On the Duality of Certain Characterizations of the Exponential and the Geometric Distributions

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Abstract

Let $\{N(t), t>0\}$ be a homogeneous Poisson process with parameter $\lambda=1$. Let Z be a non-negative random variable which is distributed independently of $\{N(t), t>0\}$ according to a mixed gamma distribution. Xekalaki and Panaretos [9] showed that the form of F (the mixing distribution) is uniquely determined by that of the distribution of $N(Z)$. They also showed that certain characterizations of $N(Z)$ can be derived through characterizations of F.

In this paper it is demonstrated that through the above mentioned result a deeper insight is gained into the relationship of the distribution duals (geometric, exponential) and (Yule, Pareto). Two characterization theorems are also shown for the exponential distribution which can be thought of as variants of Govindarajulu's [4] and Crawford's [1] characterizations of the exponential distribution as the corresponding characterizing conditions are weaker than those used by them.

1.Introduction

Engel and Zijlstra [2] in their study of problems of supply and demand proved certain results which led to characterizations of the exponential distribution from known characterizations the geometric distribution. More specifically,they showed that certain characteristic properties of these distributions which are based in the structure of the corresponding distribution functions can be thought of as equivalent. Hence, one can derive characteristic properties of one of these distributions starting from characteristic properties of the other. Xekalaki and Panaretos [9] extended the results of Engel and Zijlstra to the case where the parameters of the above mentioned distributions are random variables whose distribution belongs to the gamma family of distributions. In this way they derived characterizations of the Pareto distribution which correspond to existing characterizations of the Yule distribution. The key point in the whole argument which leads to the above mentioned results was the fact that the distributions geometric and Yule can be thought of as the images of the distributions exponential and Pareto respectively through a homogeneous Poisson process with parameter $\lambda=1$.

The aim of the present paper is to examine whether characterizations of the discrete "images" based on conditions that do not represent a fuctional relationship between their distribution functions can lead to analogous characterizations of their continuous counter parts. More specifically, we examine whether the notions of independence and conditional expectation, whenever they are characteristic, can be "transferred" through a homogeneous Poisson process from distributions defined in the set $\{0,1,2,...\}$ to corresponding characteristic properties (existing, new or variants of existing ones) of distributions defined in the interval $[0,+\infty)$.

We first give some definitions and notation necessary in the remaining of the paper.

<u>Definition 1.1</u>: A random variable (r.v.) X defined in the set $\{0,1,2,...\}$ is said to follow the <u>geometric distribution</u> with parameter q if its probability distribution is given by the formula:

$$P(X=x) = p\,q^{x}, \quad x = 0,1,2,.... \quad q = 1-p, \quad 0<p<1. \qquad (1.1)$$

<u>Definition 1.2</u>: A random variable X defined in $\{0,1,2,...\}$ is said to follow the <u>Yule distribution</u> with parameter α, $\alpha>0$ if its probability distribution is given by the formula:

$$P(X=x) = \frac{\alpha x!}{(\alpha+1)(\alpha+2)...(\alpha+x+1)}, \quad (1.2)$$
$$x=0,1,2,...;\alpha>0$$

<u>Definition 1.3</u>: A non-negative random variable X is said to follow the <u>exponential distribution</u> with parameter λ if its probability density function (p.d.f.) is given by the formula:

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0, \quad \lambda > 0.$$

Definition 1.4: A non-negative random variable Y defined in $(0,+\infty)$ is said to follow the Pareto distribution with parameter $\alpha>0$ if its p.d.f. is given by the formula:

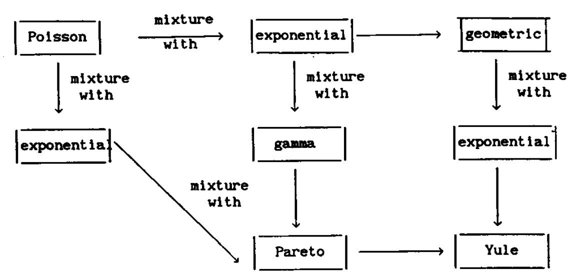
$$f_y(y) = \alpha(1+y)^{-(\alpha+1)} \quad y > 0,\ \alpha > 0.$$

2.On the duality of the pairs Exponential-Geometric and Yule -Pareto

The duality that exists between the exponential and geometric distributions is well-known in the litterature. This duality can be thought of as originating from the models that give rise to these distributions and is confirmed by the similarity of their properties, many of which are characteristic. Actually, as far as the genesis of these distributions is concerned, it is known that the geometric distribution arises as the distribution of the number of "failures" until the first "success" (or the number of "failures" between two concecutive "successes"). The exponential distribution, as it is well-known, arises as the distribution of the waiting time until an event occurs for the first time (or the time between successive events) in a sequence of events which occur with constant rate. On the other hand,characteristic properties like the well-known loss of memory property or the property of a constant hazard rate, among others, have led scientists to consider the geometric distribution as the discrete analogue of the exponential distribution or, equivalently, the exponential distribution as the continuous analogue of the geometric distribution. This association becomes even stronger in the light of certain results derived by Engel and Zijlstra [2] who reexamined and proved the known result of Feller [3] namely that through a Poisson process everyprobability distribution in the interval $[0,+\infty)$ has a uniquely defined counter part in the set of probability distributions defined in the set $\{0,1,2,\ldots\}$. Using this basic result Engel and Ziljstra derived characterizations of the exponential distribution starting from characterizations of geometric distribution.

It is interesting to observe, however, that the duality property between these two distributions is not something unique. There exists a duality of similar nature between another pair of distributions which can again be based on their genesis schemes. These are the Yule and the Pareto distributions. In fact, the Yule distribution can be derived as a gamma mixture of the geometric distribution while the Pareto distribution is arrived at as a gamma mixture of the exponential distribution. Moreover, the Yule distribution can be thought of as the discrete analogue of the Pareto distribution as suggested by the fact that the Yule distribution can be employed to describe income distributions. Xekalaki and Panaretos [9] established the duality property between these two distributions by pointing out the fact that the Yule distribution belongs to the family of the generalized negative binomial convolutions and that the Pareto distribution belongs to the family of the generalized gamma convolutions.

It is interesting to observe that these two pairs of distributions (geometric-exponential and Yule-Pareto) are connected through the fact that their discrete parts can be thought of as mixtures of the Poisson distribution. The association of these two pairs of distributions can be described by the following diagram:



Diagramme

Theorem 2.1: (Feller [3], Engel and Zijlstra [2]).

Let $\{N(t),\ t>0\}$, be a homogeneous Poisson process with $\lambda=1$ and let X and Y be two non-negative random variables. Then

$$N(X) \overset{d}{=} N(Y) \iff X \overset{d}{=} Y.$$

(i.e. the "images" of two non-negative random variables through a Poisson process have the same distribution if and only if the random variables have the same distribution).

Corollary 2.1: (Engel and Zijlstra,[2]).

If $\{N(t),\ t\geq 0\}$ is defined as in Theorem 2.1 and Y is a non-negative random variable then

$N(Y) \sim$ geometric $\iff$ $Y \sim$ exponential.

Theorem 2.2: (Xekalaki and Panaretos [9]).

Let $\{N(t),\ t\geq 0\}$ be a homogeneous Poisson process with parameter $\lambda=1$ and let Z_1, Z_2 be non-negative random variables independent of the process $\{N(t),\ t\geq 0\}$ with p.d.f.

$$h_{z_1} = \int_0^{+\infty} \frac{\alpha_1^{\beta}}{\Gamma(\beta)} z^{\beta-1} e^{-\alpha_1 z} \, dF_1(\alpha_1) \qquad (2.1)$$

Then

$$F_1 = F_2 \Longleftrightarrow N(Z_1) \stackrel{d}{=} N(Z_2)$$

i.e. There exists a one-to-one corresponde-nce between the mixing distributions F and the distribution of N(Z).

Corollary 2.2:(Xekalaki and Panaretos [9])
Let $\{N(t),\ t>0\}$, and Z be defined as in Theorem 2.2. Then N(Z) has the Yule(α) distribution if and only if F is the distribution function of a Pareto(α) random variable.

The theorems proved by Xekalaki and Panaretos [9], and Engel and Zijlstra [2] unify characterizations of the Pareto and the exponential distributions connecting them with characterizations of the Yule and the geometric distributions respectively as it becomes evident from the argument that follows.

According to corollary 2.1 Y is exponentially distributed if and only if X=N(Y) is geometrically distributed. Then, if Y is an exponential random variable, the probability functions of X=N(Y) and Y can be shown to satisfy the relationship

$$P(X\geq k) = P(Y>X_1 + \ldots + X_k). \qquad (2.2)$$

where $X_1, X_2, \ldots$ are mutually independent random variables, distributed independently of X and Y according to the exponential distribution with parameter 1. Also, according to corollary 2.2, the image $N(Z_Y)$ of Z_Y (a variable distributed as a scale mixture of the gamma distribution) is Yule distributed if and only if the scale parameter of the gamma distribution is Pareto distributed. So, if Y is a Pareto randon variable, the distribution of Y and $X=N(Z_Y)$ can be shown to also satisfy (2.2) whenever $X_1, X_2, \ldots$ are mutually independent random variables, distributed independently of X and Y according to the Pareto distribution with parameter 1.

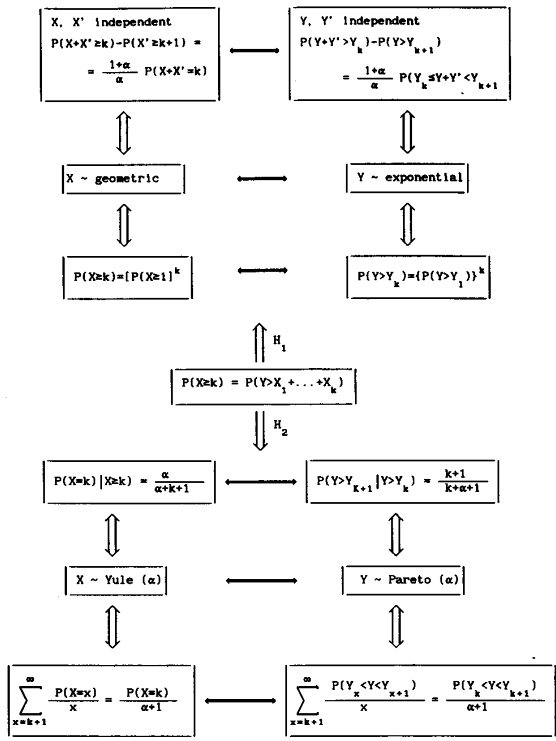
The above relationship gives the opportunity of "transferring" characteristic properties through a Poisson process. This is obvious if one observes that because

$$\text{geometric}(1/(\alpha+1)) \sim \text{Poisson}(\lambda) \underset{\lambda}{\wedge} \text{exponential}(\alpha)$$

and

$$\text{Yule}(\alpha) \sim \text{Poisson}(\lambda) \underset{\lambda}{\wedge} \text{exponential}(Y) \underset{Y}{\wedge} \text{Pareto}(\alpha).$$

With the above notation the "mappings" of the characteristic properties which were examined by Engel and Zijlstra [2] for the pair (geometric-exponential) and Xekalaki and Panaretos [9] for the pair (Yule-Pareto) can be summarised as in diagram 2.



Diagrame 2.

3.A characterization of the exponential distribution

The relationship (2.2) established in the previous section provides the basis for transferring characteristic properties of the above mentioned pairs of distributions when these properties do not represent a functional relation between the correspond-ing probability distributions of these distributions. For example, let us consider the pair of the geometric and the exponential distributions. Srivastava [8] proved that if $Z_1, Z_2, \ldots, Z_n$ are independent observations on a random variable Z and $Z_{(1)}, Z_{(2)}, \ldots, Z_{(n)}$ is the corresponding order statistics, then Z follows the geometric distribution if and only if, for $z=0,1,2,\ldots$

$$P(Z_{(1)} = z,\ Z_{(k)}-Z_{(1)} = 0, \quad k=2,3,\ldots) = P(Z_{(1)}=z)\ P(Z_{(k)}-Z_{(1)}=0, \quad k=2,3,\ldots) \tag{3.1}$$

As pointed out by Srivastava [8] (3.1) is a weaker version of the discrete analogue of an assumption of independence between $Z_{(1)}$ and $(Z_{(2)}-Z_{(1)}, Z_{(3)}-Z_{(1)}, \ldots, Z_{(n)}-Z_{(1)})$ which Govindarajulu [4] used the characteristic property of the exponential distribution.

The characterization that follows provides a variant of the characterization of the exponential distribution of Govindarajulu [4] based on a weaker assumption and can be proved using property (2.2).

Theorem 3.1: (Characterization of the exponential distribution).

Let $Y_1, Y_2, \ldots, Y_n$ be independent observations on a non-negative random variable with corresponding order statistics $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$. Let X_i, $i=1,2,\ldots,n$ be mutually independent exponential random variables with parameter $\lambda=1$, independent of Y_i, $i=1,2,\ldots,n$. Then Y follows an exponential distribution if and only if $P(S_r<Y_{(i)}<S_{r+1}, \quad i=2,3,\ldots |$

$$S_r<Y_{(1)}<S_{r+1}) = c \qquad (3.2)$$
$$r=1,2,\ldots$$

where $S_r = \sum_{i=1}^{r} X_i$.

Proof: Let $\{N(t),\ t>0\}$ be a homogeneous Poisson process with parameter $\lambda=1$ and let $Z=N(Y)$, $Z_i=N(Y_i)$, $i=1,2,\ldots$ and $Z_{(i)}$, $i=1,2,\ldots,n$ be the corresponding order statistics. Then, according to Corollary 2.1, Y follows an exponential distribution $\Longleftrightarrow$ Z follows a geometric distribution $\Longleftrightarrow$ which in return holds if and only if relationship (3.1) holds $\Longleftrightarrow$ or equivalently

$$P(Z_{(2)}=Z_{(3)}=\ldots=Z_{(n)}=r\,|\,Z_{(1)}=r) = c \qquad (3.5)$$
$$r=0,1,\ldots$$

where $c=P(Z_{(k)}-Z_{(1)}=0,\ k=2,3,\ldots)$. This last relationship is however equivelant to (3.2) according to (2.2). This proves the theorem.

Another example is Grawford's [1] characterization. If X_1, X_2 are two independent random variables then the independence between the random variables X_1-X_2 and $\min(X_1,X_2)$ characterizes the distributions of X_1, X_2 as geometric if X_1 and X_2 are discrete or as exponential if they are contineous. Starting from this characterization in the dicrete case one can show the following theorem.

Theorem 3.2:

Let Y_1, Y_2 be two independent continuous random variables and let $S_r = \sum_{i=1}^{r} X_i$ where X_i, $i=1,2,\ldots$ are mutually independent exponential random variables with parameter $\lambda=1$ and independent of Y_1 and Y_2.

Then

$$P(Y_1>S_{X+Y}\,|\,Y_1>S_Y)=P(Y_1-Y_2>S_X) \qquad (3.6)$$
$$X,Y=1,2,3,\ldots$$

if and only if the distribution of Y_1 and Y_2 is the exponential.

Proof: Let $\{N(t),\ t\geq 0\}$ a homogeneous Poisson process with parameter $\lambda=1$ and let $X_1=N(Y_1)$, $X_2=N(Y_2)$. Then from Corollary 2.1 for $i=1,2$, $Y_i\sim$exponential $\Longleftrightarrow$ $X_i\sim$geometric $\Longleftrightarrow$ (Grawford [1]) if

$$P(X_1-X_2\geq x,\ \min(X_1,X_2)\geq y) = P(X_1-X_2\geq x)\ P(\min(X_1,X_2)\geq y)$$

$$\Longleftrightarrow P(X_1-X_2\geq x,\ X_i\geq y,\ i=1,2) = P(X_1-X_2\geq x)\ P(X_1\geq y)\ P(X_2\geq y)$$

$$\Longleftrightarrow P(X_1\geq x+y) = P(X_1-X_2\geq x)\ P(X_1\geq y)$$

$$\Longleftrightarrow P(X_1\geq x+y\ |\ X_1\geq y) = P(X_1-X_2\geq x)\ P(X_2\geq y).$$

The last relationship is because of (2.2) equivalent to (3.6).

Remark: It is obvious that (3.6) is a variant of the independence condition between the random variables Y_1-Y_2 and min

(Y_1, Y_2) of Grawford [1]. Indeed, an argument similar to the one employed for the proof of Theorem 3.2 leads to the conclusion that the random variables $Y_1 - Y_2$ and $\min(Y_1, Y_2)$ are independent if and only if

$$P(Y_1 > x+y \mid Y_1 > y) = P(Y_1 - Y_2 > x), \quad x>0, \quad y>0.$$

From what has been said it becomes obvious that the "mapping" of characteristic conditions that are expressed in terms of a suitable regression function is possible, since we can always express the expected value (conditional or unconditional) of a non-negative random variable X as follows:

$$E(X \mid A) = \sum_{x>0} P(X>x \mid A)$$

for any event A.

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