Analogy Making and the Structure of Implied Volatility Skew

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An analogy based option pricing model is put forward. The model provides a new explanation for the implied volatility skew puzzle and is consistent with empirical findings regarding leverage adjusted index option returns. It also explains puzzling superior performance of covered call writing and worse-than-expected performance of zero-beta straddles. The analogy based stochastic volatility and the analogy jump diffusion models are also developed. The analogy based stochastic volatility model generates the skew even without any correlation between the stock price and volatility processes, whereas, the analogy jump diffusion does not require asymmetric jumps to generate the skew.

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Analogy Making and the Structure of Implied Volatility Skew

The existence of the implied volatility skew is perhaps one of the most intriguing anomalies in option markets. According to the Black-Scholes model (Black and Scholes (1973)), volatility inferred from prices (implied volatility) should not vary across strikes. In practice, a sharp skew in which implied volatilities fall monotonically as the ratio of strike to spot increases is observed in index options.

The Black-Scholes model assumes that an option can be perfectly replicated by a portfolio consisting of continuously adjusted proportions of the underlying stock and a risk-free asset. The cost of setting up this portfolio should equal the price of the option. Most attempts to explain the skew have naturally relaxed the assumption of perfect replication. Such relaxations have taken two broad directions: 1) Deterministic volatility models 2) Stochastic volatility models without jumps and stochastic volatility models with jumps. In the first category are the constant elasticity of variance model examined in Emanuel and Macbeth (1982), the implied binomial tree models of Dupire (1994), Derman and Kani (1994), and Rubinstein (1994). Dumas, Fleming and Whaley (1998) provide evidence that deterministic volatility models do not adequately explain the structure of implied volatility as they lead to parameters which are highly unstable through time. The second broad category is examined in papers by Chernov et al (2003), Anderson, Benzoni, and Lund (2002), Bakshi, Cao, and Chen (1997), Heston (1993), Stein and Stein (1991), and Hull and White (1987) among others. Bates (2000) presents empirical evidence regarding stochastic volatility models with and without jumps and finds that inclusion of jumps in a stochastic volatility model does improve the model, however, in order to adequately explain the skew, unreasonable parameter values are required. Empirical findings suggest that models with both stochastic volatility and jumps in returns fail to fully capture the empirical features of index returns and option prices (see Bakshi, Cao, and Chen (1997), Bates (2000), and Pan (2002)).

Highly relevant to the option pricing literature is the intriguing finding in Jackwerth (2000) that risk aversion functions recovered from option prices are irreconcilable with a representative investor. Perhaps, another line of inquiry is to acknowledge the importance of heterogeneous expectations and the impact of resulting demand pressures on option prices. Bollen and Whaley

(2004) find that changes in implied volatility are directly related to net buying pressures from public order flows. According to this view, different demands and supplies of different option series affect the skew. Lakonishok, Lee, Pearson, and Poteshman (2007) examine option market activity of several classes of investors and highlight the salient features of option market activity. They find that a large percentage of calls are written as a part of covered call strategy. Covered call writing is a strategy in which a long position in the underlying stock is combined with a call writing position. This strategy is typically employed when one is expecting slow-to-moderate growth in the price of the underlying stock. It seems that call buyers expect higher returns from the underlying stock than call writers, but call writers are not pessimistic either. They expect slow/moderate growth and not a sharp downturn in the price of the underlying stock.

Should expectations regarding the underlying stock matter for option pricing? In the Black-Scholes world where perfect replication is assumed, expectations do not matter as they do not affect the construction of the replicating portfolio or its dynamics. However, empirical evidence suggests that they do matter. Duan and Wei (2009) find that a variable closely related to the expected return on the underlying stock, its systematic risk proportion, is priced in individual equity options.

There is also strong experimental and field evidence indicating that the expected return from the underlying stock matters for call option pricing. Rockenbach (2004), Siddiqi (2012), and Siddiqi (2011) find that participants in laboratory experiments seem to value a call option in analogy with its underlying stock. They perceive the risk from a call option as similar to the risk from the underlying stock, and consequently expect similar returns from them. In fact, a call option is commonly considered a surrogate for the underlying stock by many experienced option traders and analysts[1] suggesting strong field relevance of laboratory findings.

The central prediction of asset pricing theory is:

$$E[R_i] = R_F - \frac{1}{E[u'(c_{t+1})]} Cov[u'(c_{t+1}), R_i] \qquad (0.1)$$

Where R_i and R_F denote the (gross) return on a risky asset and the return on the risk free asset respectively. Equation (0.1) shows that the return that a subjective expected utility maximizer

[1] As illustrative examples, see the following: http://ezinearticles.com/?Call-Options-As-an-Alternative-to-Buying-the-Underlying-Security&id=4274772, http://www.investingblog.org/archives/194/deep-in-the-money-options/, http://www.triplescreenmethod.com/TradersCorner/TC052705.asp, http://daytrading.about.com/od/stocks/a/OptionsInvest.htm

expects from a risky asset depends on his belief about the covariance of the asset's return with his marginal utility of consumption.

According to (0.1), an investor is required to form a judgment about the covariance of an asset's return with his marginal utility of consumption. He is required to do that for all available assets. It is reasonable to think that instead of forming such judgments in isolation for each asset, such a judgment is formed for a familiar asset and then extrapolated to another similar asset. A call option derives its existence from the underlying stock, and their payoffs are strongly related. As mentioned earlier, field and experimental evidence strongly suggest that the risk of the underlying stock is used as a reference point for forming risk judgments about a call option.

With the above in mind, an analogy maker is defined as *a subjective expected utility maximizer who uses the risk of the underlying stock as a reference point for forming risk judgments about the corresponding call options.* If one forms a judgment about call option risk in comparison with his judgment about the underlying stock risk, then one may write:

$$Cov\left(\frac{u'(c_{t+1})}{E[u'(c_{t+1})]}, R_c\right) = Cov\left(\frac{u'(c_{t+1})}{E[u'(c_{t+1})]}, R_s\right) + \epsilon \qquad (0.2)$$

Where R_c and R_s are call and stock returns respectively, and ϵ is the adjustment used to arrive at call option risk from the underlying stock risk. Almost always, assets pay more (less) when consumption is high (less), hence, the covariance between an asset's return and marginal utility of consumption is negative. That is, I assume that $Cov(u'(c_{t+1}), R_s) < 0$. So, in order to make a call option at least as risky as the underlying stock, I assume $\epsilon \leq 0$.

Such analogy making matters because it is typical for one to fail to adjust fully when starting from a reference point. That is, there is an automatic inclination to latch onto the reference point, and only deviate slightly from it. This is known as the "anchoring" heuristic. Anchoring effect is one of the most robust cognitive heuristics. 40 years of extensive research on the topic has shown it to be applicable to a wide variety of decision contexts. See Furnham, A., and Boo, H. C. (2011) for a literature review on anchoring. Anchoring implies that using the risk of the corresponding underlying stock as a reference point for forming beliefs about the risk of a call option leads to an underestimation of risk. An analogy maker understands that *a call option is a leveraged position in the underlying stock, hence is riskier. However, starting from the risk of the underlying stock, he does not fully adjust for the risk, so he underestimates the risk of a call option.*

In contrast, option pricing theory predicts that:

$$Cov\left(\frac{u'(c_{t+1})}{E[u'(c_{t+1})]}, R_c\right) = \pi. Cov\left(\frac{u'(c_{t+1})}{E[u'(c_{t+1})]}, R_s\right) \tag{0.3}$$

Where $\pi > 1$ and typically takes very large values. That is, typically $\pi \gg 1$. To appreciate, the difference between (0.3) and (0.2), note that under the Black Scholes assumptions, $\pi = \Omega$, which is call price elasticity w.r.t the underlying stock price. Ω takes very large values, especially for out-of-the-money call options. That is $Cov(u'(c_{t+1}), R_c)$ is likely to be a far bigger negative number with correct risk judgment than with analogy making. Hence, a comparison of (0.2) and (0.3) indicates that, with analogy making, one likely remains anchored to the risk of the underlying stock while forming risk judgments about the call option leading to underestimation of its risk.

Substituting (0.2) in (0.1) leads to:

$$E[R_C] = E[R_S] + |\epsilon| \tag{0.4}$$

In contrast, substituting (0.3) in (0.1) under the Black Scholes assumptions yields:

$$E[R_C] = R_F + \Omega \cdot (E[R_S] - R_F) \tag{0.5}$$

$\Omega > 1$ and typically takes very large values. Hence, expected call return is a lot smaller under analogy making when compared with the Black-Scholes predictions (including extensions to Black-Scholes such as jump diffusion and stochastic volatility).

The argument that risks are commonly misperceived in various contexts is frequently made in the literature. See Rotheli (2010), Bhattacharya, Dana, & Sood (2009), Akerlof and Shiller (2009), and Barberis & Thaler (2002) among many others. This article explores the implications of using the underlying stock's risk as a reference point to arrive at call option risk. It leads to a number of interesting findings: 1) a new explanation for the implied volatility skew puzzle, 2) an explanation for the superior historical performance of covered call writing, 3) an explanation for the worse-than-expected historical performance of zero-beta-straddles, and 4) an explanation for recent empirical findings regarding leverage adjusted index option returns. Furthermore, the model predicts that average put returns (for options held to expiry) are more negative than what the Black Scholes model predicts. This is quite intriguing given the empirical findings that average put returns (for options held to expiry) are typically more negative than what the popular option pricing models suggest. See Chambers, Foy, Liebner, & Lu (2014) and references therein. Also, with analogy making, average call returns are significantly smaller than Black Scholes prediction. Hence, the model also provides a potential explanation for the empirical findings in Coval and Shumway (2001) that call returns are a lot smaller than what they should be given their systematic risk.

In a laboratory experiment, it is possible to objectively fix the expected return available on the underlying stock and make it common knowledge, however, in the real world; people are likely to have different subjective assessments of the expected return on the underlying stock. The marginal investor in a call option is perhaps more optimistic than the marginal investor in the corresponding underlying stock. To see this, consider the following: In the market for the underlying stock, both the optimistic and pessimistic beliefs influence the belief of the marginal investor. Optimistic investors influence through demand pressure, whereas the pessimistic investors constitute the suppliers who influence through selling and short-selling. However, highly optimistic investors should favor a call option over its underlying stock due to the leverage embedded in the option. Furthermore, in the market for a call option, covered call writers are typical suppliers (see Lakonishok et al (2007)). Covered call writers are neutral to moderately bullish (and not pessimistic) on the underlying stock. Hence, due to the presence of relatively more optimistic buyers and sellers, the marginal investor in a call option is likely to be more optimistic about the underlying stock than the marginal investor in the underlying stock itself. Hence, with analogy making, $E[R_s]$ in (0.4) is likely to be larger than what the marginal investor in the underlying stock expects. Also, as more optimistic buyers are likely to self-select into higher strike calls, $E[R_s]$ is expected to rise with the striking price.

If analogy makers influence call prices, shouldn't a rational arbitrageur make money at their expense by taking an appropriate position in the call option and the corresponding replicating portfolio in accordance with the Black Scholes model? Such arbitraging is difficult if not impossible in the presence of transaction costs. Barberis and Thaler (2002) argue that the absence of "free lunch" does not imply that prices are right due to various limits to arbitrage including transaction costs. In our context, the presence of transaction costs is likely to eliminate any "free lunch" at the expense of analogy makers. It is worth mentioning that bid-ask spreads in ATM index options are typically of the order of 3 to 5% of the option price, and spreads are typically 10% of the option price for deep OTM index options.

In continuous time, no matter how small the transaction costs are, the total transaction cost of successful replication grows without bound rendering the Black-Scholes argument toothless. It is well known that there is no non-trivial portfolio that replicates a call option in the presence of transaction costs in continuous time. See Soner, Shreve, and Cvitanic (1995). In discrete time, transaction costs are bounded, however, a no-arbitrage interval is created. If analogy price lies within the interval, analogy makers cannot be arbitraged away. We show the conditions under which this happens in a binomial setting.

Analogy making is complementary to the approaches developed earlier such as stochastic volatility and jump diffusion models. Such models specify certain dynamics for the underlying stock. The idea of analogy making is not wedded to a particular set of assumptions regarding the price and volatility processes of the underlying stock. It can be applied to a wide variety of settings. In this article, first we use the setting of a geometric Brownian motion. Then, we integrate analogy making with jump diffusion and stochastic volatility approaches. Combining analogy making and stochastic volatility leads to the skew even when there is zero correlation between the stock price and volatility processes, and combining analogy making with jump diffusion generates the skew without the need for asymmetric jumps.

This article is organized as follows. Section 1 summarizes existing evidence pointing to analogy based call option pricing. Section 2 builds intuition by providing a numerical illustration of call option pricing with analogy making. Section 3 develops the idea in the context of a one period binomial model. Section 4 puts forward the analogy based option pricing formulas in continuous time. Section 5 shows that if analogy making determines option prices, and the Black-Scholes model is used to back-out implied volatility, the skew arises, which flattens as time to expiry increases. Section 6 shows that the analogy model is consistent with key empirical findings regarding returns from covered call writing and zero-beta straddles. Section 7 shows that the analogy model is consistent with empirical findings regarding leverage adjusted option returns. Section 8 puts forward an analogy based option pricing model when the underlying stock returns exhibit stochastic volatility. It integrates analogy making with the stochastic volatility model developed in Hull and White (1987). Section 9 integrates analogy making with the jump diffusion approach of Merton (1976). Section 10 concludes.

1. The Relevance of Analogy Making for Option Pricing

A call option is commonly considered a surrogate for the underlying stock by investment professionals with decades of experience (see footnote 2 for five examples). A popular strategy advocated by market professionals is called the *stock replacement strategy*. In this strategy, underlying stocks are replaced by the corresponding call options in investment portfolios. The argument underlying this strategy is based on the similarity between call and underlying stock's payoffs, and the fact that a call option is a lot cheaper than buying the underlying stock outright. A careful reading of the investment advice coming from proponents of the *stock replacement strategy*[2] suggests the following: 1) Risk of buying a call option is frequently perceived not to substantially larger than the risk of the underlying stock. Often, it is perceived to be smaller than the risk of the underlying stock because of smaller cash outlay. 2) Replacing stocks with call options is recommended as long as one expects at least the same return from the call as from the underlying stock.

Not only investment professionals with decades of experience consider a call option to be a surrogate for the underlying stock, participants in a series of laboratory experiments seem to think so too. Rockenbach (2004), Siddiqi (2012), and Siddiqi (2011) find that participants in laboratory experiments apparently perceive the risk of call as similar to the risk of the underlying stock and expect similar returns from the two assets.

Apart from opinions of professionals and experimental evidence, there is also related empirical evidence suggesting that the expected return on the underlying stock matters for pricing options. Duan and Wei (2009) find that a variable closely related to the expected return on the underlying stock, its systematic risk proportion, is priced in individual equity options.

Coval and Shumway (2001) find that the magnitudes of call option returns are too low given their systematic risk in the Black-Scholes/CAPM framework. Perhaps, this is due to analogy making as valuing a call option in analogy with its underlying stock significantly lowers the call option expected return, when compared with the expected return in the Black-Scholes/CAPM framework.

[2] As illustrative examples of this advice generated by investment professionals, see the following:
http://finance.yahoo.com/news/stock-replacement-strategy-reduce-risk-142949569.html
http://ezinearticles.com/?Call-Options-As-an-Alternative-to-Buying-the-Underlying-Security&id=4274772,
http://www.investingblog.org/archives/194/deep-in-the-money-options/
http://www.triplescreenmethod.com/TradersCorner/TC052705.asp,
http://daytrading.about.com/od/stocks/a/OptionsInvest.htm

If analogy making influences the value of call options, what are the implications for put options? A call option can be converted into a put option by combining it with the short underlying stock plus a long position in the risk free bond (in accordance with put-call parity). In this article, the analogy based pricing formula for put options is deduced from analogy based call pricing formula by using put-call parity. Even though analogy is only explicitly made between a call option and its underlying stock, there are strong implications for put option pricing due to the model-free restriction of put-call parity. Empirically observed put option returns (for options held to expiry) are a lot more negative than what is expected from popular option pricing models. This fact is known in the literature as put option pricing puzzle and is well documented. See Chambers et al (2014) and references therein. Under analogy making, put option returns are significantly more negative than what is expected from the Black Scholes model. Hence, analogy making provides a potential explanation for the empirical findings.

It has been argued in the cognitive science and psychology literature that when faced with a new situation, people instinctively search their memories for something similar they have seen before, and mentally co-categorize the new situation with the similar situations encountered earlier. This way of thinking, termed analogy making, is considered the core of cognition and the fuel and fire of thinking by prominent cognitive scientists and psychologists (see Hofstadter and Sander (2013)). Hofstadter and Sander (2013) write, *"[…] at every moment of our lives, our concepts are selectively triggered by analogies that our brain makes without letup, in an effort to make sense of the new and unknown in terms of the old and known." (Hofstadter and Sander (2013), Prologue page1).*

As a call option's payoffs are directly linked to the underlying stock's payoffs, it is reasonable to think that a call option is mentally co-categorized with the underlying stock and their risks are considered similar. As mentioned earlier, this is what experienced traders as well as subjects in laboratory experiments seem to think. This article carefully explores the implications of such analogy making, and shows that analogy making provides a new explanation for the implied volatility skew puzzle. The analogy model also provides new explanations for the puzzling historical profitability of covered call strategy, and worse-than-expected performance of zero-beta straddles. Furthermore, the model is consistent with recent empirical findings regarding leverage adjusted option returns.

2. Analogy Making: A Numerical Illustration

Consider an investor in a two state-two asset complete market world with one time period marked by two points in time: 0 and 1. The two assets are a stock (S) and a risk-free zero coupon bond (B). The stock has a price of $140 today (time 0). Tomorrow (time 1), the stock price could either go up to $200 (the red state) or go down to $94 (the blue state). Each state has a 50% chance of occurring. There is a riskless bond (zero coupon) that has a price of $100 today. Its price stays at $100 at time 1 implying a risk free rate of zero. Suppose a new asset "A" is introduced to him. The asset "A" pays $100 in cash in the red state and nothing in the blue state. How much should the investor be willing to pay for this new asset?

Finance theory provides an answer by appealing to the principle of no-arbitrage: *assets with identical state-wise payoffs must have the same price* or equivalently *assets with identical state-wise payoffs must have the same state-wise returns.* Consider a portfolio consisting of a long position in 0.943396 of S and a short position in 0.886792 of B. In the red state, 0.943396 of S pays $188.6792 and one has to pay $88.6792 due to shorting of 0.886792 of B earlier resulting in a net payoff of $100. In the blue state, 0.943396 of S pays $88.6792 and one has to pay $88.6792 on account of shorting 0.886792 of B previously resulting in a net payoff of 0. That is, payoffs from 0.943396S-0.886792B are identical to payoffs from "A". As the cost of 0.943396S-0.886792B is $43.39623, it follows that the no-arbitrage price for "A" is $43.39623. Note, that the expected return from asset "A" is 15.22%. Asset "A" is equivalent to a call option on "S" with a striking price of $100. The expected return from "S" is 5%. From (0.5), one can infer that $\Omega = 3.044$. That is, the call option is slightly over 3 times as risky as the underlying stock.

An analogy maker uses *the risk of the underlying stock as a reference point for forming risk judgments about call options*. This leads to a call option being priced in accordance with (0.4). As the expected stock return is 5%, $|\epsilon|$ has to be slightly greater than 10%, if an analogy maker is to arrive at the correct expected return. More likely, due to the anchoring heuristic, $|\epsilon|$ is likely to be a lot smaller. We consider three cases: 1) $|\epsilon| = 0$, 2) $|\epsilon| = 2.5\%$, and 3) $|\epsilon| = 5\%$.

With $|\epsilon| = 0$, the analogy price of the call option is $47.6190. That creates a gap of $4.22277 between the no-arbitrage price and the analogy price. Rational investors should short "A" and buy "0.943396S-0.886792B". However, transaction costs are ignored in the example so far.

Let's see what happens when a symmetric proportional transaction cost of only 1% of the price is applied when assets are traded. That is, both a buyer and a seller pay a transaction cost of 1% of the price of the asset traded. Unsurprisingly, the composition of the replicating portfolio changes. To successfully replicate a long call option that pays $100 in cash in the red state and 0 in the blue state with transaction cost of 1%, one needs to buy 0.952925 of S and short 0.878012 of B. In the red state, 0.952925S yields $188.6792 net of transaction cost $(200 \times 0.952925 \times (1 - 0.01))$, and one has to pay $88.6792 to cover the short position in B created earlier $(0.878012 \times 100 \times (1 + 0.01))$. Hence, the net cash generated by liquidating the replicating portfolio at time 1 is $100 in the red state. In the blue state, the net cash from liquidating the replicating portfolio is 0. Hence, with a symmetric and proportional transaction cost of 1%, the replicating portfolio is "0.952925S-0.878012B". The cost of setting up this replicating portfolio inclusive of transaction costs at time 0 is $47.82044, which is larger than the price the analogy makers are willing to pay: $47.6190. Hence, arbitrage profits cannot be made at the expense of analogy makers by writing a call and buying the replicating portfolio. The given scheme cannot generate arbitrage profits unless the call price is greater than $47.82044

Suppose one in interested in doing the opposite. That is, buy a call and short the replicating portfolio to fund the purchase. Continuing with the same example, the relevant replicating portfolio (that generates an outflow of $100 in the red state and 0 in the blue state) is "-0.934056S +0.89575B". The replicating portfolio generates $41.1928 at time 0, which leaves $38.98937 after time 0 transaction costs in setting up the portfolio are paid. Hence, in order for the scheme to make money, one needs to buy a call option at a price less than $38.98937.

Effectively, transaction costs create a no-arbitrage interval $(38.98937, 47.82044)$. As the analogy price lies within this interval, arbitrage profits cannot be made at the expense of analogy makers in the example considered. With $|\epsilon| = 2.5\%$, the analogy price is $46.51, and with $|\epsilon| = 5\%$, the analogy price is $45.45. Clearly, such analogy makers cannot be arbitraged away even at transaction costs that are much smaller than 1%.

2.1 Analogy Making: A Two Period Binomial Example with Delta Hedging

Consider a two period binomial model. The parameters are: Up factor=2, Down factor=0.5, Current stock price=$100, Risk free interest rate per binomial period=0, Strike price=$30, and the probability of up movement=0.5. It follows that the expected gross return from the stock per binomial period is 1.25 ($0.5 \times 2 + 0.5 \times 0.5$).

The call option can be priced both via analogy as well as via no-arbitrage argument. The no-arbitrage price is denoted by C_R whereas the analogy price is denoted by C_A. Define $x_R = \frac{\Delta C_R}{\Delta S}$ and $x_A = \frac{\Delta C_A}{\Delta S}$ where the differences are taken between the possible next period values that can be reached from a given node. For simplicity, $|\epsilon| = 0$ is assumed. One can easily re-do the calculations for higher values of $|\epsilon|$.

Figure 1 shows the binomial tree and the corresponding no-arbitrage and analogy prices. Two things should be noted. Firstly, in the binomial case considered, before expiry, the analogy price is always larger than the no-arbitrage price. Secondly, the delta hedging portfolios in the two cases $Sx_R - C_R$ and $Sx_A - C_A$ grow at different rates. The portfolio $Sx_A - C_A$ grows at the rate equal to the expected return on stock per binomial period (which is 1.25 in this case). In the analogy case, the value of delta-hedging portfolio when the stock price is 100 is 17.06667 $(100 \times 0.98667 - 81.6)$. In the next period, if the stock price goes up to 200, the value becomes 21.33333 $(200 \times 0.98667 - 176)$. If the stock price goes down to 50, the value also ends up being equal to 21.33333 $(50 \times 0.98667 - 28)$. That is, either way, the rate of growth is the same and is equal to 1.25 as $17.06667 \times 1.25 = 21.33333$. Similarly, if the delta hedging portfolio is constructed at any other node, the next period return remains equal to the expected return from stock. It is easy to verify that the portfolio $Sx_R - C_R$ grows at a different rate which is equal to the risk free rate per binomial period (which is 0 in this case).

Put option values can be calculated easily via put-call parity. It is easy to check that the expected return from a put option (held to expiry) in this case has a value of -89.224% under analogy making. The corresponding value under portfolio replication/Black-Scholes is -43.75%. In general, under analogy making, expected return from a put option (held to expiry) is significantly more negative then what is expected from the Black Scholes model.

In the next section, implications of analogy making are explored in a general binomial model, which helps in building intuition before discussing the continuous time case.

Exp. Ret	1.25
Up Prob.	0.5
Up	2
Down	0.5
Risk-Free r	0
Strike	30

Stock Price	400
$Call_R$	370
$Call_A$	370

Stock Price	200
x_R	1
B	-30
$Call_R$	170
x_A	1
$Call_A$	176

Stock Price	100
x_R	0.977778
B	-25.5556
$Call_R$	72.22222
x_A	0.986667
$Call_A$	81.6

Stock Price	100
$Call_R$	70
$Call_A$	70

Stock Price	50
x_R	0.933333
B	-23.3333
$Call_R$	23.33333
x_A	0.933333
$Call_A$	28

Stock Price	25
$Call_R$	0
$Call_A$	0

Figure 1

3. Analogy Making: The Binomial Case

Consider a two state world. The equally likely states are Red, and Blue. There is a stock with prices $X_1, and\ X_2$ corresponding to states Red, and Blue respectively, where $X_1 > X_2$. The state realization takes place at time T. The current time is time t. We denote the risk free discount rate by r. That is, there is a riskless zero coupon bond that has a price of B in both states with a price of $\frac{B}{1+r}$ today. For simplicity and without loss of generality, we assume that $r = 0$ and $T - t = 1$. The current price of the stock is S such that $X_1 > S > X_2$. We further assume that $S < \frac{X_1+X_2}{2}$. That is, the stock price reflects a positive risk premium. In other words, $S = f \cdot \frac{X_1+X_2}{2}$ where $f = \frac{1}{1+r+\delta}$.[3] δ is the risk premium reflected in the price of the stock.[4] As we have assumed $r = 0$, it follows that $f = \frac{1}{1+\delta}$.

Suppose a new asset which is a European call option on the stock is introduced. By definition, the payoffs from the call option in the two states are:

$$C_1 = max\{(X_1 - K), 0\}, C_2 = max\{(X_2 - K), 0\} \tag{3.1}$$

Where K is the striking price, and $C_1, and\ C_2,$ are the payoffs from the call option corresponding to Red, and Blue states respectively.

How much is an analogy maker willing to pay for this call option?

There are two cases in which the call option has a non-trivial price: 1) $X_1 > X_2 > K$ and 2) $X_1 > K > X_2$

The analogy maker infers the price of the call option, P_c , from (0.4):

$$\frac{\{C_1 - P_c\} + \{C_2 - P_c\}}{2 \times P_c} = \frac{\{X_1 - S\} + \{X_2 - S\}}{2 \times S} + \epsilon \tag{3.2}$$

For simplicity, $|\epsilon| = 0$ is assumed through the remainder of this section.

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[3] In general, a stock price can be expressed as a product of a discount factor and the expected payoff if it follows a binomial process in discrete time (as assumed here), or if it follows a geometric Brownian motion in continuous time.

[4] If the marginal call investor is more optimistic than the marginal stock investor, they would perceive different values of $X_1$and X_2 so that their assessment of δ is different accordingly.

For case 1 ($X_1 > X_2 > K$), one can write:

$$P_c = \frac{C_1 + C_2}{X_1 + X_2} \times S$$

$$=> P_c = \left(1 - \frac{2K}{X_1 + X_2}\right) S \tag{3.3}$$

Substituting $S = f \cdot \frac{X_1 + X_2}{2}$ in (3.3):

$$P_c = S - Kf \tag{3.4}$$

The above equation is the one period analogy option pricing formula for the binomial case when call remains in-the-money in both states.

The corresponding no-arbitrage price P_r is (from the principle of no-arbitrage):

$$P_r = S - K \tag{3.5}$$

For case 2 ($X_1 > K > X_2$), the analogy price is:

$$P_c = S \cdot \frac{X_1}{X_1 + X_2} - \frac{K}{2} \cdot f \tag{3.6}$$

And, the corresponding no-arbitrage price is:

$$P_r = \frac{X_1 - K}{X_1 - X_2}(S - X_2) \tag{3.7}$$

Proposition 1 ***The analogy price is larger than the corresponding no-arbitrage price if a positive risk premium is reflected in the price of the underlying stock and there are no transaction costs.***

Proof.

See Appendix A ■

Suppose there are transaction costs, denoted by "c", which are assumed to be symmetric and proportional. That is, if the stock price is S, a buyer pays $S(1+c)$ and a seller receives $S(1-c)$. Similar rule applies when the bond or the option is traded. That is, if the bond price is B, a buyer pays $B(1+c)$ and a seller receives $B(1-c)$. We further assume that the call option is cash settled. That is, there is no physical delivery.

Introduction of the transaction cost does not change the analogy price as the expected returns on call and on the underlying stock are proportionally reduced. However, the cost of replicating a call option changes. The total cost of successfully replicating a long position in the call option by buying the appropriate replicating portfolio and then liquidating it in the next period to get cash (as call is cash settled) is:

$$\left(\frac{X_1-K}{X_1-X_2}\right)\left\{\frac{S}{1-c}-\frac{X_2}{1+c}\right\}+c\left\{\frac{S}{1-c}+\frac{X_2}{1+c}\right\}\ if\ X_1>K>X_2 \tag{3.8}$$

$$\left\{\frac{S}{1-c}-\frac{K}{1+c}\right\}+c\left\{\frac{S}{1-c}+\frac{K}{1+c}\right\}\ if\ X_1>X_2>K \tag{3.9}$$

The corresponding inflow from shorting the appropriate replicating portfolio to fund the purchase of a call option is:

$$\left(\frac{X_1-K}{X_1-X_2}\right)\left\{\frac{S}{1+c}-\frac{X_2}{1-c}\right\}-c\left\{\frac{S}{1+c}+\frac{X_2}{1-c}\right\}\ if\ X_1>K>X_2 \tag{3.10}$$

$$\left\{\frac{S}{1+c}-\frac{K}{1-c}\right\}-c\left\{\frac{S}{1+c}+\frac{K}{1-c}\right\}\ if\ X_1>X_2>K \tag{3.11}$$

Proposition 2 shows that if transaction costs exist and the risk premium on the underlying stock is within a certain range, the analogy price lies within the no-arbitrage interval. Hence, riskless profit cannot be earned at the expense of analogy makers.

Proposition 2 ***In the presence of symmetric and proportional transaction costs, analogy makers cannot be arbitraged out of the market if the risk premium on the underlying stock satisfies:***

$$\mathbf{0 \le \delta \le \frac{(1-c)(1+c)}{(1-c)^2 - 2\frac{S}{K}c(1+c)} - 1} \qquad \boldsymbol{if\ X_1 > X_2} > K \tag{3.12}$$

$$\mathbf{0 \le \delta \le}$$

$$\mathbf{\frac{\frac{K\left(X_1^2 - X_2^2\right)(1-c^2)}{2}}{X_2(X_1-K)(X_1+X_2)(1-c)^2 - S\left\{(1+c)^2\left(X_1^2-X_2^2\right) - X_1(X_1-X_2)(1-c^2)\right\}} - 1}$$

$$\boldsymbol{if\ X_1} > K > \boldsymbol{X_2} \tag{3.13}$$

Proof.

See Appendix B

■

Intuitively, when transaction costs are introduced, there is no unique no-arbitrage price. Instead, a whole interval of no-arbitrage prices comes into existence. Proposition 2 shows that for reasonable parameter values, the analogy price lies within this no-arbitrage interval in a one period binomial model. As more binomial periods are added, the transaction costs increase further due to the need for additional re-balancing of the replicating portfolio. In the continuous limit, the total transaction cost is unbounded. Reasonably, arbitrageurs cannot make money at the expense of analogy makers in the presence of transaction costs ensuring that the analogy makers survive in the market.

It is interesting to consider the rate at which the delta-hedged portfolio grows under analogy making. Proposition 3 shows that under analogy making, the delta-hedged portfolio grows at a rate $\frac{1}{f} - 1 = r + \delta$. This is in contrast with the Black Scholes Merton/Binomial Model in which the growth rate is equal to the risk free rate, r.

Proposition 3 ***If analogy making determines the price of the call option, then the corresponding delta-hedged portfolio grows with time at the rate of*** $\frac{1}{f} - 1$.

Proof.

See Appendix C

■

Corollary 3.1 ***If there are multiple binomial periods then the growth rate of the delta-hedged portfolio per binomial period is*** $\frac{1}{f} - 1$.

The continuous limit of the discrete case discussed here leads to an option pricing formula, which is presented in the next section.

4. Analogy Making: The Continuous Case

We maintain all the assumptions of the Black-Scholes model except one. We assume that the risk of the underlying stock is used as a reference point for forming judgment about the risk of a call option defined on the stock. As discussed in the introduction, this leads to expected call option return given in (0.4). For a call option with strike K, over a small time interval, dt, one may write:

$$\frac{E[dC]}{C} = \frac{E[dS]}{S} + \epsilon_K \tag{4.1}$$

To maintain full generality, it is useful to consider a couple of points:

1) The marginal investor in a higher strike call is likely to be more optimistic about the return from the underlying stock than the marginal investor in lower strike calls. If the risk free rate is r and the risk premium on the underlying stock according to the marginal investor in call with strike K is δ_K. Then, $\frac{E[dS]}{S} = \mu_K = r + \delta_K$ with δ_K rising with strike.

2) To a given investor, a higher strike call is riskier than the lower strike call. That is, ϵ_K should be rising with strike.

Hence, (4.1) may be written as:

$$\frac{E[dC]}{C} = (r + \delta_K + \epsilon_K) \tag{4.2}$$

The underlying stock price follows geometric Brownian motion:

$$dS = \mu_K S dt + \sigma S dZ$$

Where dZ is the standard Guass-Weiner process.

From Ito's lemma:

$$E[dC] = \left(u_K S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2}\right) dt \tag{4.3}$$

Substituting (4.3) in (4.2) leads to:

$$(r + \delta_K + \epsilon_K)C = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}(r + \delta_K)S + \frac{\partial^2 C}{\partial S^2}\frac{\sigma^2 S^2}{2} \tag{4.4}$$

(4.4) describes the partial differential equation (PDE) that must be satisfied if analogy making determines call option prices.

To appreciate the difference between analogy making PDE and the Black-Scholes PDE, consider the expected return under the Black-Scholes approach, which is given in (0.5). Over a small time interval, dt, one may re-write (0.5) as:

$$\frac{E[dC]}{C} = r + \Omega \cdot (\mu_K - r) \tag{4.5}$$

Substituting (4.3) in (4.5) and realizing that $\Omega = \frac{S}{C}\frac{\partial C}{\partial S}$ leads to the following:

$$rC = \frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{\partial^2 C}{\partial S^2}\frac{\sigma^2 S^2}{2} \tag{4.6}$$

(4.6) is the Black-Scholes PDE.

Comparing (4.4) and (4.6), it becomes immediately obvious that in a risk neutral world, the two PDEs are equal. After all, if expected returns from all assets are equal to the risk free rate then both δ_K and ϵ_K are zero. Risk aversion creates a wedge between the analogy price and the Black-Scholes price.

There is a closed form solution to the analogy PDE. Proposition 4 puts forward the resulting European option pricing formulas.

Proposition 4 ***The formula for the price of a European call is obtained by solving the analogy based PDE. The formula is*** $\boldsymbol{C = e^{-\epsilon_K(T-t)}\left\{SN(d_1) - Ke^{-(r+\delta_K)}N(d_2)\right\}}$ ***where***

$$\boldsymbol{d_1 = \frac{ln(S/K)+(r+\delta_K+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} \text{ ***and*** } \boldsymbol{d_2 = \frac{ln\left(\frac{S}{K}\right)+\left(r+\delta_K-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}$$

Proof.

See Appendix D.

■

Corollary 4.1 ***The formula for the analogy based price of a European put option is***

$$\boldsymbol{Ke^{-r(T-t)}\left\{1 - e^{-\delta_K(T-t)}N(d_2)e^{-\epsilon_K(T-t)}\right\} - S\left(1 - e^{-\epsilon_K(T-t)}N(d_1)\right)}$$

Proof. Follows from put-call parity. ■

As proposition 4 shows, the analogy formula differs from the corresponding Black-Scholes formulas due to the appearance of δ_K and ϵ_K .

It is interesting to analyze put option returns under analogy making. Proposition 5 shows that put option returns are more negative under analogy making when compared with put option returns in the Black Scholes model.

Proposition 5 ***Expected put option returns (for options held to expiry) under analogy making are more negative than expected put option returns in the Black Scholes model as long as the underlying stock has a positive risk premium (and*** $\boldsymbol{\epsilon_K}$ ***does not take unreasonably large values).***

Proof.

Follows directly from realizing that if the risk premium on the underlying is positive and ϵ_K is not very large, the price of a put option under analogy making is larger than the price of a put option in the Black Scholes model.

■

Proposition 5 is quite intriguing given the puzzling nature of empirical put option returns when compared with the predictions of popular option pricing models. Chambers et al (2014) analyze nearly 25 years of put option data and conclude that average put returns are, in general, significantly more negative than the predictions of Black Scholes, Heston stochastic volatility, and Bates SVJ model. Hence, analogy making offers a potential explanation.

Clearly, expected call return under analogy making is a lot less than what is expected under the Black Scholes model. Empirical call returns are found to be a lot smaller given the predictions of the Black Scholes model (see Coval and Shumway (2001)). Hence, analogy making is qualitatively consistent with the empirical findings regarding both call and put option returns.

5. The Implied Volatility Skew

If analogy making determines option prices (formulas in proposition 4), and the Black Scholes model is used to infer implied volatility, the skew is observed. Table 1 shows two examples of this. In the illustration titled "IV-Homogeneous Expectation", the perceived risk premium on the underlying stock does not vary with the striking price. The other parameters are: $r = 2\%,\ \sigma = 20\%, T - t = 30\ days, S = 100, and\ \epsilon_K = 0$. In the illustration titled "IV-Heterogeneous Expectations", the risk premium on the underlying stock is varied by 40 basis points for every 0.01 change in moneyness. That is, for a change of $5 in strike, the risk premium increases by 200 basis points. This captures the possibility that more optimistic investors self-select into higher strike calls. Other parameters are kept the same. Qualitatively similar results follow if $\epsilon_K > 0$ is assumed.

Table 1

The Implied Volatility Skew

IV-Heterogeneous Expectations						**IV-Homogeneous Expectations**		
K/S	Risk Premium	Black Scholes	Analogy Price	Implied Vol.	Implied Vol. – Historical Vol.	Risk Premium	Implied Vol.	Implied Vol. – Historical Vol.
0.9	10%	10.21	10.93	36.34%	16.34%	10%	36.34%	16.34%
0.95	12%	5.69	6.47	29.33%	9.33%	10%	27.87%	7.87%
1.0	14%	2.37	2.985	25.4%	5.4%	10%	23.78%	3.78%
1.1	18%	0.129	0.231	22.74%	2.74%	10%	21.46%	1.46%

As Table 1 shows, the implied volatility skew can be observed with both homogeneous and heterogeneous expectations. It also shows that the difference between implied volatility and realized volatility is higher with heterogeneous expectations. It is easy to see that higher the dispersion in beliefs, greater is the difference between implied and realized volatilities (as long as more optimistic investors self-select into higher strike calls). Figure 2 is a graphical illustration of Table 1.

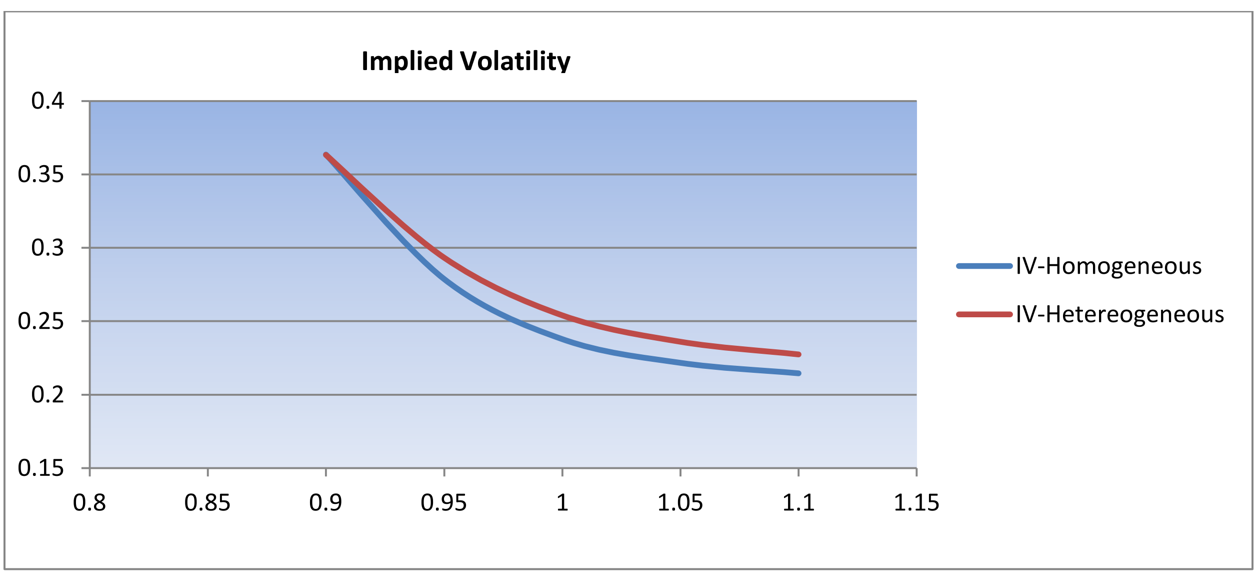

Figure 2

It is easy to illustrate that, with analogy making, the implied volatility skew gets flatter as time to expiry increases. As an example, with underlying stock price=$100, volatility=20%, risk premium on the underlying stock=5%, $\epsilon_K = 0$, and the risk free rate of 0, the flattening with expiry can be seen in Figure 3. Hence, the implications of analogy making are consistent with key observed features of the structure of implied volatility skew.

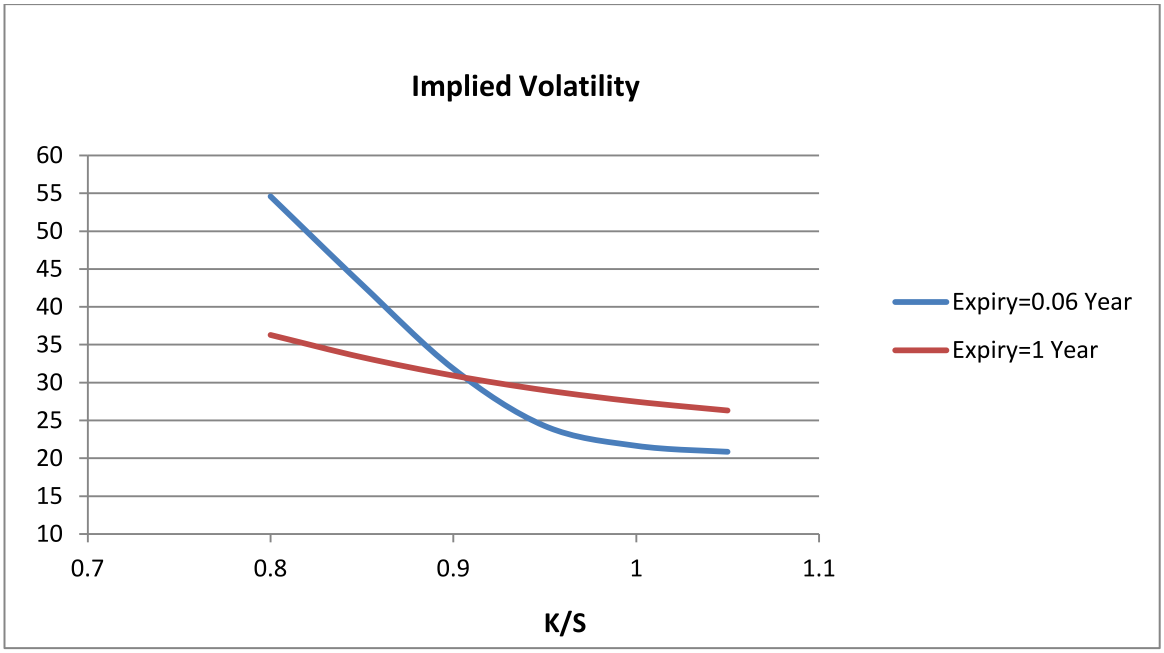

Figure 3

As an illustration of the fact that implied volatility curve flattens with expiry, Figure 4 is a reproduction of a chart from Fouque, Papanicolaou, Sircar, and Solna (2004) (Figure 2 from their paper). It plots implied volatilities from options with at least two days and at most three months to expiry. The flattening is clearly seen.

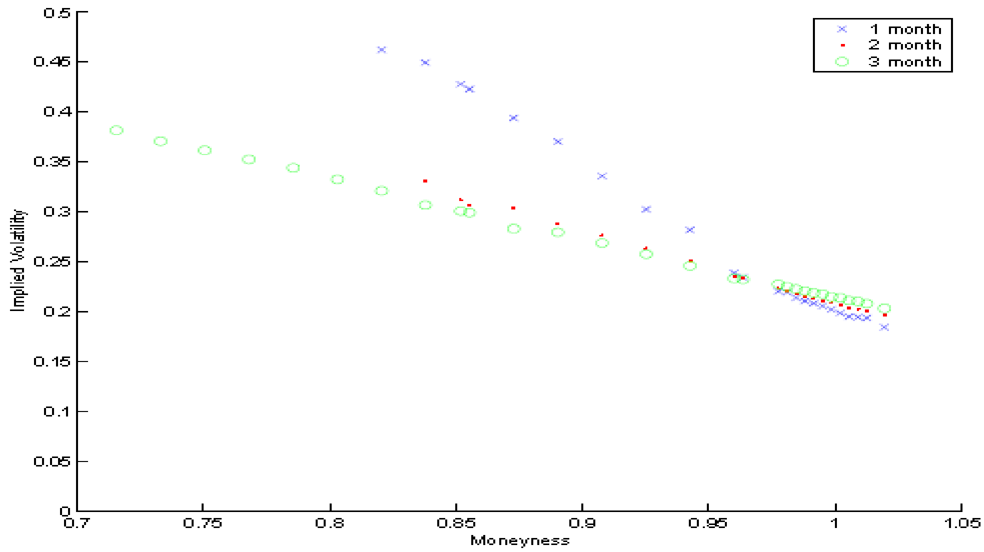

Figure 4 *Implied volatility as a function of moneyness on January 12, 2000, for options with at least two days and at most three months to expiry.*

So far, we have only considered analogy making as the sole mechanism generating the skew. Stochastic volatility and jump diffusion are other popular methods that give rise to the skew. In sections 8 and 9, we show that analogy making is complementary to stochastic volatility and jump diffusion models by integrating analogy making with the models of Hull and White (1987) and Merton (1976) respectively. In the next section, the profitability of covered call writing and zero-beta straddles are examined in the analogy model.

6. The Profitability of Covered Call Writing with Analogy Making

The profitability of covered call writing is quite puzzling in the Black Scholes framework. Whaley (2002) shows that BXM (a Buy Write Monthly Index tracking a Covered Call on S&P 500) has significantly lower volatility when compared with the index, however, it offers nearly the same return as the index. In the Black Scholes framework, the covered call strategy is expected to have lower risk as well as lower return when compared with buying the index only. See Black (1975). In fact, in an efficient market, the risk adjusted return from covered call writing should be no different than the risk adjusted return from just holding the index.

The covered call strategy (S denotes stock, C denotes call) is given by:

$$V = S - C$$

With analogy making, this is equal to:

$$V = S - e^{-\epsilon_K(T-t)}\{SN(d_1^A) - Ke^{-(r+\delta_K)(T-t)}N(d_2^A)\}$$

$$=> V = \left(1 - e^{-\epsilon_K(T-t)}N(d_1^A)\right)S + e^{-\epsilon_K(T-t)}N(d_2^A)Ke^{-(r+\delta_K)(T-t)} \tag{6.1}$$

The corresponding value under the Black Scholes assumptions is:

$$V = \left(1 - N(d_1)\right)S + N(d_2)Ke^{-r(T-t)} \tag{6.2}$$

A comparison of 6.1 and 6.2 shows that covered call strategy is expected to perform much better with analogy making when compared with its expected performance in the Black Scholes world. With analogy making, covered call strategy creates a portfolio which is equivalent to having a portfolio with a weight of $1 - e^{-\epsilon_K(T-t)}N(d_1^A)$ on the stock and a weight of $N(d_2^A)$ on a hypothetical risk free asset with a return of $r + \delta_K + \epsilon_K$. The stock has a return of $r + \delta$ plus dividend yield. This implies that, with analogy making, the return from covered call strategy is expected to be comparable to the return from holding the underlying stock only. The presence of a hypothetical risk free asset in 6.1 implies that the standard deviation of covered call returns is lower than the standard deviation from just holding the underlying stock. Hence, the superior historical performance of covered call strategy is consistent with analogy making.

6.1 The Zero-Beta Straddle Performance with Analogy Making

Another empirical puzzle in the Black-Scholes/CAPM framework is that zero beta straddles lose money. Goltz and Lai (2009), Coval and Shumway (2001) and others find that zero beta straddles earn negative returns on average. This is in sharp contrast with the Black-Scholes/CAPM prediction which says that the zero-beta straddles should earn the risk free rate. A zero-beta straddle is constructed by taking a long position in corresponding call and put options with weights chosen so as to make the portfolio beta equal to zero:

$$\theta \cdot \beta_{Call} + (1-\theta) \cdot \beta_{Put} = 0$$

$$=> \theta = \frac{-\beta_{Put}}{\beta_{Call} - \beta_{Put}}$$

Where $\beta_{Call} = N(d_1) \cdot \frac{Stock}{Call} \cdot \beta_{Stock}$ and $\beta_{Put} = (N(d_1) - 1) \cdot \frac{Stock}{Put} \cdot \beta_{stock}$

It is straightforward to show that with analogy making, where call and put prices are determined in accordance with proposition 4, the zero-beta straddle earns a significantly smaller return than the risk free rate (with returns being negative for a wide range of realistic parameter values). See Appendix G for proof. Intuitively, with analogy making, both call and put options are more expensive when compared with Black-Scholes prices. Hence, the returns are smaller.

Analogy based option pricing not only generates the implied volatility skew, it is also consistent with key empirical findings regarding option portfolio returns such as covered call writing and zero-beta straddles.

7. Leverage Adjusted Option Returns with Analogy Making

Leverage adjustment dilutes beta risk of an option by combining it with a risk free asset. Leverage adjustment combines each option with a risk-free asset in such a manner that the overall beta risk becomes equal to the beta risk of the underlying stock. The weight of the option in the portfolio is equal to its inverse price elasticity w.r.t the underlying stock's price:

$$\beta_{portfolio} = \Omega^{-1} \times \beta_{call} + (1-\Omega^{-1}) \times \beta_{riskfree}$$

where $\Omega = \frac{\partial Call}{\partial Stock} \times \frac{Stock}{Call}$ (i.e price elasticity of call w.r.t the underlying stock)

$$\beta_{call} = \Omega \times \beta_{stock}$$

$$\beta_{riskfree} = 0$$

$$=> \beta_{portfolio} = \beta_{stock}$$

Constantinides, Jackwerth and Savov (2013) uncover a number of puzzling empirical facts regarding leverage adjusted index option returns. They find that over a period ranging from April

1986 to January 2012, the average percentage monthly returns of *leverage-adjusted index call and put options are decreasing in the ratio of strike to spot.* They also find that *leverage adjusted put returns are larger than the corresponding leverage adjusted call returns*. The empirical findings in Contantinides et al (2012) are inconsistent with the Black-Scholes/CAPM framework, which predicts that the leverage adjusted returns should be equal to the return from the underlying index. That is, they should not fall with strike, and the leverage adjusted put option returns should not be any different than the leverage adjusted call returns.

If analogy making determines call prices, then the behavior of leverage adjusted call and put returns should be a lot different than their predicted behavior under the Black-Scholes assumptions. For call options (suppressing subscripts for simplicity):

$$\Omega^{-1} \cdot \frac{1}{dt}\left[\frac{E[dC]}{C}\right] + (1 - \Omega^{-1})r \tag{7.1}$$

where $E[dC] = \left((r+\delta)S\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt$ (7.2)

According to the analogy based PDE:

$$(r + \delta + \epsilon)C = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}(r+\delta)S + \frac{\partial^2 C}{\partial S^2}\frac{\sigma^2 S^2}{2} \tag{7.3}$$

Substituting (7.3) and (7.2) in (7.1) and simplifying leads to:

$$Leverage\ Adjusted\ Call\ Return = \Omega^{-1}(\delta + \epsilon) + r \tag{7.4}$$

Ω rises very rapidly as the ratio of strike to spot increases. So, the leverage adjusted call option return must fall as the ratio of strike to spot increases. Hence, analogy based option pricing is consistent with empirical evidence. The corresponding leverage adjusted put option return with analogy based option pricing is:

$$Leverage\ Adjusted\ Put\ Return = r + \left\{\frac{\delta S-(\delta+\epsilon)C}{S\left(1-N(d_1)e^{-\epsilon(T-t)}\right)}\right\} \tag{7.5}$$

The term in brackets, $\left\{\frac{\delta S-(\delta+\epsilon)C}{S\left(1-N(d_1)e^{-\epsilon(T-t)}\right)}\right\}$, falls very rapidly as the ratio of strike to spot increases. Hence, the leverage adjusted put returns fall with the ratio of strike to spot. Furthermore, (7.5) is larger than (7.4) implying that leverage adjusted put returns are larger than corresponding call returns. Hence, empirical findings in Constantinides et al (2012) regarding leverage adjusted option returns are consistent with analogy based option pricing.

8. Analogy based Option Pricing with Stochastic Volatility

In this section, we put forward an analogy based option pricing model for the case when the underlying stock price and its instantaneous variance are assumed to obey the uncorrelated stochastic processes described in Hull and White (1987):

$$dS = \mu S dt + \sqrt{V} S dw$$

$$dV = \varphi V dt + \varepsilon V dz$$

$$E[dwdz] = 0$$

Where $V = \sigma^2$ (Instantaneous variance of stock's returns), and φ and ε are non-negative constants. dw and dz are standard Guass-Weiner processes that are uncorrelated. Time subscripts in S and V are suppressed for notational simplicity. If $\varepsilon = 0$, then the instantaneous variance is a constant, and we are back in the Black-Scholes world. Bigger the value of ε, which can be interpreted as the volatility of volatility parameter, larger is the departure from the constant volatility assumption of the Black-Scholes model.

Hull and White (1987) is among the first option pricing models that allowed for stochastic volatility. A variety of stochastic volatility models have been proposed including Stein and Stein (1991), and Heston (1993) among others. Here, we use Hull and White (1987) assumptions to show that the idea of analogy making is easily combined with stochastic volatility. Clearly, with stochastic volatility it does not seem possible to form a hedge portfolio that eliminates risk completely because there is no asset which is perfectly correlated with $V = \sigma^2$.

If analogy making determines call prices and the underlying stock and its instantaneous volatility follow the stochastic processes described above, then the European call option price (no dividends on the underlying stock for simplicity) must satisfy the partial differentiation equation given below (see Appendix E for the derivation):

$$\frac{\partial C}{\partial t}+(r+\delta)S\frac{\partial C}{\partial S}+\varphi V\frac{\partial C}{\partial V}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+\frac{1}{2}\varepsilon^2 V^2\frac{\partial^2 C}{\partial V^2}=(r+\delta+\epsilon)C \tag{8.1}$$

Where δ is the risk premium that a marginal investor in the call option expects to get from the underlying stock.

By definition, under analogy making, the price of the call option is the expected terminal value of the option discounted at the rate which the marginal investor in the option expects to get from investing in the option. The price of the option is then:

$$C(S_t,\sigma_t^2,t)=e^{-(r+\delta+\epsilon)(T-t)}\int C(S_T,\sigma_T^2,T)p(S_T|S_t,\sigma_t^2)dS_T \tag{8.2}$$

Where the conditional distribution of S_T as perceived by the marginal investor is such that $E[S_T|S_t,\sigma_t^2]=S_t e^{(r+\delta)(T-t)}$ and $C(S_T,\sigma_T^2,T)$ is $max(S_T-K,0)$.

By defining $\bar{V}=\frac{1}{T-t}\int_t^T\sigma_\tau^2 d\tau$ as the means variance over the life of the option, the distribution of S_T can be expressed as:

$$p(S_T|S_t,\sigma_t^2)=\int f(S_T|S_t,\bar{V})\,g(\bar{V}|S_t,\sigma_t^2)d\bar{V} \tag{8.3}$$

Substituting (8.3) in (8.2) and re-arranging leads to:

$$C(S_t,\sigma_t^2,t)=\int\left[e^{-(r+\delta+\epsilon)(T-t)}\int C(S_T)f(S_T|S_t,\bar{V})dS_T\right]g(\bar{V}|S_t,\sigma_t^2)d\bar{V} \tag{8.4}$$

By using an argument that runs in parallel with the corresponding argument in Hull and White (1987), it is straightforward to show that the term inside the square brackets is the analogy making price of the call option with a constant variance $\bar{V}$. Denoting this price by $Call_{AM}(\bar{V})$, the price of the call option under analogy making when volatility is stochastic (as in Hull and White (1987)) is given by (proof available from author):

$$C(S_t, \sigma_t^2, t) = \int Call_{AM}(\bar{V}) g(\bar{V}|S_t, \sigma_t^2)\, d\bar{V} \tag{8.5}$$

Where $Call_{AM}(\bar{V}) = e^{-\epsilon(T-t)}\{SN(d_1^M) - Ke^{-(r+\delta)(T-t)}N(d_2^M)\}$

$$d_1^M = \frac{ln\left(\frac{S}{K}\right)+\left(r+\delta+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\ ;\ d_2^M = \frac{ln\left(\frac{S}{K}\right)+\left(r+\delta-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Equation (8.5) shows that the analogy based call option price with stochastic volatility is the analogy based price with constant variance integrated with respect to the distribution of mean volatility.

8.1 Option Pricing Implications

Stochastic volatility models require a strong correlation between the volatility process and the stock price process in order to generate the implied volatility skew. They can only generate a more symmetric U-shaped smile with zero correlation as assumed here. In contrast, the analogy making stochastic volatility model (equation 8.5) can generate a variety of skews and smiles even with zero correlation. What type of implied volatility structure is ultimately seen depends on the parameters δ and ε. It is easy to see that if $\varepsilon = 0$ and $\delta > 0$, only the implied volatility skew is generated, and if $\delta = 0$ and $\varepsilon > 0$, only a more symmetric smile arises. For positive δ, there is a threshold value of ε below which skew arises and above which smile takes shape. Typically, for options on individual stocks, the smile is seen, and for index options, the skew arises. The approach developed here provides a potential explanation for this as ε is likely to be lower for indices due to inbuilt diversification (giving rise to skew) when compared with individual stocks.

9. Analogy based Option Pricing with Jump Diffusion

In this section, we integrate the idea of analogy making with the jump diffusion model of Merton (1976). As before, the point is that the idea of analogy making is independent of the distributional assumptions that are made regarding the behavior of the underlying stock. In the previous section, analogy making is combined with the Hull and White stochastic volatility model to illustrate the same point.

Merton (1976) assumes that the stock returns are a mixture of geometric Brownian motion and Poisson-driven jumps:

$$dS = (\mu - \gamma\beta)Sdt + \sigma Sdz + dq$$

Where dz is a standard Guass-Weiner process, and $q(t)$ is a Poisson process. dz and dq are assumed to be independent. γ is the mean number of jump arrivals per unit time, $\beta = E[Y-1]$ where $Y-1$ is the random percentage change in the stock price if the Poisson event occurs, and E is the expectations operator over the random variable Y. If $\gamma = 0$ (hence, $dq = 0$) then the stock price dynamics are identical to those assumed in the Black Scholes model. For simplicity, assume that $E[Y] = 1$.

The stock price dynamics then become:

$$dS = \mu Sdt + \sigma Sdz + dq$$

Clearly, with jump diffusion, the Black-Scholes no-arbitrage technique cannot be employed as there is no portfolio of stock and options which is risk-free. However, with analogy making, the price of the option can be determined as the return on the call option demanded by the marginal investor is equal to the return he expects from the underlying stock plus an adjustment term.

If analogy making determines the price of the call option when the underlying stock price dynamics are a mixture of a geometric Brownian motion and a Poisson process as described earlier, then the following partial differential equation must be satisfied (see Appendix F for the derivation):

$$\frac{\partial C}{\partial t} + (r+\delta)S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \gamma E[C(SY,t) - C(S,t)] = (r+\delta+\epsilon)C \qquad (9.1)$$

If the distribution of Y is assumed to log-normal with a mean of 1 (assumed for simplicity) and a variance of v^2 then by using an argument analogous to Merton (1976), the following analogy based option pricing formula for the case of jump diffusion is easily derived (proof available from author):

$$Call = \sum_{j=0}^{\infty} \frac{e^{-\gamma(T-t)}\left(\gamma(T-t)\right)^j}{j!} Call_{AM}\left(S, (T-t), K, r, \delta, \sigma_j\right) \qquad (9.2)$$

$$Call_{AM}\big(S,(T-t),K,r,\delta,\sigma_j\big) = e^{-\epsilon(T-t)}\left\{SN(d_1^M) - Ke^{-(r+\delta)(T-t)}N(d_2^M)\right\}$$

$$d_1^M = \frac{ln\left(\frac{S}{K}\right) + \left(r+\delta+\frac{\sigma_j^2}{2}\right)(T-t)}{\sigma_j\sqrt{T-t}} \qquad d_2^M = \frac{ln\left(\frac{S}{K}\right) + \left(r+\delta-\frac{\sigma_j^2}{2}\right)(T-t)}{\sigma_j\sqrt{T-t}}$$

$\sigma_j = \sqrt{\sigma^2 + v^2\left(\frac{j}{T-t}\right)}$ and $v^2 = \frac{f\sigma^2}{\gamma}$

Where f is the fraction of volatility explained by jumps.

The formula in (9.2) is identical to the Merton jump diffusion formula except for one parameter, δ, which is the risk premium that a marginal investor in the call option expects from the underlying stock.

9.1 Option Pricing Implications

Merton's jump diffusion model with symmetric jumps around the current stock price can only produce a symmetric smile. Generating the implied volatility skew requires asymmetric jumps (jump mean becomes negative) in the model. However, with analogy making, both the skew and the smile can be generated even when jumps are symmetric. In particular, for low values of δ, a more symmetric smile is generated, and for larger values of δ, skew arises.

Even if we one assumes an asymmetric jump distribution around the current stock price, Merton formula, when calibrated with historical data, generates a skew which is a lot less pronounced (steep) than what is empirically observed. See Andersen and Andreasen (2002). The skew generated by the analogy formula (with asymmetric jumps) is typically more pronounced (steep) when compared with the skew without analogy making. Hence, analogy making potentially adds value to a jump diffusion model.

10. Conclusions

The observation that people tend to think by analogies and comparisons has important implications for option pricing that are thus far ignored in the literature. Prominent cognitive scientists argue that analogy making is the way human brain works (Hofstadter and Sander (2013)). There is strong experimental evidence that a call option is valued in analogy with the underlying stock (see Rockenbach (2004), Siddiqi (2012), and Siddiqi (2011)). A call option is commonly considered to be a surrogate for the underlying stock by experienced market professionals, which lends further support to the idea of analogy based option valuation. In this article, the notion that a call option is valued in analogy with the underlying stock (the underlying stock risk is used as a reference point for forming risk judgments regarding call options) is explored and the resulting option pricing model is put forward. The analogy option pricing model provides a new explanation for the implied volatility skew puzzle. The analogy based explanation complements the existing explanation as it is possible to integrate analogy making with stochastic volatility and jump diffusion approaches. The paper does that and puts forward analogy based option valuation models with stochastic volatility and jumps respectively. In contrast with other stochastic volatility and jump diffusion models in the literature, analogy making stochastic volatility model generates the skew even when there is zero correlation between the stock price and volatility processes, and analogy based jump diffusion can produce the skew even with symmetric jumps.

The analogy model differs minimally from the Black Scholes model due to the introduction of one additional variable (all the results derived in this paper follow even if $\epsilon = 0$ implying that they follow from non-zero δ). The additional variable captures the risk premium (subjective) that the marginal call option investor expects to get from the underlying stock. It is surprising that such a minimally different model can explain a wide variety of phenomena such as implied volatility skew, leverage adjusted option returns, superior performance of covered call writing, and worse-than-expected performance of zero-beta straddles.

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Appendix A

Proof of Proposition 1

For case 1, when $X_1 > X_2 > K$, the results follow from a direct comparison of (3.4) and (3.5).

For case 2, when $X_1 > K > X_2$, the spectrum of possibilities is further divided into three sub-classes and the results are proved for each sub-class one by one. The three sub-classes are: (i) $K = \frac{X_1+X_2}{2}$, (ii) $X_2 < K < \frac{X_1+X_2}{2}$, and (iii) $X_1 > K > \frac{X_1+X_2}{2}$.

Case 2 sub-class (i): $\boldsymbol{K = \frac{X_1+X_2}{2}}$

If we assume that $S \cdot \frac{X_1}{X_1+X_2} - \frac{K}{2} \cdot f \leq \frac{X_1-K}{X_1-X_2}(S - X_2)$, we arrive at a contradiction as follows:

Substitute $S = f \cdot \frac{X_1+X_2}{2}$ and $K = \frac{X_1+X_2}{2}$ above and simplify, it follows that $f \geq 1$, which is a contradiction as $f < 1$ if the risk premium is positive.

Case 2 sub-class (ii): $\boldsymbol{X_2 < K < \frac{X_1+X_2}{2}}$ or equivalently $\boldsymbol{K = g\frac{X_1+X_2}{2}}$ where $\boldsymbol{\frac{2X_2}{X_1+X_2} < g < 1}$

If we assume that $S \cdot \frac{X_1}{X_1+X_2} - \frac{K}{2} \cdot f \leq \frac{X_1-K}{X_1-X_2}(S - X_2)$, we arrive at a contradiction as follows:

Substitute $S = f \cdot \frac{X_1+X_2}{2}$ and $K = g\frac{X_1+X_2}{2}$ above and simplify, it follows that $X_1 \leq X_2$, which is a contradiction.

Case 2 sub-class (iii): $\boldsymbol{X_1 > K > \frac{X_1+X_2}{2}}$ or equivalently $\boldsymbol{K = g\frac{X_1+X_2}{2}}$ where $\boldsymbol{1 < g < \frac{2X_1}{X_1+X_2}}$

Similar logic as used in the case above leads to a contradiction: $X_1 \leq X_2$.

Hence, the analogy price must be larger than the no-arbitrage price if the risk premium is positive and there are no transaction costs.

Appendix B

Proof of Proposition 2

If $X_1 > X_2 > K$ then there is no-arbitrage if the following holds:

$$\left\{\frac{S}{1+c} - \frac{K}{1-c}\right\} - c\left\{\frac{S}{1+c} + \frac{K}{1-c}\right\} \leq S - Kf \leq \left\{\frac{S}{1-c} - \frac{K}{1+c}\right\} + c\left\{\frac{S}{1-c} + \frac{K}{1+c}\right\}$$

Realizing that $S - Kf \geq S - K > \left\{\frac{S}{1+c} - \frac{K}{1-c}\right\} - c\left\{\frac{S}{1+c} + \frac{K}{1-c}\right\}\ if\ \delta \geq 0$ and simplifying

$S - Kf \leq \left\{\frac{S}{1-c} - \frac{K}{1+c}\right\} + c\left\{\frac{S}{1-c} + \frac{K}{1+c}\right\}$ leads to inequality (3.12).

If $X_1 > K > X_2$ then there is no-arbitrage if the following holds:

$$\left(\frac{X_1 - K}{X_1 - X_2}\right)\left\{\frac{S}{1+c} - \frac{X_2}{1-c}\right\} - c\left\{\frac{S}{1+c} + \frac{X_2}{1-c}\right\} \leq S \cdot \frac{X_1}{X_1 + X_2} - \frac{K}{2} \cdot f$$
$$\leq \left(\frac{X_1 - K}{X_1 - X_2}\right)\left\{\frac{S}{1-c} - \frac{X_2}{1+c}\right\} + c\left\{\frac{S}{1-c} + \frac{X_2}{1+c}\right\}$$

Realizing that

$$\left(\frac{X_1 - K}{X_1 - X_2}\right)\left\{\frac{S}{1+c} - \frac{X_2}{1-c}\right\} - c\left\{\frac{S}{1+c} + \frac{X_2}{1-c}\right\} \leq$$

$$\frac{X_1 - K}{X_1 - X_2}(S - X_2) \leq S \cdot \frac{X_1}{X_1 + X_2} - \frac{K}{2} \cdot f\ \ if\ \delta \geq 0$$

And simplifying $S \cdot \frac{X_1}{X_1+X_2} - \frac{K}{2} \cdot f \ \leq \left(\frac{X_1-K}{X_1-X_2}\right)\left\{\frac{S}{1-c} - \frac{X_2}{1+c}\right\} + c\left\{\frac{S}{1-c} + \frac{X_2}{1+c}\right\}$ leads to (3.1).

Appendix C

Proof of Proposition 3

Case 1: $\boldsymbol{X_1 > X_2} > K$

Delta-hedged portfolio is $Sx - C$. In this case, $x = 1, S = f \cdot \frac{X_1+X_2}{2}$, and $C = S - Kf$

If the red state is realized, $S - C$ changes from Kf to K. If the blue state is realized $S - C$ also changes from Kf to K. Hence, the growth rate is equal to $\frac{1}{f} - 1$ in either state.

Case 2: $\boldsymbol{X_1} > K > \boldsymbol{X_2}$

Delta-hedged portfolio is $Sx - C$. In this case, $x = \frac{X_1-K}{X_1-X_2}, S = f \cdot \frac{X_1+X_2}{2}$, and

$$C = S \cdot \frac{X_1}{X_1 + X_2} - \frac{K}{2} \cdot f$$

Consider three sub-classes and prove the result for each: (i) $K = \frac{X_1+X_2}{2}$, (ii) $X_2 < K < \frac{X_1+X_2}{2}$, and (iii) $X_1 > K > \frac{X_1+X_2}{2}$. For the first sub-class the delta-hedged portfolio changes from the initial value of $f\frac{X_2}{2}$ to $\frac{X_2}{2}$ in both the red and the blue states. Hence, the growth rate is equal to $\frac{1}{f} - 1$ in either state. For the second and third sub-classes, the delta-hedged portfolio changes from $\frac{f\{(2-g)X_1X_2 - gX_2^2\}}{2(X_1-X_2)}$ to $\frac{\{(2-g)X_1X_2 - gX_2^2\}}{2(X_1-X_2)}$ in both red and blue states. Hence, the growth rate is equal to $\frac{1}{f} - 1$.

Appendix D

The analogy based PDE derived in Appendix D can be solved by converting to heat equation and exploiting its solution. The steps are identical to the derivation of the Black Scholes model with the risk free rate r, replaced with $r + \delta$, and an additional term due to ϵ.

Start by making the following transformations in (4.4):

$$\tau = \frac{\sigma^2}{2}(T - t)$$

$$x = ln\frac{S}{K} \Rightarrow S = Ke^x$$

$$C(S,t) = K \cdot \ c(x,\tau) = K \cdot c\left(ln\left(\frac{S}{K}\right), \frac{\sigma^2}{2}(T-t)\right)$$

It follows,

$$\frac{\partial C}{\partial t} = K \cdot \frac{\partial c}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = K \cdot \frac{\partial c}{\partial \tau} \cdot \left(-\frac{\sigma^2}{2}\right)$$

$$\frac{\partial C}{\partial S} = K \cdot \frac{\partial c}{\partial x} \cdot \frac{\partial x}{\partial S} = K \cdot \frac{\partial c}{\partial x} \cdot \frac{1}{S}$$

$$\frac{\partial^2 C}{\partial S^2} = K \cdot \frac{1}{S^2} \cdot \frac{\partial^2 C}{\partial x^2} - \ K \cdot \frac{1}{S^2}\frac{\partial C}{\partial x}$$

Plugging the above transformations into (4.4) and writing $\tilde{r} = \frac{2(r+\delta)}{\sigma^2}$, and $\tilde{\epsilon} = \frac{2\epsilon}{\sigma^2}$ we get:

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + (\tilde{r}-1)\frac{\partial c}{\partial x} - (\tilde{r}+\tilde{\epsilon})c \qquad (D1)$$

With the boundary condition/initial condition:

$$C(S,T) = max\{S-K,0\}\ becomes\ c(x,0) = max\{e^x - 1,0\}$$

To eliminate the last two terms in (D1), an additional transformation is made:

$$c(x,\tau) = e^{\alpha x + \beta \tau}u(x,\tau)$$

It follows,

$$\frac{\partial c}{\partial x} = \alpha e^{\alpha x + \beta \tau}u + e^{\alpha x + \beta \tau}\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 c}{\partial x^2} = \alpha^2 e^{\alpha x + \beta \tau}u + 2\alpha e^{\alpha x + \beta \tau}\frac{\partial u}{\partial x} + e^{\alpha x + \beta \tau}\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial c}{\partial \tau} = \beta e^{\alpha x + \beta \tau}u + e^{\alpha x + \beta \tau}\frac{\partial u}{\partial \tau}$$

Substituting the above transformations in (D1), we get:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (\alpha^2 + \alpha(\tilde{r} - 1) - (\tilde{r} + \tilde{\epsilon}) - \beta)u + \left(2\alpha + (\tilde{r} - 1)\right)\frac{\partial u}{\partial x} \qquad \text{(D2)}$$

Choose $\alpha = -\frac{(\tilde{r}-1)}{2}$ and $\beta = -\frac{(\tilde{r}+1)^2}{4} - (\tilde{\epsilon})$. (D2) simplifies to the Heat equation:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad (D3)$$

With the initial condition:

$$u(x_0, 0) = max\left\{\left(e^{(1-\alpha)x_0} - e^{-\alpha x_0}\right), 0\right\} = max\left\{\left(e^{\left(\frac{\tilde{r}+1}{2}\right)x_0} - e^{\left(\frac{\tilde{r}-1}{2}\right)x_0}\right), 0\right\}$$

The solution to the Heat equation in our case is:

$$u(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{4\tau}} u(x_0, 0) dx_0$$

Change variables: $= \frac{x_0 - x}{\sqrt{2\tau}}$, which means: $dz = \frac{dx_0}{\sqrt{2\tau}}$. Also, from the boundary condition, we know that $u > 0\ iff\ x_0 > 0$. Hence, we can restrict the integration range to $z > \frac{-x}{\sqrt{2\tau}}$

$$u(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\pi}}}^{\infty} e^{-\frac{z^2}{2}} \cdot e^{\left(\frac{\tilde{r}+1}{2}\right)(x+z\sqrt{2\tau})} dz - \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{z^2}{2}} \cdot e^{\left(\frac{\tilde{r}-1}{2}\right)(x+z\sqrt{2\tau})} dz$$

$$=: H_1 - H_2$$

Complete the squares for the exponent in H_1:

$$\frac{\tilde{r}+1}{2}\left(x + z\sqrt{2\tau}\right) - \frac{z^2}{2} = -\frac{1}{2}\left(z - \frac{\sqrt{2\tau}(\tilde{r}+1)}{2}\right)^2 + \frac{\tilde{r}+1}{2}x + \tau\frac{(\tilde{r}+1)^2}{4}$$

$$=: -\frac{1}{2}y^2 + c$$

We can see that $dy = dz$ and c does not depend on z. Hence, we can write:

$$H_1 = \frac{e^c}{\sqrt{2\pi}} \int_{-x/\sqrt{2\pi}-\sqrt{\tau/2}(\tilde{r}+1)}^{\infty} e^{-\frac{y^2}{2}} dy$$

A normally distributed random variable has the following cumulative distribution function:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{y^2}{2}} dy$$

Hence, $H_1 = e^c N(d_1)$ where $d_1 = {}^{x}/_{\sqrt{2\pi}} + \sqrt{\tau/2}\,(\tilde{r}+1)$

Similarly, $H_2 = e^f N(d_2)$ where $d_2 = {}^{x}/_{\sqrt{2\pi}} + \sqrt{\tau/2}\,(\tilde{r}-1)$ and $f = \frac{\tilde{r}-1}{2}x + \tau\frac{(\tilde{r}-1)^2}{4}$

The analogy based European currency call pricing formula is obtained by recovering original variables:

$$C = e^{-(\epsilon)(T-t)}\{SN(d_1) - Ke^{-(r+\delta)(T-t)}N(d_2)\}$$

Where $\boldsymbol{d_1} = \frac{\boldsymbol{ln(S/K)+(r+\delta+\frac{\sigma^2}{2})(T-t)}}{\boldsymbol{\sigma\sqrt{T-t}}}$ and $\boldsymbol{d_2} = \frac{\boldsymbol{ln\left(\frac{S}{K}\right)+\left(r+\delta-\frac{\sigma^2}{2}\right)(T-t)}}{\boldsymbol{\sigma\sqrt{T-t}}}$

Appendix E

By Ito's Lemma (time subscript is suppressed for simplicity):

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial V}dV + \frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2}dt + \frac{1}{2}V^2\varepsilon^2\frac{\partial^2 C}{\partial V^2}dt \tag{E1}$$

With analogy making:

$$\frac{E[dC]}{C} = r + \delta + \epsilon \tag{E2}$$

It follows,

$$\frac{\partial C}{\partial t} + (r+\delta)S\frac{\partial C}{\partial S} + \varphi V\frac{\partial C}{\partial V} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} + \frac{1}{2}\varepsilon^2 V^2\frac{\partial^2 C}{\partial V^2} = (r+\delta+\epsilon)C \tag{E3}$$

Appendix F

By following a very similar argument as in appendix E, and using Ito's lemma for the continuous part and an analogous Lemma for the discontinuous part, the following is obtained:

$$\frac{\partial C}{\partial t} + (r+\delta)S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \gamma E[C(SY,t) - C(S,t)] = (r+\delta+\epsilon)C$$

Appendix G

Following Coval and Shumway (2001) and some algebraic manipulations, the return from a zero-beta-straddle can be written as:

$$r_{straddle} = \frac{-\Omega_c C + S}{\Omega_c P - \Omega_c C + S} \cdot r_{call} + \frac{\Omega_c P + S}{\Omega_c P - \Omega_c C + S} \cdot r_{put}$$

Where C and P denote call and put prices respectively, r_{call} is expected call return and r_{put} is expected put return.

Under analogy making:

$$r_{call} = \mu + \epsilon$$

$$r_{put} = \frac{(\mu+\epsilon)C - \mu S + rPV(K)}{P}$$

Substituting r_{call} and r_{put} in the expression for $r_{straddle}$, and simplifying implies that as long as the risk premium on the underlying is positive and ϵ is not very large (not large enough to reach the correct risk judgment), it follows that:

$$r_{straddle} < r$$