Replenishing Stock Under Uncertainty

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ABSTRACT

A stock replenishing model is considered whereby not only the demand for the item, but also the stock in hand and the lead time period are considered to be random variables. The interrelations of these three item characteristics are then studied in the framework of a scheme for deciding when to place an order for additional material. The effect of a Pareto/Yule type distributed demand in determining the stock level at which to reorder is then examined and the results are subsequently looked upon in terms of the lead time distribution.

Keywords and Phrases: reorder point system; inventory model; stock replenishment; Pareto distribution; Yule distribution; demand distribution.

1. INTRODUCTION

In the context of inventory decision models the question of when to replenish the on hand stock is of dominant importance. Naturally, the time-to-order very much depends on the fluctuations of demand as the former is very often specified by the stock level. In other words, in many inventory situations an order is placed when stock reaches a specified position and various decision rules have been considered in the literature. Such replenising schemes are known as reorder point systems.

Prichard and Eagle (1965) established a decision rule based on a function associated with the fraction of lead time during which the on hand stock will fall short of demand: Let X be a non-negative integer valued random variable representing the demand for an item in units ordered and let λ be a fixed constant representing the lead time period. Assume that the fraction of λ out of stock is represented by the fluctuations of a random variable T. Then, according to Prichard and Eagle's decision rule an order for a quantity is placed when stock reaches a level y for which E(T), the expected fraction of lead time out of stock, does not exceed a given length λ_0.

By inspecting figure 1.1, one can easily deduce the similarity of the triangles $\overset{\Delta}{\lambda o X}$ and $\overset{\Delta}{AXy}$ and hence conclude that

$$\frac{X-y}{T} = \frac{X}{\lambda} \qquad (1.1)$$

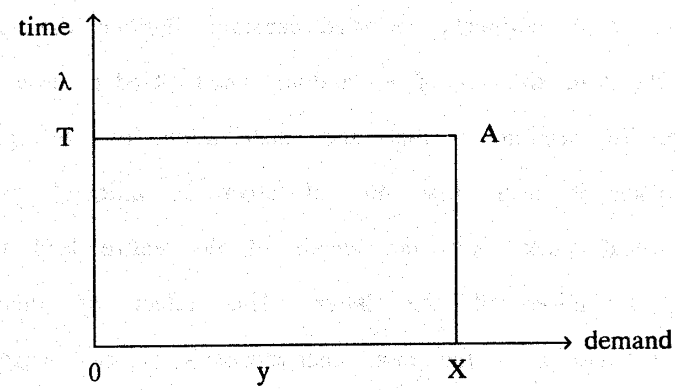

Figure 1.1

which implies that

$$E(T) = \lambda \sum_{x=y+1}^{\infty} \frac{x-y}{x} P(X=x) \tag{1.2}$$

Hence, an inventory manager who wishes to ensure that the item will not be out of stock longer than a specified length of time λ_0 in each lead time will have to choose the reorder point y so that

$$\lambda \sum_{x=y+1}^{\infty} \frac{x-y}{x} P(X=x) \leq \lambda_0 \tag{1.3}$$

or, equivalently so that

$$P(X>y) - y \sum_{x=y+1}^{\infty} \frac{P(X=x)}{x} \leq \lambda_0 / \lambda . \tag{1.4}$$

Note that in this set up only the demand for an item has been considered random. All the other item characteristics, i.e. the lead time and the stock level have been assumed fixed. Indeed this is the case with most inventory problems of this nature.

In this paper a more realistic view is taken. By contract to usual assumptions, the on hand inventory is assumed to be a random variable instead of a continually reviewed constant. Further, the assumption is made that the lead time is of a random length. Under these assumptions, it is shown in section 2 that the distribution of the proportion of the replenishment lead time out of stock is uniquely determined by the conditional stock given the length of the entire lead time regardless of the distribution of the latter. The effect of this result on the interrelationship of the item characteristics is then examined (section 2). Such inventory situations are subsequently studied in the context of certain probability distributions considered to model the demand and Prichard and Eagle's decision rule is shown to lead to an almost exact determination of the reorder point. Specifically, starting from Xekalaki's (1983) result demonstrating the potential use of the Yule distribution as a demand distribution, a model is suggested that gives rise to a Yule distribution of demand (section 3). The effect of this distribution on the determination of the reorder point is then studied. Results of similar nature are derived in the case where demand is regarded to be a continuous random variable (section 4). As demonstrated, the Yule distribution can be regarded as the discrete analogue of the Pareto distribution and as shown by Xekalaki and Panaretos (1988) there exists a duality between these two distributions analogous to that existing between the geometric and the exponential distributions. It would therefore be interesting to examine the problem of determining when to place the order in the context of this duality. Indeed this is done in section 5 where the reorder point decision rule is looked upon in terms of the distribution of lead time.

2. ON THE INTERRELATIONSHIP BETWEEN THE DEMAND AND THE ON HAND STOCK WHEN THE LEAD TIME IS VARIABLE

Let us consider that the demand for an item is described by the fluctuations of a non-negative random variable X with a distribution function $F_X(x)$ defined on $[k, +\infty)$, $k>0$ (Demands less than k are regarded as negligible, i.e., practically non-existent). The variable X can be discrete or continuous. In the first case k will be considered as a non-negative integer and X will be assumed to take values in $\{k, k+1, ...\}$, while in the second case, k will be assumed to have a non-negative real value and X will be assumed to take values in $(k, +\infty)$.

Let Y, L and T be non-negative random variables representing the on hand stock of the warehouse, the replenishment lead time and the fraction of replenishment lead time during which demand exceeds the stock in hand respectively. L is assumed to be independent of X and Y and such that $E(L)<+\infty$. It is obvious from figure 2.1 that that

$$\frac{X-Y}{T} = \frac{X}{L}. \tag{2.1}$$

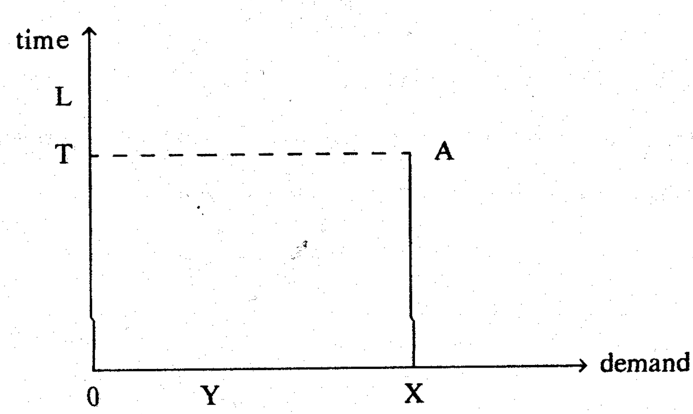

Figure 2.1

By the definition of the variables X and Y, the event $\{X-Y \geq k\}$ can be regarded as describing shortage as it is indicative of the situation where the fraction of demand that cannot be met is not negligible. It is obvious on the other hand, that (2.1) is meaningful only in cases of shortage $(X \geq Y+k)$ whence $T>0$. Then, for $y \geq 0$.

$$E(T \mid Y=y) = E(L)\, E\left(\frac{X-Y}{X} \mid Y=y \right) =$$

$$= E(L)\, E\left(\frac{X-Y}{X} \mid Y=y, X \geq Y+k \right) =$$

$$= E(L) \int_{y+k}^{\infty} \frac{x-y}{x}\, dF_{X \mid (X \geq y+k)}(x) \,.$$

But,

$$F_{X \mid (X>y)}(x) = F_X(x)/P(X>y), \quad x>y \,.$$

Then,

$$E(T \mid Y=y) = E(L) \left\{ 1 - \frac{y}{P(X>y+k)} \int_{y+k}^{\infty} \frac{1}{x}\, d\, F_X(x) \right\} =$$

$$= E(L) \left\{ 1 - \frac{y}{\bar{F}_X(y+k)} \int_{y+k}^{\infty} \frac{1}{x}\, dF_X(x) \right\}, \qquad (2.2)$$

where $\bar{F}_X(x) = P(X>x) = 1-F_X(x)$.

Therefore, to ensure that the expected fraction of replenishment lead time during which stock falls short of demand will not exceed an administratively set constant λ_0, an order should be placed when the stock in hand reaches a level y_0 such that

$$1 - \frac{y_0}{\bar{F}_X(y_0+k)} \int_{y_0+k}^{\infty} \frac{1}{x}\, d\, F_X(x) \leq \lambda_0 \; / \, \lambda \,,$$

where $\lambda = E(L)$.

Theorem 2.1: Let L be a positive random variable with an arbitrary distribution function $F_L(\tau)$, $\tau>0$ belonging to a complete family of distributions and let T be another non-negative random variable. Then, the distribution of the random variable $Z=T/L$ is uniquely determined by the form of the conditional distribution of T given that $L=\tau$ whenever τ is a scale parameter of the latter.

Proof: From the definition of Z it follows that

$$F_Z(z) = \int_0^\infty F_{T|(L=\tau)}(z\tau)d\, F_L(\tau),\ z\in(0,1). \tag{2.4}$$

On the other hand, by the assumptions of the theorem, the distribution funcion $F_{T|(L=\tau)}(t)$ must be of the form

$$F_{T|(L=\tau)}(t) = g(t/\tau),\quad t\in(0,\tau)\ , \tag{2.5}$$

where $g(\cdot)$ is an arbitrary function. Therefore, if $g(\cdot)$ is given, it follows from (2.4) that $F_Z(z)=g(z)$, $z\in(0,1)$. To prove the converse, assume that the form of $F_Z(z)$ is given. Then, because of (2.5), $F_Z(t/\tau)$, $t\in(0,\tau)$ is one solution to (2.4). But this is unique by the completeness of $F_L(\tau)$. Hence the theorem has been established.

Using the above result one can show the following theorem.

Theorem 2.2. Let L,T and Z be defined as in theorem 2.1. Then $T|(L=\tau)$ has the power function distribution with distribution function

$$F_{T|(L=\tau)}(t) = (t/\tau)^\varrho\ ,\quad \varrho>0\ ,\quad t\in(0,\tau)$$

if and only if Z has the power function distribution with distribution function

$$F_Z(z) = z^\varrho\ ,\quad \varrho>0\ ,\quad t\in(0,1)\ .$$

Corollary 2.1. Let L, T and Z be defined as in theorem 2.1. Then T/L is uniformly distributed in (0,1) if and only if $T|(L=\tau)$ is uniformly distributed in $(0,\tau)$.

Put in the context of the inventory model considered the result of corollary 2.1 becomes of great practical value. It implies that if the fraction T of a lead time L during which the item will be out stock for a given length τ of L is uniformly distributed in $(0,\tau)$, then Y/X is uniformly distributed in (0,1). This, taking into account the fact that a uniform distribution for $T|(L=\tau)$ is a rather natural assumption, implies in turn that (X-Y)/X is a uniform random variable taking values in (0,1). As a result, Y/X is also uniformly distributed in (0,1). Therefore, whenever the on hand stock Y falls short of the demand X during a lead time it would not be unreasonable to assume that

$$Y = \begin{cases} RX & \text{if X and Y are continuous} \\ [RX] & \text{if X and Y are discrete} \end{cases} \tag{2.6}$$

where R is a uniform random variable distributed independently of X in (0,1). Here [a] represents the integral part of a. Hence, in the continuous case we have for the distribution function of $Y|(X=x)$ that

$$\begin{aligned} F_{Y|(X=x)}(y) &= P(Y\le y|X=x) = P(RX\le y|X=x) \\ &= P(R\le y/x|X=x) = P(R\le y/x) \\ &= F_R(y/x) = y/x\,, \qquad y\in(0,x). \end{aligned}$$

Hence the distribution of $Y|(X=x)$ is the uniform in (0,x).

On the other hand, we have for the discrete case that

$$
\begin{aligned}
F_{Y|(X=x)}(y) &= P([RX] \le y \mid X=x) \\
&= \sum_{r=0}^{y} P([RX] = r \mid X=x) \\
&= \sum_{r=0}^{y} P(r \le RX < r+1 \mid X=x) \\
&= \sum_{r=0}^{y} \left(\frac{r}{x} \le R < \frac{r+1}{x} \mid X=x \right) \\
&= \sum_{r=0}^{y} P\left(\frac{r}{x} \le R < \frac{r+1}{x} \right) \\
&= \sum_{r=0}^{y} \frac{1}{x} \\
&= \frac{y+1}{x} .
\end{aligned}
$$

So, in this case too, the distribution of $Y|(X=x)$ is the (discrete) uniform in $\{0,1,2,\ldots,x-1\}$. This is turn implies that during a replenishment lead time whence $P(X>Y)=1$,

$$
F_Y(y) = F_{Y|(Y<X)}(y) = \int_y^{\infty} F_{Y|(X=x)}(y)\, dF_X(x) = y \int_y^{\infty} \frac{1}{x}\, dF_X(x), \quad y>k.
$$

Therefore, the integral in the right hand side of (2.2) represents the probability function (or probability density function) of Y that describes fluctuations of the on hand stock beyond the value k and (2.3) may be rewritten as

$$
1 - y_0 f_Y(y_0+k) \,/\, \bar{F}_X(y_0+k) \le \lambda_0/\lambda , \qquad (2.7)
$$

in the continuous case $\left(f_Y(y) = \frac{d}{dy} F_Y(y)\right)$ and

$$1 - y_0\ P(Y=y_0+k)\ /\ P(X>y_0+k) \le \lambda_0/\lambda \tag{2.8}$$

in the discrete case.

3. DETERMINING THE REORDER POINT ON THE ASSUMPTION OF A YULE DISTRIBUTED DEMAND

It is obvious that any algorithm for the determination of the critical level y_0 through (2.2) or either of (2.7) or (2.8) can be very tedious depending on the form of the distributions of X and Y. The theorems that follow however show that a substantial simplification can be achieved if the demand distribution is discrete and of the Yule type with probability function

$$p_x = \frac{\varrho(k+1)_{(x-k)}}{(\varrho+k+1)_{(x-k+1)}}, \tag{3.1}$$

$$x = k,\ k+1,\dots, \qquad a_{(r)} = \Gamma(a+r)/\Gamma(a), \quad a>0, \quad r\in\mathbb{R}$$

or if it is continuous and of the Pareto type with probability density function

$$f_X(x) = \varrho k^{0} x^{-(\varrho+1)}, \quad x\ge k>0, \quad \varrho>0 \tag{3.2}$$

Xekalaki (1983) showed that the fluctuations of demand can be described by the Yule distribution with probability function

$$P(X=x) = \frac{\varrho x!}{(\varrho+1)_{(x+1)}}, \quad x=0,1,2,\dots; \quad \varrho>0 \tag{3.3}$$

Note that this is the untruncated version of (3.1) as (3.3) follows from (3.1) for $k=0$. The derivation of (3.3) was based upon the following hypotheses.

Model A (Xekalaki, 1983):

Let Z, the number of orders arriving at a warehouse, be Poisson distributed with parameter θ characteristic of the buyer's behaviour and let $X_1, X_2, \ldots$, the numbers of item units ordered by the various customers, be independent and identically distributed indepently of Z according to a logarithmic distribution with probability function

$$P(X_i=r) = \frac{1}{\theta}\,\frac{\left(1-e^{-\theta}\right)^r}{r}, \qquad r=1,2,\ldots .$$

Then $X=X_1+X_2+\ldots+X_Z$, the demand for the item in terms of the total number of item units ordered follows for a given buyer (fixed θ) a distribution with probability generating function

$$G_{X|\theta}(s) = \exp\left\{\theta\left(\frac{\ln\left(1-(1-e^{-\theta})s\right)}{-\theta} - 1\right)\right\}$$

$$= \left[e^{\theta}-(e^{\theta}-1)s\right]^{-1}$$

In other words, for a given buyer whose behaviour is reflected by the parameter θ the distribution of demand is the geometric distribution with parameter $e^{-\theta}$. If now differences in the buying behaviour from buyer to buyer are effected through an exponential distribution for θ with parameter $\varrho>0$ the resulting distribution of X has probability generating function.

$$G_X(s) = \varrho \int_0^\infty G_\theta(s)\, e^{-\varrho\theta}\, d\theta$$

$$= \varrho \sum_{r=0}^{\infty} s^r \int_0^1 \theta^\varrho\, (1-\theta)^r d\theta$$

$$= \varrho \sum_{r=0}^{\infty} \frac{r!}{(\varrho+1)_{(r+1)}}\, s^r$$

i.e. the distribution of demand is the Yule with parameter ϱ. This obviously leads to (3.1) which is the (k-1) truncated version of the obtained distribution which arises if one considers that demands less than k are negligible.

In the sequel, an alternative model giving rise to a Yule demand distribution is suggested.

Model B:

Assume that during a lead time of given length t orders for an item occur according to a homogenous Poisson process $\{N(t),\ t>0\}$ with parameter $\lambda=1$. Then for the given length of time the distribution of orders has probability generating function

$$G_{N(t)}(s) = e^{t(s-1)}$$

Assume that t follows a distribution that is a scale mixture of the exponential distribution i.e. a distribution defined by

$$dF_L(t) = \left\{ \int_0^\infty \frac{1}{\alpha}\, e^{-t/\alpha}\, dF(\alpha) \right\} dt, \qquad \alpha>0 \tag{3.4}$$

where $F(\alpha)$, $\alpha>0$ is a proper distribution function. Then, the distribution of the demand X for the item will coincide with the distribution of N(L). Therefore, the probability generating function $G_X(s)$ of X will be given by

$$G_X(s) = G_{N(L)}(s) = E_L\left(G_{N(t)}(s)\right)$$

$$= \int_0^\infty G_{N(t)}(s) \; d\,F_L\,(t)$$

$$= \int_0^\infty \int_0^\infty G_{N(t)}(s) \, \frac{1}{\alpha} \, e^{-t/\alpha} \; d\,F(\alpha)\; dt$$

i.e.

$$G_X(s) = \int_0^\infty \int_0^\infty \frac{1}{\alpha} \, e^{-t[1+\alpha(1-s)]/\alpha} \; dF(\alpha)\; dt \tag{3.5}$$

If now α follows a Pareto distribution with parameter ϱ (Pearson type VI), i.e. if

$$dF(\alpha) = \varrho \, (1+\alpha)^{-(\varrho+1)} \, d\alpha \tag{3.6}$$

we obtain from (3.5)

$$G_X(s) = \varrho \int_0^\infty \int_0^\infty \frac{1}{\alpha} \, e^{-t[1+\alpha(1-s)]/\alpha} \, (1+\alpha)^{-(\varrho+1)} \, d\alpha \; dt$$

$$= \varrho \int_0^\infty \left[\frac{1}{\alpha} \int_0^\infty e^{-t[1+\alpha(1-s)]/\alpha} \, dt \right] (1+\alpha)^{-(\varrho+1)} \, d\alpha$$

$$= \varrho \int_0^\infty [1+\alpha(1-s)]^{-1} \, (1+\alpha)^{-(\varrho+1)} \, d\alpha$$

$$= \varrho \sum_{r=0}^\infty s^r \int_0^\infty \alpha^r (1+\alpha)^{-(\varrho+r+2)} \, da$$

$$= \varrho \sum_{r=0}^\infty s^r \, \frac{\Gamma(r+1)\,\Gamma(\varrho+1)}{\Gamma(\varrho+r+2)}$$

$$= \varrho \sum_{r=0}^\infty \frac{r!}{(\varrho+1)(\varrho+2)\ldots(\varrho+r+1)} \, s^r$$

But this is the probability generating function of the Yule distribution as defined by (3.3) and leading to (3.1) by truncation below the value k. Let us now examine the effect of the assumption of a Yule distributed demand on the determination of y_0 through (2.8).

Xekalaki (1983, 1984) obtained some results connecting the distributions the demand X and the stock Y.

Theorem 3.1. (Xekalaki, 1983): Let X,Y be non-negative integer valued r.v's such that $P(Y=r|X=x) = \frac{1}{x}$, $r=0,1,...,x-1$. Then X and Y are identically distributed if and only if X has a Yule distribution with probability function given by (3.3).

Theorem 3.2. (Xekalaki, 1984): Let X be a non-negative, integer-valued random variable. Then X is Yule distributed with probability function given by (3.1) if and only if

$$P(X>r) - \frac{1}{\varrho}(r+1)\, P(X=r)\ , \quad r = k,k+1,... \tag{3.7}$$

Consider now an inventory situation where the Yule distribution with probability function as given by (3.3) may be appropriate for modeling the demand fluctuations and shortage can be regarded to be effected through (2.6). Then theorems 3.1 and 3.2 imply that the decision rule will select the reorder point y_0 so that

$$\frac{(y_0+\varrho+1)}{\varrho(\varrho+1)}\, p_{y_0} \leq c\ , \quad y_0 = k,k+1,...$$

or equivalently so that

$$\frac{\Gamma(y_0+\varrho+1)}{\Gamma(y_0+1)} \geq c_0\,, \quad y_0 = k, k+1, \ldots \tag{3.8}$$

where $c_0 = \Gamma(k+\varrho+1)/[c(\varrho+1)\Gamma(k+1)]$.

4. THE CASE OF A CONTINUOUS DEMAND DISTRIBUTION

The role of discrete distributions in the description and interpretation of practical situations has in general been very basic in the sense that discrete distributions can be used to find all continuous solutions of a particular problem: Most continuous distributions used to describe practical situations are merely approximations of appropriate discrete models. These approximations can be very useful as often the available data are in a grouped form especially for large values of the random variable whose fluctuations describe the particular situation.

Our inventory model deserves, therefore a treatment in terms of an appropriate approximation of the Yule distribution as defined by (3.1). Indeed, the probability function in (3.1) can be alternatively written as

$$p_x = \frac{\varrho}{\varrho+k+1}\,\frac{(k+1)_{(\varrho+1)}}{(x+1)_{(\varrho+1)}} = \frac{\varrho\,(k+1)_{(\varrho)}}{(x+1)_{(\varrho+1)}}$$

$$= \frac{\varrho}{k}\,\frac{k_{(\varrho+1)}}{(x+1)_{(\varrho+1)}} = \frac{\varrho}{k}\,\frac{k}{x+1}\,\frac{k+1}{x+2}\cdots\frac{k+\varrho}{x+\varrho+1}\,,$$

$$x = k, k+1, \ldots$$

So, if we consider a change of scale by multiplying the numerator and denominator by $c^{\varrho+1}$ and then let $c \to 0$, the above equation is written

$$p_x \propto f_X(x) = \frac{\varrho}{k}\left(\frac{k}{x+1}\right)^{\varrho+1}$$

i.e.

$$f_X(x) = \varrho\, k^{\varrho}(1+x)^{-(\varrho+1)}, \quad x \geq k \tag{4.1}$$

But this is the probability density function of the Pareto distribution as given by (3.2). Therefore, the Pareto distribution can be regarded as the continuous analogue of the Yule distribution. The latter fact can provide a theoretical justification as to why the Pareto distribution might be thought of a representing the mechanism that generates the demand through the models that give rise to the Yule distribution considered in section 3.

As mentioned in the introduction there exists a duality of properties between the Yule distribution as given by (3.1) and its continuous analogue as given by (3.2). Hence, it would be interesting to examine our inventory model in the context of this duality.

So, let us assume that the demand is described by the fluctuations of a continuous random variable with probability density function given by (3.2). Then, as shown by the theorems that follow there hold results that bear a striking analogy to those obtained for the case of a discrete demand of the Yule type.

Theorem 4.1. Let X be a random variable on $[k, +\infty)$, $k>0$. Then the condition

$$P(X>x) = cx\, f_X(x), \quad c<1, \quad x>k, \tag{4.2}$$

uniquely determines the distribution of X as Pareto with parameters k and $(1-c)/c$.

Proof: The necessity part is obvious. For sufficiency, assume that (4.2) is valid. Then

$$F_X(x) = 1\text{-}cx\ f_X(x)$$

or equivalently

$$cx \frac{df_X(x)}{dx} = -f_X(x)$$

whose unique solution under the initial condition $\int_k^\infty f_X(x)dx = 1$ is given by (4.1) for $\alpha=(1\text{-}c)/c$.

Theorem 4.2. (Krishnaji, 1970): Let X be a random variable with probability density function $f_X(x)$, $x \geq k > 0$. Then

$$\int_y^\infty \frac{f_X(x)}{x} dx = c\ f_X(y), \quad y>k, \quad c<1$$

if and only if X has a Pareto distribution with parameters k and (1-c)/c.

Put in the context of the inventory model of section 2 the results of these two theorems lead to the following simplified expression of (2.7).

$$1\text{-}y_0/(y_0+k) \leq \lambda_0/\lambda$$

or equivalently

$$y_0 \leq k\ (\lambda/\lambda_0\text{-}1)\ .$$

In other words, under the assumption of a Pareto distributed demand Prichard and Eagle's (1965) rule will indicate the need for the placement of an order when the on hand stock first falls at or below the level $k(\lambda/\lambda_0\text{-}1)$.

5. DECIDING IN TERMS OF THE LEAD TIME DISTRIBUTION

In this section, we will look at the problem of determining when to place an order in terms of the probability distribution of the lead time starting with a discretized demand distributed according to a Yule distribution in the context of model B. For simplicity we treat the case $k=0$.

It can be shown that in the case $k=0$ (2.8) can take the form

$$P(X>y_0) - y_0\,(1-P(X=0)) \cdot P(Y=y_0) \leq C \tag{5.1}$$

where $C=\lambda_0/E(L)$ and λ_0 and L are defined as in section 2.

Note, further, that the derivation of the Yule distribution on the assumptions of model B establishes an association between the distribution of lead time and that of demand. Specifically, it implies that if the distribution of lead time L is a Pareto mixture of the exponential distribution, the distribution of demand X is the Yule distribution. The converse is also true, i.e., if X is Yule distributed, the distribution of L is a Pareto scale mixture of the exponential distribution. This follows from a more general result shown by Xekalaki and Panaretos (1988). In fact, this result goes even further as it leads to a one-to-one correspondence between the mixing distribution $F(\alpha)$ in (3.4) and the distribution of the demand X whenever X can be regarded as the image N(L) of the lead time L through a homogeneous Poisson process $\{N(t),\ t>0\}$ with parameter $\lambda=1$ as indicated by the theorem that follows.

Theorem 5.1. (Xekalaki and Panaretos, 1988): Let $\{N(t),\ t>0\}$ be a homogeneous Poisson process with parameter $\lambda=1$. Let Z_1,Z_2 be two independent non-negative random variables that are distributed independently of $\{N(t),\ t>0\}$

with probability density functions satisfying

$$h_{Z_i}(z) = \int_0^\infty \frac{\alpha_i^\beta}{\Gamma(\beta)} z^{\beta-1} \; e^{\alpha_i z} \, dF_i(a_i) \qquad z,\beta>0, \quad i=1,2.$$

where $F_i(\cdot)$ is a proper distribution function, $i=1,2$. Then $F_1=F_2$ if and only if $N(Z_1)$ and $N(Z_2)$ are identically distributed.

The implication of this result in the context of the model considered will become obvious if one observes that the distribution $F(\alpha)$, $\alpha>0$ in (3.4) represents the distribution of the mean of an exponential lead time. Indeed, we have from (3.4) that conditional on α the mean lead time is given by

$$E(L\,|\,\alpha=a) = \int_0^\infty t \, \frac{1}{a} \, e^{-t/a} \, dt = a \, .$$

Hence the result of theorem 5.1 brought within the framework of the model considered leads to the following conclusion.

The distribution of demand X is the Yule distribution with parameter ϱ if and only if the distribution of the lead time L is exponential with a mean $E(L\,|\,\alpha)$ that has a Pareto distribution with the same parameter ϱ.

Theorem 5.2. (Xekalaki and Panaretos, 1988): Let X be a random variable having the Yule distribution with parameter ϱ as defined by (3.3). Let U be another random variable distributed according to the Pareto distribution with parameter ϱ as defined by (3.2) for $k=1$ and consider U_1, U_2,... to be a sequence of mutually independent Pareto(1) random variables independent of U. Then

$$P(X\geq k) = P(U>U_1U_2+...+U_k), \qquad k=1,2,... \tag{5.2}$$

Theorem 5.3. (Xekalaki and Panaretos, 1988). Let $U, U_1, U_2, \ldots$ be independent positive random variables such that $U_1, U_2, \ldots$ have the Pareto distribution with parameter 1. Then U has the Pareto distribution with parameter $\varrho > 0$ if and only if either of the following conditions is satisfied

(i)
$$P\left(U > U_1 + \ldots + U_k + U_{k+1} \mid U > U_1 + \ldots + U_k \right) = \frac{k+1}{k+\varrho+1}, \qquad k = 0,1,2,\ldots \tag{5.3}$$

(ii)
$$\sum_{r=k+1}^{\infty} \frac{P(U_1 + \ldots + U_r < U < U_1 + \ldots + U_{r+1})}{r} = \frac{1}{\varrho+1} P(U_1 + \ldots + U_k < U < U_1 + \ldots + U_{k+1}) \tag{5.4}$$

From the definition of the random variable X and the derivation of its distribution it is obvious that the random variable $E(L \mid \alpha)$ plays the role of the random variable U in (5.2). So, if we let

$$U = E(L \mid \alpha)$$

and define

$$U_i = E(L_i \mid \alpha), \qquad i = 1,2,\ldots$$

with L_i being a continuous random variable representing the lead time for the i-th ordered unit (a unit is ordered at the time the previous ordered unit is delivered) then (5.2) states that the events {at least k item units are ordered} and {the mean lead time exceeds the aggregate of the mean lead times for the k units if ordered individually-each at the time of delivery of the previous item} are equiprobable. Then, the decision rule for determining the stock position at which an order should be placed can be expressed in terms of the distribution of the mean lead time.

Consider now the results of the theorems that follow:

Theorem 5.4. (Xekalaki, 1983): Let X,Y be non-negative integer valued r.v's such that $P(Y=r|X=x) = \frac{1}{x}$, $r=0,1,...,x-1$. Then X and Y are identically distributed if and only if X has a Yule distribution with probability function given by

$$P(X=x) = \frac{\varrho x!}{(\varrho+1)(\varrho+2)...(\varrho+x+1)}, \quad x=0,1,..., \quad \varrho>0 \tag{5.5}$$

Theorem 5.5. (Xekalaki, 1984): Let X be a non-negative, integer-valued random variable. Then X is Yule distributed with probability function given by (3.2) if and only if

$$P(X>r) = \frac{1}{\varrho}(r+1)\ P(X=r), \qquad r=0,1,2,... \tag{5.6}$$

It becomes evident therefore that theorems 5.2, 5.3 and 5.4 imply that if X, Y and U represent the demand for an item, the stock on hand and the mean lead time respectively, then

$$P(X=r) = P(Y=r) = P(U_1+...+U_r \le U \le U_1+...+U_{r+1}) .$$

Therefore the inequality in (5.1) becomes

$$y_0\ P\left(U_1+...+U_{y_0} \le U \le U_1+...+U_{y_0+1}\right) \ge c(\varrho+1)\ P\left(U>U_1+...+U_{y_0+1}\right)$$

or, equivalently

$$y_0\ P\left(U > U_1+...+U_{y_0}\right) - \left(y_0 + c(\varrho+1)\right)\ P\left(U > U_1+...+U_{y_0+1}\right) \ge 0$$

Dividing both sides of the inequality by $P(U>U_1+...+U_{y_0})$ yields

$$y_0 - \left(y_0+c(\varrho+1)\right)\ P\left(U>U_1+...+U_{y_0+1} \mid U>U_1+...+U_{y_0}\right) \ge 0 .$$

By condition (5.3) of theorem 5.3 this inequality implies that

$$y_0\ (y_0+\varrho+1) - (y_0+1)\ (y_0+c(\varrho+1)) \geq 0\ .$$

This leads to

$$y_0(\varrho-c(\varrho+1)) \geq c(\varrho+1)\ .$$

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