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SOME PROPERTIES AND APPLICATIONS OF THE STUTTERING GENERALIZED WARING DISTRIBUTION

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1. INTRODUCTION

With the aim of preventing accidents, accident theory has received much attention. In the framework of some of the various hypotheses that have been developed the generalized Waring distribution (GWD) was obtained as the distribution of accidents (see e.g. Irwin [2], Xekalaki [5], [6], [7]). Generalizing this distribution Panaretos and Xekalaki [4] introduced the stuttering generalized Waring distribution (SGWD) in the context of an urn scheme .

This is an intermingling of generalized Waring streams and is defined by the probability function (p.f)

$$P(X=x)=\frac{c_{(\Sigma m_i)}}{(\alpha+c)_{(\Sigma m_i)}}\sum_{\sum_{j=1}^{k} j\, x_j=x}\frac{\alpha_{(\Sigma x_i)}}{(\alpha+c+\Sigma m_i)_{(\Sigma x_i)}}\prod_{i=1}^{k}\frac{(m_i)_{(x_i)}}{x_i!} \tag{1.1}$$

where $\alpha_{(\beta)}$ denotes the ratio $\Gamma(\alpha+\beta)/\Gamma(\alpha)$, $\alpha>0$, $\beta\in R$, $x=0,1,2,..$

The probability generating function (p.g.f) of this distribution is given by

$$G(s)=\frac{c_{(\Sigma m_i)}}{(\alpha+c)_{(\Sigma m_i)}}\; F_D(\alpha;m_1,\dots,m_k;\alpha+\Sigma m_i+c;s,s^2,\dots,s^k) \tag{1.2}$$

where F_D denotes Lauricella's hypergeometric series of type D defined by

$$F_D(\alpha;\beta_1,\beta_2,\dots,\beta_k;\alpha+\Sigma\beta_i+\gamma;s_1,s_2,\dots,s_k) =$$

$$=\sum_{r_1=0}^{\infty}\cdots\sum_{r_k=0}^{\infty}\frac{\alpha_{(\Sigma r_i)}\,(\beta_1)_{(r_1)}\cdots(\beta_k)_{(r_k)}}{(\alpha+\Sigma\beta_i+\gamma)_{(\Sigma r_i)}}\,\frac{s_1^{r_1}}{r_1!}\cdots\frac{s_k^{r_k}}{r_k!}\,,$$

$|s_i| \le 1$, $i=1,\dots,k$. For $k=1$ one obtains the GWD. So the definition of this distribution enhances the application potential of the GWD as the underlying mechanism causing accidents as well as various other phenomena in many diverse areas ranging from linguistcs to inventory control.

The reason lies in the fact that (1.1) can be employed in situations where single events, pairs of events, triplets of events,..., k-plets of events can be thought of as beeing jointly distibuted according to the k-variate GWD. In the context of car accident statistics this implies that the SGWD would be expected to describe the distibution of the total number of cars involved in accidents if it is reasonable to assume that the joint distribution of the numbers $X_1,X_2,..,X_k$ of accidents involving one, two,...,k cars simultaneously is the k-variate GWD.

The ordinary GWD (case k=1) can be obtained through mixing from a Poisson distribution. In particular, it can arise as a mixture of a Poisson distribution whose parameter λ is itself a random variable that follows a distribution which is a scale mixture of gamma distributions. Moreover, the GWD tends to a Poisson distribution for certain limiting values of its parameters.

One would therefore expect that a similar connection exists between its generalization as given by (1.1) and the Poisson or the generalized Poisson distribution. Indeed it has been shown (Panaretos [3]) that the SGWD can be obtained as a mixture of generalized Poisson distibutions when the mixing distribution is a scale mixture of gamma distibutions. In the next section Panaretos's [3] result is restated and then some limiting cases of the SGWD are examined. Specifically, it is shown that for certain limiting values of its parameters the SGWD tends to a generalized Poisson distribution as well as to a negative binomial type of distribution. Finally, it is demonstrated in section 3 that the SGWD can arise in the context of an accident proneness hypothesis.

2. SOME PROPERTIES OF THE SGWD.

As is well known, a generalized Poisson distribution is a distribution whose p.g.f. can be put in the form

$$G(s) = \exp\{\lambda(g(s)-1)\},\ \lambda>0 \tag{2.1}$$

where g(s) is a valid p.g.f.. It can be shown (Feller, [1], p.291) that G(s) in (2.1) can alternatively be represented by

$$G(s) = \exp\left\{\sum_{i=1}^{m} \lambda_i(s^i-1)\right\} \tag{2.2}$$

where $\lambda_i=\lambda g^{(i)}(0)/i!$, $m \in I^+ \cup \{+\infty\}$, i.e. by the p.g.f. of the random variable (r.v.) $Z = \sum_{i=1}^{m} iZ_i$ with $Z_1,Z_2,\dots,Z_m$ as independent Poisson (λ) variables.

Theorem 2.1 (Panaretos [3]) Let $X|(\lambda_1,\dots,\lambda_k)$ be a non-negative integer valued r.v. whose distribution contitional on $\lambda_1,\lambda_2,\dots,\lambda_k$ is the generalized Poisson distribution with p.g.f. given by (2.2) for $m=k<+\infty$. Assume that $\lambda_1,\lambda_2,\dots,\lambda_k$ are independent gamma r.v.'s with probability density functions (p.d.f.)

$$f_i(\lambda_i) = \frac{h^{-m_i}}{\Gamma(m_i)} \lambda_i^{m_i-1} \quad e^{-\lambda_i/h} \tag{2.3}$$

where $m_i > 0$, $i=1,2,\dots,k$ and h is itself a r.v. with p.d.f.

$$f(h) = \frac{\Gamma(a+c)}{\Gamma(a)\Gamma(c)} \; h^{a-1} \; (1+h)^{-(a+c)} \qquad a,c > 0 \tag{2.4}$$

Then the distribution of X is the SGWD given by (1.1).

Theorem 2.2 The SGWD with parameters $k,a,m_1,m_2,\dots,m_k$ and c tends to the distribution of $\sum_{i=1}^{k} iY_i$ where $Y_1,\dots,Y_k$ are independent negative binomial r.v.'s with parameters $\left(m_i,\ p_i = \frac{c}{a+c}\right)$, $i=1,2,\dots,k$ if $a\to+\infty$, $c\to+\infty$ so that $a/(a+c)<+\infty$.

Proof: Let $\lim_H$ stand for limit as $a\to+\infty$, $c\to+\infty$, $a/(a+c)<+\infty$.

Then, using (1.2) we have

$$\lim_H G(s) = \lim_H \frac{c_{(\Sigma m_i)}}{(a+c)_{(\Sigma m_i)}} F_D\,(a;m_1,m_2,\dots,m_k;a+\Sigma m_i+c;s,s^2,\dots,s^k)$$

$$=\left[c/(a+c)\right]^{\Sigma m_i} \sum_{r_1,\dots,r_k} \left[a/(a+c)\right]^{\Sigma r_i} \frac{(m_1)_{(r_1)}\cdots(m_k)_{(r_k)}s^{\Sigma ir_i}}{r_1!\dots r_k!}$$

$$= \prod_{i=1}^{k} \left\{\left[c/(a+c)\right]^{m_i} \sum_{r_i=0}^{\infty} \left[a/(a+c)\right]^{r_i} \frac{(m_i)_{(r_i)}}{r_i!} (s^i)^{r_i}\right\}$$

$$= \prod_{i=1}^{k} \left[1 + (a/c)(1-s^i)\right]^{-m_i} \tag{2.5}$$

Hence the result.

Theorem 2.3 The SGWD with parameters k, $m_1,m_2,\dots,m_k$ and c tends to the generalized Poisson distribution with p.g.f. given by (2.2) where $m=k<+\infty$, $\lambda_i=am_i/(c+a)$, $i=1,2,\dots,k$ if $a\to+\infty$, $m_i\to+\infty$, $i=1,2,\dots,k$ $c\to+\infty$ so that $am_i/(a+c)<+\infty$ and $a/(a+c)\to 0$.

Proof: Let $\lim_{H'}$ stand for limit as $a\to+\infty$, $m_i\to+\infty$, $i=1,2,\dots,k$, $a/(a+c)\to\infty$, $am_i/(a+c)<+\infty$. Then from (1.2) we obtain $\lim_{H'} G(s)$ =

$$\lim_{H'} \frac{c_{(\Sigma m_i)}}{(a+c)_{(\Sigma m_i)}} \lim_{H'} F_D(a; m_1, m_2, \ldots, m_k; a+\Sigma m_j+c; s, s^2, \ldots, s^k) =$$

$$= \lim_{H'} \frac{c_{(\Sigma m_i)}}{(a+c)_{(\Sigma m_i)}} \sum_{r_1,\ldots,r_k} \prod_{i=1}^{k} \left[\frac{am_i s^i}{a+c}\right]^{r_i} \Big/ r_i!$$

Observe that

$$\lim_{H'} \frac{c_{(\Sigma m_i)}}{(a+c)_{(\Sigma m_i)}} = \lim_{H'} \left[\frac{c}{a+c}\right]^{\Sigma m_i}$$

$$= \lim_{H'} e^{\Sigma m_i \ln(c/(a+c))} .$$

However, since

$$-\Sigma m_i \left[1 - \frac{a+c}{c}\right] \le \Sigma m_i \ln \frac{c}{a+c} \le \left[\frac{c}{a+c} - 1\right] \Sigma m_i$$

it follows that

$$\Sigma m_i \frac{a}{a+c} \le \lim_{H'} \Sigma m_i \ln \frac{a}{a+c} \le -\Sigma m_i \frac{a}{a+c} .$$

Hence

$$\lim_{H'} \left[\frac{c}{a+c}\right]^{\Sigma m_i} = e^{-a\Sigma m_i/(a+c)}$$

which implies that

$$\lim_{H'} G(s) = e^{-a\Sigma m_i/(a+c)} \sum_{r_1,\ldots,r_k} \prod_{i=1}^{k} \left[\frac{am_i s^i}{a+c}\right]^{r_i} \Big/ r_i!$$

$$= e^{-a\Sigma m_i/(a+c)} \prod_{i=1}^{k} \sum_{r_i=0}^{\infty} \left[\frac{am_i s^i}{a+c}\right]^{r_i} \Big/ r_i!$$

$$= \exp\{-\alpha\Sigma m_i (s^i-1)/(\alpha+c)\}.$$

This establishes the proof of the theorem.

3. ACCIDENT THEORY AND THE STUTTERING GENERALIZED WARING DISTRIBUTION.

One of the various hypotheses that have been developed in the area of accident analysis is that of accident proneness-accident risk: An accident is the yield of factors that can be attributed to chance, exposure of the individual to external risk and psychology of the individual. In this context the ordinary GWD was shown (Irwin, [2]) to arise as the distribution of accidents incurred by an accident prone population exposed to varying external risk on suitable assumptions concerning the forms of the distribution of the proneness and risk parameters. In particular it arises on the assumption that for a given individual of proneness h the accident experience is Poisson with parameters $(\lambda|h)$ where $\lambda|h$ refers to the effect of the individual risk exponure. Then if $\lambda|h$ and h vary from individual to individual according to a gamma and a beta distribution of the second kind respectively, the GWD is arrived at as the distribution of accidents. It becomes obvious, therefore, that the results of theorem 2.1 can be put in a similar perspective thus leading or a generalization of Irwin's accident proneness hypothesis giving rise to the SGWD as an accident distribution .

Consider a population of individuals and let $X|(\underset{\sim}{\lambda},h)$ be the number of accidents experienced by an individual of proneness h exposed to an enviromental risk indexed by a parameter vector $(\underset{\sim}{\lambda}|h)=((\lambda_1,\lambda_2,\dots,\lambda_k)|h)$, where the parameters $(\lambda_1,\lambda_2,\dots,\lambda_k)$ may be considered to be reflecting the effects of different types of hazards. Assume that $X|(\underset{\sim}{\lambda},h)$ follows a generalized Poisson distribution with p.f. given by (2.2) and that differences in risk exposure from individual to individual are effected through an uncorrelated multivariate gamma distribution with p.d.f. given by (2.3). Then for individuals of the same proneness h the distribution of accidents is given by (2.5). If we further assume that differences in proneness manifest themselves in the form of the beta distribution defined by (2.4), then the final distribution of accidents will be the SGWD.

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