On a Functional Equation for the Generating Function of the Logarithmic Series Distribution

Panaretos, John

1987

ON A FUNCTIONAL EQUATION FOR THE GENERATING FUNCTION OF THE LOGARITHMIC SERIES DISTRIBUTION

JOHN PANARETOS

1. INTRODUCTION

The solution of a functional equation is one of the oldest topics in mathematical analysis. This topic has attracted increasing interest recently in mathematical statistics and probability, especially in the area of characterization of distributions. In this area the problem of deciding whether a property is unique for a particular distribution is frequently reduced to the problem of finding the unique solution of a functional equation. Some of the interesting results in this area, mainly connected with the Poisson distribution, can be found in the works of Aczél ([1][, [2]) and Chatterji [4].

In this note we study a functional equation of the form

$$G\left(\frac{pt}{1-qt}\right) = G(2pt) - G(pt)$$

where $1/2 < p < 1$, $q = 1 - p$, and $G(t)$ is a probability generating function (p.g.f.). The problem of solving this functional equation arose when the author was investigating damage and generating models using the logarithmic series distribution (LSD). In fact, in [5] a probabilistic model was introduced, where an original observation generates another observation according to a Pascal distribution. It is then observed that, if the original observation follows the LSD, a relationship leading to the above functional equation holds. In this note it is shown that the unique solution of this functional equation is the p.g.f. of LSD. The result is of interest in statistical inference because the LSD is a widely used distribution with many applications. (For the definition and properties of the LSD the reader may for example refer to Boswell and Patil [3].)

2. THE RESULT

THEOREM. *Let $G(t)$ denote the p.g.f. of the probability distribution g_n, $n = 1, 2, \ldots$. Consider also the functional equation*

$$G\left(\frac{pt}{1-qt}\right) = G(2pt) - G(pt), \quad \frac{1}{2} < p < 1, \; q = 1 - p. \tag{1}$$

REV. ROUMAINE MATH. PURES APPL. 32(1987), 4, 365–367

(Clearly this equation is meaningful at least for $|t| \leqslant 1$*). Then, the unique solution of* (1) *is*

$$G(t) = \frac{\log(1 - at)}{\log(1 - a)}; \qquad a = \frac{q}{p} \tag{2}$$

which is the p.g.f. of the LSD (a).

Proof. From (1) we have

$$\sum_{n=1}^{\infty} g_n \left(\frac{pt}{1-qt}\right)^n = \sum_{n=1}^{\infty} g_n p^n (2^n - 1) t^n \Leftrightarrow$$

$$\sum_{n=1}^{\infty} g_n (pt)^n \sum_{r=n}^{\infty} \binom{r-1}{n-1} (qt)^{r-n} = \sum_{n=1}^{\infty} g_n p^n (2^n - 1) t^n \Leftrightarrow \tag{3}$$

$$\sum_{r=1}^{\infty} \sum_{n=1}^{r} g_n \binom{r-1}{n-1} (p/q)^n (qt)^r = \sum_{r=1}^{\infty} g_r p^r (2^r - 1) t^r.$$

Equating the coefficients of t^r on both sides of (3) yields

$$\sum_{n=1}^{r-1} g_n \binom{r-1}{n-1} a^{r-n} = (2^r - 2) g_r, \; r = 2, 3, \ldots$$

where $a = q/p$, which is equivalent to

$$g_r = \frac{1}{2^r - 2} \sum_{n=1}^{r-1} g_n \binom{r-1}{n-1} a^{r-n}, \qquad r = 2, 3, \ldots \tag{4}$$

We will show that the solution of the functional equation (4) is of the form

$$g_n = g_1 \frac{a^{n-1}}{n}, \qquad n = 2, 3, \ldots \tag{5}$$

For $r = 2$ equation (4) gives

$$g_2 = g_1 a/2$$

which is of the form (5).

Assume that this is also true for $r \leqslant k$ for some $k \geqslant 2$, i.e., assume that the solution of (4), for $r \leqslant k$, $k \geqslant 2$, is of the form

$$g_r = g_1 \frac{a^{r-1}}{r}. \tag{6}$$

We will show that this is also the case for $r = k + 1$ i.e., that (4) and (6) imply

$$g_{k+1} = g_1 \frac{a^k}{k+1} \qquad k \geqslant 2. \tag{7}$$

From (4) for $r = k + 1$ we have

$$g_{k+1} = \frac{1}{2^{k+1} - 2} \sum_{n=1}^{k} g_n \binom{k}{n-1} a^{k+1-n}$$

and by using (6)

$$g_{k+1} = \frac{g_1}{2^{k+1} - 2} \sum_{n=1}^{k} \binom{k}{n-1} \frac{a^k}{n} =$$

$$= g_1 \frac{a^k}{2^{k+1} - 2} \sum_{n=1}^{k} \frac{k!}{(n-1)!\,(k-n+1)!\,n}.$$

By the binomial theorem, though

$$\sum_{n=1}^{k} \frac{k!}{n!\,(k-n+1)!} = \frac{1}{k+1} (2^{k+1} - 2).$$

Hence

$$g_{k+1} = g_1 \frac{a^k}{k+1}, \qquad k \geqslant 2.$$

This completes the argument.

Since $\{g_n,\ n = 1, 2, \ldots\}$ is a probability distribution we have that $\sum_{n=1}^{\infty} g_n = 1$. Hence

$$g_1^{-1} = \frac{1}{a} \log \frac{1}{1-a}$$

and so, finally

$$g_n = d \frac{a^n}{n}, \quad n = 1, 2, \ldots; \qquad d = -\frac{1}{\log(1-a)}, \; a < 1,$$

i.e., $\{g_n,\ n = 1, 2, \ldots\}$ is LSD (a) with p.g.f. as in (2).

Received February 11, 1983

Dept. of Statistics and Actuarial Science
The University of Iowa
Iowa City, Iowa 52242, USA

REFERENCES

1. J. Aczél, *On a characterization of Poisson distributions*. J. Appl. Probab. 9 (1972), 852–856.
2. J. Aczél, *On two characterizations of Poisson distributions*. Abh. Math. Sem. Univ. Hamburg 44(1975), 91–100.
3. M. T. Boswell, and G. P. Patil, *Chance mechanisms generating the logarithmic series distribution used in the analysis of number of species and individuals*. In: G. P. Patil, E. C. Pielou and W. E. Waters (Eds.), Statistical Ecology, Vol. 1, p. 99–130. Penn. State University Press, 1971.
4. S. D. Chatterji, *Some elementary characterizations of the Poisson distribution*. Amer. Math. Monthly 70(1963), 958–964.
5. J. Panaretos, *A generating model involving Pascal and logarithmic series distributions*. Comm. Statist. A—Theory Methods 12(1983), 841–848.