathcal{Q}}$ is non-increasing, this correspondence is monotone and almost everywhere single valued. Abusing notations, we will use $\overline{\vartheta}^{\mathcal{Q}}(q)$ as an arbitrary non-increasing selection from this correspondence when integrating.

We construct the individual tariffs of each principal $i \in N$ as follows:

$$\overline{t}_i^{\mathcal{Q}}(q) = \int_{\overline{q}^{\mathcal{Q}}(\hat{\theta}_i)}^{q} \beta_i(\overline{\vartheta}^{\mathcal{Q}}(x))dx.$$

Note that $\overline{t}_i^{\mathcal{Q}}$ is nonnegative by construction and $\overline{t}_i^{\mathcal{Q}}(q) = 0$ for $q = \overline{q}^{\mathcal{Q}}(\hat{\theta}_i)$. Because β_i and $\overline{\vartheta}^{\mathcal{Q}}(q)$ are non-increasing mappings, each constructed tariff is convex by construction. Denote the aggregates by $\overline{T}^{\mathcal{Q}} = \sum_{i \in N} \overline{t}_i^{\mathcal{Q}}(q)$ and $\overline{T}_{-i}^{\mathcal{Q}} = \sum_{j \neq i} \overline{t}_j^{\mathcal{Q}}(q)$. It follows that the aggregates are also convex.

What remains to be shown is (i) the aggregate transfer $\overline{T}$ induces the agent to choose $\overline{q}$, and (ii) each principal i, facing the rivals' aggregate $\overline{T}_{-i}$, finds it optimal to implement $\overline{q}$.

Incentive compatibility. Consider the agent's problem when facing aggregate payment, $\overline{T}^{\mathcal{Q}}$. For any pair (θ, q), the following conditions hold:

$$\begin{aligned} S_0(\overline{q}^{\mathcal{Q}}(\theta)) + \overline{T}^{\mathcal{Q}}(\overline{q}^{\mathcal{Q}}(\theta)) + (\beta(\theta) - \theta)\overline{q}^{\mathcal{Q}}(\theta) \quad &\geq \quad S_0(q) + \overline{T}^{\mathcal{Q}}(\overline{q}^{\mathcal{Q}}(\theta)) + (\beta(\theta) - \theta)q \\ &\geq S_0(q) + \overline{T}^{\mathcal{Q}}(q) + \beta(\overline{\vartheta}^{\mathcal{Q}}(q))(\overline{q}^{\mathcal{Q}}(\theta) - q) + (\beta(\theta) - \theta)q \end{aligned}$$

where the first inequality follows from the definition of $\overline{q}^{\mathcal{Q}}(\theta)$ and the second uses the convexity of $\overline{T}^{\mathcal{Q}}$. Simplifying further, we obtain

$$S_0(\overline{q}^{\mathcal{Q}}(\theta)) + \overline{T}^{\mathcal{Q}}(\overline{q}^{\mathcal{Q}}(\theta)) - \theta\overline{q}^{\mathcal{Q}}(\theta) \geq S_0(q) + \overline{T}^{\mathcal{Q}}(q) - \theta q + \left[(\beta(\overline{\vartheta}^{\mathcal{Q}}(q)) - \beta(\theta))(\overline{q}^{\mathcal{Q}}(\theta) - q)\right].$$

Because $\beta(\overline{\vartheta}^{\mathcal{Q}}(q))$ is non-increasing in q, the bracketed difference is always non-negative. Incentive compatibility is implied, as desired.

Principals' optimality. Consider principal i's program in light of Theorem 1 and Lemma 2. $\overline{q}^{\mathcal{Q}}$ is an optimal allocation for principal i if and only if

$$\overline{q}^{\mathcal{Q}}(\theta) \in \arg\max_{q \in \mathcal{Q}} \; S_0(q) + \overline{T}^{\mathcal{Q}}(q) - \overline{t}_i^{\mathcal{Q}}(q) + (\beta_i(\theta) - \theta)q, \quad a.e. \tag{26}$$

Remember that each tariff $\overline{t}_i^{\mathcal{Q}}$ is convex and therefore $\overline{T}_{-i}^{\mathcal{Q}}$ is convex. Now observe that for all pairs (θ, q), the following sequence of relationships holds:

$$\begin{aligned} & S_0(\overline{q}^{\mathcal{Q}}(\theta)) + \overline{T}_{-i}^{\mathcal{Q}}(\overline{q}^{\mathcal{Q}}(\theta)) + (\beta_i(\theta) - \theta)\overline{q}^{\mathcal{Q}}(\theta) \\ & \quad = \quad S_0(\overline{q}^{\mathcal{Q}}(\theta)) + \overline{T}_{-i}^{\mathcal{Q}}(\overline{q}^{\mathcal{Q}}(\theta)) + (\beta(\theta) - \beta_{-i}(\theta) - \theta)\overline{q}^{\mathcal{Q}}(\theta) \\ & \quad \geq \quad S_0(q) + (\beta(\theta) - \theta)q + \overline{T}_{-i}^{\mathcal{Q}}(\overline{q}^{\mathcal{Q}}(\theta)) - \beta_{-i}(\theta)\overline{q}^{\mathcal{Q}}(\theta) \\ & \quad \geq \quad S_0(q) + (\beta(\theta) - \beta_{-i}(\theta) - \theta)q + \overline{T}_{-i}^{\mathcal{Q}}(q) + \left[(\beta_{-i}(\overline{\vartheta}(q)) - \beta_{-i}(\theta))(\overline{q}^{\mathcal{Q}}(\theta) - q)\right] \\ & \quad = \quad S_0(q) + (\beta_i(\theta) - \theta)q + \overline{T}_{-i}^{\mathcal{Q}}(q) + \left[(\beta_{-i}(\overline{\vartheta}(q)) - \beta_{-i}(\theta))(\overline{q}^{\mathcal{Q}}(\theta) - q)\right] \\ & \quad \geq \quad S_0(q) + (\beta_i(\theta) - \theta)q + \overline{T}_{-i}^{\mathcal{Q}}(q). \end{aligned}$$

Both of the equalities above follow from the definition of β_i. The first inequality uses the fact that $\overline{q}^{\mathcal{Q}}(\theta)$ solves (12), while the second inequality follows from the convexity of $\overline{T}_{-i}^{\mathcal{Q}}$. The final inequality follows from the fact that, $\beta(\overline{\vartheta}(q))$ is non-increasing in q, and therefore the bracketed difference is always non-negative. This proves that (26) holds and that principal i desires to implement $\overline{q}^{\mathcal{Q}}$

when facing a rival aggregate of $\overline{T}_{-i}^{Q}$. Because $\bar{t}_i^Q$ is zero for $\overline{q}^Q(\hat{\theta}_i)$, the constructed tariff $\bar{t}_i^Q$ is the least-cost (nonnegative) transfer that accomplishes this end. $\square$

PROOF OF COROLLARY 3:

Let θ_0 be a point of discontinuity of $\overline{q}$. Such point is isolated because $\overline{q}$ is non-increasing and thus almost everywhere differentiable. Moreover $\overline{q}$ admits right- and left-hand side limits at θ_0, denoted respectively by $\overline{q}(\theta_0^+)$ and $\overline{q}(\theta_0^-)$ with $\overline{q}$ being continuous and differentiable both on a right- and a left-neighborhoods of θ_0. We also deduce from monotonicity that $\overline{q}(\theta_0^-) > \overline{q}(\theta_0^+)$ by incentive compatibility. The optimality conditions (11) at θ_0 imply that (13) must hold. Because S_0 is strictly concave, $S_0(q) + (\beta(\theta_0) - \theta_0)q$ has a unique maximum at $\overline{q}^Q(\theta_0)$, and we thus have $\overline{q}(\theta_0^-) > \overline{q}^Q(\theta_0) > \overline{q}(\theta_0^+)$.

It was established in the necessity proof of Theorem 1 that

$$S_0(\overline{q}(\theta)) + (\beta(\theta) - \theta)\overline{q}(\theta) = V(\theta) + (n-1)\overline{U}(\theta)$$

is absolutely continuous itself and thus almost everywhere differentiable. Using (25), the following condition holds at any point of differentiability of $\overline{q}$:

$$\dot{\overline{q}}(\theta)\left(S_0'(\overline{q}(\theta)) + \beta(\theta) - \theta\right) = 0. \tag{27}$$

From this, it follows that $\dot{\overline{q}}(\theta) = 0$ whenever $\overline{q}(\theta) \neq \overline{q}^Q(\theta)$ at a point of differentiability.

Using (27) on the right- and a left-neighborhoods of θ_0, we deduce that $\dot{\overline{q}}(\theta) = 0$ on such neighborhoods. By assumption, $\overline{q}(\Theta) \subset \overline{q}^Q(\Theta)$. Therefore, there exist θ_1 and θ_2 such that $\theta_2 < \theta_0 < \theta_1$ and $\overline{q}(\theta_0^-) = \overline{q}^Q(\theta_2) = q_2$ and $\overline{q}(\theta_0^+) = \overline{q}^Q(\theta_1) = q_1$. Because the allocation $\overline{q}$ must be non-decreasing, it can only be constant on the whole intervals $[\theta_2, \theta_0)$ and $(\theta_0, \theta_1]$. $\square$

PROOF OF PROPOSITION 1:

1. We shall prove this result by choosing an interval (q_1, q_2) containing $\hat{q}$ and constructing equilibrium tariffs $(\tilde{t}_1, \ldots, \tilde{t}_n)$ that induce the agent to select $\tilde{q}$.

   In what follows, we choose the open interval (q_1, q_2) sufficiently small, $q_2 - q_1 = \varepsilon$, such that each principal is either inactive for all $q \in (q_1, q_2)$ in the original allocation, or is active over the entire interval, $q \in (q_1, q_2)$. By hypothesis, there are at least two active principals over any sufficiently small interval (q_1, q_2). For any such interval, (q_1, q_2), define the corresponding type interval (θ_2, θ_1) such that in the original allocation $q_1 = \overline{q}(\theta_1)$ and $q_2 = \overline{q}(\theta_2)$.

By hypothesis, we can choose ε sufficiently small such that $\overline{q}$ is continuous and strictly decreasing over (θ_2, θ_1). As a result, Theorem 1 provides that $\overline{q}(\theta) = \overline{q}^Q(\theta)$ for all $\theta \in (\theta_2, \theta_1)$.

Because $\overline{q}(\theta) = \overline{q}^Q(\theta)$ is strictly decreasing and continuous over the interval (θ_2, θ_1), Corollary 2 implies that $\overline{T}_{-i}$ is differentiable on (q_1, q_2). From here, it follows that for each $i \in N$, for $\theta \in (\theta_2, \theta_1)$

$$\beta_i(\theta) = \bar{t}_i'(\overline{q}(\theta)).$$

Using again the fact that $\overline{q}(\theta) = \overline{q}^Q(\theta)$ on (θ_2, θ_1), we can use the inverse function of $\overline{q}^Q$, denoted $\overline{\vartheta}^Q(q)$, and integrate to obtain the result

$$\bar{t}_i(q_2) - \bar{t}_i(q_1) = \int_{q_1}^{q_2} \beta_i(\overline{\vartheta}^Q(x))dx. \tag{28}$$

2. Construction of $\{\tilde{t}_1, \ldots, \tilde{t}_n\}$, for a given interval, (q_1, q_2). For an arbitrary open interval (q_1, q_2), we construct the following tariffs:

$$\tilde{t}_i(q) = \begin{cases} \bar{t}_i(q) + \tau_{1,i} & q \leq q_1 \\ 0 & q \in (q_1, q_2) \\ \bar{t}_i(q) + \tau_{2,i} & q \geq q_2, \end{cases}$$

where

$$\tau_{1,i} = \begin{cases} 0 & i \in \mathcal{A} \\ -\beta_i(\hat{\theta})(q_2 - q_1) + \int_{q_1}^{q_2} \beta_i(\overline{\vartheta}^Q(x))dx & i \in \mathcal{B} \end{cases}$$

and

$$\tau_{2,i} = \begin{cases} \beta_i(\hat{\theta})(q_2 - q_1) - \int_{q_1}^{q_2} \beta_i(\overline{\vartheta}^Q(x))dx & i \in \mathcal{A} \\ 0 & i \in \mathcal{B}.[23] \end{cases}$$

By construction, these tariffs satisfy a few key properties. First, the constructed tariffs are

[23] In the case in which types are uniformly distributed, $\tau_{1,i} = \tau_{2,i} = 0$ for every i, and the construction is simple:

$$\tilde{t}_i(q) = \begin{cases} \bar{t}_i(q) & q \notin (q_1, q_2) \\ 0 & q \in (q_1, q_2). \end{cases}$$

nonnegative. To see this for $i \in \mathcal{A}$, note that we have $\tilde{t}_i(q_1) = \bar{t}_i(q_1) \geq 0$ and

$$\begin{aligned} \tilde{t}_i(q_2) &= \bar{t}_i(q_2) + \tau_{2,i} \\ &= \bar{t}_i(q_1) + \int_{q_1}^{q_2} \beta_i(\bar{\vartheta}^{Q}(x))dx + \tau_{2,i} \quad \text{(by (28))} \\ &= \bar{t}_i(q_1) + \beta_i(\hat{\theta})(q_2 - q_1) \\ &\geq 0 \quad \text{(because } \bar{t}_i(q_1) \geq 0 \text{ and } \beta_i \geq 0 \text{ for } i \in \mathcal{A}\text{)}, \end{aligned}$$

with strict inequality for every principal i that is active over the interval (q_1, q_2) in the original equilibrium.

A similar argument establishes nonnegativity for $i \in \mathcal{B}$.

A second property is that the constructed tariffs weakly increase over the interval for $i \in \mathcal{A}$ (i.e., $\tilde{t}_i(q_2) \geq \tilde{t}_i(q_1)$) and weakly decrease for $i \in \mathcal{B}$ (i.e., $\tilde{t}_i(q_2) \leq \tilde{t}_i(q_1)$). This follows for $i \in \mathcal{A}$ from the third line in the above nonnegativity argument. Thus, even if $\tau_{2,i}$ is negative, it is sufficiently small that principal i's tariff remains nondecreasing. A similar argument holds for $i \in \mathcal{B}$.

The third key property is that the marginal action for which principal i becomes active under the original tariff $\bar{t}_i$ (e.g., for $i \in \mathcal{A}$, the value of q^b such that $\bar{t}_i(q^b) = 0$ and $\bar{t}_i(q) > 0$ for all $q > q^b$) coincides with the marginal action under the newly constructed tariff. In the case of an active principal $i \in \mathcal{A}$, the marginal action under the original tariff lies to the left of q_1; because we chose $\tau_{1,i} = 0$, it follows that $\tilde{t}_i(q) = \bar{t}_i(q)$ for all $q < q_1$. A similar argument establishes that the marginal action is unchanged under the new tariffs for $i \in \mathcal{B}$.

3. Choice of (q_1, q_2). Corollary 3 gives the precise structure of $\tilde{q}$ that we wish to prove is an equilibrium allocation.

$$\tilde{q}(\theta) = \begin{cases} \bar{q}(\theta) & \theta \in [\theta_1, \bar{\theta}] \\ \bar{q}_1 & \theta \in (\theta_0, \theta_1) \\ \bar{q}_2 & \theta \in (\theta_2, \theta_0) \\ \bar{q}(\theta) & \theta \in [\underline{\theta}, \theta_2], \end{cases}$$

where θ_0 is the unique agent type such that

$$S_0(q_1) + (\beta(\theta_0) - \theta)q_1 = S_0(q_2) + (\beta(\theta_0) - \theta)q_2.$$

We have so far required only that the interval (q_1, q_2) contain $\hat{q}$ and that its length be sufficiently small such that a principal's activity is uniform over the interval and $\bar{q}(\theta) = \bar{q}^{Q}(\theta)$

for $\theta \in (\theta_2, \theta_1)$. We now impose the requirement that that (q_1, q_2) be chosen so that $\theta_0 = \hat{\theta}$ in the above indifference relation. That is, we choose (q_1, q_2) so that

$$S_0(q_1) + (\beta(\hat{\theta}) - \hat{\theta})q_1 = S_0(q_2) + (\beta(\hat{\theta}) - \hat{\theta})q_2 \tag{29}$$

is satisfied by construction. Given $\hat{\theta}$ and given $\varepsilon = q_2 - q_1$, there is a unique such choice of (q_1, q_2) which has this property.

We have now fully described the proposed equilibrium tariffs and allocation. Below we will demonstrate that such tariffs induce the agent to select $\tilde{q}$ (incentive compatibility) and that each principal finds the constructed tariff, $\tilde{t}_i$, to be a best response against the other constructions, $\tilde{T}_{-i}$.

4. Incentive compatibility. Because all tariffs are nonnegative, the agent will accept the profile of constructed offers. For the moment, suppose that the agent is restricted to choose $q \notin (q_1, q_2)$. Suppose also that we can establish that the marginal agent type indifferent between q_1 and q_2 under the new tariffs coincides with $\hat{\theta}$:

$$S_0(q_1) - \hat{\theta}q_1 + \overline{T}(q_1) + \sum_i \tau_{1,i} = S_0(q_2) - \hat{\theta}q_2 + \overline{T}(q_2) + \sum_i \tau_{2,i}. \tag{30}$$

In such a case, the aggregate tariffs would coincide (up to a constant) outside of the gap; i.e., the "margins" of these tariffs are equal outside of the gap. Because the agent cannot choose an action inside of the gap and the marginal agent is $\hat{\theta}$, it follows that $\tilde{q}$ will be the agent's choice as required.

The work is in establishing that (30) will indeed hold given the choice of τ's in the proposed construction. Given (29), proving (30) is equivalent to proving

$$\beta(\hat{\theta})(q_2 - q_1) = \overline{T}(q_2) - \overline{T}(q_1) + \sum_i (\tau_{2,i} - \tau_{1,i}).$$

Because (28) holds for all i, the required expression reduces to

$$\beta(\hat{\theta})(q_2 - q_1) - \int_{q_1}^{q_2} \beta(\overline{\vartheta}^{\mathcal{Q}}(x))dx = \sum_i (\tau_{2,i} - \tau_{1,i}).$$

But this expression is true by construction, given the formulae for each $\tau_{k,i}$.

Lastly, we remove the restriction that the agent must select $q \notin (q_1, q_2)$ and show that for ε sufficiently small, the agent would nonetheless never choose an action in the gap. This requires

$$\max_{q \in \overline{\mathcal{Q}} \setminus (q, q_2)} S_0(q) - \theta q + \tilde{T}(q) \geq \sup_{q \in (q_1, q_2)} S_0(q) - \theta q.$$

By construction, $\tilde{T}(q_1)$ and $\tilde{T}(q_2)$ are both bounded away from zero. As such, the continuity of S_0 implies that the inequality is satisfied for ε sufficiently small. With a sufficiently small gap, we have therefore established that $\{\tilde{t}_1, \dots, \tilde{t}_n\}$ implements $\tilde{q}$.

5. Principal optimality. To prove that $\tilde{t}_i$ is optimal, given $\tilde{T}_{-i}$, we apply Lemma 1 and confirm that for each i we satisfy the following two conditions for almost every $\theta \in \Theta$:

$$\tilde{U}(\theta) = \tilde{U}_{-i}(\theta) \iff \beta_i(\theta) = 0,$$

$$\tilde{q}(\theta) \in \arg\max_{q \in \mathcal{Q}} \; S_0(q) + (\beta_i(\theta) - \theta)q + \tilde{T}_{-i}(q).$$

We begin with the first requirement. Suppose that $\hat{\theta}_i$ is the lowest type such that $\beta_i(\theta) = 0$; i.e., $\hat{\theta}_i$ is the boundary type in principal i's activity set in the original equilibrium. Thus, in the original equilibrium, $\bar{t}_i(\bar{q}(\theta)) = 0$ for all $\theta \geq \hat{\theta}_i$ and $\bar{t}_i(\bar{q}(\theta)) > 0$ for all $\theta < \hat{\theta}_i$. We need to verify that a similar condition holds for the new equilibrium tariffs: $\tilde{t}_i(\tilde{q}(\theta)) = 0$ for all $\theta \geq \hat{\theta}_i$ and $\tilde{t}_i(\tilde{q}(\theta)) > 0$ for all $\theta < \hat{\theta}_i$. There are two cases to consider for $i \in \mathcal{A}$:

- $\hat{\theta}_i < \theta_2$. In this case, principal i is inactive over the gap in the original equilibrium. The tariff construction has $\tau_{1,i} = \tau_{2,i} = 0$ because $\beta_i(\theta) = 0$ for $\theta \in (\theta_2, \theta_1)$. Hence, $\tilde{t}_i(\tilde{q}(\theta)) = \bar{t}_i(\bar{q}(\theta)) = 0$ for $\theta \geq \hat{\theta}_i$ and $\tilde{t}_i(\tilde{q}(\theta)) = \bar{t}_i(\bar{q}(\theta)) > 0$ for all $\theta < \hat{\theta}_i$.
- $\hat{\theta}_i > \theta_1$. In this case, principal i is active over the gap in the original equilibrium.

  The tariff construction has $\tau_{1,i} = 0$, so $\tilde{t}_i = \bar{t}_i$ for $q \leq q_1$. In particular, this implies that $\tilde{t}_i(\tilde{q}(\theta)) = \bar{t}_i(\bar{q}(\theta)) = 0$ for $\theta \geq \hat{\theta}_i$ and $\tilde{t}_i(\tilde{q}(\theta)) = \bar{t}_i(\bar{q}(\theta)) > 0$ for $\theta \in [\theta_1, \hat{\theta}_i)$. Above, we established that $\tilde{t}_i(q_2) \geq \tilde{t}_i(q_1)$ for $i \in \mathcal{A}$. Because $\bar{t}_i$ is nondecreasing in q for $i \in \mathcal{A}$ in the original equilibrium, so may we conclude that $\tilde{t}_i(\tilde{q}(\theta)) > 0$ for all $\theta < \theta_1$.
- Note that the third possible case of $\hat{\theta}_i \in (\theta_2, \theta_1)$ is ruled out by choice of sufficiently small ε.

A similar argument establishes that $\tilde{U}(\theta) = \tilde{U}_{-i}(\theta) \iff \beta_i(\theta) = 0$ holds for $i \in \mathcal{B}$ under the constructed tariffs.

Suppose for the moment that principal i is restricted to choose $q \notin (q_1, q_2)$ and that the following indifference condition is satisfied for principal i:

$$S_0(q_1) + (\beta_i(\hat{\theta}) - \hat{\theta})q_1 + \tilde{T}_{-i}(q_1) = S_0(q_2) + (\beta_i(\hat{\theta}) - \hat{\theta})q_2 + \tilde{T}_{-i}(q_2). \tag{31}$$

Given $\tilde{T}_{-i}$ differs from $\overline{T}_{-i}$ by only a constant to the left and right of the interval, we have for $\theta \leq \theta_2$

$$\arg\max_{q \geq q_2} S_0(q) + (\beta_i(\theta) - \theta)q + \overline{T}_{-i}(q) = \arg\max_{q \geq q_2} S_0(q) + (\beta_i(\theta) - \theta)q + \tilde{T}_{-i}(q),$$

and for $\theta \geq \theta_1$

$$\arg\max_{q \leq q_1} S_0(q) + (\beta_i(\theta) - \theta)q + \overline{T}_{-i}(q) = \arg\max_{q \leq q_1} S_0(q) + (\beta_i(\theta) - \theta)q + \tilde{T}_{-i}(q).$$

If (31) is also satisfied, then we may conclude

$$\arg\max_{q \in \mathcal{Q} \setminus (q_1, q_2)} S_0(q) + (\beta_i(\theta) - \theta)q + \overline{T}_{-i}(q) = \arg\max_{q \in \mathcal{Q} \setminus (q_1, q_2)} S_0(q) + (\beta_i(\theta) - \theta)q + \tilde{T}_{-i}(q).$$

We therefore seek to establish (31). Note that (31) is equivalent to

$$S_0(q_1) + (\beta_i(\hat{\theta}) - \hat{\theta})q_1 + \tilde{T}(q_1) - \tilde{t}_i(q_1) = S_0(q_2) + (\beta_i(\hat{\theta}) - \hat{\theta})q_2 + \tilde{T}(q_2) - \tilde{t}_i(q_2).$$

Using (30), we have the simpler condition

$$\beta_i(\hat{\theta})(q_2 - q_1) = \tilde{t}_i(q_2) - \tilde{t}_i(q_1) = \bar{t}_i(q_2) - \bar{t}_i(q_1) + (\tau_{2,i} - \tau_{1,i}).$$

Using our construction for $\tilde{t}_i$, this is equivalent to

$$\beta_i(\hat{\theta})(q_2 - q_1) = \bar{t}_i(q_2) - \bar{t}_i(q_1) + \beta_i(\hat{\theta})(q_2 - q_1) - \int_{q_1}^{q_2} \beta_i(\overline{\vartheta}^{\mathcal{Q}}(x))dx.$$

Using (28), we conclude that (31) holds. Hence,

$$\arg\max_{q \in \mathcal{Q} \setminus (q_1, q_2)} S_0(q) + (\beta_i(\theta) - \theta)q + \overline{T}_{-i}(q) = \arg\max_{q \in \mathcal{Q} \setminus (q_1, q_2)} S_0(q) + (\beta_i(\theta) - \theta)q + \tilde{T}_{-i}(q).$$

What remains to show is that if principal i were allowed to choose $q \in (q_1, q_2)$, that for ε sufficient small, such a choice is not attractive. But given that there are at least two principals active over (q_1, q_2), it follows that for all $i \in N$, $\tilde{T}_{-i}(q_1)$ and $\tilde{T}_{-i}(q_2)$ are both positive and bounded away from zero. Thus, for ε sufficiently small

(32) $$\max_{q \in \mathcal{Q} \setminus (q_1, q_2)} S_0(q) + (\beta_i(\theta) - \theta)q + \tilde{T}_{-i}(q) \geq \sup_{q \in (q_1, q_2)} S_0(q) + (\beta_i(\theta) - \theta)q.$$

6. Lastly, note that the aggregate tariff under the new equilibrium has the property that $\tilde{T}(q) = 0$ for $q \in (q_1, q_2)$ and $\tilde{T}(q) = \overline{T}(q)$ otherwise. Thus, all agent types are weakly worse off under the new equilibrium tariffs, and those agents who previously chose $q \in (q_1, q_2)$ are strictly worse off by revealed preference.

□

PROOF OF PROPOSITION 2: Recall (15) that

$$\overline{q}^{\mathcal{Q}}(\theta) = \arg\max_{q\in\mathcal{Q}} \; S_0(q) - \theta q + \left(\sum_{i\in N} \max\left\{s_i - \frac{F(\theta)}{f(\theta)}, 0\right\}\right) q.$$

Because

$$\sum_{i\in N} \max\left\{s_i - \frac{F(\theta)}{f(\theta)}, 0\right\}$$

is convex in s_i, it weakly higher under $\tilde{\boldsymbol{s}}$ compared to $\boldsymbol{s}$. Define $\hat{\theta}_i$ by $s_i f(\hat{\theta}_i) = F(\hat{\theta}_i)$ and define $\tilde{\theta}_i$ by $\tilde{s}_i f(\tilde{\theta}_i) = F(\tilde{\theta}_i)$. Choose i such that $s_i < \tilde{s}_i$, and thus $\hat{\theta}_i < \tilde{\theta}_i$. Then for any $\theta \in (\hat{\theta}_i, \tilde{\theta}_i)$, the argmax above is strictly higher under $\tilde{\boldsymbol{s}}$ compared to $\boldsymbol{s}$. It follows that the maximal allocation under $\tilde{\boldsymbol{s}}$ is weakly higher than that under $\boldsymbol{s}$ (and it is strictly higher for some types). □

PROOF OF PROPOSITION 5: We first remind the expressions of rents and payments in the maximal equilibrium when $s_1 = -s_2 = 1 < 2\delta$ (which ensures that both $\hat{\theta}_1$ and $\hat{\theta}_2$ are interior) and the distribution is uniform on $\Theta = [-\delta, \delta]$ with $\mathcal{Q} = [-1-\delta, 1+\delta]$. From Proposition 4, we know that, on the interval $[-\min(1-\delta, \delta), \min(1-\delta, \delta)]$ that contains $\theta_0 = 0$, the maximal equilibrium policy is given by $\overline{q}^{\mathcal{Q}}(\theta) = -3\theta$ for $\theta \in [-1+\delta, 1-\delta]$. The individual equilibrium schedules and the aggregate payment are respectively

$$\overline{t}_1^{\mathcal{Q}}(q) = \overline{t}_2^{\mathcal{Q}}(-q) = \begin{cases} 0 \text{ for } q \leq -3(1-\delta), \\ \frac{1}{6}(q+3(1-\delta))^2 \text{ for } q \in [-3(1-\delta), 3(1-\delta)] \\ \frac{9}{4}(1-\delta)^2 + \frac{(1-\delta)}{2}q + \frac{q^2}{4} \text{ for } q \in [3(1-\delta), 1+\delta] \end{cases}$$

and $\overline{T}^{\mathcal{Q}}(q) = \overline{t}_1^{\mathcal{Q}}(q) + \overline{t}_2^{\mathcal{Q}}(q)$, while the agent's rent writes as

$$\overline{U}^{\mathcal{Q}}(\theta) = \begin{cases} \frac{9}{4}(1-\delta)^2 + \frac{1}{4}(1-\delta-2\theta)^2 \text{ for } \theta \in [-\delta, -1+\delta] \\ 3(1-\delta)^2 + \frac{3}{2}\theta^2 \text{ for } \theta \in [-1+\delta, 1-\delta] \\ \frac{9}{4}(1-\delta)^2 + \frac{1}{4}(1-\delta+2\theta)^2 \text{ for } \theta \in [1-\delta, \delta]. \end{cases}$$

We now construct an equilibrium with a discontinuity at $\theta_0 = 0$ so that the discontinuity gap $[-q_0, q_0]$ remains in $\overline{q}^{\mathcal{Q}}([-1+\delta, 1-\delta])$, i.e., on an area where principals' activity sets overlap in the maximal equilibrium which implies $q_0 \leq 3(1-\delta)$. In particular, we have $\overline{T}^{\mathcal{Q}}(q_0) = \overline{T}^{\mathcal{Q}}(-q_0) = \frac{1}{6}(q_0+3(1-\delta))^2 + \frac{1}{6}(-q_0+3(1-\delta))^2 = \frac{q_0^2}{3} + 3(1-\delta)^2$. Following the proof of Proposition 1 and using the specificity of the uniform distribution so that the construction in Footnote 23 applies, the so-constructed discontinuous equilibrium preserves aggregate and individual payments beyond the discontinuity gap:

$$\overline{T}(q) = \begin{cases} 0 \text{ for } q \in (-q_0, q_0) \\ \overline{T}^{Q}(q) \text{ for } q \geq q_0 \text{ and } q \leq q_0. \end{cases}$$

This yields the following expression of the agent's rent in the discontinuous equilibrium:

$$\overline{U}(\theta) = \min\left\{\overline{U}^{Q}(\theta), -\theta q_0 - \frac{q_0^2}{2} + \overline{T}^{Q}(q_0), \theta q_0 - \frac{q_0^2}{2} + \overline{T}^{Q}(-q_0)\right\}. \tag{33}$$

Following notations in the proof of Proposition 1, we denote $\theta_2 = -\theta_1 = -\frac{q_0}{3}$. To find out the maximal value of the q_0 that can be sustained, we again closely follow the proof of Proposition 1. The first condition to be checked is that the agent does not want to choose a decision in the discontinuity gap. This condition rewrites in this specific context as:

$$\overline{U}(\theta) = \max_{q \in \bar{q}^{Q}(\Theta) \setminus (-q_0, q_0)} -\theta q - \frac{q^2}{2} + \overline{T}^{Q}(q) \geq \sup_{q \in (-q_0, q_0)} -\theta q - \frac{q^2}{2} \equiv \frac{\theta^2}{2} \quad \forall \theta \in [\theta_2, \theta_1]. \tag{34}$$

Using (33) to express the lefthand side and symmetry of the rent profile in θ around the origin, this condition holds when $\overline{U}(\theta) = \theta q_0 - \frac{q_0^2}{6} + 3(1-\delta)^2 \geq \frac{\theta^2}{2}$ for all $\theta \in [0, \theta_1]$ which is always true if it holds at $\theta = 0$, i.e., $\overline{U}(0) = -\frac{q_0^2}{6} + 3(1-\delta)^2 > 0$ but this latter inequality is always true for all $q_0 \leq 3(1-\delta)$.

The second condition to be checked is (32) for each principal. Taking into account symmetry, it suffices to verify that this condition holds for principal 1 which gives:

$$\max_{q \in \bar{q}(\Theta) \setminus (-q_0, q_0)} -\frac{q^2}{2} + (1-\delta-2\theta)q + \bar{t}_2^{Q}(q) \geq \sup_{q \in (-q_0, q_0)} -\frac{q^2}{2} + (1-\delta-2\theta)q \quad \forall \theta \in [\theta_2, \theta_1]. \tag{35}$$

When $q_0 \leq 3(1-\delta)$, the max on the lefthand side is achieved either at $-q_0$ (for $\theta \in [0, \theta_1]$) or at q_0 (for $\theta \in [\theta_2, 0]$). Again using symmetry, we focus on the case $\theta \in [0, \theta_1]$ and note that the sup on the righthand side can be rewritten so that (35) becomes:

$$\frac{3}{2}(1-\delta)^2 - \frac{q_0^2}{3} \geq \mathcal{R}(\theta) = -2\theta q_0 + \max_{q \in [-q_0, q_0]} -\frac{q^2}{2} + (1-\delta-2\theta)q \quad \forall \theta \in [0, \theta_1]. \tag{36}$$

Because the maximum of linear functions of θ is convex, $\mathcal{R}$ is also convex in θ. Using the envelope theorem to evaluate the derivative of this max, it is immediate that $\mathcal{R}$ is also decreasing. Hence, the condition always holds when it holds at $\theta = 0$. We compute

$$\mathcal{R}(0) = \begin{cases} \frac{(1-\delta)^2}{2} & \text{if } q_0 \in [1-\delta, 3(1-\delta)], \\ -\frac{q_0^2}{2} + (1-\delta)q_0 & \text{if } q_0 \in [0, 1-\delta]. \end{cases}$$

Hence, (36) holds when $\frac{3}{2}(1-\delta)^2 - \frac{q_0^2}{3} \geq \mathcal{R}(0)$ which is true when $q_0 \leq \sqrt{3}(1-\delta)$.

Welfare comparison. Fix $q_0 \in [0, \sqrt{3}(1-\delta)]$ (the case $q_0 = 0$ corresponding to the maximal equilibrium). We know from Proposition 1 that the agent always prefers the maximal equilibrium to any discontinuous equilibrium keeping aggregate payments the same outside the discontinuity gap. Turning now overall expected payoff of the principals in a discontinuous equilibrium, we observe that, because of opposite interests, this expected payoff is the opposite of their overall expected payment. This expected payment writes as:

$$\mathcal{T}(q_0) = \frac{1}{2\delta}\left(\int_{-\delta}^{-\frac{q_0}{3}} \overline{T}^{\mathcal{Q}}(\overline{q}^{\mathcal{Q}}(\theta))d\theta + \int_{-\frac{q_0}{3}}^{0} \overline{T}^{\mathcal{Q}}(q_0)d\theta + \int_0^{\frac{q_0}{3}} \overline{T}^{\mathcal{Q}}(-q_0)d\theta + \int_{\frac{q_0}{3}}^{\delta} T^{\mathcal{Q}}(\overline{q}^{\mathcal{Q}}(\theta))d\theta\right).$$

Observe that:

$$\frac{d\mathcal{T}}{dq_0}(q_0) = \frac{1}{2\delta}\left(\int_{-\frac{q_0}{3}}^{0} \frac{d}{dq_0}(\overline{T}^{\mathcal{Q}}(q_0))d\theta + \int_0^{\frac{q_0}{3}} \frac{d}{dq_0}(\overline{T}^{\mathcal{Q}}(-q_0))d\theta\right) = \frac{2q_0^2}{9\delta}.$$

Henceforth, $\mathcal{T}(q_0)$ is convex for $q_0 \geq 0$ and minimized at $q_0 = 0$, i.e., the maximal equilibrium is also preferred by the principals. Since both the principals and the agent prefers the maximal equilibrium, welfare is higher at that equilibrium. □

PROOF OF PROPOSITION 6: This result follows from an immediate application of the arguments in Proposition 2 for the case of $\mathcal{A}$. For the case of $\mathcal{B}$, note that the maximal allocation function is a concave function of s_B, and hence the inequalities (both weak and strict) are reversed. □

PROOF OF PROPOSITION 7: This result follows from an application of the arguments proving Proposition 3 in the main text. □

PROOF OF PROPOSITION 8: Given the assumptions of symmetry, $\overline{q}^{\mathcal{Q}}$ is symmetric around $\theta = 0$ and the mean policy chosen by the agent is 0. Furthermore, any pairwise spread preserves symmetry and is mean preserving. It is sufficient that we establish that any component pairwise spread for some pair i results in a mean-preserving spread in $\overline{q}^{\mathcal{Q}}$.

For $\theta \leq 0$, note that the introduction of any increment Δ_i can only increase $\overline{q}^{\mathcal{Q}}(\theta)$. There are three cases to consider to establish this claim. Fixing $\theta \leq 0$, after the increment is introduced, either both principals are active, neither principal is active, or only principal $i \in \mathcal{A}$ is active.

For the first two outcomes, the increment has no effect on $\overline{q}^{\mathcal{Q}}$ for the type at θ. When only $i \in \mathcal{A}$ is active, however, the increment increases the marginal virtual preference for $i \in \mathcal{A}$, which in turn increases $\overline{q}^{\mathcal{Q}}$ (given that S_0 is differentiable and strictly concave, as maintained). Under our assumptions on the distribution of θ, there is always some region of inactivity. Hence, there is a positive measure of types for which $\overline{q}^{\mathcal{Q}}$ strictly increases. For $\theta \geq 0$, a reverse argument establishes that $\overline{q}^{\mathcal{Q}}$ must weakly decrease (and strictly decrease on a set of positive measure). Furthermore, given our symmetry assumptions, such changes in $\overline{q}^{\mathcal{Q}}$ are mean preserving, and hence the resulting allocation leads to a greater dispersion in policy choices.

$\square$

REFERENCES

AIDT (1998). Political internalization of economic externalities and environmental policy. *Journal of Public Economics* **69** 1-16.

ALONSO, N. AND R. MATOUSCHEK (2008). Optimal Delegation. *Review of Economic Studies* **75** 259-293.

BARON, D. AND R. MYERSON (1982). Regulating a Monopolist with Unknown Costs. *Econometrica* **50** 911-930.

BELLETTINI, G. AND G. OTTAVIANO (2005). Special interests and Technological Change. *Review of Economic Studies* **72** 43-56.

BERGSTROM, T., L. BLUME AND H. VARIAN (1986). On the Private Provision of Public Goods. *Journal of Public Economics* **29** 25-49.

BERNHEIM, D. AND M. WHINSTON (1986). Menu Auctions, Resource Allocations and Economic Influence. *Quarterly Journal of Economics* **101** 1-31.

BERNHEIM, D. AND M. WHINSTON (1998). Exclusive Dealing. *Journal of Political Economy* **106** 64-103.

CALZOLARI, G. AND V. DE NICOLO (2013). Competition With Exclusive Contracts and Market-Share Discounts. *American Economic Review* **103** 2384-2411.

DIXIT, A., G. GROSSMAN AND E. HELPMAN (1997). Common Agency and Coordination: General Theory and an Application to Government Decision-Making. *Journal of Political Economy* **105** 752-769.

GROSSMAN, G. AND E. HELPMAN (1994). Protection for Sale. *American Economic Review* **84** 833-850.

GROSSMAN, G. AND E. HELPMAN (1995). The politics of free-trade agreements. *American Economic Review* **85** 667-690.

GROSSMAN, G. AND E. HELPMAN (2001). *Special-interest Politics.* MIT Press (Cambridge, MA).

HOERNIG, S. AND T. VALLETTI (2011). When Two-Part Tariffs Are Not Enough: Mixing With Nonlinear Pricing. *The B.E. Journal of Theoretical Economics* 11(1).

JULLIEN, B. (2000). Participation Constraints in Adverse Selection Models. *Journal of Economic Theory* **93** 1-47.

KROSZNER, R. AND T. STRATMANN (1998). Interest-Group Competition and the Organization of Congress - Theory and Evidence from Financial Services' Political Action Committees. *American Economic Review*

88 1163–1187.

KIRCHSTEIGER, G. AND A., PRAT (2001). Inefficient Equilibria in Lobbying. *Journal of Public Economics* **82** 349-375.

LAFFONT, J.J. AND J. TIROLE (1991). Privatization and Incentives. *Journal of Law, Economics and Organization* **7** 84-105.

LAUSSEL, D. AND M. LEBRETON (1998). Efficient Private Production of Public Goods under Common Agency. *Games and Economic Behavior* **25** 194-218.

LAUSSEL, D. AND M. LEBRETON (2001). Conflict and Cooperation: The Structure of Equilibrium Payoffs in Common Agency. *Journal of Economic Theory* **100** 93-128.

LEBRETON, M. AND F. SALANIÉ (2003). Lobbying under Political Uncertainty. *Journal of Public Economics* **87** 2589-2610.

MAILATH, G. AND A. POSTLEWAITE (1990). Asymmetric Information Bargaining Problems with Many Agents. *Review of Economic Studies* **57** 351-367.

MARTIMORT, D. (1996). The Multi-Principal Nature of Government. *European Economic Review* **40** 673-685.

MARTIMORT, D. (2007). Multicontracting Mechanism Design. *Advances in Economic Theory, Proceedings of the 2005 World Congress of the Econometric Society*, ed. by R. Blundell, W. Newey and T. Persson. Cambridge: Cambridge University Press.

MARTIMORT, D. AND A. SEMENOV (2006). Continuity in Mechanism Design without Transfers. *Economic Letters*, **93** 182-189.

MARTIMORT, D. AND A. SEMENOV (2008). Ideological Uncertainty and Lobbying Competition. *Journal of Public Economics* **92** 456-481.

MARTIMORT, D. AND L. STOLE (2009). Market Participation in Delegated and Intrinsic Common Agency Games. *RAND Journal of Economics* **40** 78-102.

MARTIMORT, D. AND L. STOLE (2012). Representing Equilibrium Aggregates in Aggregate Games with Applications to Common Agency. *Games and Economic Behavior* **76** 753-772.

MARTIMORT, D. AND L. STOLE (2013). Public Contracting in Delegated Agency Games. *Working paper.* April 2013.

MARTIMORT, D. AND L. STOLE (2014). Necessary and sufficient Conditions for Optimal Control Problems with Linear-State, Semicontinuous Lagrangians. *Working paper.* January 24, 2014.

MCCARTY, N., K. POOLE AND H. ROSENTHAL (2006). *Polarized America.* MIT Press (Cambridge, MA).

MELUMAD, N. AND T. SHIBANO (1991). Communication in Settings with no Transfers. *RAND Journal of Economics* **22** 173-198.

MILGROM, P. (2007). Package Auctions and Exchanges. *Econometrica* **75** 935-965.

PERSSON, T. AND G. TABELLINI (2002). Political Economics and Public Finance. *Handbook of Public Economics, Volume 3.* Edited by A.J. Auerbach and M. Feldstein. Elsevier Science Press. 1549-1659.

OLSON, M. (1965). *The Logic of Collective Action.* Harvard University Press (Cambridge, MA).

STIGLER, G. (1974). Free Riders and Collective Action: An Appendix to Theories of Economic Regulation. *Bell Journal of Economics* **5** 359-365.