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RATIONAL EXPECTATIONS EQUILIBRIA: EXISTENCE AND REPRESENTATION

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Abstract. In this paper, we continue to explore the equilibrium theory under ambiguity. For a model of a pure exchange and asymmetric information economy with a measure space of agents whose exogenous uncertainty is described by a complete probability space, we establish a representation theorem for a Bayesian or maximin rational expectations equilibrium allocation in terms of a state-wise Walrasian equilibrium allocation. This result also strengthens the theorems on the existence and representation of a (Bayesian) rational expectations equilibrium or a maximin rational expectations equilibrium in the literature.

1. Introduction

In modern economics, it has been desirable to introduce uncertainty to general equilibrium theory, because modeling the market with uncertainty is of importance for both academic significance and realistic decision making. Toward this direction, the Arrow-Debreu state contingent model in [11], which is an extension of the deterministic Arrow-Debreu-McKenzie model in [5, 19], allows the state of nature of the world to be involved in the initial endowments and payoff functions. For this model, the issue of incentive compatibility does not arise, as all the information available to agents is symmetric. However, for this model to make sense one must assume that there exists an exogenous organization that forces the contract ex post.

Radner [21] extended the analysis of Arrow and Debreu by introducing asymmetric information so that each agent is characterized by his own private information, a state-dependent initial endowment, a state-dependent utility function and a prior. The private information of an agent is modeled as a partition of the finite state space, the initial endowment and allocations of each agent are assumed to be measurable with respect to his own private information. This means that each agent only knows the atom of his partition including the true state, but cannot distinguish those states within the same atom when he makes decisions. Since each agent is a maximizer of his ex ante expected utility function with respect to his prior, the notion of a competitive equilibrium in this model is called a Walrasian expectations equilibrium (WEE). Along this line, Yannelis [26] proposed a core concept, called the private core. It was proved by Einy et al. in [14] that if we allow for free disposal in the market clearing (feasibility) constraints then an irreducible economy has a WEE and moreover, the set of competitive equilibrium allocations coincides with the private core. However, Angeloni and Martin-da-Rocha [4] pointed out

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that when feasibility is defined with free disposal, WEE allocations may not be incentive compatible and contracts may not be enforceable. To resolve these problems, it is desirable to consider a framework without free disposal. In this direction, Angeloni and Martin-da-Rocha [4] showed that results of Einy et al. in [14] are still valid without free-disposal provided that prices are allowed to take negative value. Moreover, they also proved that every Pareto optimal exact feasible allocation is incentive compatible, implying that contracts of competitive or core allocations are enforceable. However, whether Vind's theorem in [25] is still valid in this framework emerged as a question in [20]. Recently, this question was solved by Bhowmik and Cao in [9].

In Radner's model in [21], the information revealed to agents by market prices and possibility of agents' expectations of how equilibrium prices are related to initial information are not considered. In [22], Radner refined this model and introduced the concept of a rational expectations equilibrium by imposing on agents the Bayesian (subjective expected utility) decision doctrine. Under the Bayesian decision making, agents maximize their subjective expected utilities conditioned on their own private information and also on the information that the equilibrium prices generate. The resulting equilibrium allocations are measurable with respect to the private information of each individual and also with respect to the information the equilibrium prices generate and clear the market for every state of nature. Conditions on the existence of a Bayesian rational expectations equilibrium (REE) were studied in [2, 3, 22], where some generic existence results were proved. In this line, Einy et al. [13] studied the relationship between the set of REE allocations and the ex-post core of exchange economies with asymmetric information, and established a core-Walras equivalence theorem in terms of REE allocations and the ex-post core. However, Kreps [18] provided an example that shows a Bayesian REE may not exist universally. In addition, a Bayesian REE may fail to be fully Pareto optimal and incentive compatible and may not be implementable as a perfect Bayesian equilibrium of an extensive form game, refer to [15] for more details.

The lack of existence and misbehavior of a Bayesian REE make it not a desirable solution concept. Sun et al. [24] resolved those problems by providing a new model. More precisely, they considered an asymmetric information economy with a continuum of agents whose private signals are independent conditioned on the macro states of nature. For such an economy, they proved the existence, incentive compatibility and efficiency for their new REE concept. In a recent paper [12], de Castro et al. introduced another new notion of REE by a carefully examining Krep's example of the nonexistence of a Bayesian REE. In this formulation, the Bayesian decision making adopted in [2, 22] was abandoned and replaced by the maximin expected utility (MEU), and agents maximize their MEU conditioned on their own private information and also on the information the equilibrium prices have generated. Contrary to a Bayesian REE allocation, the resulting maximin REE allocation may not be measurable with respect to the private information of each individual or the information that the equilibrium prices generate. The existence of a maximin REE was established in [12] for an economy having finitely many agents and states of nature. Solving an open problem posed in [12], Bhowmik et al. [10] recently proved the existence of a maximin REE in an asymmetric information economy whose space of agents is a finite measure space and the space of states of nature is a complete probability measure space.

In this paper, we continue to investigate the concepts of a Bayesian and a maximin REE in pure exchange economies with asymmetric information. Our economic model is the same as that in [10], which has finitely many commodities, a measure space with a complete, finite and positive measure as the space of agents and a complete probability space as the space of states of nature. For such an economic model, we provide a representation for an assignment to be a Bayesian or maximin REE allocation in terms of a state-wise Walrasian equilibrium allocation. This result also strengthens the theorems on the existence and representation of a (Bayesian) REE or a maximin REE in [10], [12] and [13]. The rest of paper is organized as follows. In Section 2, we set up our model, introduce the concepts of a Bayesian REE and a maximin REE, and investigate basic relationships among them. In Section 3, we discuss major techniques that lead to the representation of a Bayesian or a maximin REE. The main result and its proof are also provided in this section. However, we leave the detailed proofs of three technical lemmas in Appendix B. In the last section, we summarize what we have achieved in this paper, compare our results with some results in the literature, and also point out some future research. For the reader's convenience, we provide some necessary mathematical preliminaries in Appendix A.

2. Bayesian and Maximin Relational Expectation Equilibrium

In this section, we study two concepts of a rational expectations equilibrium for a model of pure exchange economies with asymmetric information, namely a Bayesian rational expectations equilibrium and a maximin rational expectations equilibrium. In Subsection 2.1, we set up the economic model. The equilibrium concepts are introduced in Subsection 2.2. Furthermore, we also provide some basic results on relationships between these equilibrium concepts.

2.1. **The model setup.** In this subsection, we introduce a model of a pure exchange economy $\mathscr{E}$ with asymmetric information. The exogenous uncertainty is described by a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where Ω is the set of all possible states of nature, $\mathscr{F}$ is the σ-algebra denoting possible events, and $\mathbb{P}$ is a complete probability measure defined on $\mathscr{F}$. The space of agents is a measure space (T, Σ, μ) with a complete, finite and positive measure μ, where T is the set of agents, and Σ is the σ-algebra of measurable subsets of T denoting coalitions whose economic weights on the market are given by μ. The commodity space is the ℓ-dimensional Euclidean space $\mathbb{R}^\ell$. In each state, the *consumption set* for every agent $t \in T$ is $\mathbb{R}^\ell_+$. Each agent $t \in T$ is characterized by a quadruple $(\mathscr{F}_t, U(t,\cdot,\cdot), a(t,\cdot), \mathbb{P}_t)$, where

- $\mathscr{F}_t$ is the σ-algebra generated by a partition $\Pi_t \subseteq \mathscr{F}$ of Ω representing the *private information* of agent t. It is interpreted as follows: if $\omega \in \Omega$ is the state of nature that is going to be realized, agent t observes $\Pi_t(\omega)$, the unique element of Π_t that contains ω.

- $U(t,\cdot,\cdot) : \Omega \times \mathbb{R}^\ell_+ \to \mathbb{R}$ is the *state-dependent utility function* of agent t, representing his (ex post) preference.

- $a(t,\cdot) : \Omega \to \mathbb{R}^\ell_+$ is the *state-dependent initial endowment* of agent t, representing his physical resources.

- $\mathbb{P}_t$ is a probability measure on $\mathscr{F}$, representing the *prior belief* of agent t.

Thus, we can express the economy $\mathscr{E}$ as follows

$$\mathscr{E} = \left\{(\Omega, \mathscr{F}, \mathbb{P}); (T, \Sigma, \mu); \mathbb{R}^{\ell}_{+}; (\mathscr{F}_t, U(t,\cdot,\cdot), a(t,\cdot), \mathbb{P}_t)_{t\in T}\right\}.$$

The quadruple $(\mathscr{F}_t, U(t,\cdot,\cdot), a(t,\cdot), \mathbb{P}_t)$ is called the *characteristics of agent* t. Note that for each given $\omega \in \Omega$, there is always a deterministic economy $\mathscr{E}(\omega)$ associated with $\mathscr{E}$, which is given by

$$\mathscr{E}(\omega) = \left\{(T, \Sigma, \mu); \mathbb{R}^{\ell}_{+}; (U(t,\omega,\cdot), a(t,\omega))_{t\in T}\right\}.$$

In the deterministic Arrow-Debreu-McKenzie model, prices are vectors in $\mathbb{R}^{\ell}_{+} \setminus \{0\}$. Following standard treatment in the literature (e.g., [8]), price vectors are normalized so that their sum is 1. In this paper, we use the symbol Δ to denote the interior of the simplex of normalized price vectors, i.e.,

$$\Delta = \left\{p \in \mathbb{R}^{\ell}_{++} : \|p\|_1 = 1\right\}.$$

A *price system* of $\mathscr{E}$ is an $\mathscr{F}$-measurable function $\pi : \Omega \to \Delta$, where Δ is equipped with the Borel σ-algebra. Let $\sigma(\pi)$ be the smallest σ-algebra contained in $\mathscr{F}$ and generated by a price system π. Intuitively, $\sigma(\pi)$ represents the information revealed by π. The combination of agent t's private information $\mathscr{F}_t$ and the information revealed by the price system π is given by the smallest σ-algebra $\mathscr{G}_t$ that contains both $\mathscr{F}_t$ and $\sigma(\pi)$. Formally, $\mathscr{G}_t = \mathscr{F}_t \vee \sigma(\pi)$. For any $\omega \in \Omega$, let $\mathscr{G}_t(\omega)$ denote the smallest element of $\mathscr{G}_t$ that contains ω.

Following Einy et al. in [13], we call a function $f : T \times \Omega \to \mathbb{R}^{\ell}_{+}$ an *assignment* if for every $\omega \in \Omega$, the function $f(\cdot,\omega)$ is μ-integrable on T. Note that, distinguishing from the definition of an assignment in [13], here in our definition we do not require that for every $t \in T$, the function $f(t,\cdot)$ is $\mathscr{F}$-measurable. If an assignment f is also *feasible*, i.e., for all $\omega \in \Omega$,

$$\int_T f(\cdot,\omega)d\mu = \int_T a(\cdot,\omega)d\mu,$$

it is called an *allocation.* For an assignment (resp. allocation) $f : T \times \Omega \to \mathbb{R}^{\ell}_{+}$, the function $f_t = f(t,\cdot) : \Omega \to \mathbb{R}^{\ell}_{+}$ is called an *assignment* (resp. *allocation*) *of agent* t. Let L_t be the set of all assignments of agent t. Indeed, L_t is precisely the set of functions from Ω to $\mathbb{R}^{\ell}_{+}$. Define L^{REE}_t by

$$L^{REE}_t = \{x \in L_t : \ x \text{ is } \mathscr{G}_t\text{-measurable}\}.$$

We shall call an element x in L^{REE}_t a *rational assignment* of agent t.

As interpreted in [12], the economy $\mathscr{E}$ extends over three time periods: ext ante $(\tau = 0)$, intrim $(\tau = 1)$ and ex post $(\tau = 2)$. At $\tau = 0$, the state space, the partitions, the structure of the economy and the price functional $\pi : \Omega \to \Delta$ are common knowledge. This stage does not play any role in our analysis and it is assumed just for a matter of clarity. At $\tau = 1$, each individual learns his private information and the prevailing prices $p(\omega)$, and thus learns $\mathscr{G}_t$. With these in his mind, the agent plans how much he will consume $x(\omega)$. However, his actual consumption may be contingent to the final state of the nature, which is not yet known by him. The individual agent only knows that one of the state $\omega' \in \mathscr{G}_t(\omega)$ will be realized. Therefore, he needs to make sure that he will be able to pay his consumption plan $x(\omega')$ for all $\omega' \in \mathscr{G}_t(\omega)$. Based upon this interpretation, the budget set of each agent $t \in T$ is given by

$$B^{REE}(t,\omega,\pi) = \{x \in L_t : \langle \pi(\omega'), x(\omega')\rangle \leq \langle \pi(\omega'), a(t,\omega')\rangle \text{ for all } \omega' \in \mathscr{G}_t(\omega)\}$$

for any given state $\omega \in \Omega$ and price system $\pi : \Omega \to \Delta$.

Throughout the paper, the following standard assumptions will be used.

($\mathbf{A}_1$) For each $\omega \in \Omega$, $a(\cdot, \omega)$ is μ-integrable such that $\int_T a(\cdot, \omega) d\mu \gg 0$.

($\mathbf{A}_2$) a is $\Sigma \otimes \mathscr{F}$-measurable.

($\mathbf{A}_3$) For each $x \in \mathbb{R}^\ell_+$, $U(\cdot, \cdot, x)$ is $\Sigma \otimes \mathscr{F}$-measurable.

($\mathbf{A}_4$) for each $t \in T$ and each $x \in \mathbb{R}^\ell_+$, $U(t, \cdot, x)$ is $\mathscr{F}_t$-measurable.

($\mathbf{A}_5$) For each $t \in T$, $a(t, \cdot)$ is $\mathscr{F}_t$-measurable.

($\mathbf{A}_6$) For each $t \in T$ and each $\omega \in \Omega$, $U(t, \omega, \cdot)$ is continuous and increasing.

($\mathbf{A}_7$) For each $t \in T$ and each $\omega \in \Omega$, $U(t, \omega, \cdot)$ is strictly quasi concave.

Assumption ($\mathbf{A}_1$) has been commonly used in the literature, for instance, [8], [10] [13] and [14]. It asserts that no commodity is totally absent from the market. Assumptions ($\mathbf{A}_2$) and ($\mathbf{A}_3$) are similar to those in [10]. Assumptions ($\mathbf{A}_4$) and ($\mathbf{A}_5$) require that each agent knows, respectively, his initial endowment and utility function. Finally, ($\mathbf{A}_6$) and ($\mathbf{A}_7$) impose properties on the agents' utility functions. The last four assumptions were used in [12] and [13].

2.2. **Rational Expectations Equilibrium and Some Basic Facts.** In this subsection, we first introduce the concepts of a Bayesian rational expectations equilibrium and a maximin rational expectations equilibrium. Then, we provide some basic results on these concepts.

We start with the concept of Bayesian expected utility of an agent. Given a price system $\pi : \Omega \to \Delta$ of $\mathscr{E}$, the *Bayesian expected utility* of an agent $t \in T$ with respect to $\mathscr{G}_t$ at $x \in L_t^{REE}$ is defined by the conditional expectation $\mathbb{E}^{\mathbb{P}_t}\left[U(t, \cdot, x(\cdot)) | \mathscr{G}_t\right]$. When Ω is finite, $\sigma(\pi)$ is generated by a partition Π_π of Ω, and thus $\mathscr{G}_t$ is generated by the partition $\Pi_t \vee \Pi_\pi$. In this case, the value of $\mathbb{E}^{\mathbb{P}_t}\left[U(t, \cdot, x(\cdot)) | \mathscr{G}_t\right](\omega)$ in a state $\omega \in \Omega$ can be expressed as follows:

$$\mathbb{E}^{\mathbb{P}_t}\left[U(t, \cdot, x(\cdot)) | \mathscr{G}_t\right](\omega) = \sum_{\omega' \in \Pi_t \vee \Pi_\pi(\omega)} U(t, \omega', x(\omega')) \times \frac{\mathbb{P}_t(\omega')}{\mathbb{P}_t(\Pi_t \vee \Pi_\pi(\omega))},$$

where $\Pi_t \vee \Pi_\pi(\omega)$ is the unique member of $\Pi_t \vee \Pi_\pi$ containing ω. In general, when Ω is finite, we assume that for each $t \in T$ and each $\omega \in \Omega$, $\mathbb{P}_t(\omega) > 0$. The *budget correspondence* $B : T \times \Omega \times \Delta \rightrightarrows \mathbb{R}^\ell_+$ is defined by

$$B(t, \omega, p) = \left\{x \in \mathbb{R}^\ell_+ : \langle p, x \rangle \le \langle p, a(t, \omega) \rangle \right\}.$$

Obviously, B is a non-empty and closed-valued correspondence.

The following general definition, given in [10], is just an extension of the corresponding concept in [2] and [22] from the case when Ω is finite to the case when Ω may be infinite.

Definition 2.1 ([10])**.** Given an allocation $f : T \times \Omega \to \mathbb{R}^\ell_+$ and a price system $\pi : \Omega \to \Delta$ of $\mathscr{E}$, the pair (f, π) is called a *Bayesian rational expectations equilibrium* (abbreviated as *Bayesian REE*) of $\mathscr{E}$ if
(1) for each $t \in T$, $f(t, \cdot)$ is $\mathscr{G}_t$-measurable;

(2) for each $(t, \omega) \in T \times \Omega$, $f(t, \omega) \in B(t, \omega, \pi(\omega))$;
(3) for each $(t, \omega) \in T \times \Omega$,

$$\mathbb{E}^{\mathbb{P}_t}\left[U(t, \cdot, f(t, \cdot)) | \mathscr{G}_t\right](\omega) = \max_{x \in B^{REE}(t, \omega, \pi) \cap L_t^{REE}} \mathbb{E}^{\mathbb{P}_t}\left[U(t, \cdot, x(\cdot)) | \mathscr{G}_t\right](\omega).$$

In this case, f is called a *Bayesian REE allocation*, and the set of such allocations of $\mathscr{E}$ is denoted by $REE(\mathscr{E})$.

Next, we introduce the concepts of maximin utility and a maximin rational expectations equilibrium. Given a price system $\pi : \Omega \to \Delta$ of $\mathscr{E}$, the *maximin utility* of an agent $t \in T$ with respect to $\mathscr{G}_t$ at $x \in L_t$ in state $\omega \in \Omega$, denoted by $\underline{U}^{REE}(t, \omega, x)$, is defined by

$$\underline{U}^{REE}(t, \omega, x) = \inf\left\{U(t, \omega', x(\omega')) : \omega' \in \mathscr{G}_t(\omega)\right\}.$$

Comparing with $\underline{U}^{REE}(t, \cdot, \cdot)$, the function $U(t, \cdot, \cdot)$ is sometimes called the *ex post* utility of agent t.

Definition 2.2 ([10], [12])**.** Given an allocation $f : T \times \Omega \to \mathbb{R}_+^\ell$ and a price system $\pi : \Omega \to \Delta$ of $\mathscr{E}$, (f, π) is called a *maximin rational expectations equilibrium* (abbreviated as *maximin REE*) of $\mathscr{E}$ if
(1) $f(t, \omega) \in B(t, \omega, \pi(\omega))$;
(2) for each $(t, \omega) \in T \times \Omega$,

$$\underline{U}^{REE}(t, \omega, f(t, \cdot)) = \max_{x \in B^{REE}(t, \omega, \pi)} \underline{U}^{REE}(t, \omega, x),$$

i.e., $f(t, \cdot)$ maximizes $\underline{U}^{REE}(t, \omega, \cdot)$ on $B^{REE}(t, \omega, \pi)$. In this case, f is called a *maximin rational expectations allocation*, and the set of such allocations is denoted by $MREE(\mathscr{E})$.

In what follows, we provide some basic results on relationships among Bayesian REE, maximin REE and Walrasian equilibria of $\mathscr{E}(\omega)$ for a given state $\omega \in \Omega$. Our first result establish relations between Bayesian REE and maximin REE.

Theorem 2.3. *Let (f, π) be a Bayesian REE of $\mathscr{E}$. Under* $(\mathbf{A}_4)$-$(\mathbf{A}_6)$*, (f, π) is a maximin REE of $\mathscr{E}$.*

Proof. We need to verify that for each agent $t \in T$ and each state $\omega \in \Omega$, $f(t, \cdot)$ maximizes $\underline{U}^{REE}(t, \omega, \cdot)$ on $B^{REE}(t, \omega, \pi)$. Suppose that there exists an assignment $x \in B^{REE}(t, \omega, \pi)$ for some $(t, \omega) \in T \times \Omega$ such that

$$\underline{U}^{REE}(t, \omega, x) > \underline{U}^{REE}(t, \omega, f(t, \cdot)).$$

Since $f(t, \cdot)$ is $\mathscr{G}_t$-measurable, under $(\mathbf{A}_4)$ and $(\mathbf{A}_6)$, $U(t, \cdot, f(t, \cdot))$ must be $\mathscr{G}_t$-measurable. This means that $U(t, \cdot, f(t, \cdot))$ is constant on $\mathscr{G}_t(\omega)$, and thus

$$\underline{U}^{REE}(t, \omega, f(t, \cdot)) = U(t, \omega, f(t, \omega)),$$

which implies $U(t, \omega, x(\omega)) > U(t, \omega, f(t, \omega))$. Define $h : \Omega \to \mathbb{R}_+^\ell$ by

$$h(\omega') = \begin{cases} x(\omega), & \text{if } \omega' \in \mathscr{G}_t(\omega); \\ a(t, \omega'), & \text{otherwise.} \end{cases}$$

The definition of h and $(\mathbf{A}_5)$ imply $h \in B^{REE}(t, \omega, \pi) \cap L_t^{REE}$. Again, under $(\mathbf{A}_4)$ and $(\mathbf{A}_6)$, $U(t, \cdot, x(\cdot))$ is constant on $\mathscr{G}_t(\omega)$ for any $\mathscr{G}_t$-measurable function

$x : \Omega \to \mathbb{R}^\ell_+$. Thus, in this case, the Bayesian expected utility is equal to the ex post utility. It follows that

$$\mathbb{E}^{\mathbb{P}_t}\left[U(t,\cdot,f(t,\cdot))|\mathscr{G}_t\right](\omega) = U(t,\omega,f(t,\omega))$$

and

$$\mathbb{E}^{\mathbb{P}_t}\left[U(t,\cdot,h(\cdot))|\mathscr{G}_t\right](\omega) = U(t,\omega,x(\omega)),$$

which implies that

$$\mathbb{E}^{\mathbb{P}_t}\left[U(t,\cdot,h(\cdot))|\mathscr{G}_t\right](\omega) > \mathbb{E}^{\mathbb{P}_t}\left[U(t,\cdot,f(t,\cdot))|\mathscr{G}_t\right](\omega).$$

However, the last inequality contradicts with the fact that (f,π) is a Bayesian REE of the economy $\mathscr{E}$. □

Finally, we establish relationships between Bayesian or maximin REE of $\mathscr{E}$ and Walrasian equilibria of $\mathscr{E}(\omega)$, refer to [16] for the definition of a Walrasian equilibrium in a deterministic economy.

Theorem 2.4. *Let (f,π) be a Bayesian or maximin REE of $\mathscr{E}$. Under $(\mathbf{A}_4)$-$(\mathbf{A}_6)$, for each $\omega \in \Omega$, $(f(\cdot,\omega),\pi(\omega))$ is a Walrasian equilibrium of $\mathscr{E}(\omega)$.*

Proof. By Theorem 2.3, we only need to verify the case that (f,π) is a maximin REE of $\mathscr{E}$. Fix an $\omega \in \Omega$. Let $t \in T$ and $x \in B(t,\omega,\pi(\omega))$. Define a function $g : \Omega \to \mathbb{R}^\ell_+$ by $g(\omega') = x$ for all $\omega' \in \Omega$. It is obvious that π is $\mathscr{G}_t$-measurable. Under $(\mathbf{A}_5)$, $a(t,\cdot)$ is $\mathscr{F}_t$-measurable and thus is $\mathscr{G}_t$-measurable. Thus, π and $a(t,\cdot)$ are constant on atoms of $\mathscr{G}_t$. It follows that $a(t,\omega') = a(t,\omega)$ and $\pi(\omega') = \pi(\omega)$ for all $\omega' \in \mathscr{G}_t(\omega)$. Thus $B(t,\omega',\pi(\omega')) = B(t,\omega,\pi(\omega))$ for all $\omega' \in \mathscr{G}_t(\omega)$, which means $g \in B^{REE}(t,\omega,\pi)$. Since (f,π) is a maximin REE of $\mathscr{E}$,

$$\underline{U}^{REE}(t,\omega,f(t,\cdot)) \geq \underline{U}^{REE}(t,\omega,g).$$

Under $(\mathbf{A}_4)$ and $(\mathbf{A}_6)$, $U(t,\cdot,h(\cdot))$ is $\mathscr{G}_t$-measurable and thus is constant on $\mathscr{G}_t(\omega)$ for any $\mathscr{G}_t$-measurable function $h : \Omega \to \mathbb{R}^\ell_+$. It follows that

$$U(t,\omega,f(t,\omega)) \geq \underline{U}^{REE}(t,\omega,f(t,\cdot)) \geq \underline{U}^{REE}(t,\omega,g) = U(t,\omega,x).$$

This means $f(t,\omega)$ maximizes $U(t,\omega,\cdot)$ on $B(t,\omega,\pi(\omega))$. Thus, $(f(\cdot,\omega),\pi(\omega))$ is a Walrasian equilibrium of $\mathscr{E}(\omega)$. □

For each $\omega \in \Omega$, let $\mathscr{W}(\mathscr{E}(\omega))$ denote the set of all Walrasian equilibrium allocations of $\mathscr{E}(\omega)$. For convenience, let us define

$$\mathscr{W}\mathscr{A}(\mathscr{E}) = \{f : f \text{ is an assignment and } f(\cdot,\omega) \in \mathscr{W}(\mathscr{E}(\omega)) \text{ for all } \omega \in \Omega\}.$$

By Theorems 2.3 and 2.4, under $(\mathbf{A}_4)$-$(\mathbf{A}_6)$, we have the following relations:

$$REE(\mathscr{E}) \subseteq MREE(\mathscr{E}) \subseteq \mathscr{W}\mathscr{A}(\mathscr{E}).$$

Einy et al. [13] showed that under assumptions similar to ours, for an economy $\mathscr{E}$ with finitely many states of nature, then $REE(\mathscr{E}) = \mathscr{W}\mathscr{A}(\mathscr{E})$. This means that for economies with finitely many states of nature, Bayesian rational expectations equilibrium allocations can be represented by assignments which are state-wise Walrasian equilibrium allocations. Recently, de Castro et al. [12] showed that under assumptions similar to ours, for an economy $\mathscr{E}$ with finitely many agents and finitely many states of nature, every assignment in $\mathscr{W}\mathscr{A}(\mathscr{E})$ is in fact a maximin rational expectations equilibrium allocation. These motivate us to explore the problem whether we can use state-wise Walrasian equalibrium allocations to represent Bayesian or maximin expectations equilibrium allocations in our model. We shall consider this problem in the next section.

3. Representation of REE Allocations

In this section, we continue to study relationships among the concepts of rational expectations equilibrium in Section 2. We establish the equivalence between Bayesian REE and maximin REE. Furthermore, we also provide a representation theorem for Bayesian or maximin REE allocations of an economy $\mathscr{E}$ in terms of Walrasian equilibrium allocations of associated deterministic economies $\mathscr{E}(\omega)$. To achieve this goal, we need to establish (joint) measurability of a Bayesian REE allocation in Subsection 3.1.

3.1. Measurability of Bayesian REE Allocations.

Assumption $(\mathbf{A}_2)$ says that a is $\Sigma \otimes \mathscr{F}$-measurable. Thus, there exists a sequence of $\Sigma \otimes \mathscr{F}$-measurable simple functions $\{a_n : n \geq 1\} : T \times \Omega \to \mathbb{R}^\ell_+$ which converges pointwise to a almost everywhere. Define a function $\delta : \Delta \to \mathbb{R}_{++}$ by

$$\delta(p) = \min\{p^k : 1 \leq k \leq \ell\}$$

for every $p = (p^1, ..., p^k, ..., p^\ell) \in \Delta$. For any $(t, \omega, p) \in T \times \Omega \times \Delta$, let

$$\gamma(t, \omega, p) = \frac{1}{\delta(p)} \, \|a(t, \omega)\|_1 \text{ and } \gamma_n(t, \omega, p) = \frac{1}{\delta(p)} \, \|a_n(t, \omega)\|_1,$$

and then define

$$b(t, \omega, p) = \gamma(t, \omega, p)\mathbf{1} \text{ and } b_n(t, \omega, p) = \gamma_n(t, \omega, p)\mathbf{1},$$

where $\mathbf{1}$ denotes the vector in $\mathbb{R}^\ell$ whose components are 1. By definition, we can see that $\{b_n : n \geq 1\}$ converges to b almost everywhere with respect to $\|\cdot\|_1$. Now, for each $n \geq 1$, we define three correspondences $B_n, B, C : T \times \Omega \times \Delta \rightrightarrows \mathbb{R}^\ell_+$ such that for any $(t, \omega, p) \in T \times \Omega \times \Delta$,

$$B_n(t, \omega, p) = \left\{x \in \mathbb{R}^\ell_+ : \langle p, x\rangle \leq \langle p, a_n(t, \omega)\rangle\right\},$$
$$B(t, \omega, p) = \left\{x \in \mathbb{R}^\ell_+ : \langle p, x\rangle \leq \langle p, a(t, \omega)\rangle\right\},$$

and

$$C(t, \omega, p) = \left\{y \in \mathbb{R}^\ell_+ : U(t, \omega, x) \leq U(t, \omega, y) \text{ for all } x \in B(t, \omega, p)\right\}.$$

For any assignment f, we further define two more correspondences $B^f, D^f : T \times \Omega \times \Delta \rightrightarrows \mathbb{R}^\ell$ by

$$B^f(t, \omega, p) = B(t, \omega, p) - f(t, \omega),$$

and

$$D^f(t, \omega, p) = B(t, \omega, p) \cap C(t, \omega, p) - f(t, \omega)$$

for all $(t, \omega, p) \in T \times \Omega \times \Delta$. Obviously, $B^0 = B$, where 0 denotes the zero function on $T \times \Omega$.

To establish the $\Sigma \otimes \mathscr{F}$-measurability of a Bayesian REE allocation, we shall need the following three lemmas, whose proofs are presented in Appendix B.

Lemma 3.1. *Under $(\mathbf{A}_2)$, for each $(t, \omega, p) \in T \times \Omega \times \Delta$,*

$$\mathrm{Li}B_n(t, \omega, p) = \mathrm{Ls}B_n(t, \omega, p) = B(t, \omega, p).$$

Lemma 3.2. *Let $f : T \times \Omega \to \mathbb{R}^\ell_+$ be an $\Sigma \otimes \mathscr{F}$-measurable assignment of $\mathscr{E}$. Under $(\mathbf{A}_2)$, $B^f : T \times \Omega \times \Delta \rightrightarrows \mathbb{R}^\ell$ is lower $\Sigma \otimes \mathscr{F} \otimes \mathscr{B}(\Delta)$-measurable.*

Lemma 3.3. *Let $f : T \times \Omega \to \mathbb{R}^\ell_+$ be an $\Sigma \otimes \mathscr{F}$-measurable assignment of $\mathscr{E}$. Under $(\mathbf{A}_2)$, $(\mathbf{A}_3)$ and $(\mathbf{A}_6)$, $D^f : T \times \Omega \times \Delta \rightrightarrows \mathbb{R}^\ell$ is lower $\Sigma \otimes \mathscr{F} \otimes \mathscr{B}(\Delta)$-measurable.*

Theorem 3.4. *Let (f, π) be a maximin REE of $\mathscr{E}$. Under $(\mathbf{A}_2)$-$(\mathbf{A}_7)$, f is $\Sigma \otimes \mathscr{F}$-measurable.*

Proof. It is given that $a(t, \cdot)$ and $U(t, \cdot, x)$ are $\mathscr{G}_t$-measurable for all $t \in T$ and $x \in \mathbb{R}^\ell_+$. Thus, analogous to Lemma 3.2, one can show that $B(t, \cdot, p)$ is lower $\mathscr{G}_t$-measurable for all $(t, p) \in T \times \Delta$. Since $B(t, \omega, \cdot)$ is Hausdorff continuous and π is $\mathscr{G}_t$-measurable, $B^f(t, \cdot, \pi(\cdot))$ is lower $\mathscr{G}_t$-measurable. Thus, an argument similar to Lemma 3.3 shows that $D^0(t, \cdot, \pi(\cdot))$ is lower $\mathscr{G}_t$-measurable for all $t \in T$. Under $(\mathbf{A}_7)$, $D^0(t, \cdot, \pi(\cdot))$ is a single-valued $\mathscr{G}_t$-measurable function for all $t \in T$. It is claimed that $D^0(t, \cdot, \pi(\cdot)) = f(t, \cdot)$ for all $t \in T$. If not, there exists some $(t_0, \omega_0) \in T \times \Omega$ such that $D^0(t_0, \omega_0, \pi(\omega_0)) \neq f(t_0, \omega_0)$. It follows that

$$U(t_0, \omega_0, D^0(t_0, \omega_0, \pi(\omega_0))) > U(t_0, \omega_0, f(t_0, \omega_0)).$$

Since $D^0(t_0, \cdot, \pi(\cdot))$ is $\mathscr{G}_t$-measurable,

$$\underline{U}^{REE}(t_0, \omega_0, D^0(t_0, \cdot, \pi(\cdot))) = U(t_0, \omega_0, D^0(t_0, \omega_0, \pi(\omega_0))).$$

Thus,

$$\underline{U}^{REE}(t_0, \omega_0, D^0(t_0, \cdot, \pi(\cdot))) > \underline{U}^{REE}(t_0, \omega_0, f(t_0, \cdot)),$$

which contradicts the fact that (f, π) is a maximin REE of $\mathscr{E}$. Consequently, $f(t, \omega) = D^0(t, \omega, \pi(\omega))$ for all $(t, \omega) \in T \times \Omega$. As above, $B(\cdot, \cdot, \pi(\cdot))$ is lower $\Sigma \otimes \mathscr{F}$-measurable. Thus, similar to Lemma 3.3, one can show that f is $\Sigma \otimes \mathscr{F}$-measurable. □

3.2. **A Representation Theorem.** In this subsection, we identify a subset of $\mathscr{W}\mathscr{A}(\mathscr{E})$ to represent Bayesian and maximin REE allocations in our economic model. First, we present a converse of Theorem 2.3, with the appearance of an additional assumption.

Theorem 3.5. *Let (f, π) be a maximin REE of $\mathscr{E}$. Under $(\mathbf{A}_4)$-$(\mathbf{A}_7)$, (f, π) is a Bayesian REE of $\mathscr{E}$.*

Proof. First, we claim that for all $(t, \omega) \in T \times \Omega$ and all $\omega' \in \mathscr{G}_t(\omega)$, we have

$$U(t, \omega, f(t, \omega)) = U(t, \omega' f(t, \omega')).$$

If not, there must exist some $(t_0, \omega_0) \in T \times \Omega$ and $\omega_1, \omega_2 \in \mathscr{G}_{t_0}(\omega_0)$ such that

$$U(t_0, \omega_1, f(t_0, \omega_1)) > U(t_0, \omega_2, f(t_0, \omega_2)).$$

It is clear that $a(t_0, \omega_0) = a(t_0, \omega')$ and $\pi(\omega_0) = \pi(\omega')$ for all $\omega' \in \mathscr{G}_{t_0}(\omega_0)$. Thus, $B(t_0, \omega_0, \pi(\omega_0)) = B(t_0, \omega', \pi(\omega'))$, which means $f(t_0, \omega_1) \in B(t_0, \omega', \pi(\omega'))$ for all $\omega' \in \mathscr{G}_{t_0}(\omega_0)$. Define $g : \Omega \to \mathbb{R}^\ell_+$ by letting $g(\omega) = f(t_0, \omega_1)$ for all $\omega \in \Omega$. Since $U(t_0, \cdot, g(\cdot))$ is constant on $\mathscr{G}_{t_0}(\omega_0)$,

$$\underline{U}^{REE}(t_0, \omega_0, g) > \underline{U}^{REE}(t_0, \omega_0, f(t_0, \cdot)).$$

This is a contradiction to the fact that (f, π) is a maximin rational expectations equilibrium of $\mathscr{E}$.

Similar to Theorem 3.4, $D^0(t, \cdot, \pi(\cdot))$ is a single-valued $\mathscr{G}_t$-measurable function. By the definition of a maximin REE of $\mathscr{E}$, one has

$$\underline{U}^{REE}(t, \omega, f(t, \cdot)) \geq \underline{U}^{REE}(t, \omega, D^0(t, \cdot, \pi(\cdot)))$$

for all $\omega \in \Omega$. Hence, it follows from the claim that

$$U(t, \omega', f(t, \omega')) \geq U(t, \omega', D^0(t, \omega', \pi(\omega')))$$

for all $\omega' \in \mathscr{G}_t(\omega)$. Since $D^0(t, \omega', \pi(\omega'))$ is the unique maximizer of $B(t, \omega', \pi(\omega'))$, one has $f(t, \omega') = D^0(t, \omega', \pi(\omega'))$ for all $\omega' \in \Omega$. Consequently, $f(t, \cdot)$ is $\mathscr{G}_t$-measurable. To show that f is also a Bayesian REE allocation, let $x \in B^{REE}(t, \omega, \pi) \cap L_t^{REE}$ be fixed. Obviously,

$$\mathbb{E}^{\mathbb{P}_t}\left[U(t, \cdot, f(t, \cdot))|\mathscr{G}_t\right](\omega) = U(t, \omega, f(t, \omega))$$

and

$$\mathbb{E}^{\mathbb{P}_t}\left[U(t, \cdot, x(\cdot))|\mathscr{G}_t\right](\omega) = U(t, \omega, x(\omega)).$$

Since (f, π) is a maximin REE of $\mathscr{E}$, under $(\mathbf{A}_4)$ and $(\mathbf{A}_6)$, one has

$$U(t, \omega, f(t, \omega)) = \underline{U}^{REE}(t, \omega, f(t, \cdot)) \geq \underline{U}^{REE}(t, \omega, x) = U(t, \omega, x(\omega)).$$

It follows that

$$\mathbb{E}^{\mathbb{P}_t}\left[U(t, \cdot, f(t, \cdot))|\mathscr{G}_t\right](\omega) \geq \mathbb{E}^{\mathbb{P}_t}\left[U(t, \cdot, x(\cdot))|\mathscr{G}_t\right](\omega)$$

holds, which shows that (f, π) is a Bayesian REE of $\mathscr{E}$. □

The following lemma, called *Aumann's measurable selection theorem* in the literature, can be found in [1, p.608].

Lemma 3.6. *Let (T, Σ, μ) be a complete finite measure space. If a correspondence $F : T \rightrightarrows \mathbb{R}^\ell$ has a measurable graph with $F(t) \neq \emptyset$ for all $t \in T$, then F admits a measurable selection $f : T \to \mathbb{R}^\ell$.*

Now, with the help of all previous lemmas, we are able to present and prove our main theorem in this paper.

Theorem 3.7. *Let f be an assignment of an economy $\mathscr{E}$. Under $(\mathbf{A}_2)$-$(\mathbf{A}_7)$, the following statements are equivalent:*

(i) *$f \in REE(\mathscr{E})$, i.e., f is a Bayesian REE allocation.*
(ii) *$f \in MREE(\mathscr{E})$, i.e., f a maximin REE allocation.*
(iii) *$f \in \mathscr{W}\mathscr{A}(\mathscr{E})$ and f is $\Sigma \otimes \mathscr{F}$-measurable.*

Proof. The equivalence of (i) and (ii) has been established by Theorem 2.3 and Theorem 3.5. That (ii) implies (iii) is given by Theorem 2.4 and Theorem 3.4.

To complete the proof, we need to verify (iii) implies (ii). For this purpose, let $f \in \mathscr{W}\mathscr{A}(\mathscr{E})$ be $\Sigma \otimes \mathscr{F}$-measurable, and $q : \Omega \to \Delta$ be a price system such that $(f(\cdot, \omega), q(\omega))$ is a Walrasian equilibrium of the economy $\mathscr{E}(\omega)$ for all $\omega \in \Omega$. Define the correspondence $G : \Omega \times \Delta \rightrightarrows L_1(\mu, \mathbb{R}^\ell)$ by $G(\omega, p) = \mathscr{S}_{D^f(\cdot, \omega, p)}$ for all $\omega \in \Omega$ and all $p \in \Delta$. By Lemma 3.3, we have that D^f is lower $\Sigma \otimes \mathscr{F} \otimes \mathscr{B}(\Delta)$-measurable. Thus, $D^f(\cdot, \omega, p)$ admits a measurable selection, which is also integrably bounded by a function $h : T \to \mathbb{R}_+$ defined by

$$h(t) = \|b(t, \omega, p) + f(t, \omega)\|_2.$$

This implies that $G(\omega, p) \neq \emptyset$ for all $(\omega, p) \in \Omega \times \Delta$.

Claim. *G is lower $\mathscr{F} \otimes \mathscr{B}(\Delta)$-measurable.*

Proof of Claim. To verify this claim, let $\Theta : L_1\left(\mu, \mathbb{R}^\ell\right) \times \Omega \times \Delta \to \mathbb{R}_+$ be the function such that for all $g \in L_1\left(\mu, \mathbb{R}^\ell\right)$ and $(\omega, p) \in \Omega \times \Delta$,

$$\Theta(g, \omega, p) = \operatorname{dist}(g, G(\omega, p)).$$

By Theorem 8.1.4 in [6, p.310], we are done if we can show that $\Theta(g, \cdot, \cdot)$ is $\mathscr{F} \otimes \mathscr{B}(\Delta)$-measurable for all $g \in L_1\left(\mu, \mathbb{R}^\ell\right)$. To this end, it suffices to show that $\Theta(g, \cdot, \cdot) : \Omega \times \Delta \to \mathbb{R}_+$ is $\mathscr{F} \otimes \mathscr{B}(\Delta)$-measurable for every simple measurable function $g \in L_1\left(\mu, \mathbb{R}^\ell\right)$. Given a simple measurable function $g \in L_1(\mu, \mathbb{R}^\ell)$, we consider $\Gamma : T \times \Omega \times \Delta \to \mathbb{R}_+$ defined by

$$\Gamma(t, \omega, p) = \operatorname{dist}\left(g(t), D^f(t, \omega, p)\right)$$

for all $(t, \omega, p) \in T \times \Omega \times \Delta$. Then, Γ is $\Sigma \otimes \mathscr{F} \otimes \mathscr{B}(\Delta)$-measurable, and for any $(t, \omega, p) \in T \times \Omega \times \Delta$ and any $\xi(\cdot, \omega, p) \in G(\omega, p)$, we have

$$\Gamma(t, \omega, p) \leq \|g(t) - \xi(t, \omega, p)\|_2.$$

Thus, $\Gamma(\cdot, \omega, p)$ is integrably bounded for all $(\omega, p) \in \Omega \times \Delta$. We shall verify that $\int_T \Gamma(\cdot, \omega, p) d\mu$ is equal to $\Theta(g, \omega, p)$ for all $(\omega, p) \in \Omega \times \Delta$. Indeed, if $\int_T \Gamma(\cdot, \omega_0, p_0) d\mu < \Theta(g, \omega_0, p_0)$ holds for some $(\omega_0, p_0) \in \Omega \times \Delta$, we can pick some $\varepsilon > 0$ such that the following inequality

$$\int_T \Gamma(\cdot, \omega_0, p_0) d\mu + \varepsilon \mu(T) < \Theta(g, \omega_0, p_0)$$

holds. Next, we define $H : T \rightrightarrows \mathbb{R}^\ell$ and $\alpha : T \times \mathbb{R}^\ell \to \mathbb{R}$ by

$$H(t) = \left\{y \in D^f(t, \omega_0, p_0) : \|g(t) - y\|_2 \leq \Gamma(t, \omega_0, p_0) + \varepsilon\right\}$$

and

$$\alpha(t, y) = \|g(t) - y\|_2 - \Gamma(t, \omega_0, p_0).$$

It is easy to see that α is $\Sigma \otimes \mathscr{B}(\mathbb{R}^\ell)$-measurable. Further, it follows that

$$\mathrm{Gr}_H = \left\{(t, y) \in T \times \mathbb{R}^\ell : \alpha(t, y) \leq \varepsilon\right\} \cap \mathrm{Gr}_{D^f(\cdot, \omega_0, p_0)}$$

is $\Sigma \otimes \mathscr{B}(\mathbb{R}^\ell)$-measurable. By Lemma 3.6, H admits a measurable selection $h : T \to \mathbb{R}^\ell$, which satisfies

$$\|g - h\|_{L_1} \leq \int_T \Gamma(\cdot, \omega_0, p_0) d\mu + \varepsilon \mu(T).$$

This is a contradiction, which means that $\int_T \Gamma(\cdot, \omega, p) d\mu = \Theta(g, \omega, p)$ for all $(\omega, p) \in \Omega \times \Delta$. Hence, G is lower $\mathscr{F} \otimes \mathscr{B}(\Delta)$-measurable. □

Define another correspondence $F : \Omega \rightrightarrows \Delta$ by $F(\omega) = \{p \in \Delta : 0 \in G(\omega, p)\}$. Since $q(\omega) \in F(\omega)$, we conclude that $F(\omega) \neq \emptyset$ for all $\omega \in \Omega$ and $\mathrm{Gr}_F \in \mathscr{F} \otimes \mathscr{B}(\Delta)$. By Lemma 3.6, F has an $\mathscr{F}$-measurable selection $\pi : (\Omega, \mathscr{F}, \mathbb{P}) \to \Delta$. Since $0 \in G(\omega, \pi(\omega))$, $f(t, \omega) \in C(t, \omega, \pi(\omega))$ and $f(t, \omega) \in B(t, \omega, \pi(\omega))$. It follows that

$$\langle \pi(\omega), f(t, \omega) \rangle \geq \langle \pi(\omega), a(t, \omega) \rangle$$

and

$$\langle \pi(\omega), f(t, \omega) \rangle \leq \langle \pi(\omega), a(t, \omega) \rangle$$

for all $(t, \omega) \in T \times \Omega$. Hence, $\langle \pi(\omega), f(t, \omega) \rangle = \langle \pi(\omega), a(t, \omega) \rangle$ for all $(t, \omega) \in T \times \Omega$. Finally, it remains to verify that $f(t, \cdot)$ maximizes $\underline{U}^{REE}(t, \omega, \cdot)$ on $B^{REE}(t, \omega, \pi)$

for all $(t,\omega) \in T \times \Omega$. To this end, let $g : \Omega \to \mathbb{R}^{\ell}_{+}$ be a function such that $g \in B^{REE}(t,\omega,\pi)$ and

$$\underline{U}^{REE}(t,\omega,g) > \underline{U}^{REE}(t,\omega,f(t,\omega))$$

for some $(t,\omega) \in T \times \Omega$. Then, we have

$$U(t,\omega',g(\omega')) > U(t,\omega',f(t,\omega'))$$

for some $\omega' \in \mathscr{G}_t(\omega)$, which contradicts with $f(t,\omega') \in C(t,\omega',\pi(\omega'))$. $\square$

4. Conclusion

Since 1970's, the study of equilibria of an economic system under uncertainty has attracted many economic theorists and mathematicians. Several equilibrium concepts have been introduced in the literature, e.g., a Walrasian expectations equilibrium, a (Bayesian) REE, and a maximin REE. However, the question whether there is an equilibrium of these types and relevant questions have puzzled researchers in this field for a long time. Only in recent years, the answers to these questions have been getting clear.

In this paper, we study the relationships among the set of Bayesian REE allocations, the set maximin REE allocations and the set of state-wise Walrasian equilirium allocations in a fairly general model of pure exchange economies. Our model of a pure exchange economy has a measure space of agents with asymmetric information, and finitely many commodities.The exogenous uncertainty of agents is described by a complete probability space. Each agent is characterized by a state-dependent utility, a state-dependent initial endowment, his private information and prior belief. Our main result claims that under appropriate assumptions, the set of Bayesian REE allocations coincides with the set of maximin REE allocations. Furthermore, these sets can be represented by a subset of state-wise Walrasian equilibrium allocations, consisting of those jointly measurable allocations.

Our main result is similar to Theorem 4.3 in [13]. However, the economic model in [13] allows free-disposal, while our model does not allows free-disposal. For the economic model of a pure exchange asymmetric information economy with finitely many agents, finitely many states of nature and finitely many commodities considered in [12], each state-wise Walrasian equilibrium allocation is jointly measurable with respect to the counting measure. Thus, our main result extends the existence theorem, i.e., Theorem 4.1, in [12]. In addition, our main result not only claims the existence of an MREE, but also identifies MREE allocations. Thus, it also strengthens Theorem 5.5 in [10]. However, it remains an open and interesting question whether the set of Bayesian (maximin) REE allocations coincides with the ex-post core in our economic model. An affirmative answer to this question will allow us to establish a core-Walras type theorem in our economic model. We leave this question for our future research.

Appendix A. Mathematical Preliminaries

Let $\mathbb{R}^{\ell}$ be the ℓ-dimensional Euclidean space, and $\mathscr{K}_0\left(\mathbb{R}^{\ell}\right)$ be the family of non-empty compact subsets of $\mathbb{R}^{\ell}$. In this paper, we use two equivalent norms $\|\cdot\|_1$ and

$\|\cdot\|_2$ on $\mathbb{R}^\ell$, which are defined by

$$\|x\|_1 = \sum_{1\le i\le \ell} |x_i| \text{ and } \|x\|_2 = \sqrt{\sum_{1\le i\le \ell} |x_i|^2}$$

for each $x = (x_1, ..., x_i, ..., x_\ell) \in \mathbb{R}^\ell$. The pointwise order on $\mathbb{R}^\ell$ is denoted by $\le$ and $\mathbb{R}^\ell_+$ denotes the positive cone of $\mathbb{R}^\ell$. For any two vectors $x = (x_1, ..., x_\ell)$ and $y = (y_1, ..., y_\ell)$ in $\mathbb{R}^\ell$, the symbol $x \le y$ (or $y \ge x$) means that $x_k \le y_k$ for all $1 \le k \le \ell$. We write $x < y$ (or $y > x$) when $x \le y$ and $x \ne y$, and $x \ll y$ (or $y \gg x$) when $x_k < y_k$ for all $1 \le k \le \ell$. Let $\mathbb{R}^\ell_{++} = \{x \in \mathbb{R}^\ell_+ : x \gg 0\}$. A function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ is called *increasing* if $u(x) < u(y)$ for any $x, y \in \mathbb{R}^\ell_+$ with $x < y$, and it is *strictly quasi concave* if

$$u(\alpha x + (1-\alpha)y) > \min\{u(x), u(y)\}$$

for any $x, y \in \mathbb{R}^\ell_+$ with $x \ne y$ and any $0 < \alpha < 1$.

Let X be a non-empty set and (Y, ϱ) be a metric space. A *correspondence* $F : X \rightrightarrows Y$ from X to Y assigns to each $x \in X$ a subset $F(x)$ of Y. Meanwhile, F can also be viewed as a function $F : X \to 2^Y$, where 2^Y denotes the power set of Y. Furthermore, F is called *non-empty valued* (resp. *closed-valued*, *compact-valued*) if $F(x)$ is non-empty (resp. closed, compact) subset of (Y, ϱ) for all $x \in X$. The *graph* of F, denoted by Gr_F, is defined by

$$\mathrm{Gr}_F = \{(x, y) \in X \times Y : y \in F(x) \text{ and } x \in X \text{ with } F(x) \ne \emptyset\}.$$

For a point $y \in Y$ and a set $A \in 2^Y \setminus \{\emptyset\}$, we define

$$\mathrm{dist}(y, A) = \inf\{\varrho(y, a) : a \in A\}.$$

The *Hausdorff metric* H on $\mathscr{K}_0\left(\mathbb{R}^\ell\right)$ is defined by

$$H(A, B) = \max\left\{\sup_{a\in A} \mathrm{dist}(a, B),\ \sup_{b\in B} \mathrm{dist}(b, A)\right\}.$$

for any two $A, B \in \mathscr{K}_0\left(\mathbb{R}^\ell\right)$, where $\mathbb{R}^\ell$ is equipped with the Euclidean metric. For equivalent definitions of H, refer to [6]. The *Hausdorff metric topology* $\mathscr{T}_H$ on $\mathscr{K}_0(\mathbb{R}^\ell)$ is the topology generated by H. For a closed subset M of $\mathbb{R}^\ell$, $\mathscr{K}_0(M)$ and the Hausdorff metric H on $\mathscr{K}_0(M)$ can be defined similarly. If Z is a topological space, a correspondence $F : Z \rightrightarrows \mathbb{R}^\ell$ such that $F(x)$ is nonempty and compact for all $x \in Z$ is called *Hausdorff continuous* if $F : Z \to \left(\mathscr{K}_0\left(\mathbb{R}^\ell\right), \mathscr{T}_H\right)$ is continuous. This also holds when $\mathbb{R}^\ell$ is replaced by a closed subset M of $\mathbb{R}^\ell$.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability measure space and $\mathscr{G}$ a σ-algebra contained in $\mathscr{F}$. The *conditional expectation* of an integrable random variable U with respect to $\mathscr{G}$ denoted by $\mathbb{E}^{\mathbb{P}}(U|\mathscr{G})$, is a $\mathscr{G}$-measurable random variable such that

$$\int_A \mathbb{E}^{\mathbb{P}}(U|\mathscr{G})\ d\mathbb{P} = \int_A U\ d\mathbb{P}$$

for any $A \in \mathscr{G}$. It is a well known fact in Measure Theory that the conditional expectation of an integrable random variable with respect to a given σ-algebra exists and is unique, refer to [23].

Let $\{A_n : n \ge 1\}$ be a sequence of non-empty subsets of $\mathbb{R}^\ell$. A point $x \in \mathbb{R}^\ell$ is called a *limit point* of $\{A_n : n \ge 1\}$ if there exist $N \ge 1$ and $x_n \in A_n$ for each $n \ge N$ such that $\{x_n : n \ge N\}$ converges to x. The set of limit points of $\{A_n : n \ge 1\}$ is denoted by $\mathrm{Li}A_n$. Similarly, a point $x \in \mathbb{R}^\ell$ is called a *cluster point* of $\{A_n : n \geq 1\}$ if there exist positive integers $n_1 < n_2 < \cdots$ and for each k an $x_k \in A_{n_k}$ such that $\{x_k : k \geq 1\}$ converges to x. The set of cluster points of $\{A_n : n \geq 1\}$ is denoted by $\mathrm{Ls}A_n$. It is clear that $\mathrm{Li}A_n \subseteq \mathrm{Ls}A_n$, and both $\mathrm{Ls}A_n$ and $\mathrm{Li}A_n$ are closed (possibly empty) sets. If $\mathrm{Ls}A_n \subseteq \mathrm{Li}A_n$, then $\mathrm{Li}A_n = \mathrm{Ls}A_n = A$ is called the *limit* of the sequence $\{A_n : n \geq 1\}$. Note that $\mathrm{Ls}A_n = \mathrm{Ls}\overline{A}_n$ and $\mathrm{Li}A_n = \mathrm{Li}\overline{A}_n$, where $\overline{A}_n$ is the closure of A_n with respect to the Euclidean topology. Hence, if A is the limit of $\{A_n : n \geq 1\}$, then A is also the limit of $\{\overline{A}_n : n \geq 1\}$. If A and all A_n's are closed and contained in a compact subset $M \subseteq \mathbb{R}^\ell$, then $\mathrm{Li}A_n = \mathrm{Ls}A_n = A$ if and only if $\{A_n : n \geq 1\}$ converges to A in the Hausdorff metric topology on $\mathscr{K}_0(M)$, refer to [1].

Let (T, Σ, μ) be a measure space and $\{F_n : n \geq 1\}, F : (T, \Sigma, \mu) \rightrightarrows \mathbb{R}^\ell$ be correspondences. Recall that F is said to be *lower measurable* if

$$F^{-1}(V) = \{t \in T : F(t) \cap V \neq \emptyset\} \in \Sigma$$

for every open subset V of $\mathbb{R}^\ell$. It is well known that a non-empty closed-valued correspondence $F : (T, \Sigma, \mu) \rightrightarrows \mathbb{R}^\ell$ is lower measurable if and only if there exists a sequence of measurable selections $\{f_n : n \geq 1\}$ of F such that for all $t \in T$,

$$F(t) = \overline{\{f_n(t) : n \geq 1\}}.$$

If all $F_n's$ are non-empty closed-valued, lower measurable and at least one of $F_n's$ is compact-valued, then $\bigcap_{n \geq 1} F_n$ is lower measurable, refer to [17]. If $(S, \mathscr{S}, \nu)$ is another measure space and $f : (T, \Sigma, \mu) \times (S, \mathscr{S}, \nu) \to \mathbb{R}^\ell$ is jointly measurable, then it is well known that $\int_T : f(\cdot, \cdot) d\mu : (S, \mathscr{S}, \nu) \to \mathbb{R}^\ell$ is measurable. A *selection* of F is a single-valued function $f : (T, \Sigma, \mu) \to \mathbb{R}^\ell$ such that $f(t) \in F(t)$ for all $t \in T$. If a selection f of F is measurable (resp. integrable), then it is called a *measurable* (resp. an *integrable*) *selection*. Let $\mathscr{S}_F$ denote the set of integrable selections of F, and the *integration* of F over T in the sense of [7] is a subset of $\mathbb{R}^\ell$, defined as

$$\int_T F d\mu = \left\{ \int_T f d\mu : f \in \mathscr{S}_F \right\}.$$

Moreover, F is called *integrably bounded* if there exists an integrable function $g : (T, \Sigma, \mu) \to \mathbb{R}^\ell$ such that $-g(t) \leq y \leq g(t)$ for all $t \in T$ and $y \in F(t)$. It is a known that fact that if F is non-empty closed-valued and integrably bounded, then $\int_T F d\mu$ is compact, [16]. Let $M \subseteq \mathbb{R}^\ell$ be endowed with the relative Euclidean topology, and (Y, ϱ) be a metric space. It is known that a function $f : (T, \Sigma, \mu) \times M \to (Y, \varrho)$ is jointly measurable with respect to the Borel structure on M, if $f(\cdot, x)$ is measurable for all $x \in M$, and $f(t, \cdot)$ is continuous for all $t \in T$.

Appendix B. Proof of lemmas in Section 3.1

In this appendix, we present detailed proofs for Lemma 3.1, Lemma 3.2 and Lemma 3.3.

Proof of Lemma 3.1. It is evident that $\mathrm{Ls}B_n(t, \omega, p) \subseteq B(t, \omega, p)$. To show that $B(t, \omega, p) \subseteq \mathrm{Li}B_n(t, \omega, p)$, let $x \in B(t, \omega, p)$. If $a(t, \omega) = 0$ then $x = 0$, and thus there is nothing to prove. Assume that $a(t, \omega) \neq 0$. There are two cases for us to consider. First, $\langle p, x \rangle < \langle p, a(t, \omega) \rangle$. In this case, $\langle p, x \rangle < \langle p, a_n(t, \omega) \rangle$ for $n \geq 1$ sufficiently large. Thus, $x \in B_n(t, \omega, p)$ for $n \geq 1$ sufficiently large, which means $x \in \text{Li}B_n(t,\omega,p)$. The second case is that $\langle p,x\rangle = \langle p,a(t,\omega)\rangle$. In this case, $\langle p,\left(1-\frac{1}{k}\right)x\rangle < \langle p,a(t,\omega)\rangle$ for all $k \geq 1$. Thus, for any $k \geq 1$, $\left(1-\frac{1}{k}\right)x \in B_n(t,\omega,p)$ for $n \geq 1$ sufficiently large. It follows that for any $k \geq 1$, $\left\{\text{dist}\left(\left(1-\frac{1}{k}\right)x, B_n(t,\omega,p)\right) : n \geq 1\right\}$ converges to 0 as $n \to \infty$. Since $\{(1-\frac{1}{k})x : k \geq 1\}$ converges to x as $k \to \infty$, then $\{\text{dist}(x, B_n(t,\omega,p)) : n \geq 1\}$ converges to 0 as $n \to \infty$. This means that $x \in \text{Li}B_n(t,\omega,p)$. □

Proof of Lemma 3.2. Suppose that $\{f_n : n \geq 1\}$ is a sequence of simple $\Sigma \otimes \mathscr{F}$-measurable functions converging pointwise to f almost everywhere with respect to $\|\cdot\|_1$. For each $n \geq 1$, consider a correspondence $B_n^{f_n} : T \times \Omega \times \Delta \rightrightarrows \mathbb{R}^\ell$ defined by $B_n^{f_n}(t,\omega,p) = B_n(t,\omega,p) - f_n(t,\omega)$. By Lemma 3.1,

$$\text{Li}B_n^{f_n}(t,\omega,p) = \text{Ls}B_n^{f_n}(t,\omega,p) = B^f(t,\omega,p)$$

for all $(t,\omega,p) \in T \times \Omega \times \Delta$. Note that $B_n^{f_n}$ is lower $\Sigma \otimes \mathscr{F} \otimes \mathscr{B}(\Delta)$-measurable for all $n \geq 1$. Fix an $(t,\omega,p) \in T \times \Omega \times \Delta$. Since $\{b_n + f_n : n \geq 1\}$ converges to $b + f$ almost everywhere with respect to $\|\cdot\|_1$, we can choose an $N \geq 1$ such that

$$\|b_n(t,\omega,p) + f_n(t,\omega)\|_1 \leq \|b(t,\omega,p) + f(t,\omega)\|_1 + 1$$

for all $n \geq N$. Put

$$\eta = \max\left\{\|b_n(t,\omega,p) + f_n(t,\omega)\|_1,\ \|b(t,\omega,p) + f(t,\omega)\|_1 + 1 : 1 \leq n < N\right\}.$$

It can be verified readily that for all $n \geq 1$,

$$B_n^{f_n}(t,\omega,p),\ B^f(t,\omega,p) \subseteq B(0,\eta),$$

where $B(0,\eta)$ denotes the closed ball in $\mathbb{R}^\ell$ centered at 0 and with radius η with respect to the norm $\|\cdot\|_1$. Thus, $B_n^{f_n}(t,\omega,p)$ converges to $B^f(t,\omega,p)$ in the Hausdorff metric topology on $\mathscr{K}_0(\mathbb{R}^\ell)$. Since $B_n^{f_n} : T \times \Omega \times \Delta \to \mathscr{K}_0(\mathbb{R}^\ell)$ is $\Sigma \otimes \mathscr{F} \otimes \mathscr{B}(\Delta)$-measurable, its pointwise limit $B^f : T \times \Omega \times \Delta \to \mathscr{K}_0(\mathbb{R}^\ell)$ is $\Sigma \otimes \mathscr{F} \otimes \mathscr{B}(\Delta)$-measurable. Consequently, $B^f : T \times \Omega \times \Delta \rightrightarrows \mathbb{R}^\ell$ is lower $\Sigma \otimes \mathscr{F} \otimes \mathscr{B}(\Delta)$-measurable. □

Proof of Lemma 3.3. By Lemma 3.2, B and B^f are lower $\Sigma \otimes \mathscr{F} \otimes \mathscr{B}(\Delta)$-measurable. Thus, there is a sequence $\{\varphi_n : n \geq 1\}$ of $\Sigma \otimes \mathscr{F} \otimes \mathscr{B}(\Delta)$-measurable functions from $T \times \Omega \times \Delta$ to $\mathbb{R}_+^\ell$ such that for all $(t,\omega,p) \in T \times \Omega \times \Delta$,

$$B(t,\omega,p) = \overline{\{\varphi_n(t,\omega,p) : n \geq 1\}}.$$

For each $x \in \mathbb{R}^\ell$, let

$$\Theta(x) = \left\{(t,\omega) \in T \times \Omega : x + f(t,\omega) \in \mathbb{R}_+^\ell\right\}.$$

Note that $\Theta(x) \in \Sigma \otimes \mathscr{F}$ and $\Theta(x) = T \times \Omega$ if $x \geq 0$. It is possible to have $\Theta(x) = \emptyset$ for some $x < 0$. Let $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$. Define $\psi : T \times \Omega \times \mathbb{R}^\ell \to \mathbb{R}^*$ by

$$\psi(t,\omega,x) = \begin{cases} U(t,\omega,x + f(t,\omega)), & \text{if } (t,\omega) \in \Theta(x); \\ \infty, & \text{otherwise.} \end{cases}$$

Thus, for all $x \in \mathbb{R}^\ell$, $\psi(\cdot,\cdot,x)$ is $\Sigma \otimes \mathscr{F}$-measurable when $\mathbb{R}^*$ is equipped with the σ-algebra $\mathscr{B}^* = \mathscr{B}(\mathbb{R}) \cup \{B \cup \{\infty\} : B \in \mathscr{B}(\mathbb{R})\}$. It follows that $\psi(\cdot,\cdot,x)|_{\Theta(x)} : \Theta(x) \to \mathbb{R}$ is measurable for any $x \in \mathbb{R}^\ell$ with $\Theta(x) \neq \emptyset$. For each $n \geq 1$, define the function $\xi_n : T \times \Omega \times \Delta \times \mathbb{R}^\ell \to \mathbb{R}^*$ by

$$\xi_n(t,\omega,p,x) = \begin{cases} U(t,\omega,\varphi_n(t,\omega,p)) - \psi(t,\omega,x), & \text{if } (t,\omega) \in \Theta(x); \\ \infty, & \text{otherwise,} \end{cases}$$

and define $Z_n : T \times \Omega \times \Delta \rightrightarrows \mathbb{R}^\ell$ by

$$Z_n(t,\omega,p) = \left\{x \in \mathbb{R}^\ell : \xi_n(t,\omega,p,x) \leq 0\right\}.$$

As done above, one can show that $\xi_n(\cdot,\cdot,\cdot,x)|_{\Theta\times\Delta} : \Theta \times \Delta \to \mathbb{R}$ is $\Sigma \otimes \mathscr{F} \otimes \mathscr{B}(\Delta)$-measurable for all $x \in \mathbb{R}^\ell$ with $\Theta(x) \neq \emptyset$. Let W be an open subset of $\mathbb{R}^\ell$ and put $W \cap \mathbb{Q}^\ell = \{r_m : m \geq 1\}$. By $(\mathbf{A}_6)$, $Z_n(t,\omega,p) \cap W \neq \emptyset$ yields $r_m \in Z_n(t,\omega,p)$ for some $m \geq 1$. Consequently, we have

$$Z_n^{-1}(W) = \bigcup_{m\geq 1} \{(t,\omega,p) \in T \times \Omega \times \Delta : \xi_n(t,\omega,p,r_m) \in (-\infty, 0]\}$$

and thus $Z_n^{-1}(W) \in \Sigma \otimes \mathscr{F} \otimes \mathscr{B}(\Delta)$. Thus, Z_n is lower $\Sigma \otimes \mathscr{F} \otimes \mathscr{B}(\Delta)$-measurable. It is now claimed that

$$C(t,\omega,p) - f(t,\omega) = \bigcap_{n\geq 1} Z_n(t,\omega,p)$$

for all $(t,\omega,p) \in T \times \omega \times \Delta$. Obviously, $C(t,\omega,p) - f(t,\omega) \subseteq \bigcap_{n\geq 1} Z_n(t,\omega,p)$. Let $x \in \bigcap_{n\geq 1} Z_n(t,\omega,p)$ be such that $x \notin C(t,\omega,p) - f(t,\omega)$. So, $(t,\omega) \in \Theta(x)$ and $U(t,\omega,y) > \psi(t,\omega,x)$ for some $y \in B(t,\omega,p)$. Thus, $U(t,\omega,\varphi_n(t,\omega,p)) > \psi(t,\omega,x)$ for some $n \geq 1$, which is a contraction and the claim is verified. Since B^f is compact-valued, Z_n is closed-valued and

$$D^f(t,\omega,p) = (C(t,\omega,p) - f(t,\omega)) \cap B^f(t,\omega,p),$$

D^f is lower $\Sigma \otimes \mathscr{F} \otimes \mathscr{B}(\Delta)$-measurable. □

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