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Abstract

In this paper I present a new single factor stochastic volatility model for asset return observed in discrete time and its latent volatility. This model unifies the feedback effect and return skewness using a common factor for return and its volatility. Further, it generalizes the existing stochastic volatility framework with constant feedback to one with time varying feedback and as a consequence time varying skewness follows. However, presence of dynamic feedback effect violates the weak-stationarity assumption usually considered for the latent volatility process. The concept of bounded stationarity has been proposed in this paper to address the issue of non-stationarity. A characterization of the error distributions for returns and volatility is provided on the basis of existence of conditional moments. Finally, an application of the model has been explained using

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S&P100 daily returns under the assumption of Normal error and half Normal common factor distribution.

1 Introduction

Research in financial econometrics has seen a surge in the area of time-varying volatility models for asset returns over last three decades. Stochastic volatility (SV) model ((Taylor 1982)) has been one of the key instruments to address this issue. In addition SV model explains some interesting aspects of asset returns observed empirically and known as "stylized facts". Some of the important stylized facts are mean reversion of returns, volatility clustering indicating periods of similar volatility occurring together and the negative relation between the return and its volatility divulging their movement in opposite direction. The work of (Taylor 1982) models the time varying volatility of financial returns as a latent auto-regressive process to account for the volatility clustering. Since then multitude of SV models have been developed to explain different stylized facts about asset returns. A comprehensive review of the SV models can be obtained from the works of (Shephard & Andersen 2009) and (Chib, Omori & Asai 2009).

Recent works in this context emphasize on two important aspects of return-volatility relation *viz.* the correlation between current volatility and future returns (or the feedback effect) and the negative correlation between current asset return and its future volatility (or leverage effect). Different types of SV models have been developed to explain the time varying volatility of asset return in presence of leverage effect. (Renault 2009) provides a comprehensive account of feedback and leverage effect in SV models. Another related stylized fact, *viz.* return skewness, has gained importance due to its role in asset and option pricing ((Christoffersen, Heston & Jacobs 2006), (Renault 2009)). However, no work has been done so far to establish the connection between feedback effect, return skewness and leverage effect in discrete time general SV models to the best of the knowledge of the present author. In this paper, I develop a parsimonious generalized single factor SV model to explain the relation between conditional feedback and return skewness and extend it to an SV model with time varying feedback and skewness.

Time series data on asset returns provide evidences of its correlation with its volatility ((Nelson 1991)) and skewness in asset returns ((Harvey & Siddique 1999)). The negative correlation between current volatility and future return (or feedback) may be attributed to the fact that an anticipated increase in volatility results in immediate price fall ((French, Schwert & Stambaugh 1987)). (Bollerslev, Litvinova & Tauchen 2006) shows that a stronger signal of the feedback effect is reflected through the contemporaneous correlation between asset return and its volatility. They conclude in favor of the contemporaneous correlation as a measure of volatility feedback effect. On the other hand, the fact that a decline in current price would lead to increase in future volatility, could be attributed to changes in financial leverage ((Nelson 1991)) and such a correlation is called the leverage effect. (Renault 2009) points out the possibility of an alternative explanation to the feedback using the return skewness. The intuition behind such possibility could be justified by the following argument. The magnitude of volatility increase due to price fall is much higher than the magnitude by which volatility decreases in case of price increase. Thus the conditional volatility for negative returns is more compared to the same for positive returns. This fact leads to the skewness in the return distribution. (Tsiotas 2012) and (Feunou & Tédongap 2012) provide an account of SV models developed so far with leverage and skewed return distributions.

In this paper I propose a parsimonious representation of such an interlocked explication of feedback effect and skewness. Both of them could be looked upon as the resultant of a common positive stochastic factor acting on both return and volatility shocks which are symmetric. A financial justification of the presence of such a stochastic factor could be in assuming the presence of a market sentiment influencing both return and volatility in different magnitudes. Directions of market sentiment impact could be similar or opposite. I model the influence of market sentiment on return and volatility as product of a common positive latent random variable and corresponding weights. This mechanism generates perturbation to symmetric return shocks by a positive random variable and generates asymmetry in returns whereas the shared factor generates the feedback effect.

(Bollerslev, Sizova & Tauchen 2012) provides empirical evidence of the dynamic nature of the correlation between return and volatility. On the other hand (Harvey & Siddique 1999) and recently (Boyer, Mitton & Vorkink 2010) provide evidences of time dependent conditional return skewness. Based on the above findings I assume the weights of the factor on return and volatility to be time varying so that the feedback effect and conditional skewness are dynamic. Individual impact of the stochastic factor on return and volatility are measured by the corresponding time dependent coefficients which will be referred to as impact parameters here onwards. The underlying reason of different directions and magnitudes of the time varying conditional skewness and the feedback effect could be then comprehended in terms of the impact parameters.

The main complexity of the proposed model is that it violates weak-stationarity condition of the volatility process. Weak stationarity is crucial to a stochastic process as it restricts the process to increase indefinitely in expectation with time lag. In this paper I introduce the concept of bounded stationarity in terms of 1^{st} and 2^{nd} order moments of a stochastic process to relax the existing weak-stationarity condition yet ensure that the process does not explode. I also provide here a characterization of the auto-regressive volatility process of order one, which is most commonly used to describe volatility process in SV models, in the light of bounded stationarity.

The proposed model is developed under general distributions for return, volatility and the common factor. Many of the existing SV models has been shown to be particular cases of this generalized SV model. An immediate characterization of the plausible distributions for return, volatility and the common factor has been given based on the existence of return moments and the feedback effect. Further, I provide explanation of the skewness and feedback effect in terms of the influence of market sentiment assuming the usual Gaussian framework. The affine combination of Normal return shocks and Half-Normal common factor distribution used in the proposed model results in a variant of a general class of distributions containing standard Normal known as skew-normal distribution ((Azzalini & Dalla Valle 1996)).

Parameter estimation in SV model is a challenging task due to non-identically and non-independently distributed (non-iid) returns with complicated likelihood and high dimension of the parameter space. (Jacquier, Polson & Rossi 1994) developed a Monte Carlo Markov Chain (MCMC) method to estimate the SV model parameters. As pointed out by (Eraker, Johanners & Polson 2003), MCMC method provides an edge over other methods of estimation in different ways. First, MCMC provides estimates of latent volatilities which are as many in numbers as the number of observed returns. Second, it is computationally efficient and easily implementable ((Meyer & Yu 2000)). Another major advantage of the MCMC method is that it allows to incorporate restrictions on the model parameters through selection of relevant prior distributions. Hence, I use MCMC method in this paper to estimate the model parameters and latent volatilities.

Two major concerns in MCMC simulation are to measure the convergence and model adequacy. (Gelman & Rubin 1992) suggested potential scale reduction factor (psrf) as a measure of convergence which is calculated from more than one parallel MCMC simulations. In this paper, Gelman-Rubin psrf has been used to measure convergence of MCMC simulations. On the other hand, recent progress in MCMC estimation and its implementation has become compelling for researchers to fit models with large number of parameters to explore real life complexities more closely. As a result, measurement of model adequacy and complexity has become increasingly important. (Spiegelhalter, Best, Carlin & Van Der Linde 2002) proposed the deviance information criteria (DIC) as a measure of model adequacy. DIC is calculated based on separate measures of model fit and model complexity.s The discrepancy between data and model could be measured by the posterior mean of log-likelihood in an MCMC simulation, which is a measure of model fit. Model complexity could be measured using the log-likelihhod at the posterior means of the model parameters. DIC combines the two measures to arrive at a measure of model adequacy ((Spiegelhalter et al. 2002)). Observing that the number of unknown quantities involved in the model (including latent volatility) it is crucial to measure the complexity in the model. In this paper I report both DIC and measure of model complexity to gauge the usefulness of the proposed model.

Rest of the paper is organized as follows. In section 2 the general framework for SV model with common factor and time dependent impact parameters is described. The concept of bounded stationarity is introduced in this section to tackle non-stationarity. Section 3 presents an example of generalized SVDF model with half-normal and Gaussian distributions. The expressions for the dynamic feedback, leverage and skewness are presented here. Also necessary and sufficient conditions for negative feedback has been discussed. In section 4 SV models with constant and time-varying feedback effect have been tested with S&P100 daily returns. Estimation results from the proposed models have been compared with the same from some comparable models. Section 5 concludes the paper with a discussion on the proposed model and its applications.

2 Dynamic Feedback SV Model with Common Market Factor

Let P_t be the daily price of an asset and $\log \frac{P_t}{P_{t-1}}$ be the log return. The time series of mean-corrected daily log returns is denoted by y_t and the underlying latent volatilities by θ_t. Let us start with the SV model proposed by (Jacquier et al. 1994) which is given as follows:

$$y_t = e^{\frac{\theta_t}{2}} \epsilon_t, \tag{2.1}$$

$$\theta_t = \alpha + \phi(\theta_{t-1} - \alpha) + \eta_t, t = 1, ..., T \tag{2.2}$$

ϵ_t and η_t being independent sequences of independently and identically distributed (iid) random shocks (or innovations) with 0 means and variances 1 and σ^2 respectively. ϕ is the volatility clustering parameter which reflects the stylized fact that volatility pattern (high or low) cluster together. Subsequently SV models with contemporaneous correlation (ρ) between ϵ_t and η_t has been discussed by, *e.g.* (Jacquier, Polson & Rossi 2004) among others. SV model with the feedback effect ρ relates the changes in volatility to the sign and magnitude of price changes which helps in pricing the options more accurately.

In this paper, I consider a new SV model with independent symmetric random shocks

ϵ_t and η_t and a general positive common factor for market sentiment, say γ_t, which impacts the return and its latent volatility at each time point. However, such impact on return and its volatility may be different in magnitude and direction and may vary over time ((Boyer et al. 2010)). Let $\lambda_{y,t} \in \mathcal{R}$ and $\lambda_{\theta,t} \in \mathcal{R}$ be the dynamic impacts of the market factor on the return and its latent volatility respectively. Thus the new single factor SV model with time varying feedback (SVDF) is given as

$$y_t = \mu_{y,t} + e^{\frac{\theta_t}{2}}(\lambda_{y,t}\gamma_t + \epsilon_t) \tag{2.3}$$

$$\theta_t = \alpha + \phi(\theta_{t-1} - \alpha) + \mu_{\theta,t} + (\lambda_{\theta,t}\gamma_t + \eta_t) \tag{2.4}$$

where $y_t,\ \theta_t$ are same as in equations (2.1)-(2.2) and $\{\gamma_t\}$ is a sequence of iid positive random variables. $\mu_{y,t}$ and $\mu_{\theta,t}$ are so selected that $E[y_t \mid \boldsymbol{\mathcal{F}_{t-1}}] = 0$ and $E[\theta_t \mid \boldsymbol{\mathcal{F}_{t-1}}] = \alpha + \phi(\theta_t - \alpha)$ preserving mean reversion of the returns and the memory effect in volatility respectively. Further ϵ_t and η_t are two sequences of symmetric random variables independent to each other contemporaneously as well as inter-temporally.

The affine combination of positive factor with symmetric innovation results in a skewed family of distributions. The impact parameters determine the amount and direction of conditional skewness in the corresponding process and hence will be interchangeably called as skewness parameters and impact parameters here onwards. The presence of common factor in both return and volatility induces the correlation or the feedback effect. The time-dependent impact parameters cause the feedback to be dynamic. It may be remarked here that considering $\lambda_{y,t} = \lambda_y$ and $\lambda_{\theta,t} = \lambda_\theta$, constant feedback model (SVCF) can be obtained. Clearly the volatility asymmetry can now be interpreted in terms of the market sentiment impacts which has been discussed in detail in subsection 2.3.

The SVDF model postulated in equations (2.3)-(2.4) describes a robust class of parametric SV models. Different distributions has been used in SV model to capture the leverage, feedback and skewness in return ((Tsiotas 2012)). Such models can be obtained as special cases of the proposed SVDF model. Some of important ones are described below:

1. Let $\lambda_{y,t} = \lambda_{\theta,t} = 0$, $\epsilon_t \sim N(0,1)$ and $\eta_t \sim N(0,\sigma^2)$ be independent processes to obtain

the usual SV model with Gaussian errors ((Jacquier et al. 1994)).

2. Let $\lambda_{y,t} = \lambda_{\theta,t} = 0$, $\epsilon_t \sim t_\nu$, $\eta_t \sim N(0, \sigma^2)$ and they are independent which leads to the SV model with t-errors (SVt) in return ((Harvey, Ruiz & Shephard 1994)).

3. Let $\lambda_{\theta,t} = 0$ and γ_t be standard half-normal variate. Further, let $\epsilon_t \sim N(0, 1)$ and independent of $\eta_t \sim N(0, \sigma^2)$ and both ϵ_t and η_t are independent of γ_t which results in the SV model with returns distributed as a variant of Skew-Normal distribution ((Tsiotas 2012)).

4. Set $\lambda_{\theta,t} = 0$, γ_t as half-t_ν variate. In addition $\epsilon_t \sim t_\nu$ and $\eta_t \sim N(0, \sigma^2)$ and are independent of each other as well as γ_t. This leads to the SV model with Skew-t returns ((Tsiotas 2012)).

5. Let $\lambda_{\theta,t} = 0$ and γ_t be distributed as Generalized Inverse Gaussian distribution. Further, let $\epsilon_t = \sqrt{\gamma_t}\epsilon_t^*$, where ϵ_t^* are $NID(0, 1)$ variates independent of γ_t and $\eta_t \sim N(0, \sigma^2)$ to obtain the SV model with generalized hyperbolic Skew-t returns ((Aas & Haff 2006)).

I assume the independence between γ_t, ϵ_t and η_t, $\forall t = 1, 2, \ldots$ for the rest of this paper.

The assumption of dynamic nature of impact parameters ($\lambda_{y,t}$ and $\lambda_{\theta,t}$) in SVDF model immediately results in a serious issue of violating the weak stationarity of the auto-regressive volatility process as stated in (2.4). This in turn may lead the process to explode as its future variance may increase indefinitely with time lag. To avoid this issue and yet to incorporate the time varying impact parameters I first introduce the concept of bounded stationarity in the following subsection and then describe some characteristics of θ_t with respect to bounded stationarity.

2.1 Bounded Stationarity For Non-Stationary Process

The bounded stationarity of a discrete time stochastic process is defined as follows.

Bounded Stationarity: Let X_t be a discrete time stochastic process such that its 1^{st} and 2^{nd} moments exist. The process is defined to be *bounded stationary* if $E[X_t] < M$ and $Cov(X_t, X_{t-k}) < V$; M and V being finite real numbers and k is any integer.

Taking $k = 0$ in the above definition we get the condition $V(y_t) < V$ on the variance for bounded stationarity.

Remark: Notice that, if the 1^{st} and 2^{nd} order moments of a bounded stationary time series are constant, then the series is weak stationary. Further suppose the 1^{st} and 2^{nd} moments of a locally weak stationary series, *viz.* $y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_T}$, $\tau \in I$(an index set), be given by μ_τ and σ^2_τ. If μ_τ and σ^2_τ are finite for all $\tau \in I$, then setting $M = \sup_{\tau \in I} \mu_\tau$ and $V = \sup_{\tau \in I} \sigma_\tau$ we observe that a locally weak stationary series is bounded stationary.

Based on the above definition, the conditions of bounded stationarity for the volatility process in (2.4) is derived in the following theorem.

Theorem 2.1 *Let us define $a_t = \mu_{\theta_t} + \lambda_{\theta_t}\gamma_t + \eta_t$ in the auto-regressive volatility equation 2.4. Also let $\{\gamma_t\}$ be a sequence of iid positive random variables and $\{\eta_t\}$ be a sequence of iid random variables with zero mean and constant variance independent of γ_t, $\forall t$. Further the sequence $\{a_t\}$ is assumed to be independent of $\theta_{t'}$, $\forall t' < t$. Assuming that the 2^{nd} moment of γ_t exists, the following results hold*

1. *$E[\theta_t]$ is finite $\forall\, t$ if $\mid \phi \mid < 1$.*

2. *$V(\theta_t)$ is given by*

$$\varUpsilon_t(0) = V(\theta_t) = \sigma^2(1 + \phi^2 + \phi^4 + \ldots) + \delta^2 \sum_{k=1}^{\infty} \phi^{2k} \lambda^2_{t-k}, \tag{2.5}$$

where $\delta = V(\gamma_t)$. Further $\varUpsilon_t(0)$ is non-negative and bounded if $\mid \phi \mid < 1$ and $\mid \lambda_t \mid \leq \lambda$, $\lambda > 0\, \forall t$, in which case

$$\varUpsilon_t(0) \leq \frac{\sigma^2 + \delta^2\lambda^2}{1 - \phi^2} \tag{2.6}$$

3. The auto-covariance function of lag k is given by $\Upsilon_t(k) = Cov(\theta_{t+k}, \theta_t)$

$$\Upsilon_t(k) = Cov(\theta_{t+k}, \theta) = \phi^k \Upsilon_t(0) \le \phi^k \frac{\sigma^2 + \delta^2\lambda^2}{1-\phi^2} \; \forall k. \tag{2.7}$$

Proof. Let $V(\eta_t) = \sigma^2$ and observe that $E[a_t] = 0$ and $V(a_t) = \sigma^2 + \lambda_t^2\delta^2, \; \forall t = 1, 2, \ldots$. The proof of the results are given as below.

1. Notice that ,

$$\begin{aligned} E[\theta_t] &= \alpha(1-\phi) + \phi E[\theta_{t-1}] \\ &= \alpha(1-\phi)(1 + \phi + \phi^2 + \phi^3 + \ldots) \end{aligned}$$

so that $E[\theta_t]$ exists finitely if $| \phi |< 1$.

2.

$$\begin{aligned} \Upsilon_t(0) &= \phi^2 V(\theta_{t-1}) + [\sigma^2 + \delta^2\lambda_t^2] \\ &= \sigma^2(1 + \phi^2 + \phi^4 + \ldots) + \delta^2[\lambda_t^2 + \phi^2\lambda_{t-1}^2 + \phi^4\lambda_{t-1}^4 \ldots] \end{aligned}$$

and if $| \phi |< 1$ then

$$= \frac{\sigma^2}{1-\phi^2} + \delta^2 \sum_{k=0}^{\infty} \phi^{2k}\lambda_{t-k}^2.$$

Further if for a finite $\lambda \in \mathcal{R}$, the condition $| \lambda_t | \le \lambda$ hold for all t, then the bound is immediate from the expression of $\Upsilon_t(0)$.

3. The autocovariance function $\Upsilon_t(k)$ is given by

$$\begin{aligned} \Upsilon_t(k) &= E[(\theta_{t+k} - \alpha)(\theta_t - \alpha)] \\ &= \phi E[(\theta_t - \alpha)E[(\theta_{t+k-1} - \alpha)]] \end{aligned}$$

since a_{t+k} is independent of $\theta_j \; \forall j < t+k$. Thus, by repeated substitutions we get

$$\Upsilon_t(k) = \phi^k \Upsilon_t(0).$$

The bound on the auto-covariance function follows from (2.6).

Remark: 1. The condition for bounded stationarity in this case is given by $\mid \lambda_t \mid \leq \lambda,\ \lambda \in \mathcal{R}$.

2. The auto-correlation function is time invariant and depends only on the lag which is similar to the weak stationary time series.

3. The upper bound of the auto-covariance function dampens to zero as the lag increases. Thus, similar to weakly stationary series, the impact of the past realizations decreases with the time horizon. However, unlike the weak stationary series, the auto-covariance of a bounded stationary AR process may not reduce to a time invariant constant with increasing lag.

4. The k-period ahead forecast for such a series is given by $\hat{\theta}_{t+k} = \alpha + \phi^k(\theta_t - \alpha)$ so that $\lim_{k\to\infty} \hat{\theta}_{t+k} = \alpha$. The forecast error is given by

$$\hat{e}_t(k) = \sum_{j=0}^{k-1} \phi^j a_{t+k-j}$$

.

5. The forecast error variance is given by $V(\hat{e}_t(k)) = \sum_{j=0}^{k-1} \phi^{2j} V(a_{t+k-j})$. The bounded stationarity condition on λ leads to the following upper bound on forecast error variance.

$$V(\hat{e}_t(k)) \leq \frac{1-\phi^{2k}}{(1-\phi^2)^2}(\sigma^2 + \delta^2\lambda)$$

Notice that the bound tends to $\frac{\sigma^2+\delta^2\lambda}{(1-\phi^2)^2}$, as $k \to \infty$, *i.e* the bound increases with lag. Further, if $\lambda \to 0$, the bound on forecast error reduces to $\frac{\sigma^2}{(1-\phi^2)^2}$. The error variance also increases with the volatility persistence parameter ϕ.

The above discussion ensures that although the auto-regressive volatility process in the proposed SV model is not weakly stationary but the first two moments of the process are bounded and hence the process and its forecast does not explode with increasing lag. In the following section we discuss on the feedback effect for the proposed SVDF model with general innovation distribution under the assumption of bounded stationarity.

2.2 Time Varying Feedback in SVDF Model

The following lemma provides means and variances of the return and volatility under the model postulated in (2.3) and (2.4).

Lemma 2.2 *Let y_t and θ_t be the return and volatility at time t and the stochastic volatility model describing the evolution of y_t and θ_t be given as in (2.3-2.4). Suppose ϵ_t is distributed with mean 0 and variance unity and η_t is distributed independent of ϵ_t with mean 0 and variance σ^2. Further suppose that the moment generating functions (MGF) of γ_t (denoted by $M_{\gamma_t}(u)$, $\forall\, u \in \mathcal{R}$) and η_t (denoted by $M_{\eta_t}(u)$, $\forall\, u \in \mathcal{R}$) exist and the first two derivatives of the MGFs are denoted as $M'_X(u)$ and $M''_X(u)$ respectively, $X \in \{\gamma_t, \eta_t\}$, $t = 1, 2, \dots T$. Under the above postulates the following results hold given the information set $\boldsymbol{\mathcal{F}_{t-1}}$ available up to time $t-1$:*

$$\mu_{y,t} = -A_{t-1}\lambda_{y,t} M_{\eta_t}\left(\frac{1}{2}\right) M'_{\gamma_t}\left(\frac{\lambda_{\theta_t}}{2}\right), \text{ where } A_{t-1} = e^{\frac{\alpha+\phi(\theta_{t-1}-\alpha)+\mu_{\theta,t}}{2}} \tag{2.8}$$

$$V(y_t \mid \boldsymbol{\mathcal{F}_{t-1}}) = A_{t-1}^2\left[M_{\eta_t}(1)\left\{\lambda_{y,t}^2 M''_{\gamma_t}(\lambda_{\theta,t}) + M_{\gamma_t}(\lambda_{\theta,t})\right\} - \lambda_{y,t}^2 M_{\eta_t}^2\left(\frac{1}{2}\right) {M'_{\gamma_t}}^2\left(\frac{\lambda_{\theta,t}}{2}\right)\right] \tag{2.9}$$

$$\mu_{\theta,t} = -\lambda_{\theta,t} E(\gamma_t) \tag{2.10}$$

$$V(\theta_t \mid \boldsymbol{\mathcal{F}_{t-1}}) = \lambda_{\theta,t}^2 V(\gamma_t) + \sigma^2 \tag{2.11}$$

Proof. To prove (2.8), first define $Z_{y,t} = \lambda_{y,t}\gamma_t + \epsilon_t$ and $Z_{\theta,t} = \lambda_{\theta,t}\gamma_t + \eta_t$. Notice that the mean reversibility of y_t implies that $E[y_t \mid \boldsymbol{\mathcal{F}_{t-1}}] = 0$. Denoting $e^{\frac{\alpha+\phi(\theta_{t-1}-\alpha)+\mu_{\theta,t}}{2}}$ by A_{t-1}, the expression of $\mu_{y,t}$ is given as follows

$$\begin{aligned}\mu_{y,t} &= -A_{t-1}E[e^{\frac{Z_{\theta,t}}{2}} Z_{y,t} \mid \boldsymbol{\mathcal{F}_{t-1}}] \\ &= -A_{t-1}\lambda_{y,t} M_{\eta_t}\left(\frac{1}{2}\right) M'_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right)\end{aligned}$$

where $M'_{\gamma_t}(u) = \frac{d}{du} M_{\gamma_t}(u)$.

To obtain the return variance, denote $e^{\frac{\theta_t}{2}} Z_{y,t}$ in (2.3) by R_t, $\forall t = 1, 2, \dots T$, and observe that

$$
\begin{aligned}
V(y_t \mid \boldsymbol{\mathcal{F}_{t-1}}) &= E[y_t^2 \mid \boldsymbol{\mathcal{F}_{t-1}}] \\
&= E[R_t^2 \mid \boldsymbol{\mathcal{F}_{t-1}}] - \mu_{y,t}^2 \\
&= A_{t-1}^2 M_{\eta_t}(1)[\lambda_{y,t}^2 M''_{\gamma_t}(\lambda_{\theta,t}) + M_{\gamma_t}(\lambda_{\theta,t})] - \mu_{y,t}^2 \\
&= A_{t-1}^2 \left[M_{\eta_t}(1) \left\{ \lambda_{y,t}^2 M''_{\gamma_t}(\lambda_{\theta,t}) + M_{\gamma_t}(\lambda_{\theta,t}) \right\} - \lambda_{y,t}^2 M_{\eta_t}^2 \left(\frac{1}{2}\right) {M'_{\gamma_t}}^2 \left(\frac{\lambda_{\theta,t}}{2}\right) \right]
\end{aligned}
$$

where $M''_{\gamma_t}(u) = \frac{d^2}{du^2} M_{\gamma_t}(u)$.

Observing that $E[\theta_t \mid \boldsymbol{\mathcal{F}_{t-1}}] = \alpha + \phi(\theta_{t-1} - \alpha)$ the result (2.10) is immediate. The proof of (2.11) is trivial.

Corollary 2.3 *Observing that $M_{\eta_t}(1) \geq M_{\eta_t}^2\left(\frac{1}{2}\right)$, the following lower bound can be obtained from (2.9):*

$$
V(y_t \mid \boldsymbol{\mathcal{F}_{t-1}}) \geq A_{t-1}^2 M_{\eta_t}(1) \left[\lambda_{y,t}^2 \left\{ M''_{\gamma_t}(\lambda_{\theta,t}) - {M'_{\gamma_t}}^2 \left(\frac{\lambda_{\theta,t}}{2}\right) \right\} + M_{\gamma_t}(\lambda_{\theta,t}) \right] \tag{2.12}
$$

Further, letting $\lambda_{\theta,t} \to 0$ the bound in (2.12) reduces to

$$
e^{\alpha+\phi(\theta_{t-1}-\alpha)} M_{\eta_t}(1) \left[\lambda_{y,t}^2 V(\gamma_t) + 1 \right] \tag{2.13}
$$

The above corollary may be helpful in determining the minimum risk premium for options based on returns y_t. Next I provide an expression for the dynamic feedback effect for SVDF model

Theorem 2.4 *Under the model and the assumptions postulated in lemma 2.2, the dynamic feedback ρ_t is given by*

$$
\rho_t = \frac{\lambda_{y,t} \left[\lambda_{\theta,t} M_{\eta_t}\left(\frac{1}{2}\right) \left\{ M''_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) - M'_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) E(\gamma_t) \right\} + E\left(\eta_t e^{\frac{\eta_t}{2}}\right) M'_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) \right]}{\sqrt{M_{\eta_t}(1) \left\{ \lambda_{y,t}^2 M''_{\gamma_t}(\lambda_{\theta,t}) + M_{\gamma_t}(\lambda_{\theta,t}) \right\} - \lambda_{y,t}^2 M_{\eta_t}^2\left(\frac{1}{2}\right) {M'_{\gamma_t}}^2\left(\frac{\lambda_{\theta,t}}{2}\right)} \sqrt{\lambda_{\theta,t}^2 \delta^2 + \sigma^2}}. \tag{2.14}
$$

Proof. The conditional covariance between y_t and θ_t given the information set $\boldsymbol{\mathcal{F}_{t-1}}$ can be derived as follows:

$$
\begin{aligned}
Cov_{t-1}(y_t, \theta_t) &= Cov(y_t, \theta_t \mid \boldsymbol{\mathcal{F}_{t-1}}) \\
&= E[A_{t-1} e^{\frac{\lambda_{\theta,t}\gamma_t + \eta_t}{2}} (\lambda_{\theta,t}\gamma_t + \eta_t)(\lambda_{y,t}\gamma_t + \epsilon_t)] - \mu_{y,t}\mu_{\theta,t} \\
&= \lambda_{y,t} A_{t-1} E_{\gamma_t} E\left[e^{\frac{\lambda_{\theta,t}\gamma_t + \eta_t}{2}} (\lambda_{\theta,t}{\gamma_t}^2 + \gamma_t \eta_t) \right] - \mu_{y,t}\mu_{\theta,t} \\
&= \lambda_{y,t} A_{t-1} \left\{ \lambda_{\theta,t} M_{\eta_t}\left(\frac{1}{2}\right) E_{\gamma_t}\left[{\gamma_t}^2 e^{\frac{\lambda_{\theta,t}\gamma_t}{2}} \right] + E_{\gamma_t}\left[\gamma_t e^{\frac{\lambda_{\theta,t}\gamma_t}{2}} \right] E\left[\eta_t e^{\frac{\eta_t}{2}} \right] \right\} - \mu_{y,t}\mu_{\theta,t} \\
&= \lambda_{y,t} A_{t-1} \left[\lambda_{\theta,t} M_{\eta_t}\left(\frac{1}{2}\right) M''_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) + E\left(\eta_t e^{\frac{\eta_t}{2}}\right) M'_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) \right] - \mu_{y,t}\mu_{\theta,t} \\
&= \lambda_{y,t} A_{t-1} \left[\lambda_{\theta,t} M_{\eta_t}\left(\frac{1}{2}\right) \left\{ M''_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) - M'_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) E(\gamma_t) \right\} \right. \\
&+ \left. E\left(\eta_t e^{\frac{\eta_t}{2}}\right) M'_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) \right]
\end{aligned} \tag{2.15}
$$

Notice that existence of MGF of η_t ensures existence of $E\left(\eta_t e^{\frac{\eta_t}{2}}\right)$ and hence the expression for feedback in (2.14) is immediate.

As a consequence of the above theorem the distributions of return and volatility shocks can be characterized as follows.

Corollary 2.5 *The class of distributions that can be considered to model the market factor γ_t and the volatility shock η_t, $\forall t = 1, 2, \ldots T$, are the ones admitting MGF so that the feedback effect and hence the conditional or unconditional return moments exist. However, ϵ_t need not be restricted by such property.*

Remark: Notice that the feedback will not vanish even if there is no impact of market sentiment on volatility. If the market sentiment has influence only on returns, then the future marginal distribution of return or volatility does not change whereas the future marginal distribution of volatility changes in presence of market sentiment impact on volatility. SV models with only non-zero return impact may be classified as transient feedback SV models where as the same with non-zero impact on both return and volatility could be classified as SV model with persistent feedback.

In the remaining part of this paper I assume that the MGF of γ_t and η_t exists $\forall\ t = 1, 2, \ldots T$. Next I describe the leverage effect under the proposed model.

2.3 Time Varying Leverage in SVDF Model

The presence of conditional leverage effect in the proposed model is reflected through the impact of current return on future volatility ((Renault 2009)). The following theorem shows that the conditional expectation of future volatility depends linearly on the current asset return and the direction of the dependence is determined by the return impact parameter as well as the volatility clustering parameter (ϕ).

Theorem 2.6 *In the model described by (2.3-2.4) along the assumptions described in lemma 2.2, the conditional expectation of future volatility given current return is given as follows:*

$$E[\theta_{t+1} \mid y_t, \boldsymbol{\mathcal{F}_{t-1}}] = C_t + D_t \phi \rho_t y_t \tag{2.16}$$

where ρ_t is the dynamic feedback effect, $u_t = \lambda_{y,t}\gamma_t + \epsilon_t,\ v_t = \lambda_{\theta,t}\gamma_t + \eta_t,\ \omega_{x,t} = \sqrt{Var(x_t)},\ x \in \{u, v\}$ and

$$C_t = \alpha + \phi^2(\theta_{t-1} - \alpha) - \frac{\omega_{v,t}}{\omega_{u,t}}\phi\rho_t E[u_t \mid \boldsymbol{\mathcal{F}_{t-1}}] - A_{t-1}\frac{\omega_{v,t}}{\omega_{u,t}}\phi\rho_t\mu_{y,t}M_{\gamma_t}\left(-\frac{\lambda_{\theta,t}}{2}\right)M_{\eta_t}\left(-\frac{1}{2}\right)$$

$$D_t = A_{t-1}\frac{\omega_{v,t}}{\omega_{u,t}}M_{\gamma_t}\left(-\frac{\lambda_{\theta,t}}{2}\right)M_{\eta_t}\left(-\frac{1}{2}\right) > 0$$

The sign of the conditional leverage is determined by the same of the volatility clustering parameter and the direction of the feedback effect.

Proof. Notice that,

$$\begin{aligned} E[\theta_{t+1} \mid y_t, \boldsymbol{\mathcal{F}_{t-1}}] &= E_{\theta_t}\left[E(\theta_{t+1} \mid \theta_t) \mid y_t, \boldsymbol{\mathcal{F}_{t-1}}\right] \\ &= \alpha + \phi^2(\theta_{t-1} - \alpha) + \phi\mu_{\theta,t} + \phi E[(v_t) \mid y_t, \boldsymbol{\mathcal{F}_{t-1}}], \end{aligned}$$

since γ_{t+1} is independent of y_t and its marginal expectation exists. Let $v'_t = \frac{v_t - E(v_t)}{\omega_{v,t}}$ and $u'_t = \frac{u_t - E(u_t)}{\omega_{u,t}}$ where $\omega_{v,t} = \sqrt{Var(v_t)}$, $\omega_{u,t} = \sqrt{Var(u_t)}$. Further, define $w_t = \frac{v'_t - \rho_t u'_t}{\sqrt{1-\rho_t^2}}$, where

ρ_t is the correlation between u_t and v_t. Notice that w_t and u_t' are uncorrelated and $E[w_t] = 0$. Hence,

$$
\begin{aligned}
E[\theta_{t+1} \mid y_t, \boldsymbol{\mathcal{F}_{t-1}}] &= \alpha_{t-1} + \phi\omega_{v,t} E\left[v_t' \mid y_t, \boldsymbol{\mathcal{F}_{t-1}}\right], \\
&\quad (\text{ where } \alpha_{t-1} = \alpha + \phi^2(\theta_{t-1} - \alpha) \text{ and } \mu_{\theta,t} = -E[v_t \mid \boldsymbol{\mathcal{F}_{t-1}}]) \\
&= \alpha_{t-1} - \frac{\omega_{v,t}}{\omega_{u,t}}\phi\rho_t E[u_t \mid \boldsymbol{\mathcal{F}_{t-1}}] + \frac{\omega_{v,t}}{\omega_{u,t}}\phi\rho_t E\left[u_t \mid y_t, \boldsymbol{\mathcal{F}_{t-1}}\right] \\
&= \alpha_{t-1} - \frac{\omega_{v,t}}{\omega_{u,t}}\phi\rho_t E[u_t \mid \boldsymbol{\mathcal{F}_{t-1}}] + A_{t-1}\frac{\omega_{v,t}}{\omega_{u,t}}\phi\rho_t(y_t - \mu_{y,t}) E\left[e^{-\frac{v_t}{2}} \mid \boldsymbol{\mathcal{F}_{t-1}}\right] \\
&\qquad\qquad (\text{ where } A_{t-1} \text{ is defined above }) \\
&= C_t + A_{t-1}\frac{\omega_{v,t}}{\omega_{u,t}}\phi\rho_t y_t M_{\gamma_t}\left(-\frac{\lambda_{\theta,t}}{2}\right) M_{\eta_t}\left(-\frac{1}{2}\right) \\
&= C_t + D_t\phi\rho_t y_t \qquad (2.17)
\end{aligned}
$$

where $C_t = \alpha_{t-1} - \frac{\omega_{v,t}}{\omega_{u,t}}\phi\rho_t E[u_t \mid \boldsymbol{\mathcal{F}_{t-1}}] - A_{t-1}\frac{\omega_{v,t}}{\omega_{u,t}}\phi\rho_t\mu_{y,t} M_{\gamma_t}\left(-\frac{\lambda_{\theta,t}}{2}\right) M_{\eta_t}\left(-\frac{1}{2}\right)$ and $D_t = A_{t-1}\frac{\omega_{v,t}}{\omega_{u,t}} M_{\gamma_t}\left(-\frac{\lambda_{\theta,t}}{2}\right) M_{\eta_t}\left(-\frac{1}{2}\right) > 0$. Hence the sign of the dynamic leverage depends on the sign of feedback effect and the volatility clustering parameter. In particular if the feedback effect and the volatility clustering parameter are of opposite sign then the future volatility is negatively correlated to the current return.

2.4 Time Varying Skewness in SVDF Model

This subsection attempts to explain the conditional return skewness in terms of the impact parameters. The following theorem provides an expression for the conditional skewness.

Theorem 2.7 *In the model described in (2.3-2.4) along the assumptions described in lemma 2.2, the conditional skewness of return is given as follows:*

$$Sk_t = \frac{\Psi_t^3 + 3\Lambda_t\Psi_t + 3\Delta_t}{\Lambda_t^{\frac{3}{2}}} \qquad (2.18)$$

where $\Psi_t = \frac{\mu_{y,t}}{A_{t-1}}$, $\Lambda_t = \frac{V\left(y_t \mid \boldsymbol{\mathcal{F}_{t-1}}\right)}{A_{t-1}^2}$ *and*

$$\frac{\Delta_t}{A_{t-1}^3} = M_{\eta_t}\left(\frac{3}{2}\right)\left[\lambda_{y,t}^3 M_{\gamma_t}'''\left(\frac{3\lambda_{\theta,t}}{2}\right) + 3\lambda_{y,t} M_{\gamma_t}'\left(\frac{3\lambda_{\theta,t}}{2}\right)\right] \qquad (2.19)$$

$M_X'''(u) = \frac{d^3}{du^3} M_X(u)$.

Proof. Simple algebraic manipulation will show that

$$E\left[y_t^3 \mid \boldsymbol{\mathcal{F}_{t-1}}\right] = 3E\left[e^{\frac{3Z_{\theta,t}}{2}} Z_{y,t}^3 \mid \boldsymbol{\mathcal{F}_{t-1}}\right] + 3\mu_{y,t} V\left(y_t \mid \boldsymbol{\mathcal{F}_{t-1}}\right) + \mu_{y,t}^3$$

where $Z_{\theta,t}$ and $Z_{y,t}$ are defined as in theorem 2.2. Further,

$$\begin{aligned} E\left[e^{\frac{3Z_{\theta,t}}{2}} Z_{y,t}^3 \mid \boldsymbol{\mathcal{F}_{t-1}}\right] &= A_{t-1}^3 E\left[e^{\frac{3Z_{\theta,t}}{2}} \left(\lambda_{y,t}^3 \gamma_t^3 + 3\lambda_{y,t}\gamma_t\epsilon_t^2\right) \mid \boldsymbol{\mathcal{F}_{t-1}}\right] \\ &= M_{\eta_t}\left(\frac{3}{2}\right)\left[\lambda_{y,t}^3 M_{\gamma_t}'''\left(\frac{3\lambda_{\theta,t}}{2}\right) + 3\lambda_{y,t} M_{\gamma_t}'\left(\frac{3\lambda_{\theta,t}}{2}\right)\right] \end{aligned}$$

In the above expression we notice that the conditional skewness is not dependent on the expected volatility or the persistence. Only the impact parameters and the variance of the volatility distribution contributes to the conditional return skewness. Thus the model disentangles the effect of past volatility from the return skewness.

It is difficult to gain further insight on the dynamic leverage effect without assuming particular distributions for γ_t, ϵ_t and η_t. In the following section we make specific assumptions about the distributions of the market factor and return and volatility innovations.

3 SVDF Model with Gaussian Error Distributions

The SVDF model proposed above aims to capture the skewness in returns and the dynamic nature of the feedback effect together. In this section I first inspect the SVDF model for skewed returns. In particular, I provide the expression for the feedback effect and conditional skewness and their interpretation in terms of the impact parameters using a Gaussian framework.

3.1 Gaussian SVDF Model

I assume that $\gamma_t \sim HN(0,1)$, $HN(0,1)$ being the standard half-normal distribution and $\epsilon_t \sim N(0,1)$, $\eta_t \sim N(0,\sigma^2)$, $\forall t = 1,2,\ldots$ in addition to the assumptions made in the SVDF model. The expression of feedback effect (ρ_t) can be derived in a similar manner as in theorem 2.4. The following useful lemma provides the expression of the moment generating function of standard half normal distribution.

Lemma 3.1 *Let us consider a standard half normal distribution random variable X and let, for any $u \in \mathbf{R}$, $M_X(u)$ be the moment-generating function (MGF) of X. Then,*

$$M'_X(u) = \frac{d}{du} M_X(u) = u M_X(u) + \sqrt{\frac{2}{\pi}} \tag{3.20}$$

$$M''_X(u) = \frac{d^2}{du^2} M_X(u) = \{1 + t^2\} M_X(u) + t\sqrt{\frac{2}{\pi}} \tag{3.21}$$

Proof. The MGF of X, $M_X(u),\ u \in \mathbf{R}$ is given by

$$\begin{aligned} M_X(u) &= E[e^{uX}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{ux} e^{-\frac{x^2}{2}} dx \\ &= \sqrt{\frac{2}{\pi}} e^{\frac{u^2}{2}} \int_{-u}^{\infty} e^{-\frac{z^2}{2}} dx, \\ &= e^{\frac{u^2}{2}} \left[1 + \frac{2}{\sqrt{\pi}} \int_0^{\frac{u}{\sqrt{2}}} e^{-w^2} dw \right] \\ &= e^{\frac{u^2}{2}} \left[1 + erf\left(\frac{u}{\sqrt{2}} \right) \right], \text{ where } erf(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-w^2} dw \\ &= 2e^{\frac{u^2}{2}} \Phi(u), [\text{ since } erf(u) + 1 = 2\Phi(u\sqrt{2})] \end{aligned} \tag{3.22}$$

and hence,

$$M'_X(u) = u M_X(u) + \sqrt{\frac{2}{\pi}} = 2ue^{\frac{u^2}{2}} \Phi(u) + \sqrt{\frac{2}{\pi}}. \tag{3.23}$$

Further, differentiating (3.20) with respect to u and substituting the expression for $M'_X(u)$ we get

$$M''_X(u) = \{1 + u^2\} M_X(u) + u\sqrt{\frac{2}{\pi}}. \tag{3.24}$$

In the following theorem the expression for dynamic leverage (ρ_t) is derived under the model postulated in (2.3-2.4) and the distributional assumptions stated above and state some sufficient conditions in terms of impact parameters for negative leverage.

Theorem 3.2 *Let y_t and θ_t be the return and volatility at time t and the stochastic volatility model describing the evolution of y_t and θ_t be given as in (2.3-2.4) where γ_t follows standard half normal distribution. Further γ_t is assumed to be independent of the normal variates*

ϵ_t and η_t which are independent among themselves with mean 0 and variances 1 and σ^2 respectively, $\forall t = 1, 2, \ldots n$. Under this model the dynamic leverage effect ρ_t is given by

$$\rho_t = \frac{\lambda_{y,t}\left[\lambda_{\theta,t} M_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right)\left\{\frac{\lambda_{\theta,t}^2}{4} + 1 + \frac{\sigma^2}{4} - \frac{\lambda_{\theta,t}}{\sqrt{2\pi}}\right\} + \frac{1}{\sqrt{2\pi}}\left(\sigma^2 + \lambda_{\theta,t}^2\right) - \frac{2\lambda_{\theta,t}}{\pi}\right]}{\sqrt{e^{\frac{\sigma^2}{4}}\left\{M_{\gamma_t}(\lambda_{\theta,t})[\lambda_{y,t}^2(\lambda_{\theta,t}^2+1)+1] + \sqrt{\frac{2}{\pi}}\lambda_{\theta,t}\lambda_{y,t}^2\right\} - \lambda_{y,t}^2 {M'_{\gamma_t}}^2\left(\frac{\lambda_{\theta,t}}{2}\right)}\sqrt{\lambda_{\theta,t}^2\left(1-\frac{2}{\pi}\right)+\sigma^2}} \tag{3.25}$$

where $M'_{\gamma_t}(u)$ and $M''_{\gamma_t}(u)$ are as defined in (3.20-3.21).

Proof. Notice that, here $M_{\eta_t}\left(\frac{1}{2}\right) = e^{\frac{\sigma^2}{8}}$ and hence from (2.8)

$$\mu_{y,t} = -A_{t-1}\lambda_{y,t} e^{\frac{\sigma^2}{8}} M'_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) \tag{3.26}$$

Further, observing that $M_{\eta_t}(1) = e^{\frac{\sigma^2}{2}}$, expressions in (2.11) and (2.9) leads to

$$V(\theta_t \mid \boldsymbol{\mathcal{F}_{t-1}}) = \lambda_{\theta,t}^2\left(1 - \frac{2}{\pi}\right) + \sigma^2 \text{ and} \tag{3.27}$$

$$\begin{aligned} V(y_t \mid \boldsymbol{\mathcal{F}_{t-1}}) &= A_{t-1}^2 e^{\frac{\sigma^2}{2}}\left\{M_{\gamma_t}(\lambda_{\theta,t})[\lambda_{y,t}^2(\lambda_{\theta,t}^2+1)+1] + \sqrt{\frac{2}{\pi}}\lambda_{\theta,t}\lambda_{y,t}^2\right\} - \mu_{y,t}^2 \\ &= A_{t-1}^2 e^{\frac{\sigma^2}{4}}\left[e^{\frac{\sigma^2}{4}}\left\{M_{\gamma_t}(\lambda_{\theta,t})[\lambda_{y,t}^2(\lambda_{\theta,t}^2+1)+1] + \sqrt{\frac{2}{\pi}}\lambda_{\theta,t}\lambda_{y,t}^2\right\}\right. \\ &\left. -\lambda_{y,t}^2 {M'_{\gamma_t}}^2\left(\frac{\lambda_{\theta,t}}{2}\right)\right]. \end{aligned} \tag{3.28}$$

Further, $E\left[\eta_t e^{\frac{\eta_t}{2}}\right] = \frac{\sigma^2}{2} e^{\frac{\sigma^2}{8}}$ and hence from (2.15) :

$$\begin{aligned} Cov_{t-1}(y_t, \theta_t) &= A_{t-1} e^{\frac{\sigma^2}{8}}\left[\lambda_{\theta,t}\lambda_{y,t} M''_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) + \lambda_{y,t}\frac{\sigma^2}{2} M'_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right)\right] - \mu_{y,t}\mu_{\theta,t} \\ &= A_{t-1} e^{\frac{\sigma^2}{8}}\left[\lambda_{y,t}\lambda_{\theta,t}\left\{\left(\frac{\lambda_{\theta,t}^2}{4}+1\right) M_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) + \sqrt{\frac{2}{\pi}}\frac{\lambda_{\theta,t}}{2}\right\}\right. \\ &+ \left.\lambda_{y,t}\frac{\sigma^2}{2}\left\{\frac{\lambda_{\theta,t}}{2} M_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) + \sqrt{\frac{2}{\pi}}\right\}\right] - \mu_{y,t}\mu_{\theta,t} \\ &= A_{t-1} e^{\frac{\sigma^2}{8}}\lambda_{y,t}\left[\lambda_{\theta,t} M_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right)\left\{\frac{\lambda_{\theta,t}^2}{4} + 1 + \frac{\sigma^2}{4} - \frac{\lambda_{\theta,t}}{\sqrt{2\pi}}\right\}\right. \\ &+ \left.\frac{1}{\sqrt{2\pi}}\left(\sigma^2 + \lambda_{\theta,t}^2\right) - \frac{2\lambda_{\theta,t}}{\pi}\right] \end{aligned} \tag{3.29}$$

Hence the expression for leverage in (3.25) is immediate.

Remark: The correlation coefficient varies with respect the impact parameters as well as the variance of the volatility. A necessary and sufficient condition for the feedback effect to be negative is that $\lambda_{y,t}$ and $\kappa_t = \lambda_{\theta,t} M_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right)\left\{\frac{\lambda_{\theta,t}^2}{4} + 1 + \frac{\sigma^2}{4} - \frac{\lambda_{\theta,t}}{\sqrt{2\pi}}\right\} + \frac{1}{\sqrt{2\pi}}\left(\sigma^2 + \lambda_{\theta,t}^2\right) - \frac{2\lambda_{\theta,t}}{\pi}$ are of opposite sign ($\forall \sigma > 0$). It can be shown that κ_t has a minimum at $\lambda_{\theta,t}^{min}$ for each $\sigma > 0$ with minimum value κ_t^{min}, $\forall t = 1, 2, \ldots, T$. Figure 3(a) in the Appendix plots κ_t^{min} against σ for any t.

As evident from figure 3(a), κ_t^{min} exceeds zero and numerical computation shows that the corresponding $\sigma = 0.74182$. Thus κ_t can take negative values only for $\sigma \in (0, 0.7419819)\{= I_{min}\}$. Thus, to find the range of $\lambda_{\theta,t}$ so that $\kappa_t < 0$, we restrict σ within this interval. Further, numerically it can be verified that there are only two roots to $\kappa_t = 0$, say $\lambda_{\theta,t}^1 < \lambda_{\theta,t}^2$, for $\sigma \in I_{min}$. Figure 3(b) in Appendix A show the plots of $\lambda_{\theta,t}^1$ and $\lambda_{\theta,t}^2$ against $\sigma \in I_{min}$.

It is clear from the above figures that the interval within which $\kappa_t < 0$ reduces with increasing σ. Thus necessary and sufficient condition for feedback to be negative is translates to either $\lambda_{y,t} < 0$ and $\lambda_{\theta,t}$ lies out side the interval $(\lambda_{\theta,t}^1, \lambda_{\theta,t}^2)$ or the other way around where the limits $\lambda_{\theta,t}^i$, $i = 1, 2$ depend on the variance of the volatility process.

Remark: Figures (4-5) given in the appendix provide the feedback effect surface corresponding to impact parameters for given volatility variances.

It may be noticed from the above figures that as $\sigma \to \infty$, the impact surface closes to the constant plane at zero. Observing that very high volatility variance induces positive probability for the event that realization of conditional volatility is far away from its conditional mean. Such a case may happen in times of bubbles and crashes. One possible explanation for such minuscule feedback effect could be that during such time, market factor impacts are outperformed by the random shocks and hence feedback appears insignificant. In the particular case of no impact of market factor on volatility ($\lambda_{\theta,t} \to 0$), simple algebraic calculation will reveal that $\rho_t \to \frac{\lambda_{y,t}\sigma}{\sqrt{2\pi e^{\frac{\sigma^2}{4}}(\lambda_{y,t}^2+1) - 2\lambda_{y,t}^2}}$, which tends to 0 with increasing σ.

Remark: Notice that for a standard half-normal random variable X,

$$M_X'''(u) = \frac{d^3}{du^3} M_X(u) = u(u^2 + 3) M_X(u) + \sqrt{\frac{2}{\pi}}(u + 1) \tag{3.30}$$

Hence, from theorem 2.7, the conditional skewness could be derived.

4 Estimating Stochastic Volatility Feedback and Return Skewness

In this section, I describe likelihood based MCMC estimation of the parameters of the proposed SV model. Let $\boldsymbol{\Theta}$ indicate the set of model parameters, *viz.* $\{\lambda_{\boldsymbol{y}},\ \lambda_{\boldsymbol{\theta}},\ \alpha,\ \phi,\ \sigma^2\}$ where $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots \theta_T)'$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_T)'$. To ensure bounded stationarity of the volatility process in SVDF model, the volatility impact parameters needs to be bounded and the auto-regression parameter ϕ is restricted within (-1,1) a priori. The same could be ensured using suitable prior distributions of λ_{θ_t} with a finite support and ϕ with support (-1,1). The sample information about $\boldsymbol{\Theta}$ and the latent volatility $\boldsymbol{\theta}$ is described through the posterior distribution which combines the likelihood, $f(\boldsymbol{y} \mid \boldsymbol{\theta}, \boldsymbol{\Theta})$ and the prior $f(\boldsymbol{\Theta})$ as follows :

$$f(\boldsymbol{\Theta} \mid \boldsymbol{y}, \boldsymbol{\theta}) \propto f(\boldsymbol{y} \mid \boldsymbol{\theta}, \boldsymbol{\Theta}) f(\boldsymbol{\theta} \mid \boldsymbol{\Theta}) f(\boldsymbol{\Theta})$$

4.1 Convergence of MCMC

In such iterative simulations one critical issue is to correctly assess the convergence of the method. As proposed by (Gelman & Rubin 1992) potential scale reduction factor (psrf) based on multiple independent chains is used in this paper to measure the convergence of iterative simulation. To obtain psrf first n observations are simulated from $m(> 2)$ independent sequences after a sufficient burn-in for each parameter. The variance of the target posterior distribution (V) is then estimated by

$$\hat{V} = \frac{n-1}{n} W + \frac{B}{n}$$

where W is the average within sequence variance of the m independent chains and B is the variance of the means between chains . Notice that expectation of W asymptotically approaches V. Gelman and Rubin's psrf is defined as $\sqrt{\hat{R}} = \sqrt{\frac{n-1}{n} + \frac{B}{nW}\frac{m+1}{m}\hat{\nu}(W,B,m,n)}$ where $\hat{\nu}(W,B,m,n)$ is an adjustment factor tending to unity as $n \to \infty$ (see Eq. 4 in (Gelman & Rubin 1992)). Since $\sqrt{\hat{R}}$ declines to 1 asymptotically, psrf closer to 1 would indicate convergence of the MCMC simulation of the corresponding parameter.

4.2 Model Adequacy and Complexity

To measure how good a model fits to the data, the proposed one could be compared with a saturated model. In a saturated model a perfect fit to data is obtained by using as many parameters as the number of observations. Rooting from this concept, a frequentist measure of a model fit is defined as the departure of the model from saturated model which is known as deviance. For the SV model proposed in this paper the deviance would be given by

$$D(\mathbf{\Theta}) = -2\log\{f(\boldsymbol{y} \mid \mathbf{\Theta})\} + 2\log\{h(\boldsymbol{y})\}.$$

Here $f(\boldsymbol{y} \mid \mathbf{\Theta})$ is the conditional likelihood function of the data given the set of parameters $\mathbf{\Theta}$ and $h(\boldsymbol{y})$ is a fully specified standardizing term depending only on the observations. On the other hand, model complexity depends on the number of parameters in the model along with the data and priors. Thus large number of parameters add to the complexity of a model as well. (Spiegelhalter et al. 2002) proposed a measure of model adequacy, called deviance information criteria (DIC), based on posterior mean deviance along with a penalty for model complexity. DIC for the model proposed in this paper is given by

$$DIC = \overline{D(\mathbf{\Theta})} + p_D,$$

where $\overline{D(\mathbf{\Theta})}$ is the posterior mean of deviance and p_D is the penalty for model complexity. p_D is also interpreted as the effective number of parameters which is measured with deviance at posterior mean ($D(\overline{\mathbf{\Theta}})$) as follows:

$$p_D = \overline{D(\mathbf{\Theta})} - D(\overline{\mathbf{\Theta}}).$$

DIC and p_D can be easily computed from the MCMC output.

The number of unknowns (parameters and volatilities) in a typical SV model are more than the number of observations. Thus p_D and DIC separately play important roles in selecting most appropriate SV model from a set of candidate models ((Berg, Meyer & Yu 2004), (Abanto-Valle, Bandyopadhyay, Lachos & Enriquez 2010), (Tsiotas 2012)). In this paper, I report both DIC and p_D, the former to measure model adequacy and and the latter for model complexity.

4.3 Empirical Results

The proposed SV models have been tested with S&P100 daily returns. S&P100 data has been used earlier by (Blair, Poon & Taylor 2001), (Harvey et al. 1994) and (Berg et al. 2004) to examine its heteroskedastic volatility. The data considered here is the same as that in (Berg et al. 2004). The data contains 1516 mean corrected daily log-returns on S&P100 observed during the period January, 1993 to December, 1998. Figure 1 presents the time plot of the data and the summary statistics are given in table 1 (see Appendix A).

4.3.1 Estimation in Constant Feedback Models

In this sub-section I test the proposed SVCF model and compare it with two more similar SV models. The SVCF model (M1) is as described in (2.3-2.4) with the impact parameters being constant over time. Other two similar models which include the return skewness as well as the leverage effect are as follows:

M2 (SVF-BVSN): An SV model with feedback and skew-normal returns and volatility:

$$y_t = \mu_y + e^{\frac{\theta_t}{2}}[\lambda_1\gamma_{1,t} + \epsilon_t], \quad (4.31)$$

$$\theta_t = \mu_\theta + \alpha + \phi(\theta_{t-1} - \alpha) + \lambda_2\gamma_{2,t} + \eta_t, t = 1, ..., T \quad (4.32)$$

where $\gamma_{i,t}$ are Half standard Normal variates (i=1,2) so that both return and its volatility follow a bivariate skew-normal distribution ((Azzalini & Dalla Valle 1996)) with feedback effect $\rho = cor(\epsilon_t, \eta_t)$.

M3 (SVF-SN-N): An SV model with feedback and skew-normal returns:

$$y_t = \mu_y + e^{\frac{\theta_t}{2}}[\lambda_1\gamma_t + \epsilon_t], \quad (4.33)$$

$$\theta_t = \alpha + \phi(\theta_{t-1} - \alpha) + \eta_t, t = 1, ..., T \quad (4.34)$$

where $(\epsilon_t, \eta_t)'$ bivariate normal distribution with $\mathbf{0}$ mean and dispersion matrix $\mathbf{\Sigma} = \begin{pmatrix} 1 & \rho\sigma \\ \rho\sigma & \sigma^2 \end{pmatrix}$.

To complete the model specification for MCMC estimation the prior distributions are described next. In this paper I mostly follow (Kim, Shephard & Chib 1998) and (Berg et al. 2004) to assign priors to the model parameters. I define $\beta = 2\beta^* - 1$ and assume an informative prior with $B(20, 1.5)$ for ϕ^*. Since the 5^{th} and 95^{th} percentiles of the prior distribution are $P_5 = 0.65$ and $P_{95} = 0.98$, it incorporates strong volatility clustering a priori. I assume a flat prior N(-10,25) for α. Further, I assume the conjugate hierarchical prior Gamma($\frac{\nu}{2}, \frac{\nu}{2}$) for σ^2 with $\nu \sim U(2, 128)$ ((Chib, Nardari & Shephard 2002)). The priors for the impact parameters are assumed to be non-informative U(-2,2). The corresponding skewness interval is (-38.3,38.3) which covers the skewness intervals reported in the existing literature (*e.g* see (Boyer et al. 2010)). I also assume a non-informative U(-1,1) prior for ρ. Posterior distribution of the model parameters are obtained from 3 chains of MCMC samples. Gelman-Rubin psrf for each parameter is computed using the 3 chains of MCMC samples to check convergence on these parameters.

4.3.2 Estimation Result for SVCF

Table 2 in Appendix A reports the posterior medians of the important parameters in each of the models M1-M3. The figures in the brackets indicate respectively standard deviation and psrf of the parameter. The Gelman-Rubin psrf values indicate stronger convergence for all the parameters in M1 and M2 with psrf less than 1.05 whereas some parameters in M3 have psrf higher than 1.05. The table also contains the triplet - total number of samples generated, number of observations dropped between two consecutive sampled values in each chain (thin) and the time taken to complete under the row label "MCMC samples" for the three models. The values of the triplets show that the proposed SVCF model had a faster convergence with sample size 30000 and thin =1 compared to M2 and M3 which required sample sizes 30000 and 90000 with respective thins 40 and 20 for convergence.

The model adequacy measure (DIC) reported in the table shows that the proposed SVCF model provides the best fit among the three models considered here. Also the effective number of parameters (p_D) is lowest for the proposed model among the three. This indicates that the proposed SVCF model fits to the S&P100 data with maximum parsimony.

From the figures given in table 2 it can be seen that the posterior medians of all the parameters are comparable to the same in existing literature. In particular, all models depict high volatility persistence. The feedback effect in the SVCF model is calculated using posterior medians of λ_1 and λ_2. The estimated value is is -0.1041 whereas the same (ρ) in M2 and M3 indicates negligible feedback effect.

Posterior median of the return impact parameter (λ_1) is less in the proposed SVCF model compared to M2 and M3. Further, posterior median of return and volatility impact parameters are of opposite signs in the proposed SVCF model where as M2 indicates both the impacts in the same direction.

Next I compare the model based estimates of S&P100 volatility with the implied volatilities of the same obtained from Chicago board of options exchange (CBOE). CBOE provides annualized implied volatility percentage using S&P100 options (VXO) (CBOE 2009). I transform VXO to daily implied volatility by dividing it with 100*$\sqrt{12}$. On the other hand model based volatilities are estimated using the posterior median of log-volatilities. The model based volatility estimates have been plotted along with VXO in figure 2 after making suitable linear transformations so that they can be visually compared. Since linear transforms are monotone, the pattern (shape) of the plot remains unchanged. The figure shows that VXO has an over all increasing trend starting with a moderately volatile period especially during 1994-1995. The volatility started increasing since 1996 and after 1998 sharp rise in volatility has been observed. It could observed from the graph that estimated volatility from the proposed model (SVCF) replicates similar pattern whereas estimates obtained from M2 and M3 does not indicate any major change after 1998.

4.4 Estimation Result for SVDF

In this section I consider the dynamic feedback models and apply different extensions of the models described in the previous section incorporating dynamic feedback and skewness parameters. The proposed common factor model is easily obtained from 2.3-2.4 by making the impact parameters time dependent. Model M4 describes time dependent feedback through time varying return impact parameters ($\lambda_{1,t}$) while the volatility impact parameter is considered to be constant. Model M5, on the other hand, describes the time varying feedback effect setting both the impact parameters time dependent. Model M6 is obtained from M3 considering the feedback to be time varying (ρ_t) where as model M7 considers both the return impact parameter and feedback parameter as time varying ones. We have excluded the extensions of M2 as the convergence was too slow in the constant feedback model. Prior distributions remain same as in the SVCF models. Posterior medians of important parameters in all the four models and their respective sd and psrf are shown in table 3.

The psrf figures in the table indicate that simulation has converged for M4 and M5 whereas M6 and M7 could not achieve the same level of convergence. Notice that M4 provides best fit to S&P100 daily returns with lowest DIC and p_D among the four variants considered here. However, model adequacy reduces in dynamic feedback models compared to the constant feedback counterparts and at the same time complexity increases in terms of effective number of parameters (p_D). Observing the fact that convergence may not have been achieved in models M6 and M7, the results should be interpreted with caution.

Similar to the constant feedback model, posterior medians are considered to be the point estimates of the unknown parameters. Table 3 shows the posterior medians along with sd and psrf in brackets. It may be noticed that the posterior median of the persistence parameter and impact parameters are smaller in dynamic feedback models compared to their constant feedback counterparts.

The four plot in figure 6 describes the estimated return (or market) and volatility impact plots corresponding to M4, M5 and M7. Notice that the estimates of all the impact parameters vary around 0 but the amount of variation is different in different time-window. Among the return impact parameters, the estimates obtained from M5 vary more compared to M4. The volatility impact parameter plot obtained from M5 describes large positive impacts on volatility after 1997 which could be interpreted as an indication of riskier time for investment.

For the models M4 and M5, I compute the feedback effect from (3.25) using the posterior median of the impact parameters and the variance of the volatility. It may be remarked here that out of 1516 trading days, MCMC simulation for ρ_t did not converge for 251 days in M6 and 434 days in M7 with a sample of size 15000 for each day. Figures 7 in the appendix presents the time-varying feedback effects only on the time points where convergence has occurred. It could be observed from the graphs that feedback effect fluctuates about 0 but amount of variation in different time-window are different. Large movements in estimated feedbacks form all the models are found to be frequent after 1997. Further, the plot corresponding to M5 shows higher range of feedback effects compared to the other three. A similar pattern is found for the time varying skewness which are computed from (2.7) for M4 and M5 using the posterior medians of the impact parameters and variance of the log-volatility distribution. In terms of magnitude of skewness, estimates based on M5 are larger compared to the other two models.

5 Discussion

This paper presents a parsimonious single factor SV model that leads to four major insights related to return skewness and feedback.

1. The inter connection between feedback effect and return skewness has been established. Precisely, the skewness of returns has been shown as a perturbation of symmetric return error with a positive "market sentiment" factor common to both return and volatility

and the feedback is generated as a result of the shared factor.

2. The model accommodates the dynamic nature of the skewness and feedback effect as mentioned in (Boyer et al. 2010). In particular, the concept of bounded stationarity has been introduced as a generalization of weakly stationary process and the non-stationarity arising out of dynamic skewness has been tackled with the bounded stationarity which enables finite forecasts of risk.

3. The proposed model leads to a simple characterization of the admissible distributions for return and volatility shocks. The findings establish the fact that the common factor and the volatility shocks can't be characterized by a heavy-tail distribution for which the MGF does not exist.

4. The reaction of the feedback effect to the variance of the volatility process elicits from this single factor model. The interesting fact that could be noticed from the feedback surface plot is that if the volatility process itself has very high variance then the feedback effect is infinitesimal. In particular, large variance of the volatility shock leads to a non-informative distribution. As per the plot, market sensitivity looses its importance on the risk or volatility in such a condition and in turn generates an infinitesimal effect on future price.

Application of the proposed single factor SV models on S&P100 daily returns shows some additional advantages. Results from MCMC estimation method shows that the single factor model is computationally more efficient compared to some comparable models discussed in this paper in terms of time to convergence and degree of convergence as measured by psrf. Further, the single factor models provide better fit and the complexity measure p_D is lower compared to the comparable models. In-sample volatility estimates seem to replicate the implied volatility pattern over time.

The single factor model gives rise to an interesting problem. Construction of a portfolio involves understanding the risk or volatility of multiple assets and an experienced portfolio manager may have prior knowledge available about the correlation between asset returns and their volatilities within and between different industry sectors. Such expertise may help in eliciting priors for individual feedback in a Bayesian multivariate SV framework. However, assigning priors on individual elements of the correlation matrix does not ensure that at an intermediate stage of simulation the resulting correlation matrix would remain positive definite. On the other hand, if sector specific factors are shared between within-industry assets and the priors for the impact parameters are elicited from the expert knowledge the positive definiteness still may be achieved while incorporating the individual priors.

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Table 1: Summary Statistics for S&P100 returns

Series	Mean	Median	Min	Max	Skewness	Kurtosis
S&P100	0.0003191	0.0003041	-0.03264	0.02435	-0.5878	11.90426

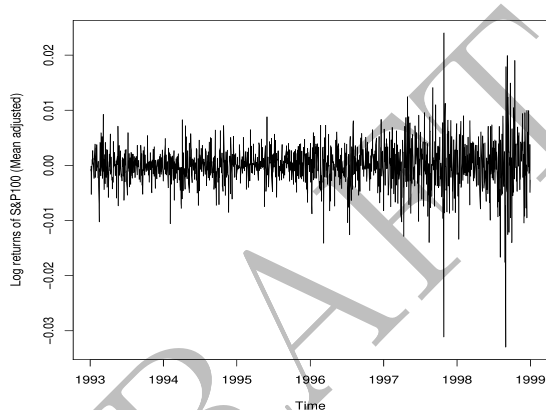

Figure 1: Time plot of S&P100 returns over 1993-1998

Table 2: Parameter Estimates for S&P100 Returns

Parameter	M1	M2	M3
α	-11.716 (0.1058, 1.0003)	-9.6294 (2.848,1.001)	-11.8 (0.1667, 1.0521)
ϕ	0.8277 (0.0306, 1.0008)	0.94193 (0.0087,1.0008)	0.8148 (0.0377, 1.0638)
σ	0.4998 (0.062, 1.0016)	0.53443 (0.05402,1.01)	0.5582 (0.074, 1.1326)
λ_1	-0.4072 (0.1911, 1.0018)	-0.6925 (0.291,1.0149)	-0.5866 (0.3378, 1.0149)
λ_2	0.098 (0.1585, 1.008)	-0.0148 (0.1053, 1.0015)	- (-)
ρ	- (-,-)	-0.0054 (0.00047, 1.0043)	0.0002 (0.0004, 1.043)
p_D	522.2921	1759.247	778.2841
DIC	-12883.86	-8941.212	-12754.21
Iterations	30000 (1,2.8 Hrs)	30000 (40,1.7 days)	90000 (20,1.7 days)

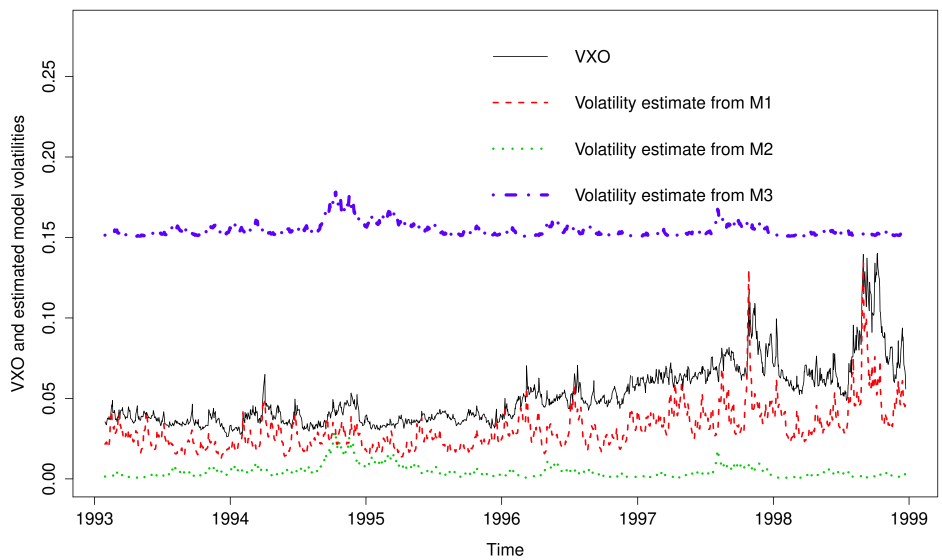

Figure 2: Implied volatility (VXO) and model based estimated volatility of of S&P100

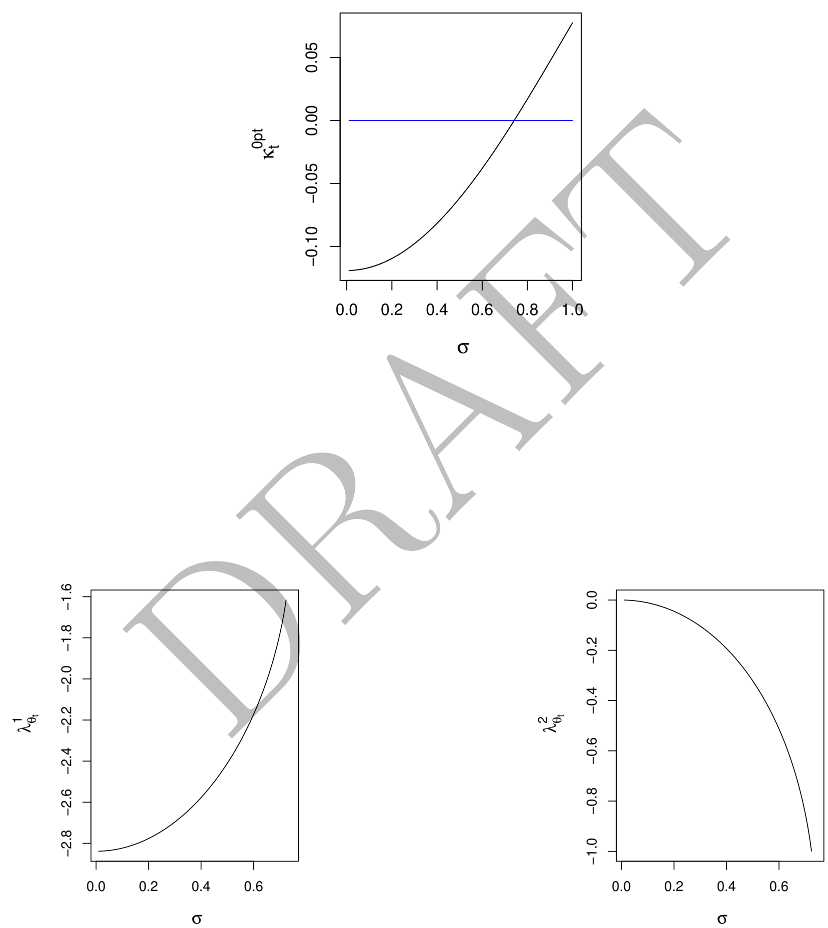

Figure 3: Minimum value of κ_t for different values of σ

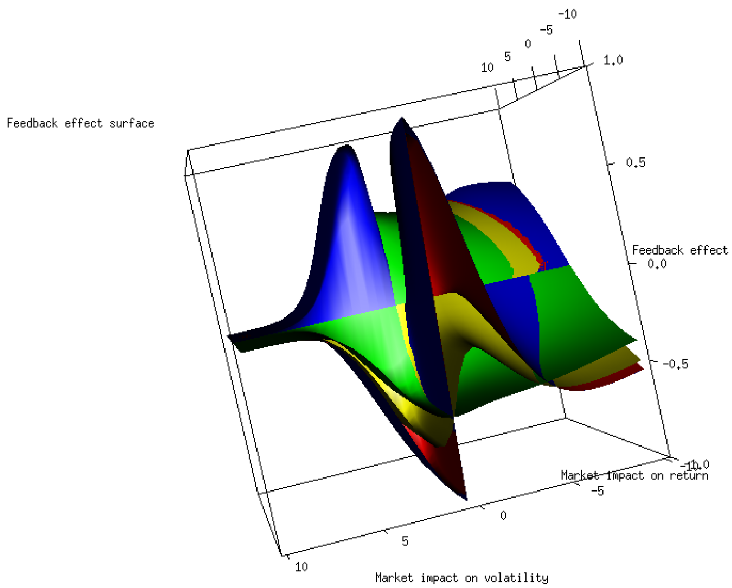

Figure 4: Feedback surface plots for σ =0.001(blue), 0.1(yellow), 1(red), 2(green)

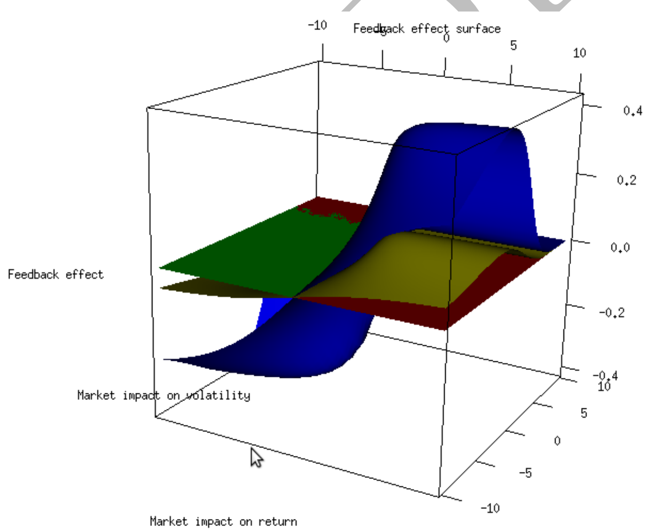

Figure 5: Feedback surface plots for σ =3 (blue), 5 (yellow), 10 (red), 20 (green)

Table 3: Parameter Estimates for S&P100 Returns with dynamic Feedback

Parameter	M4	M5	M6	M7
α	-12.004	-12.22	-10.99	-12.28
	(0.0955, 1.0)	(0.0767,1.0006)	(0.4294,1.333)	(0.5881,1.37)
ϕ	0.8174	0.5865	0.5723	0.6704
	(0.0339, 1.0034)	(0.04399,1.0008)	(0.08335, 1.7118)	(0.0351,1.60)
σ	0.5161	0.4981	0.9783	0.6019
	(0.0631,1.0026)	(0.09288,1.001)	(0.13138, 1.0758)	(0.11934, 1.15)
λ_1	-	-	-0.00305	-
	(-, -)	(-,-)	(0.1317, 1.001)	-
λ_2	-0.152	-	-	-
	(0.1584, 1.0008)	(-,-)	(-,-)	(-,-)
p_D	1012.442	3244.661	2348.584	2835.136
DIC	-12832.39	-11075.3	-10379.52	-11783.02
Sample Size	30000	10000	100000	300000
	(20,10.6 Hrs)	(5, 4.3 hrs)	(1,6.5 hours)	(1,16 hours)

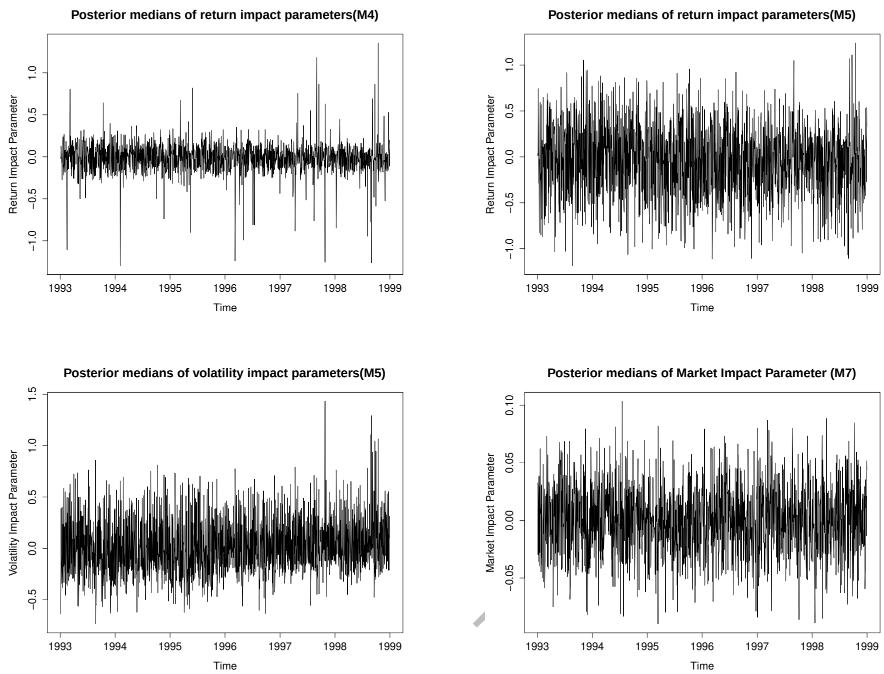

Figure 6: Plot of time-varying impact parameters in M4-M7.

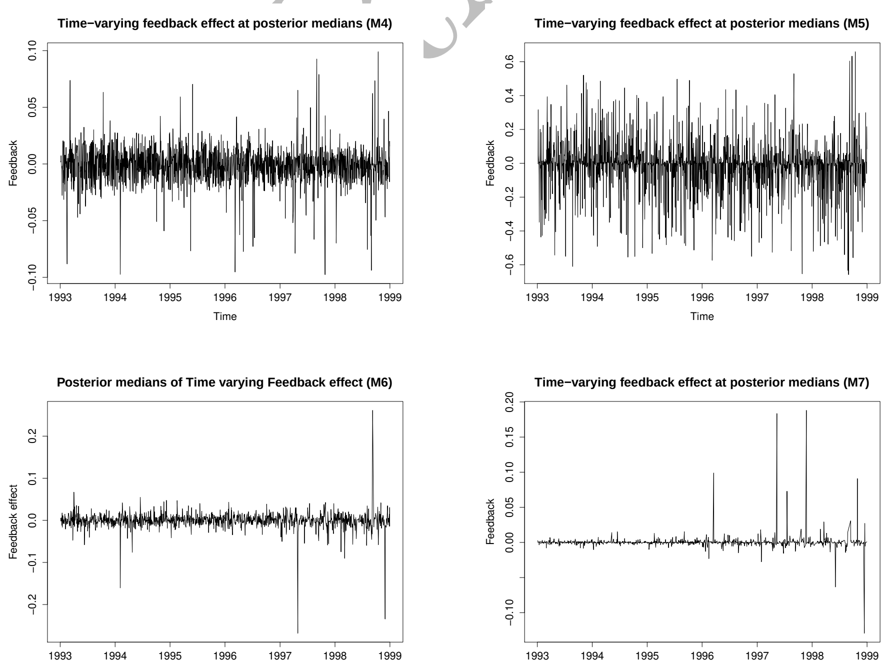

Figure 7: Plot of ρ_t in M4-M7.

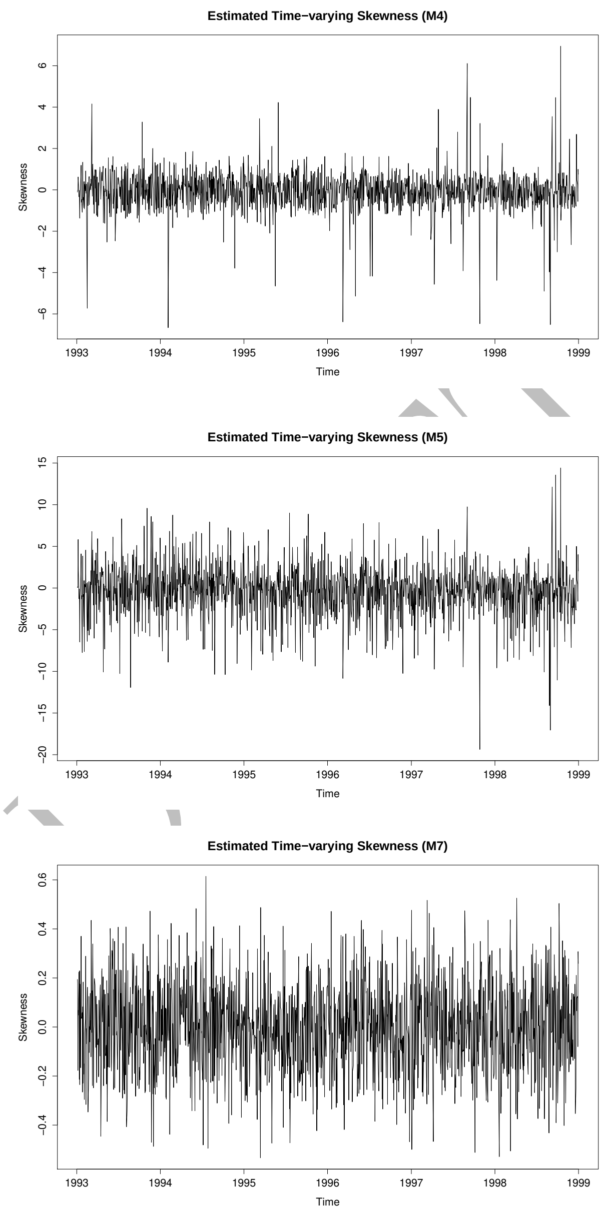

Figure 8: Plot of time-varying skewness in M4, M5 and M7.