Dynamic Games under Bounded Rationality

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8 March 2015

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I propose a dynamic game model that is consistent with the paradigm of bounded rationality. Its main advantages over the traditional approach based on perfect rationality are that: (1) the strategy space is a chain-complete partially ordered set; (2) the response function is certain order-preserving map on strategy space; (3) the evolution of economic system can be described by the Dynamical System defined by the response function under iteration; (4) the existence of pure-strategy Nash equilibria can be guaranteed by fixed point theorems for ordered structures, rather than topological structures. This preference-response framework liberates economics from the utility concept, and constitutes a marriage of normal-form and extensive-form games.

Among the common assumptions of classical existence theorems for competitive equilibrium, one is central. That is, individuals are assumed to have perfect rationality, so as to maximize their utilities (payoffs in game theoretic usage). With perfect rationality and perfect competition, the competitive equilibrium is completely determined, and the equilibrium depends only on their goals and their environments. With perfect rationality and perfect competition, the classical economic theory turns out to be deductive theory that requires almost no contact with empirical data once its assumptions are accepted as axioms (see Simon 1959).

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Unfortunately, this is empirically *not* the economic problem which society faces in reality (see Hayek 1945). More seriously, under the assumption of perfect rationality one can not develop a "truly" dynamic theory of games (see Aumann 1997). The difficulty is that perfectly rational players have foresight, so they can contemplate all of time from the beginning to play. Thus the situation can be seen as one-shot game, each play of which is actually a long sequence of "stage games," and then one has lost the dynamic character of the situation.[1] This conceptual difficulty can only be "solved" by eliminating the perfect foresight under the constraint of bounded rationality.

Generally, models of bounded rationality are richer in structure than models of perfect rationality, because they treat not only of equilibrium but of the process of reaching it as well. Nevertheless, if we wanted to know about the dynamic behavior of an economy before equilibrium was reached, then much more information would be required. For example, we need information not only about the structures of the strategy space, but also about the dynamics of the response function. In other words, we need to known how people *do* behave, not just how *should* behave (see Simon 1955, 1972). Lacking the empirical knowledge of the decisional processes that will be required for a definitive theory, one cannot describe the actual world in a systematic and rigorous way.

Essentially, bounded rationality means that players respond to each other's behavior in a relatively stable pattern, though not the best. Under the constraint of bounded rationality, players may quest for satisfactory outcome rather than optimal outcome (Simon 1959). Heuristic rule in the pursuit of satisfactory outcome is utilized rather than its sharp contrast, the pursuit of "maximized utility." Rather than consciously maximizing in each decision situation, people adopt rules that work well "on average", taken over all decision situations to which that rule applies (see Aumann 2008).

In this paper, I propose a dynamic game model that is consistent with bounded rationality. According to Aumann (2008), our preference-response based approach can be categorized into the idea of *rule rationality*, which may provide a synthesis between

[1] For example, if two perfectly rational players were to play chess, the game should be over as soon as they had chosen who is to play first. Each, by working backward from every terminal node of the game tree, would be able to calculate his optimal strategy, and a win or draw would be declared without playing.

rationalistic neo-classical economic theory and behavioral economics. Rule rationality is a form of bounded rationality yielded by the evolutionary paradigm.[2] From molecular biology, we have seen how a basic mechanism—the DNA double helix and the base pairing rules—can account for a wide range of complex phenomena (see Simon 1979). Historically, a significant catalyst to the development of the theory of rule rationality was the experimental work of Axelrod and Hamilton (1981), based on the idea of a computer to program for playing repeated prisoner's dilemma. And the winner of Axelrod's tournament –TIT FOT TAT – was, because of its simplicity, nicely illustrative of the rule rationality idea (see Aumann 1997).

In our dynamic game model, the preference $\geqq_i$ of each player $i \in \{1,2,\cdots,n\}$ is just a *partial order* on his strategy set S_i, i.e., a binary relation over the strategy space that is reflexive, antisymmetric, and transitive.[3] No utility concept need be hypothesized, and no topologic structure need be imposed.

The Cartesian product of the ordered strategy sets $(\mathrm{S}_i, \geqq_i)$ of all the players $i = 1,2,\cdots,n$ makes up the strategy space of the game, denoted by $S = S_1 \times S_2 \times \cdots \times S_n$. Mathematically, the strategy space S can be made into a partially ordered set by imposing the *product order* $\geqq$ defined by

$$(1) \qquad (x_1, x_2, \cdots, x_n) \geqq (y_1, y_2, \cdots, y_n) \Leftrightarrow x_i \geqq_i y_i, \forall i .$$

This product order describes the preference pattern for the society as a whole.[4]

[2] It is worth noting that our approach differs from that of evolutionary games. The evolutionary dynamics emerges as a form of rationality that is bounded, in that foresight is eliminated. One of the most fundamental ideas in evolutionary game theory is that no rationality at all is required to arrive at a Nash equilibrium. The Nash equilibrium of a strategic form corresponds precisely to the evolutionarily stable equilibrium of populations that interact in accordance with the rules of the game. For detail see Maynard Smith and Price (1973).

[3] With no lose of generality we assume that every player considers the range of alternatives to be beyond his power to alter and hence knows the strategy sets of all players. Within the game-theoretic framework it has been shown that any lack of information about the strategy set can be reduced to the ignorance about the payoff functions of some players. For detail see Harsanyi (1967/1968). Under given state, however, each player may make his choice within a set of alternatives more limited than the whole range of objectively available to it. If states change, then either the strategy set or the preference or the response function will change accordingly. For details see section IV.B.

[4] It is silly to think that there might be social preferences that are analogous to individual preferences. In general, society as a whole may not have a social ordering relation for alternative social states. We can not expect what is true of individuals is also true of groups. For details see Arrow (1950).

In order to be consistent with the relatively stable pattern of the behaviors of players with bounded rationality, the response is assumed to be a function $R : S \to S$, rather than a correspondence.[5] Further, the stable pattern of players' behavior is described in terms of order-theoretic properties of the response function, such as *continuity*, *order preservation*, and *directedness.* [6] Interestingly, it turns out that in the repeated prisoner's dilemma game, if the TIT FOT TAT strategy is adopted by both players, then the corresponding response function is order-preserving (see section III.D).

In essence, the response function represents a dynamical rule of strategy updating, where player's preferred strategy, whether optimal or satisficing, is determined by their responses to other players' strategies (see Nash 1950, 1951; Simon 1955). From the viewpoint of *dynamic system*, the response function amounts to the *evolution rule* of the dynamical system that describes what future states follow from the current state. Thus, to investigate the dynamic process of decision making we have to study the dynamic system defined by the iterations of response function.

To be specific, let the response be a function $R : S \to S$ on the strategy space. Then, to study the dynamic evolution of economic system, we must discuss the behavior of the following dynamical system defined by the response function

$$\left\{R^k \mid k \in \mathrm{N}\right\} = \left\{R^0, R^1, R^2, R^3, \cdots, R^k, \cdots\right\}. \tag{2}$$

Further, given an initial strategy it is possible to determine all its future strategies, a collection of points known as an *orbit*. Mathematically, given any initial strategy s in strategy space $(\mathrm{S}, \geqq)$ we must study the orbit generated by the iterated sequence

$$\left\{R^k(s) \mid k \in \mathrm{N}\right\} = \left\{s, R(s), R^2(s), R^3(s), \cdots, R^k(s), \cdots\right\}. \tag{3}$$

[5] It is worth emphasis that the assumption of response function is introduced not just to simplify the notation employed. In most real economic activities (such as auction and sports) the response does turn out to be a function, whether the player is of perfect rationality or not. When individuals are roughly indifferent between two actions they appear to choose more or less at random. Further, since most celebrated fixed point theorems for ordered structures had been generalized to set-valued functions, our model can be extended to the case of response correspondence without difficulty (see section IV.C).

[6] For accurate definitions see section I.C. The intuition behind order-preserving response function is that a higher strategy triggers a higher response. This fact enables order-preserving response to arise naturally in economics. For example, in supermodular games, the best response functions are order-preserving, so that the player's strategies are "strategic complements" (see Fudenberg and Tirole 1991, section 12.3).

It turns out that the dynamic evolution of the economic system can be governed by the behavior of this dynamical system defined by the response function under iteration. Typically, the long term behavior of a dynamical system can be learned from its *limit set*, namely

$$\{\lim_{k\to\infty} R^k(s) \mid s \in S\}, \tag{4}$$

which in turn determines all possible final states of the economic system.

As in classical game theory, an *equilibrium* of the economics system is defined to be a *fixed point* of given response function $R: S \to S$, i.e., $s \in S$ such that $R(s) = s$. It is worth emphasis that all fixed points will be contained in the limit set.

Geometrically, if the strategy space S is finite, then the partial preference ordering $\geqq$ can be represented by a *Hasse diagram* in which every vertex represents a strategy profile (see Davey and Priestley 2002). [7] The response function $R: S \to S$ is a deterministic rule that describes what future strategies follow from the current strategy. At any given decision node belonging to the Hasse diagram, one branch is chosen according to the rule described by the response function. Given an initial strategy $s \in S$, its orbit $\{R^k(s) \mid k \in \mathrm{N}\}$ determines a path through the Hasse diagram. In view of this, our preference-response framework constitutes a marriage of normal-form game and extensive-form game: the Hasse diagram amounts to the game tree, and the response function determines the dynamic process of decision making. [8]

However, since players are just of bounded rationality, their preferences $\geqq_i$ may not be represented by utility functions in general. Further, the strategy space $(\mathrm{S}, \geqq)$ may not have topologic structure, such as compactness and convexity. So to guarantee the dynamic process of decision making to converge, we have to impose conditions on the

[7] As an example, see section I.D for the Hasse diagram corresponding to the centipede game.

[8] In game theory, the normal-form representation of a game is a specification of players' strategy spaces and payoff functions, and the extensive-form representation describes a game as being a game tree with payoffs. While normal-form approach can be of greater use in identifying Nash equilibrium, some information is lost as compared to extensive-form representations (see Kuhn 1950, 1953). Our preference-response representation of games provides a unifying framework to synthesize these two kinds of forms.

strategy space and response function in terms of order-theoretic properties, as opposed to topologic properties.

To this end, we assume that the strategy space $(S,\geqq)$ satisfies the *chain-complete condition* (i.e. every chain has a least upper bound or a greatest lower bound or both), and the response function $R : S \to S$ satisfies certain *order-theoretic properties*, such as continuity, order-preserving, and directedness. Under these assumptions the existence of pure-strategy Nash equilibrium can be guaranteed by fixed pointed theorems for ordered structures, such as Zermelo-Bourbaki fixed point theorem (Boubaki 1949), the Knaster-Tarski fixed point theorem (Tarski 1955), the Abian-Brown fixed point theorem (Abian and Brown 1961), as well as the Markowsky fixed point theorem (Markowsky 1976). These order-theoretic fixed point theorems have been widely cited in computer science, and have been applied to the study of supermodular games (see Topkis 1978, 1979; Vives 1990; Milgrom and Roberts 1990).

It turns out that these order-theoretic assumptions are, in several respects, weaker than and closer to economic reality than classical theory of Nash (1950, 1951). Furthermore, if the preference of each player does happen to be able to be presented by a utility function and the response function was derived from utility maximization, then the equilibrium defined by the fixed point of given response function will be the same as Nash equilibrium (for details see section II.A). It is this coincidence that justifies the preference-response representation of games as a useful framework.

There does seem to have some advantages in concentrating attention at the preference and response on strategy space, rather than the utility. One of the advantages is that it is dynamic in essence. Another advantage is that it permits the behavior of players with bounded rationality. But the essential advantages of our model may be secured by successfully resolving some longstanding paradoxes in classical theory, yielding straightforward ways out of the impossibility theorem, the Keynesian beauty contest, the Bertrand Paradox, and the backward induction paradox. These applications have certain characteristics in common: they all involve important modifications in the concept of perfect rationality.

The rest of the paper proceeds as follows. Section **I** builds a dynamic game model that is consistent with the paradigm of bounded rationality. Solutions are investigated in section **II**. Applications are studied in section **III**. Extensions are discussed in section **IV**. Section **V** concludes this paper with some remarks.

I. The Model

It has long been recognized that the concept of bounded rationality was of great importance, but the lack of a formal approach impeded its progress. Indeed, there is no unified definition for bounded rationality so far (Aumann 1997).[9] However, models that are described as bounded rationality tend to have some common characteristics. Thus, for the sake of clarity, it is a good idea to begin with some definitions. What is meant by "bounded rationality", and, for that matter, by "dynamic games"?

1. Bounded Rationality.—"Bounded rationality" means that players respond to each other's behavior in a relatively stable pattern, subject to an adjustment process that is rational in some dynamic sense.

Firstly, bounded rationality means incomplete information. Individuals with bounded rationality may be uncertain about the "rules" of the game. [10] For example, Players with bounded rationality may not have precise information about the specific mathematical form of the utility functions of some other's. [11]

Secondly, bounded rationality also means uncertain foresight. Nevertheless, since we are trying to describe rationality and not some sort of irrationality, we can not assume that players make choice completely at random and without foresight, as in Alchian (1950).

[9] Despite of this, we shall try to construct definitions of "rational choice" that are modeled closely upon the actual decision processes in the behavior of individuals. Within our framework, the degree of rationality is naturally characterized by the structure of fixed points of given response function. For details see section II.D.

[10] Games with incomplete information have been systematically investigated by Harsanyi (1967/1968). However, our approach differs from that of Harsanyi in that our model need not be based on the assumption that, in dealing with incomplete information, every player will use the Bayesian approach. In contrast, under our model, it will be possible to analyze the dynamic evolution of any economic system in terms of preference ordering, rather than utility function.

[11] Most existing game theory is based on the assumption that the players *know* each other's utility functions. This essentially means that they know each other's preference as well as each other's attitudes towards risk. Bargaining Games in ignorance of the opponent's utility function has been systematically studied by Harsanyi (1961, 1962). He discussed the more general case where the players do not know (and know they no dot know) each other's utility functions.

To be more realistic, we assume that all players know each other's preference orderings. But, under bounded rationality an individual may be willing and able to arrive at preference decisions only for certain pairs of alternatives, while for others he may be unwilling or unable to arrive at a decision. In short, any two alternatives may not be comparable under bounded rationality. This means that we need to drop or modify some of the fundamental rationality axioms that govern the preference relation in the classical utility theory of von Neumann and Morgenstern (1944), especially the *completeness axiom*.[12]

As a result, to make it logically compatible with the paradigm of bounded rationality, the preference of each player should to be a *partial order* on his strategy set. This means that preference ordering of each player need not be expressed by a real-valued utility function, thus liberates us from the utility concept.

2. Dynamic Games.—"Dynamic game" means extending the mathematical model of Nash equilibrium to include strategy adjustments.

As criticized by Simon (1978), economics has largely been preoccupied with the results of rational choice rather than the process of choice. Yet as economics analysis acquires a broader concern with the dynamics of choice under uncertainty, it will become more and more essential to consider choice processes. The advances in our understanding of the process of choice could provide immense help in deepening our understanding of the dynamics of rationality. As economics moves out toward situations of increasing complexity, it becomes increasingly concerned with the ability of agents to cope with the complexity, and hence with the procedural aspects of rationality (see also Simon 1955, 1972).

According to Veblen (1898), any evolutionary science should be a theory of a process, of an unfolding sequence. Therefore, to study the dynamic evolution of economic system,

[12] Historically, the possibility of a partial preference ordering without the completeness axiom was discussed by Simon (1955), Shapley (1959) and Aumann (1962), among others. Simon introduced partial ordering of pay-offs in order to tolerate bounded rationality. Shapley set up a game-like model with vector payoffs, where the utility spaces of the players can therefore be given only a partial ordering. Aumann developed a utility theory that parallels the von Neumann-Morgenstern utility theory, but makes no use of the completeness axiom. But, our approach differs from theirs in that we do not assume that preference can be represented by a utility function, thus liberate economics from the utility concept. Instead of utility functions, our approach is directly based on response functions.

it is necessary to investigate the process of how individuals respond to each other's strategies under bounded rationality.

In practice, to predict the short-run behavior of an adaptive organism in a complex and rapidly changing environment, it is not enough to know its goals. In contrast, we must know also a great deal about its internal structure and particularly its mechanism of adaptation. As the complexity of the environment increases, we need to know more and more about the mechanism and processes that economic man uses to relate himself to that environment and achieve his goals. Similarly, in an organism having a multiplicity of goals, or afflicted with some kind of internal goal conflict, behavior could be predicted only from information about the relative strengths of the several goals and the way in which the adaptive processes respond to them. Thus to explain the behavior in the face of complexity and instability of environment, the theory must incorporate at least some description of the processes and mechanism through which the adaption takes place (see Simon 1959).

In this section, we shall formulate the process of decision making in situations where we wish to take explicit account of the "internal" as well as the "external" constraints that determine the degree of rationality for the individuals. A new approach based on the fixed point theorems for ordered structures will enable us to analysis the dynamic evolution of economic system under bounded rationality.

In describing the proposed model, we shall begin with the most general assumptions which are consistent with the classical economics. The mathematical techniques are order-theoretical. A central concept is that of a chain-complete partially ordered set, which is a generalization of the concept of a complete lattice, and includes nonempty compact subset of finite-dimensional Euclidean space as special case (see Birkhoff 1967).

A. Preference

Since the first formulation of utility analysis by Jevons, Menger and Walras, there has been much controversy for and against this concept.[13] Indeed, the immeasurability of

[13] For the development of utility theory, especially the movement of abandon utility, see Stigler (1950).

utility has forced Edgeworth and Pareto to replace the old concept of utility by the concept of an objective scale of preferences. It is shown that this change from cardinal conception of utility to ordinal utility turned out to be a change of methodology (see Hicks and Allen 1934).

As pointed out by Slutsky (1915), to place economic science upon a solid basis, we must make it completely independent of psychological assumptions and philosophical hypotheses. However, the discrediting of utility as a psychological concept robbed it of its only possible virtue as an explanation of human behavior in other than a circular sense, revealing its emptiness as even a construction. Thus, to liberate economics from the utility concept, economic models should be generalized to base upon operationally meaningful foundations in terms of preference (see Samuelson 1938 a, 1938 b, 1948, 1950). [14]

The replacement of utility by preference has this further advantage: it is possible to proceed from a utility function to a scale of preference, but it is impossible to proceed in the reverse direction. We can deduce from the utility function a scale of preference, but this theoretical construction do not enable us to proceed from the scale of preference to a particular utility function. Even if the utility function exists at all, it is by no means unique and it can serve only as an *index*, and not as *measure*, of individual welfare (see Hicks and Allen 1934).[15] In many cases it is in principle impossible to get detailed quantitative empirical information concerning the exact forms of the utility functions, even in the neighborhood of the equilibrium point.

As in Arrow (1950), the behavior of an individual is completely expressed by a well-behaved and stable preference pattern. This system of preferences guarantees the consistency in the patterns of individual choice between different pairs of alternatives, so that the choice from any collection of alternatives can be determined by knowledge of the choices which would be made from pairs of alternatives.

[14] In order to dispense with the notion of utility, Samuelson presented the revealed preference theory: starting from a few logical axioms of demand consistency, to derive the whole theory of valid utility analysis as corollaries. The fundamental axiom of revealed preference theory, namely the weak axiom of consumer's behavior, assumes that the preferences of consumers can be revealed by consistent consumption behaviors. Revealed preference theory tries to provide new foundations for utility theory.

[15] Pigou (1951) pointed out that money does not enable us to measure satisfactions in the sense we understood.

Under the constraint of bounded rationality, however, any two alternatives may not be comparable. Thus, to make it logically compatible with the paradigm of bounded rationality, the preference of each player is just assumed to be a *partial order* on his strategy set.

Mathematically, the preference of player $i \in \{1,2,\cdots,n\}$ is a partial order relation $\geqq_i$ on his strategy set S_i, the set of all possible alternatives. Player i prefers strategy x_i to strategy y_i will be symbolized by $x_i \geqq_i y_i$.[16] More specifically, the preference of player i is a binary relation $\geqq_i$ on strategy set S_i that satisfies the following properties:

(i) *Reflexivity*: $x_i \geqq_i x_i$ for all $x_i \in S_i$.

(ii) *Anti-symmetry*: $x_i \geqq_i y_i$ and $y_i \geqq_i x_i$ imply $x_i = y_i$ for all $x_i, y_i \in S_i$.

(iii) *Transitivity*: $x_i \geqq_i y_i$ and $y_i \geqq_i z_i$ imply $x_i \geqq_i z_i$ for all $x_i, y_i, z_i \in S_i$.

The strategy set S_i equipped with the preference relation $\geqq_i$ becomes a partially ordered set (**poset** for short), and will be denoted by $(S_i, \geqq_i)$.[17]

A *greatest element* of the strategy set is a strategy $x_i \in S_i$ which player i would prefer to all other strategies, i.e., $x_i \geqq_i y_i$ for all $y_i \in S_i$. The greatest element, if exists, will be unique by definition. Nevertheless, since the strategy set $(S_i, \geqq_i)$ is just partially ordered, we cannot expect a greatest element to exist in general.

Within bounded rationality framework, more attention should be paid to the question of the existence of the maximal element, a strategy that can not be dominated by other strategies. Formally, a strategy $x_i \in S_i$ is said to be a *maximal element* if there is no

[16] In the terminology adopted by Arrow (1950), the preference relation $\geqq_i$ should be understood as "preferred to or indifferent with".

[17] Dually, we can use notation $x_i \geqq_i y_i$ and $y_i \leqq_i x_i$ interchangeably, if necessary. In general, given concept about ordered sets, we can obtain the dual concept by replacing each occurrence of $\leqq_i$ by $\geqq_i$ and vice versa.

element $y_i \in S_i$ for which $y_i \geqq_i x_i$. It turns out that the question of the existence of the maximal element is a fundamental problem in optimization theory. In classical economics, this problem was often solved by imposing some topological structure on the involved strategy sets. Fortunately, in ordered sets without topological structures, the existence of the maximal element can be guaranteed by invoking the famous Zorn's Lemma (See Ok).

Zorn's Lemma: *Let* $(\mathrm{S}, \geqq)$ *be a partially ordered set in which every chain (i.e., totally ordered subset) has an upper bound in* S. *Then the set* S *contains at least one maximal element.*

Zorn's lemma is a powerful tool in modern mathematics, and is equivalent to the Axiom of Choice over Zermelo–Fraenkel Set Theory (See Ok). In a sense, Zorn's lemma is an obvious result. Start at some element of the ordered set. Take a strictly larger element, then another, then another, and so on. Of course it may be impossible to go on, in which case one already has a maximal element. Otherwise one can go through an infinite sequence of elements. These are linearly ordered, so there is an upper bound by the assumption. Take a strictly larger element, then another, then another, and so on. This process is continued as many times as necessary. Eventually one reaches an element that is maximal.

Zorn's Lemma allows one to carry out constructions that may require infinite sequences of choices, each of which depends on the preceding ones, so that one does not know initially just what choices are to be made and in what order.

B. Strategy Space

The Cartesian product $S_1 \times S_2 \times \cdots \times S_n$ of the ordered strategy sets $(\mathrm{S}_i, \geqq_i)$ of all the players $i \in \{1,2,\cdots,n\}$ makes up the strategy space of the game, denoted by $S = S_1 \times S_2 \times \cdots \times S_n$. Mathematically, the strategy space S can be made into an ordered set by imposing the *product order* $\geqq$ defined by

(5) $$(x_1, x_2, \cdots, x_n) \geqq (y_1, y_2, \cdots, y_n) \Leftrightarrow x_i \geqq_i y_i, \forall i .$$

This product order describes the preference pattern for society. Put otherwise, the whole social preference relation $\geqq$ is determined by the preference relations $\geqq_i$ of all individuals. As before, all that is needed to define such an order is to know the relative ranking of each pair of alternatives of social states.

Ordered by the product order the strategy space $(S,\geqq)$ will become into a partially ordered set in itself. If the strategy space S is finite, then the partial order $\geqq$ can be represented by a *Hasse diagram*: represents each element of S as a vertex in the plane and draws a line segment or curve that goes upward from x to y whenever y covers x (that is, whenever $y \geqq x$ and there is no $s \in S$ such that $y \geqq s \geqq x$). These curves may cross each other but must not touch any vertices other than their endpoints. Such a diagram, with labeled vertices, uniquely determines its partial order (see Davey and Priestley 2002).

In the literature, a maximal element of the strategy space $(S,\geqq)$ is said to be a (Pareto) *optimum* of the economic system, in that no player can make improvement without making other players worse off. Pareto optimum is an attainable state to which no attainable state is preferred to (see Debreu 1959, chapter 6).

Just as for a single individual, Zorn's Lemma is still the basic existence theorem for Pareto optimum.[18] However, since Zorn's Lemma is a non-constructive existence axiom in mathematics and the statement of Zorn's Lemma is not intuitive, it is not suit for providing a suitable foundation to the study of dynamic process of decision making.

Essentially, the achievement of optimum is the dynamic process of seeking to optimize the response to the other players' strategies subject to bounded rationality. Hence, unless some *a priori* restrictions are placed upon the nature of the elements involved in the strategy space, no useful results can be derived. Thus, it is necessary to introduce assumptions about the nature of the structure of the strategy space thus obtained.

[18] As with any type of behavior described by optimization, the measurability of social welfare need not be assumed; all that matters is the existence of a social ordering consistent with bounded rationality. In this respect, our approach is similar to that of Arrow (1950). However, our approach differ from that of Arrow in that the social ordering relation is simply determined by the individual ordering relations, and the problem here will be to study whether the economic system can achieve the Optimum, and how.

But since the strategy sets are just partially ordered by the preference relations, we can not assume the strategy space to have topologic structures. In contrast, to study the dynamic process of decision making we must make some order-theoretic assumptions. Our major assumption will be that the society as a whole has a well-organized and stable system of preference, so that the strategy space $(\mathrm{S}, \geqq)$ satisfies certain chain-complete conditions.

CPO: *A partially ordered set* $(\mathrm{S}, \geqq)$ *is said to be a* CPO *if it satisfies the chain-complete condition, i.e., every chain has a least upper bound (i.e., supremum) in* S.

It is worth mentioning that the chain-completeness property can be characterized in terms of directed sets (see Markowsky 1976). A nonempty subset D of a partially ordered set $(\mathrm{S}, \geqq)$ is said to be *directed* if, for every pair of elements in D there exists an upper bound in S. Directed sets arise very naturally in the context of computer science as well as economics.

CPO: *A partially ordered set* $(\mathrm{S}, \geqq)$ *is said to be a* CPO *if it satisfies the directed complete condition: every directed subset has a least upper bound (i.e., supremum) in* S.[19]

Note that the chain-complete condition in the definition of CPO is stronger than the condition of Zorn's Lemma. However, chain-completeness itself is a significantly weaker assumption. A remarkable merit of the chain-complete condition is that it does not require convexity and compactness, in the topology sense. In fact, the condition that $(\mathrm{S}, \geqq)$ is a CPO is extremely general and encompasses many (perhaps most, maybe even all) settings of economic interest.

Firstly, any finite partially ordered set is a CPO.

[19] In the literature, a partially ordered set satisfying the directed complete condition is called a DCPO (see Gierz et al. 2003, Davey and Priestley 2002).

Secondly, every complete lattice is a CPO. A partially ordered set $(\mathrm{S},\geqq)$ is a *complete lattice* if every subset has a least upper bound and greatest lower bound in S. A complete lattice need not be complete in the metric space sense even when S is a metric space.

Thirdly, every nonempty *compact* subset of finite-dimensional Euclidean space is a complete lattice, and hence is chain-complete (see Birkhoff 1967). It follows that chain-complete condition is weaker than the conditions required by the famous Brouwer fixed-point theorem, which asserts that every continuous function from a ***convex*** *compact* subset of a Euclidean space onto itself has a fixed point.

Finally, the chain-completeness property is preserved under taking Cartesian product. That is, the strategy space $(S = S_1 \times S_2 \times \cdots \times S_n, \geqq)$ is a CPO if and only if the strategy sets $(\mathrm{S}_i, \geqq_i)$ of all players are CPOs.

Intuitively, the concept of chain-completeness amounts to the process of Pareto improvement. Nevertheless, during the dynamic bargaining process we have to make concessions step by step and hence a dual concept is needed.

Dual CPO: *A partially ordered set* $(\mathrm{S},\geqq)$ *is said to be a dual CPO if it satisfies the condition that every chain has a greatest lower bound (i.e., infimum) in* S.[20]

It is worth emphasis that CPO with a least element and dual CPO with a greatest element play an indispensable role in order-theoretic fixed point theory.[21] For details section II.C.

C. Response Function

The response function describes the way in which players respond to each other's strategies. In a sense, the role of response function in dynamic games is more

[20] In the literature, a dual CPO is sometimes called a filtered CPO (see Gierz et al. 2003), a notation not customarily employed in economics, though familiar in mathematics.

[21] In the literature, a CPO with a least element is also called a pointed CPO (see Gierz et al. 2003, Davey and Priestley 2002).

fundamental than the utility function. In fact, if the argument of the present paper is correct, then it is the response function and not the utility function which are in some sense primary.

In classical game theory, the *best* response has been derived from the utility function by assuming that players are of perfect rationality so as to maximize their utilities. Nevertheless, in the absence of evidence that the classical concepts do describe the decision-making process, it seems reasonable to examine the possibility that the actual process is quite different from the ones the utility-maximizing rules describe. Our procedure will be to introduce some modifications that appear on the basis of casual empirical evidence corresponding to observed behavior processes in humans. The point is rather that these are procedures which appear often to be employed by human beings in complex choice situations.

According to Aumann (2008), our approach can be categorized as the idea of "rule rationality", as opposed to "act rationality".[22] Within the paradigm of rule rationality, people do not maximize utility in each of their acts, but rather follow rules of behavior that usually—but not always—maximize utility. Specially, rather than choosing an act that maximizes utility among all possible acts in a given situation, people adopt rules that maximize some measure of total or average or expected utility, taken over all decision situations to which that rules applies. Then, when making a decision, they choose an act in accordance with the rule they have adopted.

With rule rationality, one optimizes a rule so as to do well "in general," but not necessarily always. Rule rationality concerns a whole class of decision scenarios. Often, the rule will be described by means of a mechanism, which expresses the rule only indirectly. It is worth emphasis that rule rationality is a positive concept: it describes how people *do* behavior, rather than how they *should* behavior (see Aumann 2008). One example of such a mechanism is the TIT FOR TAT strategy in repeated prisoner's dilemma (see Axelrod and Hamilton 1981).

[22] The distinction between rule- and act-rationality is analogous to that between rule- and act-utilitarianism. See Aumann (2008).

In conclusion, bounded rationality means that the response function is such that players respond to each other's behavior in a relatively stable pattern, though not necessarily always the best. Hence, under bounded rationality any "*well-behaved*" mapping $R: S \to S$ on the strategy space $(S, \geqq)$ can serve as a response function. Where, by "well-behaved" we mean that the order-theoretic properties of the response function are consistent with rule rationality. These order-theoretic properties, such as continuity, order preserving, and directedness, described the processes that players with bounded rationality uses in reaching equilibrium.

This kind of definition of response function expresses fully the idea that all social choices are determined by individual desires as a whole. It is society as a whole that determines the final social choice.

As we shall see, if the response function $R: S \to S$ is well-behaved, then its square $R^2 : S \to S$ defined by $R^2(x) = R(R(x))$ is also well-behaved. Inductively, the k th-power of the response function $R^k : S \to S$ is also well-behaved for any $k \in \mathrm{N}$. In this way the response function $R: S \to S$ defines a dynamic system $\{R^k \mid k \in \mathrm{N}\} = \{R^0, R^1, R^2, R^3, \cdots, R^k, \cdots\}$ on the strategy space $(S, \geqq)$.

In a sense our framework constitutes a marriage of normal-form game and extensive-form game: the Hasse diagram of the strategy space $(S, \geqq)$ amounts to the game tree, and the response function determines the dynamic process of decision making. At any given decision node belonging to the Hasse diagram, one branch is chosen according to the rule described by the response function, in a deterministic way. Given an initial strategy $s \in S$, its orbit $\{R^k(s) \mid k \in \mathrm{N}\}$ determines a path through the Hasse diagram.

To proceed, let us consider more concretely some specific order-theoretic properties of response functions of players who behave in accordance with bounded rationality.

1. Scott Continuity.—Continuity of functions is one of the core concepts of topology. However, under bounded rationality, the strategy space may not have topologic structures. Hence, under bounded rationality the response function may not be continuous in the *topological* sense, as required by the Brouwer fixed point theorem.

On the other hand, since the strategy space is partially ordered by the preferences of players with bounded rationality, it is necessary to characterize the continuity of the response function in terms of order theory at first. In order theory, especially in CPO, this can be done be considering a notion of continuity known as Scott continuity (see Gierz *et al*. 2003).

Scott continuous: *Let* $(\mathrm{S},\geqq)$ *be a partially ordered set and* $R: S \to S$ be a self-map. *Then* R *is said to be Scott continuous if it preserves suprema of directed sets, that is,* $R(\sup D) = \sup R(D)$ *for every directed subsets* D *of* S *.*

Scott continuity is equivalent to the topological continuity induced by the Scott topology (see Gierz *et al*. 2003).

2. Order Preservation.—For many applications, Scott continuity is the appropriate one. But from the standpoint of computer science, only maps which are order-preserving are likely to be of computational significance. The property of order preserving generalizes the concept of Scott continuity in a natural way so as to guarantee the consistency in players' behavior with bounded rationality.

Order Preserving: *Let* $(\mathrm{S},\geqq)$ *be a partially ordered set and* $R: S \to S$ be a self-map. *Then* R *is said to be order preserving if it satisfies the property that* $x \geqq y$ implies $R(x) \geqq R(y)$ for all $x, y \in S$.

It is worth emphasis that since S is only partially ordered, there may be many pairs of x and y for which this property has no bite. However, if the response function is order preserving, then $x \geqq y$ implies that $R^k(x) \geqq R^k(y)$ for all $k \in \mathrm{N}$ by induction.

The intuition behind order-preserving response function is that a higher strategy triggers a higher response. This fact enables order-preserving response to arise naturally in economics. For example, in supermodular games, the best response functions are order-preserving, so that the player's strategies are "strategic complements." Roughly speaking, strategic complements means that the best responses of players are increasing in actions of the other players (see Topkis 1978, 1979; Vives 1990; Milgrom and Roberts 1990).

3. Directedness.—In the absence of precise quantitative information we can just infer analytically the qualitative direction of movement of a complex system. For example, agents with bounded rationality may quest for improvements or make concessions step by step. This observation leads to the following definitions.

Directed Response: *Let* $(\mathrm{S},\geqq)$ *be a partially ordered set and* $R: S \to S$ *be a self-mapping. Then* R *is said to be increasing (decreasing) if it satisfies the property that* $R(s) \geqq \mathrm{s}$ ($s \geqq \mathrm{R(s)}$) for all $s \in S$.

Given an increasing response function $R: S \to S$, its orbit $\{R^k(s) \mid k \in \mathrm{N}\}$ starting at the initial strategy *s* is an *ascending chain*

(6) $$s \leqq R(s) \leqq R^2(s) \leqq R^3(s) \leqq \cdots \leqq R^k(s) \leqq \cdots.$$

In case the strategy space is finite, then this ascending chain uniquely determines a path through the Hasse diagram. At any given node belonging to the Hasse diagram, one upward branch is chosen according to the rule described by the response function. This dynamic system provides a language to describe a Pareto improving process among players with bounded rationality.

Dually, given a decreasing response function $R: S \to S$, its orbit $\{R^k(s) \mid k \in \mathrm{N}\}$ starting at the initial strategy *s* is a *descending chain*

(7) $$s \geqq R(s) \geqq R^2(s) \geqq R^3(s) \geqq \cdots \geqq R^k(s) \geqq \cdots.$$

In case the strategy space is finite, then this descending chain uniquely determines a downward path through the Hasse diagram. This dynamic system can be used to describe the process of concession-making in dynamic bargains.

Thus far we have not assumed that anything is known concerning the form of the response function, merely knowing that the response function satisfies certain order-theoretic properties. We do not know the *actual* forms of the response functions. As we shall later show, these order-theoretic properties are sufficient to guarantee the consistency in players' behavior under bounded rationality. With this definition of response function we are able to extend the classical game theory to contain bounded rationality.

To conclude, we have formulated the hypotheses upon which the theory of dynamic games is constructed. Since we intend to treat the problem in the most general manner, both the strategy space and the response function should not be subjected to further restrictions. On *a priori* ground there is no reason why more information should be obtained, especially in a dynamic environment.

Under bounded rationality, each player is just assumed to be informed on the order structure of the strategy space and on the order-theoretic property of the response function. That is, only the minimum quantity of qualitative information necessary for the solution is retained. Other quantitative information, such as the scale of the preference and the form of the response function, are not regarded as essential information.

The only common knowledge of dynamic games under bounded rationality is the structure of the strategy space and the property of the response function, which, taken together, determine the existence of equilibrium solutions. This remark completes the formal, or mathematical, description of the model.

D. Value Space

Before we proceed to discuss the solutions of our model, we point out that there is a subtle difference between our model and classical game model (see Nash 1950, 1951). In classical theory, it is assumed that, corresponding to each player $i \in \{1,2,\cdots,n\}$, the preference $\geqq_i$ can be represented by a real-valued utility function $U_i(\cdot)$ on the strategy space $S = S_1 \times S_2 \times \cdots \times S_n$, in the sense that $x \geqq_i y$ if and only if $U_i(x) \geq U_i(y)$. So in classical theory, the preference of player $i \in \{1,2,\cdots,n\}$ are defined on the strategy space $S = S_1 \times S_2 \times \cdots \times S_n$, as opposed to the strategy set S_i.

In some cases, especially in the case of dynamic games, the strategy sets S_i themselves will be very complex. So the structure of the strategy space S will become too complex to be treated. In order to simplify notations we shall work in the value space rather than strategy space.

To this end, consider the image of utility function and denote

(8) $$U_i(S) = \{U_i(s) \mid s \in S\},$$

which is a totally ordered set induced by the ordering $\geq$ on the real numbers. In this way the payoff function naturally induces an order-epimorphism $U_i : S \to U_i(S)$. Consequently, we use vector notations to rewrite the payoff function into

(9) $$U(s) = (U_1(s), U_2(s), \cdots, U_n(s)).$$

Then this vector-valued payoff function $U(s)$ defines an order-epimorphism from strategy space S to the value space $U(S) = \{U(s) \mid s \in S\}$. If the vector-valued payoff function happens to be an order-isomorphism, then we can regard $U(S)$ as *the* strategy space, since the structure of two isomorphic ordered sets are essentially the same.

The value space $U(S)$ is a subset of n-dimensional Euclidean space. So it has inherited from n-dimensional Euclidean space the componentwise order $\geqq$, defined by the product of ordering $\geq$ on the real numbers. That is,

(10) $$(u_1, u_2, \cdots, u_n) \geqq (v_1, v_2, \cdots, v_n) \Leftrightarrow u_i \geq v_i, i = 1, 2, \cdots, n.$$

Comparing two strategy profiles for the preference $\geqq$ is therefore equivalent to comparing their images in n-dimensional Euclidean space by the function $U(\cdot)$ for the componentwise order. A strategy profile $s \in S$ is an optimum if and only if its image $U(s)$ is a maximal element in $U(S)$.

In theory, if the strategy space (S,$\geqq$) is finite, then the value space $U(S)$ is also a finite partially ordered set in terms of componentwise order. In such a case $U(S)$ itself is a CPO.

To describe a game in terms of value space, the response must be defined as a well-behaved function $R: U(S) \to U(S)$. This approach leads us to value iterations, as opposed to strategy iterations.[23]

Example: The Centipede Game

The centipede game was introduced by Rosenthal (1981) to question the logic of backward induction. Typically, the extensive form of centipede game can be illustrated in figure 1.

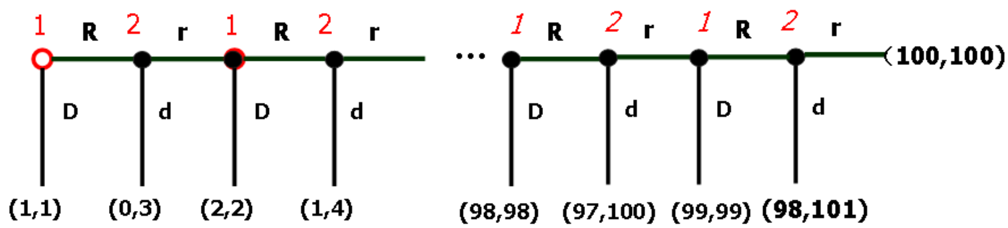

FIGURE 1. THE CENTIPEDE GAME

Since the centipede game is "deep" in the sense that it had a limit of 100 rounds, the strategy space is too complex. The value space, instead, is rather simple since it consists of pairs of real numbers

(11) $$U(S) = \{(1,1),(0,3),(2,2),(1,4),\cdots,(99,99),(98,101),(100,100)\}.$$

Inherited the componentwise order $\geqq$, the value space $(U(S),\geqq)$ becomes into a CPO that has *no* least or greatest element. It is worth emphasis that the value space does have two *maximal* elements $(100,100)$ and $(98,101)$, though they are incomparable with each other. In fact, these two maximal elements are exactly the payoffs of two terminal nodes (100^{th} round) in centipede game.

The Hasse diagram of the value space $U(S)$ of centipede game can be depicted in figure 2.

[23] Historically, Shapley's 1953 paper on stochastic games initialed the value iteration method. Value iteration and strategy iteration are two fundamental methods in solving dynamic programming.

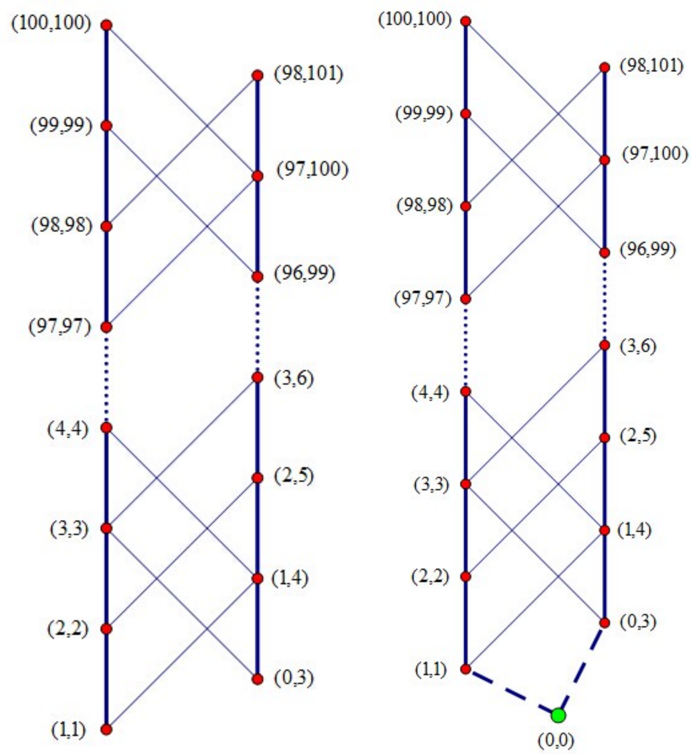

FIGURE 2. THE HASSE DIAGRAM OF THE VALUE SPACE (LEFT) AND ITS LIFTING (RIGHT)

Given the game tree depicted in figure 1, backward-induction solution predicts that the first player should pick **Down** on the firs move. As a result, the play ends at the first move and each player gets just 1. According to backward-induction algorithm, in each round both players should pick **Down**, *as if* all vertices are actually reached (see Aumann 1995, 1998).

Indeed, if backward induction was used by both players throughout the game tree, then the process of backward-induction reasoning gives rise to a *best* response function $R: U(S) \to U(S)$ that is order-preserving. Starting from the terminal node $\tau = (100{,}100)$, the orbit $\{R^k(\tau) \mid k = 1,2,\cdots,100\}$ determines a *downward* path through the Hasse diagram (figure 2 left). The backward-induction outcome amounts to that the orbit $\Re^k(\tau)$ converges to minimal element $(1{,}1)$ as $k \to \infty$.

But, since the terminal node $\tau = (100{,}100)$ is Pareto optimal, it seems strange that rational players would choose to proceed back through the game tree and get worse off.

Now that the payoffs for some amount of cooperations in the Centipede game are so much larger than immediate defection, the "rational" solutions given by backward induction seems to be paradoxical (see Aumann 1995, 1998).

Empirical evidences suggest that players with bounded rationality seldom accept the backward induction as a guide to practical behavior, as indicated by the chain store paradox (see Selten 1978). In contrast, the reasoning of *rule rationality* seems to be much more compelling for games with sufficiently large limit of rounds. In what follows, a solution to the backward induction paradox will be given based on our dynamic game model under bounded rationality

To see this, note that both players will get *nothing* if they do not play the game. Then we can add one element $\perp=(0,0)$ to the value space $U(\mathrm{S})$. In this way we have "lifted" the value space $U(\mathrm{S})$ into a CPO $U(\mathrm{S})_\perp$ with a least element $\perp=(0,0)$ (see Davey and Priestley 2002).

Now if both players choose to play the game, then they will start from $\perp=(0,0)$ and quest for improvement according to certain heuristic rules,[24] which give rise to an order-preserving function $R:U(S)_\perp \to U(S)_\perp$. That is, for any $x, y \in U(\mathrm{S})_\perp$ such that $x \geqq y$, we have $R(x) \geqq R(y)$. This order-preserving property stands for one kind of rational expectation. It is rational expectation that makes the escape from the backward induction paradox possible. However, the actual dynamic process of decision making can only be determined after the response function has been specified.

Now, starting from the least element $\perp=(0,0)$, the orbit $\{R^k(\perp) \mid k=1,2,\cdots,100\}$ determines a *upward* path through the Hasse diagram (figure 2 right). As a result of rule rationality, the Abian-Brown fixed point theorem (Abian and Brown 1961, see also section II.C) asserts that the response function $R:U(S)_\perp \to U(S)_\perp$ has a *least* fixed point, which can be approximated by the sequence $R^k(0,0)$ as $k \to \infty$. In general, equilibrium defined by this least fixed point is

[24] Note that in centipede game no player can punish defective behavior, since defection means game over.

the last vertex that will be *actually* reached. This will differ from the backward-induction solution if the limit of round is sufficiently large, and hence enables us to escape from the backward induction paradox.

Finally, if the limit of rounds is too small then the game changes its character and it is much more advantageous to use the best response given by backward induction (see Selten 1978).[25] □

II. The Solutions

In this section, the notion of equilibrium solutions under bounded rationality will be defined, and the properties of equilibrium solutions will be investigated. I shall show that the problem of existence of equilibrium solutions depends formally in an important way upon the structures of strategy space and the properties of response functions.

The notion of fixed points of the response function is the basic ingredient in our theory. It turns out that the fixed points of the response function will coincide with pure-strategy Nash equilibrium if each player's strategy happens to be optimal against those of the others; in this case all players are regarded as being of perfect rationality.

A. Equilibrium

As in the classical theory, an equilibrium of the economics system is a *fixed point* of given response function $R: S \to S$, namely, $s \in S$ such that $R(s) = s$.

This definition yields a generalization of the concept of Nash equilibrium. Indeed, if the preference does happen to be able to be presented by a utility function and the response function was derived from utility maximization, our game-like model is the same as the classical models. As we shall see, much of the properties of classical game model stay intact within our framework.

[25] Selten argued, and we agree, that the explanation of the backward induction paradox requires a limited rationality view of human decision behavior. For this purpose he developed a three-level theory of decision making, where decisions can be made on different levels of rationality. His theory is speculative rather than based on empirical facts other than circumstantial evidence, and hence cannot claim to be more than a heuristic tool for the investigation of problems of limited rationality. In contrast, our theory is based on rule rationality and is consistent with procedural rationality.

In classical game theory, the *best* response is derived from the utility function $U_i(s)$ by assuming that players are of perfect rationality so as to maximize their utilities. To be specific, given the set of other players' strategies s_{-i}, define the best response of player $i \in \{1,2,\cdots,n\}$ to be strategy $R_i(s_{-i})$ in the strategy set S_i such that it maximizes his utility if the strategies of the others are held constant. [26] In symbols,

(12) $$R_i(s_{-i}) = \arg\max_{s_i \in S_i} U_i(s_i, s_{-i}) .$$

Consequently, consider the strategy space $S = S_1 \times S_2 \times \cdots \times S_n$ as a whole and define the best response for any strategy profile $s = (s_1, s_2, \cdots, s_n) \in S$ to be

(13) $$R(s) = \prod_{i=1}^{n} R_i(s_{-i}) = \big(R_1(s_{-1}), R_2(s_{-2}), \cdots, R_i(s_{-i}), \cdots, R_n(s_{-n})\big).$$

In this way, the fixed point of the best response function $R: S \to S$, namely, $s \in S$ such that $R(s) = s$, exactly gives rise to Nash equilibrium, in the sense that each player's utility is maximized against those of the others. In fact, in the static equilibrium model of Nash (1950, 1951) the strategies that prevail "when the dust settles" are simply those that coordinate the strategies of various players.

It is this coincidence that justifies the preference-response representation of games as a useful framework. Further, if we adopt the traditional identification of perfect rationality with maximization of utilities, then our preference-response approach is exactly a genuine generalization of the model of Nash (1950) to the case of bounded rationality.

By keeping in mind these advantages, one may compare the present approach with "classical" game theory (see Nash 1950, 1951). Compared to traditional theory, our theory is more general in that it can encompass a wide range of observable phenomena and systematize a wide range of empirical knowledge.

[26] In general, the best response $R_i(s_{-i})$ may not be an element of the strategy set S_i, but a subset of it. But this problem is not essential within the framework of bounded rationality, since most celebrated fixed point theorems for ordered structures had been generalized to set-valued functions (see section IV.C)

However, there is a price to pay. That is, we have to treat qualitative information rather than quantitative information during the analysis of decision making. However, this kind of qualitative analysis has a virtue. Indeed, as economic expands beyond its central core of price theory and its central concern with quantities of commodities and money, we observe in it this same shift from a highly quantitative analysis to a much more qualitative analysis. In qualitative analyses aimed at explaining institutional structure, maximizing assumptions play a much less significant role than they typically do in the analysis of market equilibria (see Simon 1978).

B. Existence

The investigation of the existence of solutions is of interest both for descriptive and for normative economics (see Arrow and Debreu 1954).

Traditionally, to guarantee the existence of Nash equilibrium, some fundamental *topological* conditions have been imposed on both the strategy space and the response function, in the mathematical senses (see Nash 1950, 1951). The strategy space was usually assumed to be compact and convex. The response function usually was assumed to be derived from utility maximization. [27] If the utility function is continuous,[28] then the proof of the existence of competitive equilibrium of a non-cooperative game can be based upon fixed point theorems for topological structure, such as Brouwer fixed point theorem, or its generalizations (see Kakutani 1941; Glicksberg 1952).

Fixed point theory thus serves as an essential tool for the mathematical proof of the existence of an equilibrium solution. However, since under bounded rationality the strategy space is neither convex nor compact and where the response function lacks continuity in topologic sense, to prove the existence of equilibrium solutions there is a need for purely order-related fixed point theorems for ordered structures. From a mathematical point of view, the trick is the replacement of fixed point theorems for

[27] McKenzie (1959) gave a direct proof that the general equilibrium for a competitive market exists, with features that he did not define a utility function, but proceed directly form the preference ordering. However, this is not a weakening of the assumptions of Arrow and Debreu (1954), but an exercise of Occam's Razor.

[28] The assumption of a continuous utility function amounts to a continuity assumption on the preference relation. For details see Arrow and Debreu (1954).

topological structures by fixed point theorems for ordered structures in the existence proof of equilibrium solutions.

Recall that we have imposed chain-complete conditions on the structure of the strategy space and order-theoretic properties on the response function. Taken together, these conditions are sufficient for assuring the existence of equilibrium solutions in accordance with bounded rationality, based directly on the fixed point theorem for CPOs.

Zermelo-Bourbaki Fixed Point Theorem: *Let* $(S,\geqq)$ *be a CPO with a least element and* $R: S \to S$ *be an increasing map. Then* $R: S \to S$ *has a fixed point.*

The Zermelo-Bourbaki fixed point theorem is a fundamental, deep and powerful result (see Ok). The theorem itself does not depend on the Axiom of Choice. However together with the Axiom of Choice it will lead to a proof of Zorn's lemma. A classical proof of the Zermelo-Bourbaki theorem constructs the increasing sequence for any $s \in S$

$$s \leqq R(s) \leqq R^2(s) \leqq R^3(s) \leqq \cdots \leqq R^k(s) \leqq \cdots, \tag{14}$$

where chain-completeness is used at limit stages. If the sequence is indexed by a large enough ordinal, it must stabilize, giving a fixed point above s. So no matter where one starts, there is a tendency to approach some equilibrium point.

From this constructive proof we can directly watch rational individuals' thinking process as they consider their choices of strategies. If one improves his strategy relative to his former strategy, then the action taken is better than the former one, and presumably one could continue by small increment to advance an equilibrium. But there is no assurance that he can also achieve optimum (see subsection below).

In practice, instead of the quest for Pareto improvement, sometime we have to make concessions step by step, especially during the process of dynamic bargaining.

Typically, the process of two-person bargaining works as follows: given the current offer $s \in S$, both players will simultaneously or alternatively decide the reply to other players, which jointly determine the response $R(s) \in S$ to the current offer. The result of a bargain is a compromise between a theoretical ideal and practical necessity, and a common pattern in bargaining is to make concessions. In this way the bargaining process defines a *decreasing* response function $R: S \to S$ on the strategy space such that $s \geqq R(s)$ for all $s \in S$. Starting from the most preferred strategy $\tau \in S$ and making concessions step by step we obtain a descending chain

$$\tau \geqq R(\tau) \geqq R^2(\tau) \geqq R^3(\tau) \geqq \cdots \geqq R^k(\tau) \geqq \cdots, \tag{15}$$

To guarantee the process of bargaining to converge, we have to invoke the dual form of the Zermelo-Bourbaki fixed point theorem.[29]

Zermelo-Bourbaki Fixed Point Theorem (Dual Form): *Let* $(S, \geqq)$ *be a dual CPO with a greatest element and* $R: S \to S$ *a decreasing mapping. Then* $R: S \to S$ *has a fixed point.*

Thus we have obtained an equilibrium solution. It is worth noting that this equilibrium is formed by pure strategies. So we have solved the bargaining problem by directly analyzing the bargaining process, rather than attack the bargaining problem axiomatically by stating general properties that "any reasonable solution" should possess as in Nash (1950).[30] A more detailed analysis of the actual process of bargaining can be found in section II.D.

C. Optimum

Once existence has been established, it is natural to consider the relation between the existence of equilibrium solutions and the problems of welfare economics.

Ideally one would prefer the fixed point in the Zermelo-Bourbaki theorem to be Pareto optimal. But this is not necessarily the case. Since players are of bounded rationality, not all fixed points of response function are optimal. However, if we admit Zorn's Lemma, or equivalently, the Axiom of Choice (AC), then the inverse is true (see Davey and Priestley

[29] In general, given statement about ordered sets, we can obtain the dual statement by replacing each occurrence of $\leqq$ by $\geqq$ and vice versa. The Duality Principle says that given a statement about ordered sets which is true for all ordered sets, the dual statement is also true in all ordered sets (see Davey and Priestley 2002). This implies that dual forms of the all the fixed point theorems cited in this paper are also true.

[30] For a critical discussion of different approaches to the bargaining problem, see Harsanyi (1956).

2002). Due to the importance of this result in welfare economics, we shall state it more precisely as follows:

Zermelo-Bourbaki Theorem with AC: *Let the strategy space* $(S, \geqq)$ *be a CPO with least element and let* $R: S \to S$ *be an increasing map. Then every Pareto optimum is a fixed point of* $R: S \to S$.

Since every Pareto optimum is by definition a maximal element, this theorem follows instantly from Zorn's Lemma: If s is a maximal element in S guaranteed by chain-completeness, then $R(s) \geqq s$ by increasing property, which implies $s = R(s)$ by maximality. Thus, if the strategy space is a CPO with a least element and the response function is increasing, then the set of optimum is contained in the set of fixed points.

In reality, since players are assumed to be of bounded rationality, we can not require all equilibrium solution to be optimal among the entire strategy space. Rather, the best thing we can do is to single out the fixed point that is optimal among the set of all fixed points of the response function. This leads us to a kind of weak optimality.

When alternatives are examined sequentially, we may regard the *first* satisfactory alternative as the one actually selected. For example, in English auction the final offer must be the least fixed point of the response function of the underlying game. This is so if the response function is order-preserving, as declared by the Abian-Brown fixed point theorem (Abian and Brown 1961).

Abian-Brown Fixed Point Theorem: *Let* $(S, \geqq)$ *be a CPO with a least element* $\perp$. *If* $R: S \to S$ is order-preserving, *then* $R: S \to S$ *has one least fixed point.*

Intuitively, this least fixed point can be approximated by using *transfinite induction*:[31] define $x_0 = R^0(\perp)$, and $x_{\alpha+1} = R(x_\alpha)$ for every ordinal α, and $x_\beta = \sup_{\alpha<\beta} R(x_\alpha)$ for limit ordinals. This recursively defined sequence describes a kind of process of Pareto

[31] Mathematically, transfinite induction is equivalent to the Axiom of Choice. It is worth emphasis that the Abian-Brown fixed point theorem does not require the axiom of choice for its proof. A second proof avoiding transfinite reasoning can be found in Davey B. and Priestley (2002).

improvement initialed from the bottom $\perp$. As the cardinality of S is bounded, there is an ordinal γ such that $x_\gamma = x_{\gamma+1} = R(x_\gamma)$, which is precisely the least fixed point of $R: S \to S$.

The Abian-Brown fixed point theorem is in a sense optimal, because order-preserving mappings on a CPO with *no* least element may not have least fixed points in general. To illustrate, consider the set $S = (0,1]$, ordered by the ordering $\leq$ on the real numbers. Then $S = (0,1]$ is by definition a CPO with *no* least element. Now define a map $R(x) = \frac{x}{2}$ on $S = (0,1]$. It is easy to see that $R: \mathrm{S} \to \mathrm{S}$ is both decreasing and order-preserving but has *no* fixed point.

The necessity of the assumption about a bottom or a top for the validity of the existence of equilibrium solution points up an important principle: to have equilibrium, it is necessary that each player's strategy set is bounded from below or from above so that social orders emerge *spontaneously* as the result of interactive actions.[32]

However, a much weaker assumption of the same spirit was made by Amann (1979), where the existence of a least element was relaxed by requiring an element $s \in S$ with $R(s) \geqq s$.

Amann Fixed Point Theorem: *Let* $(\mathrm{S}, \geqq)$ *be a CPO and* $R: S \to S$ *be order preserving. If there exists an element* s *with* $R(s) \geqq s$, *then* $R: S \to S$ *has one least fixed point.*

To continue, let us reconsider the process of dynamic bargaining. If the response function defined by making concessions step by step turns out to be order-preserving, then the final offer must be the *greatest* fixed point of the response function. This is so even if bargainers do not start making concession from the top, as asserted by the dual form of the Amann fixed point theorem (Amann 1979).

[32] For example, in the Arrow–Debreu general equilibrium model it is assumed that the set of consumption vectors available to each individual is a closed convex subset of finite-dimensional Euclidean space which is bounded from below (see Arrow and Debreu 1954, Assumption II).

Amann Fixed Point Theorem (Dual Form): *Let* $(S, \geqq)$ *be a dual CPO and* $R: S \to S$ *be order preserving. If there exists an element* s *with* $s \geqq R(s)$, *then* $R: S \to S$ *has one greatest fixed point.*

As mentioned above, the condition that *there exists an element* s *with* $s \geqq R(s)$ works by specifying a starting position so that social orders emerge *spontaneously* as the result of interactive actions. With no suitable position to start, social orders can not emerge *spontaneously* as the result of interactive actions.

In essence, the existence of the least (or greatest) fixed point means some kind of uniqueness. We cannot be sure the least (or greatest) fixed point will be chosen as final outcome; but the chances are certainly a great deal better than if they pursued a random course of selection. It is not being asserted that players will always choose the least (or greatest) fixed point as final solution; but the chances of their doing so are ever so much greater than the bare logic of abstract random probabilities would ever suggest.

In the terminology adopted by Schelling (1957), both the least and greatest fixed points are *focal points*.[33] The spontaneous tendency to settle at focal points involves stable mutual expectations. Thus the existence of the least (or greatest) fixed point prove that expectation can be coordinated when the co-ordination of expectations is essential.

It is worth emphasis that the least (or greatest) fixed point of order-preserving response function need not be optimal in general. This shows a kind of divergence between social and individual rationalities in interactive decision making under bounded rationality. In fact, this divergence exists even under perfect rationality, as indicated by the famous prisoner's dilemma (see Axelrod and Hamilton 1981).

D. Structure

Now, for a given set of individual preferences, there is a given response function being in accordance with the degree of rationality. The dynamic process of interactive decision-

[33] Schelling suggested that it is better to adopt criteria that are as qualitative and discrete as possible in order to avoid reliance on matters of degree of judgment. It is worth noting that our preference-response approach perfectly satisfies this criterion.

making proceeds until a fixed point of the response function is reached. But, since players are just of bounded rationality, the given set of individual preferences and the given response function may not uniquely determine the final social choice if no prior assumptions are made. Any fixed point of the response function is a possible equilibrium, and the precise final social choice reached will be governed by the degree of rationality of the society as a whole.

The multiplicity of equilibrium can be considered as corresponding to different degrees of rationality. [34] For a given degree of rationality only one fixed point would be relevant. Of the multiple fixed-point equilibria, only one is in general consistent with the degree of rationality, but that one cannot be foreseen.

Observable patterns of behavior are predictable in terms of their relative probabilities of occurrence if they are played. Unfortunately, such prediction can be applied concretely only if one has full knowledge of the degree of rationality. If one does not have this knowledge, then, the best thing he can do is to investigate the structure of the set of all fixed points of given response function.

In many situations we may be interested in the precise question of whether one equilibrium solution is more rational that another, and to answer this question will usually have to consider the structure of the fixed points of given response function.

Just like the fixed point theorem of Brouwer and Kakutani, the structure of fixed points in the Zermelo-Bourbaki theorem is not clear. However, we shall see that the information about the set of fixed points can be obtained in case the strategy space is a CPO with a least element and the response function is order-preserving.

Markowsky Fixed Point Theorem: *Let* $(\mathrm{S},\geqq)$ *be a CPO with a least element and* $R: S \to S$ be order-preserving. *Then the set of fixed points* $Fix(R) = \{s \in S \mid s = R(s)\}$ *forms a CPO in itself. In particular,* $R: S \to S$ *has one least fixed point.*

[34] This interpretation differs from that of Hurwicz (1945), where the multiplicity of solutions was attributed to alternative institutional setups. The truth is, even for a given institutional framework there may still be multiple solutions under the constraints of bounded rationality.

The Markowsky fixed point theorem (Markowsky 1976) sharpens the results of Abian-Brown fixed point theorem, and is a natural generalization of the Knaster-Tarski fixed point theorem for complete lattice (see Tarski 1955).

Knaster-Tarski Fixed point Theorem: *Let* $(\mathrm{S},\geqq)$ *be a complete lattice and let* $R: S \to S$ *be order-preserving. Then the set of fixed points* $Fix(R) = \{s \in S \mid s = R(s)\}$ *forms a complete lattice in itself. In particular,* R *have both least and greatest fixed point.*

Intuitively, the set $Fix(R)$ can be *approximated* by iterations of $R: S \to S$. To see this, assume the least and greatest element of $(\mathrm{S},\geqq)$ to be $\perp$ and τ respectively, then by order-preserving property we have

(16) $$\perp \leqq \mathrm{R}(\perp) \leqq \mathrm{R}(\tau) \leqq \tau .$$

By Induction, we obtain the chain

(17) $$\perp \leqq \mathrm{R}(\perp) \leqq \mathrm{R}^2(\perp) \leqq \cdots \leqq \mathrm{R}^k(\perp) \leqq \cdots \leqq \mathrm{R}^k(\tau) \leqq \cdots \leqq \mathrm{R}^2(\tau) \leqq \mathrm{R}(\tau) \leqq \tau .$$

Now given any fixed point $s = R(s)$, we have $\perp \leqq s = \mathrm{R}(s) \leqq \tau$ and hence $R^k(\perp) \leqq s = \mathrm{R}(s) \leqq R^k(\tau)$ by induction. Thus, the set $Fix(R)$ must be contained in the interval ranging from $\sup_k \mathrm{R}^k(\perp)$ to $\inf_k \mathrm{R}^k(\tau)$. But in general there is no sense in which either $\sup_k \mathrm{R}^k(\perp)$ or $\inf_k \mathrm{R}^k(\tau)$ is a fixed point, needless to say $\perp$ or τ.

If S is finite then it is easy to see that $\mathrm{R}^k(\perp)$ converges to the least fixed point and $\mathrm{R}^k(\tau)$ converges to the greatest fixed point, respectively. If S is not finite then the situation may be more delicate. [35]

The significance of this structure theorem should be clear. For one thing, the multiplicity of equilibrium means a range of indeterminateness. The extent of the range

[35] In case the response function turns out to be Scott continuous, then the Kleene fixed point theorem asserts that the least and greatest fixed point indeed can be constructed in this way. For details see section II.D below.

of indeterminateness is determined by the degree of rationalities of players. For another, the multiplicity of equilibrium can be considered as corresponding to different degree of rationality. Of the multiple fixed-point equilibria, only one is in general consistent with the degree of rationality, but that one cannot be foreseen. Thus the structure of fixed points gives rise to a natural characterization of the degree of rationality. This relation can be summarized into the following table:

Type of choices	Degree of rationalities
The least (or greatest) fixed point	Perfect rationality
Other fixed points	Bounded rationality
Non-fixed points	Irrationality

In *ex ante* analysis, if players are of perfect rationality then the least (or dually, greatest) fixed point of the order-preserving response functions will be the uniquely determined final social choice. Thus, the crucial difficulty in selecting the most preferred social choice can be avoided under perfect rationality.

In *ex post* analysis, however, if the degree of rationality is measured on the basis of results rather than motivation, then those players whose choice happens to be the least or greatest fixed point will be "selected" as the best, in the sense of perfect foresight. Thus, even in a world of irrational men there might still be someone happens to behave *as if* they are of perfect rationality according to the *ex post* survival criterion (see Alchain 1950). Also, the greater the uncertainty of the world, the greater is the possibility. Thus individual behavior under bounded rationality does not eliminate the likelihood of observing "appropriate" decisions.

In this way we have established a unified theory of bounded rationality, and developed a meaning for formal definition of rationality. This gives rise to an answer to one open problem of Aumann (1997), which is regarded as the most challenging conception problem in the area of bounded rationality. The evolution of the dynamic system defined by the iterations of response function, namely

$$\{R^k \mid k \in \mathrm{N}\} = \{R^0, R^1, R^2, R^3, \cdots, R^k, \cdots\}, \tag{18}$$

is essential to this problem. This confirms the conjecture of Aumann that an evolution approach may eventually turn out to be the key to a general solution.

However, lacking the kinds of empirical knowledge of the decisional processes that will be required for a definitive theory, the hard facts of the actual world can, at the present stage, enter the theory only in a qualitative way.

Example: Coasian Bargaining

The Coase theorem describes the economic efficiency of an economic allocation or outcome in the presence of externalities. The theorem states that if trade in an externality is possible and there are sufficiently low transaction costs, bargaining between polluters and pollution victims will lead to an efficient outcome regardless of the initial allocation of property rights (see Coase 1960).

Now assume that there was only a single victim, and consider the bargaining between the polluter (buyer, bargainer 1) and the victim (seller, bargainer 2) on a private property right, viewed as a single unique indivisible good.[36] Mathematically, it is natural to assume that the strategy set of each player is a closed interval of positive real numbers, say, $S_i = [0,10]$ for $i = 1,2$. They alternatively or simultaneously choose price as strategies so the strategy space $\mathrm{S} = \mathrm{S}_1 \times S_2$ is a complete lattice that consists of pairs of real numbers. Since the buyer prefers lower prices and the seller prefers higher prices, we have a descending chain in $(\mathrm{S}, \geqq)$

(19) $$(0,10) \geqq (1,9) \geqq (2,8) \geqq (3,7) \geqq (4,6) \geqq (5,5) \cdots .$$

[36] Contrary to our approach, Hahnel and Sheeran (2009) argued that the only way to make sense of the traditional analysis of the Coase theorem is to imagine the polluter and the pollution victim bargaining with one another about how to divide the potential efficiency gain. In such a case, the traditional analysis of Coasian negotiations is a two-person, non-operative game called "divide-the-pie". However, in this game the polluter and the victim must both known the size of the "pie" they have to divide, i.e., the true size of the potential efficiency gain form bargaining. And the only way Coasian negotiators could know the true size of the pie would be that they are of complete information. By relaxing the assumption of complete information, Hahnel and Sheeran (2009) discovered that it is no longer likely Coasian bargains would lead to efficient outcomes. But our discussion shows that their so-called "internal" critique is unsound. In contrast, the advantage of our dynamic game model may give rise to an efficient solution to the Coasian bargaining if bargainers behave rationally, even without the assumption of complete information.

In most real-life bargaining situations, the players have very little reliable information about each other's *utility* functions.[37] In this case, however, players can obtain useful information about each other's *response* function by observing each other's bargaining behavior. So the players have to test out each other's response function by the laborious procedure of successive bids. This means that the bids made, up to the very last bid of each player, are not final but are merely tentative, and serve to test out the opponent's response functions (see Harsanyi 1962). Under the assumption of bounded rationality, the information conveyed during the bargaining process will be sufficient for determining the order-theoretic property of the response function. It is the advantage of step-by-step bargaining that enables bargainers to gathering a small amount of information—the order-theoretical properties of the response function, even if they have no reliable information about each other's *utility* functions.

Indeed, buyers would like to begin with a relative lower price and increase his bids step by step, and sellers would like to start from a relatively higher price and reduce his bids step by step. In other words, both bargainers would start from his most preferred strategy and concede step by step. This process will continue until no bargainer would like to make a further concession. Consequently, their responses give rise to a decreasing and order-preserving function $R: S \rightarrow S$ on the strategy space $(\mathrm{S}, \geqq)$. As a result, the Knaster-Tarski fixed point theorem (see Tarski A. 1955) asserts that the set of fixed points of $R: S \rightarrow S$ forms a complete lattice in itself.

If the players are of perfect rationality and the response function is derived from utility maximization, then the complete lattice of fixed points must coincide with the "range of practicable bargains," defined in terms of Edgeworth contract curve (see Harsanyi 1956). In our theory, the range of practicable bargains is made up of all strategy profiles between the least fixed point and the greatest fixed point. Consequently, any strategy profiles above the greatest fixed point and any strategy profiles below the least fixed point, which

[37] In a world where people would know each other's utility functions, there would be no need for bargaining in the usual sense because they would not have to test out each other's utility functions by the laborious procedure of successive bids. Both parties could simply state their final terms independently of each other and then compare their bids. For details see Harsanyi (1962).

lies outside the range of indeterminateness, is wholly ineffective (see Pigou 1932, Chapter VI).

Further, the actual equilibrium solution can be determined within the range of practicable bargains on the basis of information conveyed during the bargaining process. Since perfectly rational bargainers would start from his most preferred strategy and make concessions step by step, the result of mutual adjustment of their strategies during the bargaining process must be the *greatest* fixed point, which is the upper limit of the range of indeterminateness.[38]

To be specific, assume that the response function $R: S \to S$ is defined in such a way that we have

(20) $$R(0,10) = (1,9), R(1,9) = (2,8), R(2,8) = (3,7), R(3,7) = (4,6), \cdots.$$

Typically, suppose that the greatest fixed point is $(5,5)$. Then the highest price that the buyer is willing to pay is equal to the lowest price that the sell is willing to accept, and hence the market-clearing price is 5. In this case the range of practicable bargains would naturally have tended to become a single point, representing a price equal to the one which clears the market. As a result, even if the two parties started out with mutually inconsistent expectations, there would be a good chance of their reaching mutual consistency by the end of the bargaining process.

But, the solution determined by the greatest fixed point may not clear the market. To see this, suppose that the greatest fixed point is $(4,6)$, which means that the buyer would go as high as 4 but no higher and the seller would go as low as 6 but no lower. Since both players insist on mutually incompatible strategies, an agreement may never be reached, leaving the market unclear. To clear the market, one more rounds of Coasian bargaining are needed. In the extreme case, it seems that government intervention is

[38] Hahnel and Sheeran (2009) have shown that inefficient outcomes are to be expected when complete information assumption is not met. Typically, Coasian bargaining with incomplete information yields a broad range of potential solutions, making it unlikely that the efficient outcome will be the one selected. It turns out that their negative results are partly due to the lack the knowledge of the structure of the fixed points.

necessary to make sure negotiations are fair and do not suffer from a mismatch of information or the free rider problem. □

E. Stability

A great deal of work has been done on what one may call static aspects of equilibrium: its existence, optimality and structure. With regard to dynamics, especially stability of equilibrium, much remains to be done.

Historically, dynamic stability of competitive equilibrium has been systematically treated by Samuelson (1941), Arrow and Hurwizc (1958), and Arrow, Block and Hurwizc (1959). They treated the problem of dynamic stability by dealing with only one special type of dynamic system, that of set of simultaneous ordinary differential equations. But, their approach has the disadvantage of using the topological properties that are not assumed within our framework. So their special results can not cover our general case.

Fortunately, the response function $R: S \to S$ defines a dynamical system $\{R^k \mid k \in \mathrm{N}\} = \{R^0, R^1, R^2, R^3, \cdots, R^k, \cdots\}$, whose limit set $\{\lim_{k\to\infty} R^k(s) \mid s \in S\}$ in turn determines all possible final states of the economic system. Since the *qualitative* properties of the response function, such as Scott-continuity, order preservation as well as directedness, have been considered in our model, the system as a whole may still be endowed with *structural stability*, a fundamental property of a dynamical system. [39]

Unlike Lyapunov stability, which considers perturbations of initial conditions for a fixed system, structural stability deals with perturbations of the system itself. That is, to consider whether the qualitative behavior of the orbits is invariant under small perturbations of the evolution rule of the dynamical system. Specifically, we need to consider whether the structure of equilibrium set is preserved by small perturbations of the response function.

[39] Arrow, Block and Hurwizc (1959) considered the stability of systems where only qualitative features of the dynamic process are of importance. They proved the global stability for process where the price adjustment rate is a continuous and sign-preserving function of excess demand.

Obviously, structural stability of equilibrium depends on the type of the process that guides strategy changes. If the strategy adjustment process is based on response function, then it is natural to ask which type of response function can guarantee structural stability. For many applications, Scott-continuity, which is generally strictly stronger property than order-preservation, is the appropriate one.

Firstly, the result of Abian-Brown fixed point theorem can be refined if the response function is Scott-continuous. In the literature it is called the Kleene fixed point theorem (see Gierz *et al*. 2003).

Kleene Fixed Point Theorem: *Let* $(\mathrm{S}_\perp, \geqq)$ *be a CPO with least element* $\perp$ *and* $R : \mathrm{S}_\perp \to \mathrm{S}_\perp$ be Scott-continuous. *Then the least fixed point exists and can be approximated by the chain*

$$\perp \leqq R(\perp) \leqq R^2(\perp) \leqq R^3(\perp) \leqq \cdots \leqq R^k(\perp) \leqq \cdots$$

in the sense that it equals $\sup_k R^k(\perp)$.

Kleene fixed point theorem is extremely useful, especially in computer science, because it gives a construction of the least fixed point. From the standpoint of computer science, it gives an algorithmic procedure for find the least fixed point. From the viewpoint of economics, however, the process of recursively defining the Kleene chain amounts to the process of Pareto improvement initialed from the bottom.

Secondly, the least fixed point of a Scott-continuous response function is preserved by Scott-continuous maps (see Gierz *et al*. 2003).

Kleene Preservation Theorem: *Let* $(\mathrm{S}_\perp, \geqq)$ and $(\Sigma, \geqq)$ *be CPOs with least element and let*

$$\begin{array}{ccc} S_\perp & \xrightarrow{R} & S_\perp \\ h\downarrow & & \downarrow h \\ \Sigma & \xrightarrow{\Re} & \Sigma \end{array}$$

be a commuting diagram of Scott-continuous maps. Then

$$h(\sup_k R^k(\perp)) = \sup_k \Re^k(\mathrm{h}(\perp)).$$

Further, if $h(\perp)$ *is the least element of* Σ*, then* $h(\sup_k R^k(\perp))$ *is the least fixed point of* $\Re:\Sigma\to\Sigma$.

It follows that the structure of the set of fixed points of a Scott-continuous response function is preserved by strict Scott-continuous perturbations. This confirms our expectation that the equilibrium sets of Scott-continuous response functions are structurally stable. At the moment, however, the stability of other type of response function remains open.

F. Complexity

The complexity of computing equilibrium in an economy has been the subject of study in recent years. Roughly speaking, the results are qualitatively similar: finding equilibrium is hard, even finding a best response to a given strategy may be a hard problem in general (see Aumann 1997). For instance, the proof of the existence of competitive equilibrium was usually based on the Brouwer fixed point theorem or its generalizations (see Nash 1950, 1951; Arrow and Debreu 1954). However, it has been shown that *any* algorithm for computing the Brouwer fixed point of a function based on function evaluations must perform exponential complexity in the worst case (see Hirsch, Papadimitriou, and Vavasis 1989). Thus, it is less likely that the problem of finding a competitive equilibrium is solvable in polynomial time.

In short, the very complexity that has made a theory of the decision-making process essential has made its construction exceedingly difficult.

On the other hand, actual behaviors may be rational without any conscious attempt at optimization. In connection with the perfect rationality assumption, Friedman (1953) discussed the "*as if*" doctrine: that people do not consciously optimize, but only act as if they do. For example, individuals behave *as if* they were seeking rationally to maximize their expected returns and have full knowledge of the data needed to succeed in this attempt. That is, they behave *as if* they knew the complicated mathematical formulas that give the optimal solution, could make lighting mathematical calculations from the formulas, and could act according to the optimal solution indicated by the formulas. Now, of course, individuals do not actually and literally solve the optimization problem in terms of which the mathematical economist finds it convenient to express the hypothesis.

Friedman’s “*as if*” doctrine rests on the observation that, even with the powerful new tools, most real-life choices still lie beyond the reach of maximizing techniques—unless the situations are heroically simplified by drastic approximations. If man makes decisions and choices that have some appearance of rationality, rationality in real life must involve something simpler than maximization of utilities (see Simon 1959).

Complexity is deep in the nature of things. The limits on computational capacity may be important constraints entering into the definition of rational choice under particular circumstances. A theory of rationality that does not give an account of problem solving in the face of complexity is seriously misleading by providing “solutions” to economic questions that are without operational significance. Broadly stated, the task is to replace the perfect rationality of economic man with a kind of rational behavior that is compatible with the access to information and the computational capacities that are actually possessed by organisms in the kinds of environments in which such organisms exist (See Simon 1955, 1959, 1978).

In theory, the gains from greater accuracy, which depend on the purpose in mind, must be balanced against the costs of achieving it. In view of this, one interesting question is whether the complexity of finding equilibrium can be decreased by weakening the requirements for rationality—by requiring solutions only to approximate the optimum, or by replacing an optimality criterion by a satisficing criterion (see Aumann 1997). After all, it is already known that there are some cases where such modifications reduce exponential or NP-complete problem classes to polynomial classes.

In our model, players are assumed to be of bounded rationality, and the existence proof of Nash equilibrium was based on the fixed point theorems for ordered structure. So the question is whether the problem of computing fixed points for ordered structures admits polynomial-time algorithm.

As an illustration, we shall consider the complexity of the Knaster-Tarski fixed point theorem. In the literature, the oracle complexity of Tarski’s fixed point theorem has been investigated. A polynomial time algorithm for finding a Tarski's fixed point was known when the lattice $(S,\geqq)$ has a total order (see Chang *et al.* 2008). Given $(S,\geqq)$ as input and the order preserving function $R: S \to S$ as oracle, this algorithm outputs a fixed point of R in time $O(\log(|S|))$. A polynomial time algorithm for finding a Tarski's fixed point in terms of the componentwise ordering was reported by Dang, Qi and Ye (2011). They develop an algorithm of time complexity $O((\log|S|)^d)$ to find a Tarski's fixed point on the componentwise ordering lattice $(S,\geqq)$ of dimension d. This is a generalization of Chang *et al.* result for the oracle function model since a total order lattice is equivalent to a componentwise ordering lattice of dimension one.

To the best of our knowledge, the complexity of Tarski's fixed point theorem remains open on general lattices. Further, algorithmic and complexity-theoretic issues of the Abian-Brown fixed point theorem are not yet known.

So it is still open whether our preference-response based approach leads to substantial simplification of the computations in the making of choice. In principle, the complexity of decision making will depend on the empirical facts about the economic system.

III. The Applications

In this section I shall explore possible ways of formulating the process of rational choice in situations to which classical theory do not apply. These applications have certain characteristics in common: they all involve important modifications in the concept of perfect rationality. These applications justify our dynamic game model under bounded rationality as a useful framework. In an empirical science, after all, nothing ultimately counts but results (Leontief 1986).

Though the mathematical tools are simple, the application of our model is a highly complex operation. The most important first step, and one that has little appeal to the traditional theory, is the gathering and ordering of the qualitative information involved in reality.

A. Impossibility Theorem

The mechanism of voting is a method of amalgamating the preferences of many individuals in the making of social choices. In voting, the individual does not choose among *existing* but rather among *potential* alternatives. Voting will produce consistent or rational social choice only if men are able to agree on the ultimate social goals (see Buchanan 1954). The problem is whether it is formally possible to construct a procedure for passing from a set of known individual preferences which satisfy certain natural conditions to a consistent pattern of social decision-making. An illustration of the difficulty of problem is the well-known impossibility theorem of Arrow (1950), which states that any method of deriving social choices by aggregating individual preference patterns fails to satisfy the condition of rationality as we ordinarily understand it.

The impossibility theorem shows that, if no prior assumptions are made about the nature of the individual orderings, there is no satisfactory method of passing from individual preferences to social preferences (see Arrow 1950). The failure of purely individualistic assumptions to lead to a well-defined social welfare function means, in effect, that there must be a divergence between social and private benefits if we are to be able to discuss a social optimum. If we wish to make social welfare judgments which depend on all individual preferences, then we must relax some of the conditions imposed on collective rationality.

Within the framework of bounded rationality, the way out of the impossibility theorem becomes perfectly straightforward. What is characteristic of our approaches is that we can investigate various possibilities by eliminating or weakening or replacing one or more conditions that Arrow imposed.

1. The Independence of Irrelative Alternatives.

The condition of Independence of Irrelative Alternatives states that the choice made by society from a given set of alternatives depends only on the orderings of individuals among those alternatives. Alternatively stated, if we consider two sets of individual orderings such that, for each individual, his ordering of those particular alternatives under consideration is the same each time, then we require that the choice made by society be the same if individual values are given by the first set of orderings as if they are given by the second (see Arrow 1950).

However, under bounded rationality, the choices made by society from a given set of alternatives may not have consistency of certain type. Alternatively stated, the choice made by society from a given set of alternatives may not be determined by the orderings of individuals among those alternatives. In contrast, even fixed the set of alternatives the choice made by society may depend on the degree of rationality, which in turn may depend on the existence of alternatives outside the given set.

In short, even the preferences of individuals are the same on the two strategy spaces, and the response functions of individuals are the same restricted on the two strategy spaces, the choice made by society may not be the same under bounded rationality.

For example, consider arbitrary CPO $(\mathrm{S}_{\perp}, \geqq)$ with the least element $\perp$. Then by the Abian-Brown fixed point theorem we know that any order-preserving mapping $R : \mathrm{S}_{\perp} \to \mathrm{S}_{\perp}$ has one least fixed point. However, if we eliminate the least element and consider the remaining CPO S, still ordered by $\geqq$, then the same order-preserving mapping $R : \mathrm{S} \to \mathrm{S}$ may not have a least fixed point. For counterexample see section II.C.

In reality, the death of the most preferred candidate may dramatically affect the final result of an election, even through the same election procedure. This phenomenon may be sufficient to relax the condition of the Independence of Irrelative Alternatives, which is an appealing way of escaping the Arrow impossibility theorem.

But, more seriously, Sen (1970) have proved the impossibility of a Paretian liberal without imposing the condition of the Independence of Irrelative Alternatives. So this way out is not open therein.

2. The Completeness Axiom.—The completeness axiom states that any two alternatives are comparable.

Arrow's framework assumes that individuals are rational, by which is meant their preferences orderings satisfy completeness and transitivity on the set of alternatives.[40] However, under the constraint of bounded rationality, individual preferences are just assumed to be partial orders on the strategy space. Hence it may be possible to construct a *partial* preference ordering for the society as a whole. Relaxing the completeness axiom may enable us to escape from the negative conclusion of impossibility.

Needless to say, this relaxation is not without its price. Under the constraint of bounded rationality, not all equilibrium solutions remain Pareto optimal, even we admit the Axiom of Choice (see section II.C). But this price is not paid in vain, since a rejection of the Pareto principle is exactly one way out the impossibility of a Paretian Liberal (see Sen 1970). Indeed, the difficulties of achieving Pareto optimality in the presence of externalities are well known. Our result simply shows that bounded rationality is a type of externality that cannot be neglected.

B. Keynesian beauty contest

Keynesian beauty contest describes the mental process of higher-order reasoning. Historically, Keynes developed this metaphor to explain price fluctuations in equity markets (see Keynes 1936):

> ". . . professional investment may be likened to those newspaper competitions in which the competitors have to pick out the six prettiest faces from a hundred photographs, the prize being awarded to the competitor whose choice most nearly corresponds to the average preferences of the competitors as a whole; so that each competitor has to pick, not those faces which he himself finds prettiest, but those which he thinks likeliest to catch the fancy of the other competitors, all of whom are looking at the problem from the same point of view. It is not a case of choosing those which, to the best of one's judgment, are really the

[40] Unlike in the impossibility theorem of Arrow, Sen (1790) has not required transitivity of social preference, but still have impossibility.

prettiest, nor even those which average opinion genuinely thinks the prettiest. We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practice the fourth, fifth and higher degrees."

Later, more explicit scenarios were investigated in order to convey the process of a convergence to Nash Equilibrium (see Moulin 1986). In the classical *p*-beauty contest game (see Nagel 1995), n players have to simultaneously pick a number in the closed interval $[0,100]$. The winner is the person whose number is closest to p times the average of all chosen numbers.

In this game each player has to guess what the average choice is going to be and make his choice equal to p times that average. Denote the choice of player i by $s_i \in S_i = [0,100]$, and denote the choice profile of all players to be $s = (s_1, s_2, \cdots, s_n) \in S = S_1 \times S_2 \times \cdots \times S_n$, then the best response function of player i is $R_i(s_{-i}) = p\mathrm{E}(s_{-i})$, where $\mathrm{E}(s_{-i})$ is the average choice of all players except i. In this way we get the best response function of the system as a whole

$$(21) \quad R(s) = \prod_{i=1}^{n} R_i(s_{-i}) = \left(p\mathrm{E}(s_{-1}), p\mathrm{E}(s_{-2}), \cdots, p\mathrm{E}(s_{-i}), \cdots, p\mathrm{E}(s_{-n})\right).$$

If $0 \le p < 1$, then the only Nash equilibrium is for all players to choose 0, which is also the only strategy profile to survive the iterated elimination of dominated strategies. Also, for the repeated supergame, all Nash equilibria induce the same results as in the one-shot game. Thus game theory predicts unambiguous outcome (see Nagel 1995).

However, experiment studies showed that most actual players do not behave according to this solution; rather they are cleverly anticipating the behavior of others, with noise. Thus there is a clear distinction between game theoretic solution and bounded rationality (see Nagel 1995; Ho *et al.* 1998).

It is of interest to note that if $p > 0$ then the best response function $R : S \to S$ is order-preserving. In fact, $x \geqq y$ in $S = S_1 \times S_2 \times \cdots \times S_n$ implies $x_{-i} \geqq y_{-i}$, and hence $\mathrm{E}(x_{-i}) \geqq \mathrm{E}(y_{-i})$ for all i. In turn this implies $R(x) \geqq \mathrm{R}(y)$ by definition.

Under the constraint of bounded rationality, strategies of players don't have to be best reply to each other. For example, player with rule rationality may respond to other player's strategy according to his own belief about other players' believes. Thus the response function may not be the best. In general, if players are rule rational, then the response function $R : S \to S$ will still be order-preserving. In this case, the Knaster-Tarski fixed point theorem (see Tarski 1955) asserts that the set of fixed points forms a complete lattice, and hence has a least and a greatest fixed point.

In theory, the greatest fixed point may not coincide with 0. Furthermore, this greatest fixed point can be *approximated* by the descending sequence started from the top $\tau = (100, 100, \cdots, 100) \in S$, namely,

$$\tau \geqq R(\tau) \geqq R^2(\tau) \geqq R^3(\tau) \geqq \cdots \geqq R^k(\tau) \geqq \cdots, \tag{22}$$

in the sense that it equals $\inf_k R^k(\tau)$. This sequence captures the dynamic essence of iterated reasoning described by Keynes (1936).

C. Bertrand paradox

An Oligopoly is a market mechanism between monopoly and perfect competition, in which the market is completely controlled by only a few numbers of firms producing perfectly homogeneous productions.

In the typical Bertrand (1883) model, there are two firms that produce a homogeneous good at a marginal cost facing a decreasing demand curve. They simultaneously announce prices as strategic variable. The firm announcing the lowest price fulfils the entire demand. If the two firms announce the two firms announce the same price, the demand is evenly split between them.

It is shown that the Bertrand model of price competition has a unique Nash equilibrium, one where price equals marginal cost and firms earn zero profit. Put more precisely, if the choice variable is price instead of output, then the presence of only two firms is sufficient to eliminate all market power and give rise to the perfectly competitive outcome. This result is paradoxical because once the number of firms goes from one to two, the price decreases from the monopoly price to the competitive price and stays at the same level as the number of firms increases further.

Solutions to the Bertrand paradox attempt to derive solutions that are more in line with solutions from the Cournot (1838) model of output competition, where firms simultaneously choose output as strategic variable. In Cournot-Nash equilibrium, price is below the monopoly price but above marginal cost, and an increase in the number of firms is associated with a convergence of prices to marginal costs.

Within our framework, a systematic examination of the paradox can be undertaken from two different angles.

Firstly, the Bertrand model is a static game. Fisher (1898) was concerned with the static nature of the Cournot and Bertrand models, arguing that real firms engage in various forms of dynamic behavior. In a dynamic Bertrand game, one firm may choose price in the first period and the other chooses price in the second period.

Secondly, the Bertrand model is a fully rational game. In the Bertrand model it is assumed that each firm sets price to maximize profits. Under the conditions of full information all firms will try to reduce their products price, until the product is selling at no profit. But it is possible that all players are of bounded rationality so as to adopt their expectations to adjust his product's price according to some heuristic rules.

The way, or at least a way, out of the Bertrand paradox lies in the rejection of the narrowly interpreted optimal principle as synonymous with rational behavior. Instead of choosing the best response to the *next* round, each firm must choose the optimal strategy to all the *future* rounds. The underlying logic can be described as follows: if one of the proprietors will reduce his price in order to attract buyers, then the other will in turn reduce his price even more to attract buyers back to him. Thus rationality will force them to stop undercutting each other when either proprietor, even if the other abandoned the struggle, has nothing more to gain from reducing his price any further.

Mathematically, we can assume that the strategy set of each firm, S_i ($i = 1,2$), is a closed interval of positive real numbers, and hence is a complete lattice. Thus the strategy space $S = S_1 \times S_2$ is also a complete lattice.

On the other hand, the Bertrand competition naturally defines a decreasing response function $R : S \to S$ such that $s \geqq R(s)$ for all $s \in S$. Further, the response function $R : S \to S$ will be order-preserving if the step of price-reducing is sufficiently small. Together, the Knaster-Tarski fixed point theorem (see Tarski 1955) asserts that the set of fixed points forms a complete lattice, and hence has both least and greatest fixed point.

Since it is *irrational* to set price lower than marginal cost, the least fixed point must be larger than or equal to the marginal cost. So equilibrium price determined by the greatest fixed point must be above the marginal cost in general. Therefore, the Bertrand paradox can be resolved on the basis of our dynamic game model under bounded rationality. [41]

D. Repeated Prisoner's Dilemma

The best-understood class of dynamic games is that of repeated games, in which players face the same stage game in every period. Probably the best-known example of repeated games is the celebrated prisoner's dilemma. In the prisoner's dilemma game, two individuals can each either cooperate (C) or defect (D). Figure 1 shows the payoff matrix of the prisoner's dilemma in terms of real numbers with $a_1 > a_2 > a_3 > a_4$ and $b_1 > b_2 > b_3 > b_4$.

	Player 2

[41] More precisely, the Bertrand paradox is resolved by noting that in game situations, one player's irrationality requires another's super-rationality. *You* must be super-rational in order to deal with *my* irrationalities. Since this applies to all players, taking account of possible irrationalities leads to a kind of super-rationality for all. To be super-rational, one must leave the equilibrium path. Thus, a more refined concept of rationality cannot feed on itself only; it can only be defined in the context of bounded rationality. For details see Aumann (1997).

		C	D
Player 1	C	(a_2, b_2)	$(a_4, \underline{b_1})$
	D	$(\underline{a_1}, b_4)$	$(\underline{a_3}, \underline{b_3})$

FIGURE 1. THE PRISONER'S DILEMMA

If the Prisoner's Dilemma game is repeated a finite number of times, *however large it is*, the backward induction leads to a result which has the same set of equilibria as the static version. Indeed, subgame perfection requires both players to *defect* in the last period, and backward induction implies that the only subgame-perfect equilibrium is for both players to *defect* in every period. In fact, in repeated prisoner with a fixed finite horizon "always defect" is the only Nash equilibrium (see Fudenberg and Tirole 1991).

However, the set of equilibia of infinite-horizon version of prisoner's dilemma turns out be quite different. It is well known that an infinitely repeated game may have game theoretical properties which are not apparent from the analysis of the stage game. For example, in the infinitely repeated prisoner's dilemma, the Folk Theorem says that, provided players are sufficiently patient, there is a Nash equilibrium such that both players *cooperate* on the equilibrium path (see Fudenberg and Tirole 1991). Thus, infinite-horizon version of prisoner's dilemma with patient players not only makes cooperation possible, but also leads to a large set of other equilibrium outcomes.

Thus there may be a *gap* between finite-horizon and infinite-horizon versions of the same game. Nevertheless, the game theoretical reasoning for finite repeated games is inconsistent with actual human behavior, as indicated by the chain store paradox (see Selten 1978). As we shall show, this logical difficulty can be resolved based on our dynamic game model under bounded rationality.

It is easy to see that the strategy space of static prisoner's dilemma is $S = \{(C,C),(C,D),(D,C),(D,D)\}$. In order to simplify notation we shall work in the value space (see section I.D)

(23) $$U(S) = \{(a_2,b_2),(a_4,b_1),(a_1,b_4),(a_3,b_3)\}.$$

It is worth noting that the value space $U(S)$, which has inherited from two-dimensional Euclidean space the componentwise order $\geqq$, is a CPO with *no* least element.

Now consider the T-fold repetition of prisoner's dilemma, and suppose that all players can observe the realized action at the end of each period. In repeated games the overall payoff of a player is the weighted sum of his payoffs for all repetitions. So the value space of the T-repetition of prisoner's dilemma will be

$$U^T = \{\sum_{t=1}^{T} \delta^t u^t \mid u \in U(S)\}, \tag{24}$$

where $0 < \delta \leq 1$ is the discount factor. In theory, the strategy space U^T is also a CPO with *no* least element, as in the static case.

To complete the description of the repeated game, we have to specify a response function $R : U^T \to U^T$ that is consistent with bounded rationality.

Under bounded rationality both players quest for improvement according to certain heuristic rules in order to reward cooperation and punish defection. Typically, players with bounded would adopt trigger strategy that initially cooperates but punishes the opponent if a certain level of defection is observed. A typical trigger strategy is TIT FOT TAT, which is a strategy of cooperation based on reciprocity. This strategy is simply one of cooperating on the first move and then doing whatever the other player did in the preceding move. Axelrod and Hamilton (1981) showed that the TIT FOR TAT strategy is robust and stable.

Now assume that $R : U^T \to U^T$ is the response function induced by the TIT FOR TAT strategy, adopted by *both* players. Interestingly, it turns out that $R : U^T \to U^T$ is *order-preserving*. To see this it suffices to consider the stage game. That is, we need to show that its restriction to the value space, i.e., $R : U(S) \to U(S)$, is order-preserving. In fact, the TIT FOR TAT strategy, if adopted by *both* players, gives rise to a function $R : U(S) \to U(S)$ as follows

$$(25) \qquad \begin{array}{l} R(a_2,b_2)=(a_2,b_2), R(a_3,b_3)=(a_3,b_3) \\ R(a_4,b_1)=(a_1,b_4), R(a_1,b_4)=(a_4,b_1) \end{array}.$$

Note that there are only two comparable strategies in the value space, that is, $(a_2,b_2) \geqq (a_3,b_3)$, which implies $R(a_2,b_2) \geqq \mathrm{R}(a_3,b_3)$ by definition. So the response function defined by TIT FOR TAT strategy, if adopted by *both* players, is order-preserving by induction.

Also, it is worth emphasis that this response function has two fixed points: (a_2,b_2) and (a_3,b_3), which amount to two strategies—unconditional cooperation (ALL C) and unconditional defection (AAL D) respectively. Since AAL D is evolutionarily stable, this raises the problem of how an evolutionary trend to cooperative behavior can ever have started in the first place, or how to escape from the equilibrium of ALL D (see Axelrod and Hamilton 1981).

To summarize, the value space U^T of repeated prisoner's dilemma is a CPO with *no* least element, and the response function $R: U^T \to U^T$ will be order-preserving if heuristic rules are adopted. But these two conditions are not sufficient for the existence of fixed point of the response function in general (see section II.C for counterexample). As shown by the Amann fixed point theorem, to guarantee the existence of the fixed point it is required at least that $(U^{\mathrm{T}}, \geqq)$ *has an element* u *with* $u \leqq R(u)$. With no suitable position to start, social orders can not emerge *spontaneously* as the result of interactive actions. Put more precisely, there must be a strategy where both players have incentive to improve, so that we can escape from the evolutionarily stable equilibrium of always defect (see Axelrod and Hamilton 1981).

Why is the existence of such an improvement necessary (as well as sufficient) to insure unconditional cooperation? The answer is inherent in the evolutionary stability of always defect: an evolutionary trend to cooperative behavior permits that cooperation may be established between the players, but it does not exclude the possibility that no cooperation is established (see Axelrod and Hamilton 1981).

Nevertheless, since the equilibrium of always defect can not achieve optimality, player with bounded rationality always has incentive to deviate equilibrium by punishing defective behavior, say, via TIT FOR TAT. So we can safely assume that $(U^{\mathrm{T}}, \geqq)$ *has an element* u *with* $u \leqq R(u)$. In such a case, the Amann fixed point theorem asserts that the response function $R: U^{T} \to U^{T}$ has one least fixed point $u_T = R(u_T)$. In general, any fixed point above this least fixed point can be achieved in an equilibrium. This is essentially the folk theorem for finitely repeated prisoner's dilemma (see Fudenberg and Tirole 1991).

Further, this result is consistent with the empirical evidence (see Axelrod and Hamilton 1981) and hence enables us to escape from the backward induction paradox, at least as far as the repeated prisoner's dilemma is concerned.

IV. The Extensions

I have now investigated several paradoxes, all referring to the fixed point theorem for ordered structure. These applications illustrated the advantage of our framework. However, they are of significance in themselves, because each has played an important part in the history of economic science; and precisely because of their simplicity, they provide a useful illustration of the general principle involved.

In this section I shall consider how the basic model generalizes to the more realistic case. Also I shall be concerned with more complex problems.

A. Random Response

In actual human decision-making, alternatives are often examined sequentially. Thus we may not know the mechanism that determines the order of offer-making. For example, in English auction where bids are freely made and announced, each bidder, in attempting to determine at what point he should be prepared to make a bid so as to obtain the greatest expectation of gain, will need to take into account whatever information he was concerning the probable bids that might be made by others. Even we know the response function of each bidder, we can not know in advance who will make bids in the *next* round. Thus the response function of the system *as a whole* can not be known as common knowledge. [42]

Essentially, the process of English auction consists of iteratively applying one response function chosen *at random* from the system of response functions to generate a new bid. The bidding will stop at a price where no auctioneer wishes to make any further higher bid. This final price must be a *common* fixed point for the system of response functions. Thus, to guarantee the dynamic process of auctioning to converge we need a common fixed point theorem (Davey and Priestley 2002).

Common Fixed point Theorem: *Let* $(S, \geqq)$ *be a CPO with a least element. Then every family of increasing* ***and*** *order-preserving self-maps on* S *has a least common fixed point.*

To apply the common fixed point theorem to English auction, first note that the least element of the strategy space amounts to the reservation value of the seller. Further, the properties of increasing and order-preserving are automatically satisfied in English auction. It follows that the final bidding price is the least common fixed point for the system of response functions of individuals. This result is obviously optimal, in the sense of market-clearing.

Dually, in the Dutch auction, each auctioneer has to respond to the prices announced by the auctioneer in descending sequence, started from the highest reservation value of the seller. Given the announced price s, there is no buyer to make offer means that the announced price s is dominated by each player's response $R_i(s)$, in terms of preference. So the response function of each auctioneer is decreasing, i.e., $s \geqq_i R_i(\mathrm{s})$. Further, if the decrement of price-reducing is sufficiently small, then all response functions $R_i(s)$ are order-preserving. In this case, the first and only bid that concludes the transaction must be the greatest common fixed point of the response functions of all auctioneers.

[42] Within the framework of perfect rationality, auctions are modeled as non-cooperative games with imperfect information (see Vickrey 1961). The basic assumption is that the reservation value of the buyers are independent and identically distributed, drawn form the common distribution, so that the classical bidding procedure belong to the class of games of incomplete information. The basic drawback to this scheme is that it loses the dynamic character of auctions.

Common Fixed point Theorem (Dual Form): *Let* $(S,\geqq)$ *be a dual CPO with a greatest element. Then every family of decreasing* ***and*** *order-preserving self-maps on* S *has a greatest common fixed point.*

Specially, assume that the set of possible price is a complete lattice, then the least fixed point in English auction and the greatest fixed point in Dutch auction exist at the same time. But these two critical prices may not coincide, even if the bidders are the same group of buyers. As a result, revenue equivalence between English auction and Dutch auction may not hold under bounded rationality (see Vickrey 1961; Myerson 1981). It essentially depends upon the quantitative strengths of the auctioneers, but that is not the part of mathematical deductive analysis to answer this question. Which effect will be the stronger cannot be decided on a priori ground.

Taken together, in *double auctions* the system of response functions of all buyers have a *least* common fixed point and the system of response functions of all sellers have a *greatest* common fixed point. Since asking prices always exceed offering price, perhaps we should consider as the solution the interval of determinacy given by the two critical points: the least fixed point and the greatest fixed point. In general, this interval will coincide with the "range of practicable bargains," defined in terms of Edgeworth contract curve (see Pigou 1932, Chapter VI).

B. State Space

It is, perhaps, worth stressing that economic problems arise always and only in consequence of change. So long as things continue as before, or at least as they were expected to, there arise no new problems requiring a decision, no need to form a new game.

In practice players have to make decision under given state. If the state of the system is changed, then a new pattern of game would be observed. Of course, if states change, then either the strategy set or the preference or the response function will change accordingly, and a new optimal strategy will be found. The really possible social choice has changed with the changing environment.

Now assume that the state space is Ω. Under given state $\omega \in \Omega$, each player may make his choice within a set of alternatives more limited than the whole range of objectively available to it. Each player's choice is not entirely free, and the actions of all the other agents determine the subset to which his selection is restricted (see Debreu 1952). That is, the conditioned strategy set of player i is a subset $S_i(\omega)$ of the unconditional strategy set $(\mathrm{S}_i, \geqq_i)$. Essentially, $S_i(\omega)$ can be viewed as a correspondence defining for each state $\omega \in \Omega$ a subset $S_i(\omega)$ of $(\mathrm{S}_i, \geqq_i)$. In the special case where the correspondence $S_i(\omega)$ is in fact constant, say, $S_i(\omega) = S_i$ for any $i \in \{1,2,\cdots,n\}$, our model reduces to the classical game model of Nash (1950, 1951). In general, the correspondence $S_i(\omega)$ will depend on the state variable $\omega \in \Omega$.[43]

Consequently, the strategy space under given state $\omega \in \Omega$ equals the product $S(\omega) = S_1(\omega) \times S_2(\omega) \times \cdots \times S_n(\omega)$, which is a subset of the unconditional strategy space $(S = S_1 \times S_2 \times \cdots \times S_n, \geqq)$. As in the case of unconditional decision, the response function is defined to be a well-behaved function $R : S(\omega) \to S(\omega)$, which may be Scott continuous, or order preserving, or directed.

In general, the fixed point theorems for ordered can be applied to the response function $R : S(\omega) \to S(\omega)$ to guarantee the existence of equilibrium solution under given state $\omega \in \Omega$. For example, if the conditioned strategy space $(S(\omega), \geqq)$ is a complete lattice under given state $\omega \in \Omega$ and the response function $R : S(\omega) \to S(\omega)$ is order preserving, and then the Knaster-Tarski fixed point theorem (see Tarski A. 1955) says that the set of fixed points $\{s \in S(\omega) \mid s = R(s)\}$ forms a complete lattice in itself. In

[43] For simplicity we assume that $S_i(\omega)$ has inherited the preference relation $\geqq_i$ from the strategy set $(\mathrm{S}_i, \geqq_i)$. That is, preference is assumed to be exogenous and stable (see Stigler and Becker 1977). However, in practice individual's preference may change with the state. That is, $S_i(\omega)$ may have different preference relation $\geqq_i^{\omega}$ under different state $\omega \in \Omega$. In view of this our framework is also consistent with the paradigm of endogenous preferences.

particular, $R: S(\omega) \to S(\omega)$ have a least fixed point $\underline{s}(\omega)$ and a greatest fixed point $\bar{s}(\omega)$.

In many situations we may be interested in the precise question of whether one state is more preferable than another. For many purpose there is needed a *partial* ordering relation on pairs of elements of Ω, i.e., a relation $\succeq$ that states that $\omega_1 \succeq \omega_2$ if and only if ω_1 is preferred to ω_2. In this case, a natural question is, whether the least and greatest equilibrium strategies, $\underline{s}(\omega)$ and $\bar{s}(\omega)$, are order-preserving function of $\omega \in \Omega$. That is, whether $\omega_1 \succeq \omega_2$ *implies* $\underline{s}(\omega_1) \geqq \underline{s}(\omega_2)$ *and* $\bar{s}(\omega_1) \geqq \bar{s}(\omega_2)$. This problem is of considerable economic interest. Indeed, it is related to monotone comparative statics (see Milgrom and Roberts 1994; Milgrom and Shannon 1994).

Attention should be directed to the threefold distinction drawn by the definitions among the strategy space $(S, \geqq)$, the state space $(\Omega, \succeq)$, and the response functions $R: S(\omega) \to S(\omega)$ ($\omega \in \Omega$). Compare with "classical" theory, the present formulation is more general in the sense that it is based on response functions, rather than pay-off functions.[44] One element of realism is that, while the pay-off function may be known in advance, the response function may not, especially under the circumstance of bounded rationality.

C. Response Correspondence

Up until now I have considered dynamic games concerning response functions. The techniques used here are of even more fruitful applicability to games of response correspondences. In fact, this extends easily to systems of response correspondence since most celebrated fixed point theorem for ordered structures had been generalized to set-valued mappings.

[44] It is worth noting that Simon considered "simple" pay-off functions that are consistent with the paradigm of bounded rationality. The advantage of these simple pay-off functions over the more common notion is that it implies much less complex and subtle evaluation criteria. The player, instead of seeking for a "best" strategy, needs only to look for a "good" strategy. It is shown that, by the introduction of a simple pay-off function and a process for gradually improving it, the process of reaching a rational decision may be dramatically simplified from a computational standpoint.

For example, Fujimoto (1984) extended the Knaster-Tarski fixed pointed theorem to the case of set-valued mappings, and applied it to a class of complementarity problems defined by isotone set-valued operators in a complete vector lattice. Zhou (1994) generalized the Knaster-Tarski fixed pointed theorem to the case of correspondence and applied it to show that the set of Nash equilibria of a supermodular game is a complete lattice.

As another illustration, Li (2014) provided several extensions of the Abian–Brown fixed point theorem from single-valued mappings to set-valued mappings on chain-complete posets and applied it to some well-known problems in game theory.

V. Concluding Remarks

The problem of introducing dynamics into economic theory, listed into the *mathematical problems for the next century* by Steve Smale (1998), was regarded as the main problem of economic theory. This problem asks for a dynamical model that is compatible with the existing equilibrium theory. A most desirable feature is to have the time development of strategies determined by the individual actions of players. The resulting change in outlook can be compared to that of the transition from classical to quantum mechanics.

This paper is concerned with a new effort to generalize economic theory from static equilibrium to dynamic evolution, from perfect rationality to bounded rationality. Essentially it is a method of analysis that takes advantage of the relatively stable pattern of the behaviors of players with bounded rationality to bring a much more detailed dynamic process into interactive decision-making.

In advancing from the phase of static equilibrium to a broadening sphere of dynamic evolution, I proposes a game-like model with bounded rationality players, based on the fixed point theorems for ordered structures, such as Zermelo-Bourbaki fixed point theorem (Boubaki 1949), the Knaster-Tarski fixed point theorem (Tarski 1955), the Abian-Brown fixed point theorem (Abian and Brown 1961), as well as the Markowsky fixed point theorem (Markowsky 1976). These order-theoretic fixed point theorems have been widely cited in computer science, and have been applied to the study of supermodular games (see Topkis 1978, 1979; Vives 1990; Milgrom and Roberts 1990).

The connection with ordered structures may be very surprising at first, but it turns out that it is natural in problems for this kind. The immediate reason for this is the fact that the preference can be naturally described by certain order relations. Consequently, the response function can be naturally described by certain order-preserving map on the strategy space, ordered by preference.

Another important reason lies in the similarity between the market and the computer, which has long been noted (see Lange 1967). The market may be considered as one of the oldest computing device which serves to find the equilibrium solution. The market mechanism and trial and error procedure played the role of a computing device leading to the equilibrium solutions by iteration. We must look at the market as such a mechanism for communicating information if we want to understand its real function (see Hayek 1945). It is in this connection that the computer helps us, at least by analogy, to see how the market can be described by the fixed theorem for ordered structures. Since fixed point theorems for ordered structures have been widely used in computer science, this approach is not accident.

The possibility of linking up the existence of equilibrium with a fixed point problem is but one special aspect of our framework. In any event our framework is mathematically more convenient. Its main advantages over the traditional approach based on perfect rationality are that: (1) the strategy space is a chain-complete partially ordered set; (2) the response function is certain order-preserving map on strategy space; (3) the evolution of economic system can be described by the Dynamical System defined by the response function under iteration; (4) the existence of pure-strategy Nash equilibria can be guaranteed by fixed point theorems for ordered structures, rather than topological structures. This preference-response framework liberates economics from the utility concept, and constitutes a marriage of normal-form game and extensive-form game.

But the essential advantages of our model may be secured by successfully resolving some longstanding paradoxes in classical theory, yielding straightforward ways out of the impossibility theorem, the Keynesian beauty contest, the Bertrand Paradox, and the backward induction paradox. These applications have certain characteristics in common: they all involve important modifications in the concept of perfect rationality.

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