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Abstract

This paper contributes to qualifying the Basu-Mitra approach to the problem of intergenerational social choice, by analyzing the impact of the structure of the feasible set of utilities on Banerjee's (2006) impossibility theorem. We prove that if the utilities that each generation can possess lie in $\mathbb{N} \cup \{0\}$, then an *explicit* expression for a Paretian social welfare function that accounts for a strengthened form of Hammond Equity for the Future can be given.

Key words: Social welfare function, Equity, Pareto axiom, Intergenerational utility

1 Introduction

In ranking sets of infinite horizon intergenerational streams, economists are interested in postulating equity and efficiency in a variety of forms. Some of the combinations are incompatible with the use of certain devices. And of course, appropriate tools of analysis are sufficiently well-behaved on domains that are of interest to the researcher.

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With respect to the resolution of the aggregation problem two approaches can be mentioned. The first one is usually called the Diamond approach after Diamond (1965), and appeals to the use of welfare relations that are continuous with respect to suitable topologies. As is well known, this author established that Paretian welfare relations, continuous in the sup norm, can not treat all generations equally. A second approach is concerned with the possible existence of social welfare functions (SWF's). No topological consideration is made in this case, that we call the Basu-Mitra approach. This line of inquiry follows Basu and Mitra (2003), whose main result implies that one can dispense with the continuity axiom in Diamond's impossibility theorem.

Two other type of factors must be mentioned in the resolution of the conflict between infinite generations. One includes the version of the Pareto criterion that is impossed in order to account for efficiency, plus the equity-related postulate that is requested. The other factor is the domain of utilities that each generation can possess (that we assume is equal across generations) and in particular, if it is discrete or not. We call that domain the *feasible utilities.* The use of discrete sets of feasible utilities is backed by the recognition that human perception is not endlessly fine. It is a natural setting if the utilities have a well-defined smallest unit (as happens when they measure monetary amounts).

The Basu-Mitra impossibility theorem shows the incompatibility of two axioms (Anonymity and Paretianity) when we seek SWF's and the feasible set of utilities is the smallest non-trivial one (namely, $\{0, 1\}$). Likewise, Basu and Mitra (2007) shows that it is possible to combine Anonymity and a weaker form of the Pareto postulate called Weak Dominance in a SWF, irrespective of the feasible utilities. Therefore the domain restriction is not an issue when we discuss the compatibility of those two pairs of axioms with the existence of SWF's. But for other sets of axioms, the structure of the domain is crucial. For example, if we replace the Pareto criterion by Weak Pareto in the Basu-Mitra impossibility theorem then impossibility remains when the feasible utilities are $[0, 1]$ but it becomes possibility when they are $\mathbb{N}^* = \{0, 1, 2, ...\}$ (cf. Basu and Mitra, 2007).

This paper contributes to qualifying the Basu-Mitra approach to the problem of intergenerational social choice, by analyzing the impact of the structure of the feasible set of utilities on Banerjee's (2006) impossibility theorem. Now the properties under inspection are Weak Dominance and a weak equity postulate that was introduced in Asheim and Tungodden (2004), namely Hammond Equity for the Future (HEF). Banerjee (2006) proves that they are incompatible under the Basu-Mitra perspective when the feasible utilities are $[0, 1]$. Here we prove that if we consider $\mathbb{N}^*$ instead, then an *explicit* expression for a Paretian social welfare function that accounts for a strengthened form of Hammond Equity for the Future can be given.

2 Notation and Preliminaries

Let $\mathbf{X}$ denote a subset of $\mathbb{R}^{\mathbb{N}}$, that represents a domain of utility sequences or infinite-horizon utility streams. We adopt the usual notation for such utility streams: $\mathbf{x} = (x_1, ..., x_n, ...) \in \mathbf{X}$. By $(y)_{con}$ we mean the constant sequence $(y, y, ...)$, and $= (x, (y)_{con})$ holds for $(x, y, y, y, ...)$. We write $\mathbf{x} \geqslant \mathbf{y}$ if $x_i \geqslant y_i$ for each $i = 1, 2, ...$, and $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i$ for each $i = 1, 2, ...$ Also, $\mathbf{x} > \mathbf{y}$ holds for $\mathbf{x} \geqslant \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.

A *social welfare function* (SWF) is a function $\mathbf{W} : \mathbf{X} \longrightarrow \mathbb{R}$. In this paper we are concerned with two axioms of different nature on SWF's.

Axiom 1 (*Hammond Equity for the Future* $^+$, also HEF$^+$). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $\mathbf{x} = (x_1, (x)_{con})$ and $\mathbf{y} = (y_1, (y)_{con})$ $(x_1 > y_1 > y > x)$, then $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{x})$.

Observe that this statement is slightly stronger than the usual HEF, where only $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{x})$ is requested. Because we obtain a possibility result, we adopt the strongest version that we can justify.

HEF $^+$ (respectively, HEF) states the following restriction on the ranking of streams where the level of utility is constant from the second period on and the present generation is better-off than the future: if the sacrifice by the present generation conveys a higher utility for all future generations, then such trade off is preferred (respectively, weakly preferred). As is argued in Asheim et al. (2007), these conditions can be endorsed both from an egalitarian and utilitarian point of view.

Axiom 2 (*Pareto*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} > \mathbf{y}$ then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$.

A main achievement in the field of aggregation of infinite utility streams has been Basu and Mitra, 2003, Theorem 1. It states that no SWF is Paretian and Equitable or Anonymous [3] when $\mathbf{X} = Y^{\mathbb{N}}$ and $\mathbf{Y} = \{0, 1\}$. If we replace Anonymity by Hammond Equity for the Future, then Banerjee (2006) proves that the impossibility of making these requirements compatible with an SWF remains when $\mathbf{Y} = [0, 1]$ even though we only require Weak Dominance [4] instead of Pareto. Along the next Section we analyze the situation when $\mathbf{Y} = \mathbb{N}^*$.

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[3] The *Anonymity* axioms states that a finite permutation of a utility stream produces a utility stream with the same social utility.

[4] *Weak Dominance* requires that if $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and there is $j \in \mathbb{N}$ such that $x_j > y_j$, and $x_i = y_i$ for all $i \neq j$, then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$.

3 Existence of HEF and Paretian Social Welfare Functions

We begin by proving a lemma on the properties of certain auxiliary functions *on different domains.* Recall that the application

$$\psi(n) = \frac{n}{1+n} \quad \text{for each } n \in \mathbb{N}^*$$

maps $\mathbb{N}^*$ into $[0,1)$ and satisfies: $m < n$ if and only if $\psi(m) < \psi(n)$ for every possible m, n (Bridges and Mehta, 1996, p. 30).

Lemma 1 *For each* $n = 2, 3, ...$ *let* $\delta_n = \frac{2}{3}\frac{n-1}{2+n}$ *, and let* $\delta_0 = 0.1$ *,* $\delta_1 = 0.15$ *. Define the functions* $\mathcal{W}_n : \{0,1,2,....,n\} \times \mathbb{N}^* \times \mathbb{N}^* \times ..... \longrightarrow \mathbb{R}$ *by the expression: for any* $\boldsymbol{x} = (x_1, x_2, ...) \in \{0,1,2,....,n\} \times \mathbb{N}^* \times \mathbb{N}^* \times$ *.....*

$$\mathcal{W}_n(\boldsymbol{x}) = \sum_{i=1}^{+\infty} (\delta_n)^i \psi(x_i) < +\infty \tag{1}$$

Then:

(a) Each $\mathcal{W}_n$ *satisfies HEF*$^+$ *and Pareto.*

(b) If $m > m'$ *and* $(m', x_2, x_3, ...) \neq (0)_{con}$ *then* $\mathcal{W}_m(m', x_2, x_3, ...) > \mathcal{W}_{m'}(m', x_2, x_3, ...)$*.*

Proof. The construction of the $\mathcal{W}_n$'s functions ensures that they are well-defined and satisfy Pareto. Observe that HEF$^+$ holds vacuously for $\mathcal{W}_n$ when $n = 0, 1, 2$. Thus in order to complete the proof of (a) the only property that we need to check is that $\mathcal{W}_n$ satisfies HEF$^+$ when $n \geqslant 3$.

Let us fix $n \geqslant 3$ and select elements in $\{0,1,2,....,n\}$, in such way that $x_1 > y_1 > y > x$. By definition

$$\mathcal{W}_n(x_1, (x)_{con}) = \delta_n \psi(x_1) + \sum_{i=2}^{+\infty} (\delta_n)^i \psi(x) = \delta_n \psi(x_1) + \frac{(\delta_n)^2}{1-\delta_n}\psi(x)$$

$$\mathcal{W}_n(y_1, (y)_{con}) = \delta_n \psi(y_1) + \sum_{i=2}^{+\infty} (\delta_n)^i \psi(y) = \delta_n \psi(y_1) + \frac{(\delta_n)^2}{1-\delta_n}\psi(y)$$

Some trivial computations yield

$$\mathcal{W}_n(y_1, (y)_{con}) > \mathcal{W}_n(x_1, (x)_{con}) \Leftrightarrow \frac{\delta_n}{1-\delta_n} > \frac{\psi(x_1) - \psi(y_1)}{\psi(y) - \psi(x)}$$

Besides,

$$\frac{\psi(x_1) - \psi(y_1)}{\psi(y) - \psi(x)} \leqslant \frac{\psi(n) - \psi(2)}{\psi(1) - \psi(0)} = \frac{2n-4}{3+3n}$$

because the largest feasible numerator is $\psi(n) - \psi(2)$ and the least feasible denominator is $\psi(1) - \psi(0)$ across all selections of $x_1 > y_1 > y > x$ from $\{0, 1, 2, ..., n\}$. Thus one readily checks that

$$\frac{\delta_n}{1 - \delta_n} = \frac{2n - 2}{8 + n} > \frac{2n - 4}{3 + 3n} \geqslant \frac{\psi(x_1) - \psi(y_1)}{\psi(y) - \psi(x)}$$

and the proof of (a) is finished.

Regarding (b), we observe that $\delta_{n+1} - \delta_n = \frac{2}{(n+2)(n+3)} > 0$ whenever $n \geqslant 2$. Because $\delta_0 < \delta_1 < \delta_2$, the sequence $\{\delta_n\}_{n=0,1,2,...}$ is strictly increasing and the expression (1) produces (b) trivially. □

We are ready to prove our theorem.

Theorem 1 *There are SWF's on* $\boldsymbol{X} = Y^{\mathbb{N}}$, *where* $Y = \{0, 1, 2, ...\}$, *that satisfy both Hammond Equity for the Future* $^+$ *and Pareto.*

Proof. Our proof is constructive: we give an explicit expression for an SWF on $\mathbf{X}$ that satisfies HEF^+ and Pareto. We lean on the properties of the auxiliary functions studied in Lemma 1.

For any $\mathbf{x} = (x_1, x_2, x_3, ...) \in \mathbf{X}$ let

$$\mathcal{W}(\mathbf{x}) = \mathcal{W}_{x_1}(\mathbf{x}) \tag{2}$$

To check that $\mathcal{W}$ is Paretian, take $\mathbf{x} > \mathbf{y}$, $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. If $x_1 = y_1$ then because $\mathcal{W}_{x_1} = \mathcal{W}_{y_1}$ is Paretian one has $\mathcal{W}(\mathbf{x}) > \mathcal{W}(\mathbf{y})$. Otherwise $x_1 > y_1$ and we entail the following inequalities:

$$\mathcal{W}(\mathbf{x}) = \mathcal{W}_{x_1}(\mathbf{x}) > \mathcal{W}_{x_1}(y_1, x_2, x_3, ...) \geqslant \mathcal{W}_{y_1}(y_1, x_2, x_3, ...) \geqslant \mathcal{W}_{y_1}(\mathbf{y}) = \mathcal{W}(\mathbf{y})$$

The first inequality above holds because $\mathcal{W}_{x_1}$ is Paretian. The second one holds strictly when $(y_1, x_2, x_3, ...) \neq (0)_{con}$ by Lemma 1 (b), and it is an equality when $(y_1, x_2, x_3, ...) = (0)_{con}$. The last inequality holds because if $\mathbf{y} = (y_1, x_2, x_3, ...)$ then we have equality, otherwise $(y_1, x_2, x_3, ...) > \mathbf{y}$ and we use that $\mathcal{W}_{y_1}$ is Paretian to derive strict inequality.

In order to check that $\mathcal{W}$ satisfies HEF^+, take $x_1 > x_1' > y' > y$ elements from Y. Now

$$\mathcal{W}(x_1, (y)_{con}) = \mathcal{W}_{x_1}(x_1, (y)_{con}) > \mathcal{W}_{x_1}(x_1', (y')_{con}) > \mathcal{W}_{x_1'}(x_1', (y')_{con}) = \mathcal{W}(x_1', (y')_{con})$$

The first inequality above holds because $\mathcal{W}_{x_1}$ satisfies HEF^+ by Lemma 1 (a). The second one holds by Lemma 1 (b) since $x_1' \neq 0$ and $y_1' \neq 0$. □

4 Conclusions

Banerjee (2006) argued that " ... a compromise that generates a possibility of ranking infinite utility streams is open to debate and does not necessarily call for abandoning the appealing equity postulate, Hammond Equity for the Future." Here we provide support for such possibility: if the feasible set of utilities is $\mathbb{N}^*$ then it is possible to strengthen HEF even if we require the full force of the Pareto postulate under the Basu-Mitra approach. Our argument is constructive and a explicit criterion has been provided. Evidences like the discouraging Zame (2007, Theorem 4') –which implies that no Weak Paretian and Anonymous welfare relation can be "explicitly described"– make this feature especially valuable.

Table 1 gathers some of the results that have served us to motivate our discussion, and permits to compare differences in the approaches when we vary the feasible utilities.

Table 1. Summary of results for domains of utility streams $Y^{\mathbb{N}}$

	$Y = \mathbb{N}^*$	$Y = [0, 1]$
HEF^+ and Pareto	Existence $\star$	Non-existence
HEF^+ and WD	Existence	Non-existence $\diamond$
Anonymity and WD	Existence $\dagger$	Existence $\ddagger$

Fact $\star$ is proved here. Fact $\diamond$ is trivial from Banerjee, 2006. It is true even if we replace HEF^+ by the weaker HEF. Facts $\dagger$ and $\ddagger$ are proved in Basu and Mitra (2007), which shows that WD can be reinforced to Weak Pareto [5] in $\dagger$ but $\ddagger$ becomes non-existence if we replace WD with Dominance [6]. The other statements in the table derive from $\star$ and $\dagger$.

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[5] *Weak Pareto* means that if $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and either $\mathbf{x} \gg \mathbf{y}$ or there is $j \in \mathbb{N}$ such that $x_j > y_j$ and $x_i = y_i$ for all $i \neq j$, then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$.

[6] *Dominance* means that if $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ then (a) if there is $j \in \mathbb{N}$ such that $x_j > y_j$ and $x_i = y_i$ for all $i \neq j$, then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$, and (b) if $\mathbf{x} \gg \mathbf{y}$ then $\mathbf{W}(\mathbf{x}) \geqslant \mathbf{W}(\mathbf{y})$. Observe that Pareto $\Rightarrow$ Weak Pareto $\Rightarrow$ Dominance $\Rightarrow$ WD.

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