When should the distant future not be discounted at increasing discount rates?

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WHEN SHOULD THE DISTANT FUTURE NOT BE DISCOUNTED AT INCREASING DISCOUNT RATES?

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Abstract: A number of governments have already adopted the policy of applying Declining Discount Rates (DDRs) to long lived projects, a move that will significantly affect public sector investment decisions. This paper argues that such policy is misguided, and revisits the discussion that led to it. A 2009 paper by Christian Gollier and Martin L. Weitzman is widely regarded as having solved the Weitzman-Gollier Puzzle, which is that the definition of expected present value (EPV) proposed by Weitzman's in 1998 is inconsistent with the calculation of expected future values (EFV) when market interest rates are stochastic but perfectly auto-correlated. The inconsistency is actually due to the fact that Weitzman's EPV formulation is incorrect. When it is replaced by the correct formulation, the puzzle disappears, and risk neutral certainty equivalent rates (CERs) turn out to be growing, rather than declining under the assumptions of Weitzman's model. This removes the justification for the use of DDRs. This paper shows that Gollier and Weizmann (2009) fail to resolve the puzzle. Adding risk aversion to Weitzman's 1998 model to derive risk adjusted CERs cannot resolve the inconsistency between alternative methods of computing expected monetary yields, because investors' risk aversion only affects their own valuations, not market yields. If monetary CERs increase, the underlying efficiency of investment projects must generally match the growing monetary CERs of capital markets for them to be worth investing in, even for risk averse investors. The distant future should only not be discounted at increasing discount rates if Weitzman's 1998 assumption of perfectly auto-correlated interest rates fails to hold sufficiently.

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Discounting plays a central role in cost-benefit analysis and is therefore a sensitive topic in environmental economics. One strand of research that has gained considerable acceptance is the suggestion by Martin L. Weitzman, made in "Why the Far-Distant Future Should Be Discounted at its Lowest Possible Rate" (1998), that certainty equivalent discount rates should be declining for the very long run and will tend to the lowest possible interest rate. Weitzman's conclusion was based on computing the certainty equivalent interest rate (CER) implicit in the expected present value (EPV) of a future benefit known with certainty, in a two time-period model in which the interest rate is stochastic but perfectly auto-correlated in time.

"Despite some puzzles along the way, the burgeoning theoretical literature on discounting distant time horizons points more or less unanimously towards the use of a declining term structure of social discount rates (DDRs) for risk free public projects," state

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Mark C. Freeman, Ben Groom, Ekaterini Panopoulou and Theologos Pantelidis in "Declining discount rates and the 'Fisher Effect': Inflated past, discounted future?" (2014), observing that "The ENPV approach has strongly influenced the current U.K., U.S. and Norwegian governments' guidance on long-term discounting." A recent paper supporting this conclusion is "Should Governments Use a Declining Discount Rate in Project Analysis?" by Kenneth J. Arrow, et al. (2014).

In "Governments Should Not Use Declining Discount Rates in Project Analysis" Szabolcs Szekeres (2015) takes a contrary view, on the basis of the observation that Weitzman's definition of EPV is inconsistent with the definition of present value, and results in a violation of the generally accepted requirement of transitivity of preferences. It corresponds to time reversed negative compounding, rather than to discounting. When discounting is used instead, the conclusions of Weitzman's 1998 model are reversed.

Szekeres (2015) only refers briefly to the Weitzman-Gollier Puzzle, which arose when Christian Gollier (2003), using Weitzman's model, but computing expected future values (EFVs) instead of EPVs, showed that certainty equivalent interest rates should be increasing and will tend to the highest possible interest rate. He then concluded that "Clearly, we cannot be both right." (2003:5). By 2009, however, Gollier and Weitzman squarely endorsed the concept of DDRs in their joint "How Should the Distant Future be Discounted When Discount Rates are Uncertain?" (2009). Both authors, incidentally, are co-authors of Arrow *et al.* (2014).

While Szekeres (2015) solved the puzzle in its original risk neutrality context, other papers claim to have solved it in other ways. This paper analyzes two alternative offered solutions. First, it examines Wolfgang Buchholz and Jan Schumacher (2008) "Discounting the Long-Distant Future: a Simple Explanation for the Weitzman-Gollier-Puzzle." These authors avoided the trap of time reversed negative discounting, but stated that "a discount rate that is declining over time, is nevertheless reasonable, since it can be justified by assuming a plausible degree of risk aversion." Second, it examines how Gollier and Weitzman (2009) were able to conclude that "When agents optimize their consumption plans and probabilities are adjusted for risk, the [the original Weitzman (1998) and Gollier (2003)] approaches are identical" (Gollier and Weitzman, 2009:1).

This paper shows that the discount rate to which Buchholz and Schumacher (2008) refers to is not the one to which Weitzman (1998) refers to, and that therefore Buchholz and Schumacher (2008) does not provide a solution to the puzzle. The same is true of Gollier and Weitzman (2009). Its arguments do not address the discrepancy between Weitzman's calculation of monetary EPVs and EFVs. Its claim to have demonstrated the equality of the unequal rests on a morphological similarity between expressions defining risk averse CERs and Weitzman's formulation of risk neutral CER.

Finally, this paper concludes that the Weitzman model (1998) implies increasing discount rates due to the perfect auto-correlation of interest rates that it assumes, and that this conclusion is only invalid if the assumed serial auto-correlation of interest rates is not sufficiently strong.

This paper is organized as follows. Section 1 presents briefly the Weitzman-Gollier puzzle and its resolution. Section 2 reviews Buchholz and Schumacher (2008). Section 3 points out that valuation and discounting are not the same, and that introducing risk aversion does not translate into lower monetary yield requirements for long lived projects. Section 4 recapitulates the logic of Gollier and Weitzman (2009), while Section 5 presents a numerical example of the model proposed in it. Section 6 shows how EPVs should be calculated. Finally, Section 7 presents conclusions.

1. The Weitzman-Gollier Puzzle and its resolution

The decision situation on which the Weitzman model was based is described as follows in Gollier and Weitzman (2009:3):

> "In the highly stzlized model of this paper, time $t = 0, 1, 2, ...,$ is measured in discrete periods of unit length. To state loosely the issue at hand, a decision must be taken now, just before time zero (call it time 0^-), whether or not to invest a marginal cost δ that will yield a marginal benefit ε at future time t. Right now, at time 0^-, it is unknown what will be the appropriate future rate of return on capital in the economy. There are n possible future states of the economy, indexed by $i = 1, 2, ..., n$. As of now (time 0^-), future state i is viewed as having marginal product of capital r_i with probability $p_i > 0$, where $\Sigma_i p_i = 1$. A decision must be made now (at time 0^-, *just before* the "true" state of the world is revealed at time $t = 0$) about whether or not to invest δ now in order to gain payoff ε at future time t. To pose the problem sharply, it is assumed that immediately after the investment decision is made, at time 0, the true state of the world i is revealed and the marginal product of capital will thenceforth be r_i, from time $t = 0$ to time $t = \infty$."

In Weitzman (1998) the certainty equivalent discount factor used to compute the expected present value (EPV) of $1 due at time t was defined as

$$A_W = \sum p_i e^{-r_i t} \tag{1}$$

The corresponding certainty-equivalent discount rate R_W will fulfill the following condition[2]

$$A_W = e^{-R_W t} \tag{2}$$

from which it follows that

$$R_W = -\left(1/t\right)\ln\left(\sum_i p_i e^{-r_i t}\right) \tag{3}$$

which, as Weitzman (1998) shows, will tend to the lowest possible interest rate for large t. It should be clear that calculating certainty equivalents on the basis of expected monetary values, as above, is equivalent to assuming that the decision makers are risk neutral. Therefore this is the context in which Weitzman's (1998) model is to be interpreted.

In contrast, Gollier (2003) derived a certainty equivalent rate from the expected future value (EFV) of $1 invested in the present, from which the following certainty equivalent rate can be derived:

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[2] Subscripts W attribute to Weitzman, G to Gollier.

$$R_G = \left(\frac{1}{t}\right)\ln\left(\sum_i p_i e^{r_i t}\right) \tag{4}$$

which, as Gollier (2003) shows, will tend to the highest possible interest rate for large t.

This was the essence of the Weitzman-Gollier puzzle, and as Gollier (2003) pointed out, (3) and (4) cannot both be right. Indeed, a risk neutral investor computing EPVs using expression (1) would be willing to pay more for a given future value than what needs to be invested in the market to arrive at the same expected future value. Such an investor would become a "money pump," which is only possible because of the implicit violation of the generally accepted requirement of transitivity of preferences. As the EFV calculation corresponds to the definition of expectation applied to a description of the behavior of the capital markets that enjoys universal acceptance, the inconsistency could only be due to an incorrect calculation of EPV, which is an act of valuation of the computed EFV. But so deceptively plausible does expression (1) appear to be, that for a very long time nobody drew the obvious conclusion.

The discrepancy was universally regarded as troublesome. Ben Groom, Cameron Hepburn, Phoebe Koundouri and David Pearce (2005:464) characterized the puzzle as follows:

> "So, confusingly, whereas in the absence of uncertainty the two decision criteria are equivalent, once uncertainty regarding the discount rate is introduced the appropriate discount rate for use in CBA depends upon whether we choose ENPV or ENFV as our decision criterion. In the former case, discount rates are declining and in the latter they are rising through time. It is not immediately clear which of these criteria is correct."

Even though Weitzman's (1998) model is framed in the context of risk neutrality, the lack of a solution to the original puzzle drove most attempts to explain the puzzle (or to justify Weitzman's conclusions) to the use of utility functions, thus abandoning the risk neutrality assumption of the Weitzman (1998) model. Already Gollier (2003:5) stated this: "Taking the expected net future value is equivalent to assuming that all risks will be borne by the future generation."

Risk aversion by investors cannot resolve an inconsistency in the measurement of monetary (risk neutral) market yields, however. The solution of the puzzle lies in the observation made by Szekeres (2015): expression (1) does not give the correct EPV of a future sum of \$1 because it does not correspond to the accepted definition of present value, namely, that PV is the sum that compounds to the FV[3]. The EPV that expression (1) computes will not compound to \$1 in the future:

$$\left(\sum p_i e^{-r_i t}\right)\left(\sum p_i e^{r_i t}\right) \neq 1 \tag{5}$$

where the second term is the expected compound factor derived from the same interest rates and probabilities. The correct discount factor of \$1 due in year t is

[3] "The present value of an asset is obtained by calculating how much money invested today would be needed, at the going interest rate, to generate the asset's future stream of receipts." (Paul A. Samuelson and William D. Nordhaus, 1992:271.)

$$A(t) = \frac{1}{\sum p_i e^{r_i t}} \tag{6}$$

which, when multiplied by the expected compound factor, $\sum p_i e^{r_i t}$, will clearly result in an EFV equal to \$1.

While Weitzman (1998) derived its EPV calculation from a certain future amount, it did so in the context of risk neutrality, in which the certainty equivalent value of a stochastic yield is equal to its expected monetary value. Therefore, the foregoing is also true if the FV is not certain. In the risk neutral Weitzman (1998) model, it makes no difference whether the future value discounted is certain or is just an expected value.

The correct certainty equivalent rate can be derived from (6) and is

$$R = \left(\frac{1}{t}\right)\ln\left(\sum_i p_i e^{r_i t}\right) \tag{7}$$

Consequently it is (4) that is right and (3) that is wrong, as the certainty equivalent rate (7), derived from computing EPVs, is the same as that which is derived from EFVs by expression (4). The certainty equivalent rate is always the same for discounting and compounding, because the expected discount factor is the inverse of the expected compound factor.

Szekeres (2015) explains that what (1) and the certainty equivalent rate (3) correspond to, is time reversed negative compounding; it explores their characteristics, and compares them to the corresponding expressions for discounting.

The Weitzman model, described in detail by the quote from Gollier and Weitzman (2009) inserted above, is particularly conducive to mistaking time reversed negative compounding for discounting, for one is directly tempted to probability weight the alternative scenarios, which is effectively negative compounding. Doing so will appear deceptively plausible, but will lead to the wrong result. It is worth bearing this in mind when using payoff tables, decision trees or Monte Carlo simulation of interest rate uncertainties. When such formulations need to be used, the correct certainty equivalent rate R should be used instead of the r_i corresponding to each scenario. Using anything other than the certainty equivalent interest rate, computed as in (4) or (7), will lead to incorrect results.

2. Risk aversion and risk adjusted CERs

There were many attempts to resolve the Weitzman-Gollier puzzle. All involved the introduction of utility functions of some kind, and most accepted Weitzman's definition of EPV. One paper that did not, is Wolfgang Buchholz and Jan Schumacher (2008). The authors avoided the trap of time reversed negative discounting. Their paper states in its abstract: "We show that, while Weitzman's use of the present value approach may indeed seem questionable, its outcome, i.e. a discount rate that is declining over time, is nevertheless reasonable, since it can be justified by assuming a plausible degree of risk aversion."

In this section their analysis is illustrated with a numerical example. It is assumed that there are two scenarios, with interest rates r_1 = 1% and r_2 = 5%, and that the probability of the two scenarios is equal $p_1 = p_2 = 0.5$. It is further assumed that \$1 is being invested. The risk adjusted certainty equivalent compound factor F for various time horizons t is calculated as follows:

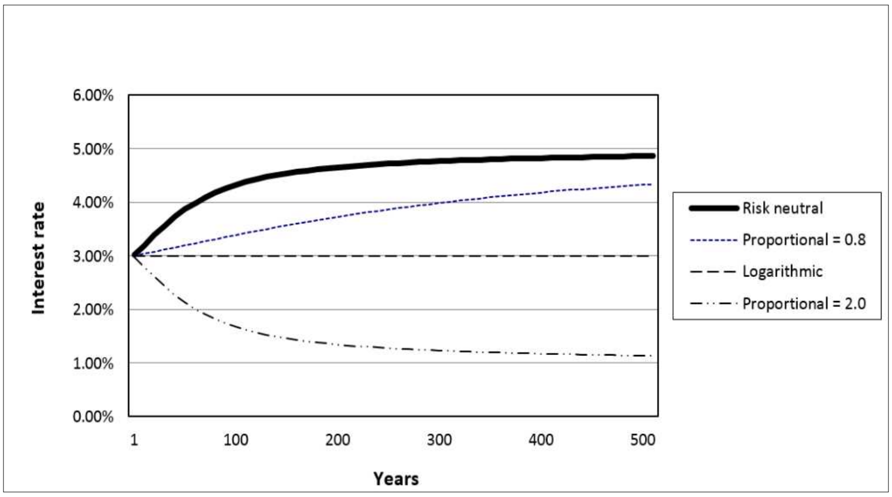
$$F = U^{-1}(p_1 U(C_t + \exp(r_1 t)) + p_2 U(C_t + \exp(r_2 t))) - C_t \tag{8}$$

where, $U^{-1}()$ is the inverse utility function, U() is the corresponding utility function, which is assumed to be of the constant-inter temporal-elasticity-of-substitution (CIES) type, which implies constant proportional risk aversion[4]. C_t is the decision maker' consumption at time t that is unrelated to the investment of \$1 and is considered certain.

C_t, which is not present in Buchholz and Schumacher (2008), was introduced to show that even though the utility function employed displays constant proportional risk aversion, the proportion that the yield of an investment of \$1 constitutes of C_t very much affects the effective risk aversion implied by the utility function.

The following Figure 1 assumes that $C_t = 0$, thus illustrating the results of Buchholz and Schumacher (2008). The behavior of the risk adjusted CERs is shown for risk aversions of $\sigma = 0.8$, $\sigma = 1$ and $\sigma = 2.0$. Of course, when $\sigma = 1$ then $U(C_t) = \ln(C_t)$ and $U^{-1}() = \exp(U(C_t))$.

Figure 1
Certainty equivalent rates when $C_t = 0$



As Buchholz and Schumacher (2008) observes, when $\sigma < 1$, risk adjusted CERs are increasing functions of time, and when $\sigma > 1$, they are decreasing. When $\sigma = 1$, the term structure of interest rates is flat. For reference, the correctly calculated market monetary CERs implicit in the original Weitzman model are also shown, and their curve is labeled "Risk neutral."

$C_t = 0$ means that all future consumption is derived from the investment and is therefore at risk. Consequently, Figure 1 shows the most risk averse behavior consistent with the value of σ for each curve. In Figure 2, C_t is set equal to the expected monetary yield of \$1 invested, that is, to $E(\exp(r_i t))$, meaning that an expected one half of total consumption is at risk. We see that in this case there is no declining risk adjusted CER, not even if $\sigma > 1$.

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[4] Its formulation is shown in Section 5 below.

Figure 2
Certainty equivalent rates when C_t = EFV of $1

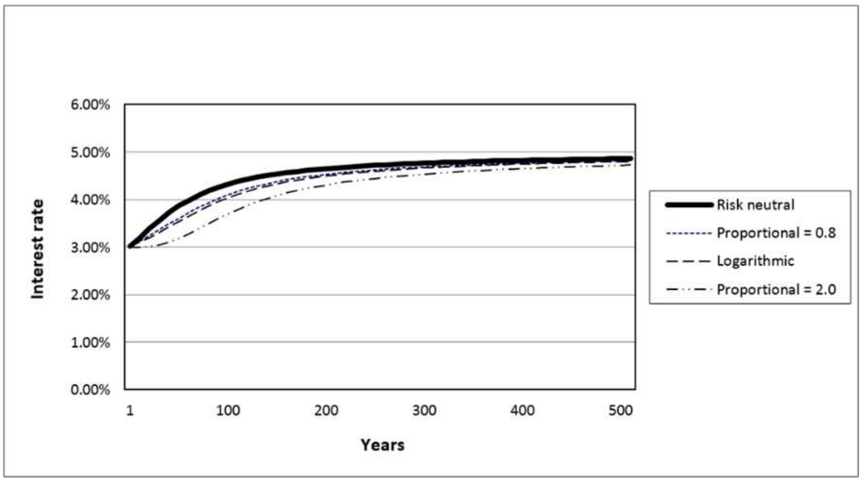
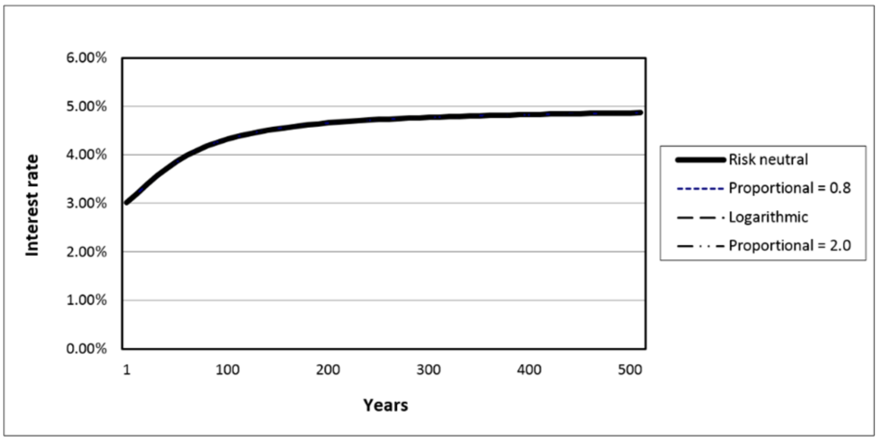


Figure 3 shows the case in which C_t is 100 times the EFV of the investment, meaning that less than 1% of total consumption is at risk.

Figure 3
Certainty equivalent rates when C_t = 100 · EFV of $1



The situation depicted in Figure 3 provides an explanation for why it is often said that public sector projects should be evaluated assuming risk neutrality. As most projects constitute only a small fraction of the public sector's budget, the degree of risk aversion assumed might make little difference, and the resulting behavior might be indistinguishable from risk neutrality for small investments. In that case, risk adjusted CERs would be virtually the same as the risk neutral CERs.

3. Discounting and valuation

The objective of Weitzman (1998) was to define a market yield CER that could serve as a hurdle rate that long lived projects would have to meet to ensure that they are at least as efficient in transferring resources from the present to the future as the capital market would be, assuming risk neutrality. Weitzman thought that the market CERs would be declining with time, which did not turn out to be the case. Buchholz and Schumacher (2008) thought that risk adjusted CERs would be declining for reasonable degrees of risk aversion, but that turns out to be unlikely for public sector projects of typical size.

While, as a practical matter, this establishes that certainty equivalent discount rates applicable to public sector projects should be increasing as a function of time, provided that interest rates are perfectly auto-correlated, as Weitzman (1998) assumes, there is another reason why risk adjusted CERs will not result in declining *monetary* hurdle rates, even under the assumptions used by Buchholz and Schumacher (2008).

Before proceeding further, it is worth examining the assumptions of the Weitzman (1998) model, the description of which is quoted in Section 1 above. The effect of the peculiar requirement that the investor decide on a very long lived project at time 0^{-}, an eye blink before clairvoyance strikes, is that it makes the yield uncertainty unique to the investor and therefore unhedgeable in a market. After clairvoyance has struck, again there is no market for risk, for then the serenity of certainty reigns.

The presence or absence of a capital market is crucial in interpreting the meaning of discounting. In textbook inter-temporal optimization, the market rate is a given for the investor, and helps define his budget constraint. It is the rate at which resources can be transferred in time in either direction, and therefore can convert the future returns of a project into present day cash. Therefore no recourse to a utility function is needed to decide on a project, for its net benefit can be readily converted into a certain present day gain or loss. If uncertainty is added to the model, but there are functioning markets, it is still possible to convert all benefits and costs to present day cash if the investor is willing to pay someone to absorb all risks. In this case, recourse to the utility function is needed only to determine if the investor might not be better off by assuming some of the risks himself. For this he will have to compute the certainty equivalents of all risks.

Interest rate risk is not a problem by itself, therefore, if there is a market for it, for then the resulting risk free transaction interest rate can be used as a discount rate directly. But if there is interest rate uncertainty, but no markets for interest rate risk, then only risk adjusted CERs are left as guides for investors.

Risk adjusted CERs are not proper discount rates, however. A proper discount rate helps define investors' budget constraints, on which they have no effect individually. The discount rate is the same for all market participants, and can be used to transfer resources from the present to the future and vice versa, because it is the interest rate at which transactions are possible. In contrast, risk adjusted CERs are different and unique for all market participants, as they depend on investors' degrees of risk aversion and on the inter-temporal distribution of the portion of their income or wealth. Most importantly, risk adjusted CERs cannot be used to transfer resources in any direction, and do not define budget constraints. They are merely valuation tools.

The Weitzman (1998) model was probably designed precisely to avoid the presence of a market for risk, perhaps to reflect the problem that in the very long term capital markets cannot fulfill all of their functions. Markets for risk for the very long term cannot exist, because the counterparties to the required risk swapping contracts are not contemporaneous. Transfer of financial resources from the future to the present is also not possible, for the same reason. The only capital market function that is left, is the transfer of resources from the present to the future, albeit at an interest rate that is uncertain.

In these circumstances, investors will have to make do with valuation: comparing the risk adjusted CERs of project returns with the risk adjusted CERs of market interest rates. Because risk adjusted market CERs are computed using utility functions, they are not directly comparable to the monetary CERs of uncertain project yields. Risk neutral investors, however, or those who effectively behave as such due to the relatively low amount at risk in their investments, can compute expected discount and compound factors conventionally, using the monetary CERs of uncertain market rates as discount rates, because for them risk adjusted CERs are the same as monetary CERs.

A few numerical examples will illustrate this. The calculations are conducted in the spirit of Buchholz and Schumacher (2008) by assuming that $\sigma = 2$ and $C_t = 0$, $t = 200$, and that the market has an equal probability of returning 1% or 5%, in which case the monetary yields of investing $1will be either $7.39 or $22,026.47, respectively. The expected value of these is $11,016.93, from which the monetary CER of the market can be computed. It is 4.65%. The certainty equivalent of the alternative monetary yields is only $14.77 for the risk averse investor, however. As the certainty equivalent amount is lower for the risk averse investor, his CER is lower as well, it is 1.35%. This does not mean that the risk adjusted CER of 1.35% can be used as a hurdle rate for the *monetary* expected yield of projects, however. It can only be used to discount their risk adjusted certainty equivalent yields. Only projects with risk adjusted CERs higher than 1.35% will be welfare enhancing, whatever their monetary CER might be.

Real life projects seldom have certain returns. In fact, it could be that the yield uncertainty of individual projects is higher than that of the market. Let's take the example of a project that yields 0% or 6% with equal probability. The expected monetary yield of this project in year 200 would be $81,377.40, the corresponding monetary project CER is 5.65%, but the risk adjusted project CER is only 0.35%. The risk averse investor would reject this project, and invest in the market instead, even though the project's monetary CER exceeds that of the market.

It is instructive also to look at a project that is acceptable. Assume it can either yield 1.6% or 6.6% with equal probability. Its monetary CER is 6.25%, while its risk adjusted CER is 1.95%. The expected monetary payout is $270,194.70, but the risk averse certainty equivalent is only $49.03. The certainty equivalent yield of $1 invested in the market is $14.77, which means that this protect yields $34.29 more than the market in certainty equivalent FV terms. Discounting this at 1.35% (we can compute its welfare equivalent present value with the CER because it is a certainty equivalent FV), we obtain the present value advantage of the project over the market, which is $2.31. Compounding this at the CER would of course yield $34.29 again, which shows that discounting and compounding are always consistent, but this is only valuation, not a possible transaction. Investing $2.31 in the market, which *is* a possible transaction, would yield $17.15 and $51,125.19 with equal probability in monetary terms, the certainty equivalent of which is $34.29. This illustrates the equivalence of the risk adjusted CER of 1.35% and the monetary CER of 4.65%.[5]

Notice that it would make no sense to discount either $17.15 or $51,125.19 (or their expected value) at the risk adjusted CER of 1.35%. Only certainty equivalents can be discounted or compounded with risk adjusted CERs, not conditional or expected monetary yields. To subject their projects to the efficiency test of the market, risk averse investors have

[5] These examples assume that project risk and market risk are perfectly correlated. This point is explained in Section 5.

to compare the CERs of their projects with those of the market. They cannot use their risk adjusted market yield CERs as hurdle rates to apply to the monetary flows of projects. Strictly speaking this is true for risk neutral investors as well, but as for them risk adjusted CERs happen to be identical to monetary CERs, they can evaluate their projects using the market monetary CERs as project hurdle rates.

The only time that a future amount can equally be discounted both by the monetary market CER and the risk adjusted market CER is to value truly risk free amounts (which could be called Divine IOUs, as opposed to certainty equivalents of risky project yields). This will result in different PVs for investors with differing degrees of risk aversion, of course, but all will interpret their PVs the same way. For instance a certain yield of $49.03 in year 200 implies a monetary return of 1.95%, and has a PV of $3.30 to the risk averse investor, discounted at 1.35%, while it is only worth $0.004482 in the present to the risk neutral investor, who discounted it at 4.65%. The risk averse investor can invest his $3.32 at the stochastic market yield and then compute the corresponding certainty equivalent, which will be $49.03. (Compounding $3.32 at his risk adjusted CER yields the same result, but that is only a valuation check, not a possible transaction.) The risk neutral investor can obtain the FV of investing his $0.004453 the same way, and it results in a certainty equivalent FV of $49.03. This could be computed even more directly, in this case, by compounding at the monetary market CER of 4.65%.

Because risk adjusted CERs cannot be used to discount monetary values, the assertion of Buchholz and Schumacher (2008) that “a discount rate that is declining over time, is nevertheless reasonable, since it can be justified by assuming a plausible degree of risk aversion” is only true for the headline risk adjusted CERs, but that does not mean that the monetary CERs of market interest rates are declining functions of time, as Weitzman (1998) asserts. Therefore Buchholz and Schumacher (2008) doesn’t address the original puzzle, as declining risk adjusted CERs are not equivalent to declining monetary CERs. Rather, they are the equivalents of *increasing* ones. As the relationship between risk adjusted CERs and risk neutral CERs is monotonic, for $\sigma > 1$, the lower the risk adjusted CER, the higher the underlying market yield, and hence the opportunity cost of capital.

Weitzman (1998) thought that monetary CERs would be declining with time. In fact, they increase, under the assumptions of his model. Projects whose monetary yield risks are the same as that of the market would have to have monetary yields at least as high as the market’s growing monetary CERs to be feasible, even while their risk adjusted CERs decline with time for some risk averse investors. Observe that in Figure 1 a growing monetary CER and a declining risk adjusted CER (for $\sigma = 2$) are derived from the same probability distribution of compound factors. Projects that have the same risk as the market will only have higher risk adjusted CERs than those of the market if their monetary CERs are higher than those of the market. Projects with risks higher than the market’s would have to yield even more. Given that the monetary CER of market rates grows with time if interest rates are perfectly auto-correlated, so does the monetary yield that risky projects must produce to be preferred to the investment of their cost in the market, even for risk averse investors.

4. Gollier and Weitzman on the Weitzman-Gollier Puzzle

Gollier and Weitzman (2009) postulate that if a decision maker optimizes his consumption path by reference to a linear budget constraint represented by interest rate r_i of state of the world i, then the optimal consumption trajectory for each scenario i must satisfy the following first order condition

$$V'_i(C_0) = V'_i(C_t)\, exp(r_i\, t) \tag{9}$$

where $V'_i(C_t)$ is the marginal utility of consumption of the period indicated by the subscript t of C, in the state of the world identified by subscript i of V'.

At time $t = 0^-$ a safe investment opportunity arises that expends marginal cost of δ in time period 0 to yield a marginal benefit of ε in time period t. The investment project will increase the expected utility of the decision maker if and only if

$$\varepsilon \sum p_i\, V'_i(C_t) \geq \delta \sum p_i\, V'_i(C_0) \tag{10}$$

Using optimality condition (9) this can be rewritten in two ways. The one called the "Weitzman approach" eliminates $V'_i(C_t)$ from (10) yielding:

$$\varepsilon \sum q^W{}_i\, exp(-r_i\, t) \geq \delta \tag{11}$$

where $q^W{}_i = p_i\, V'_i(C_0) / \sum p_i\, V'_i(C_0)$

According to Gollier and Weitzman (2009) this is equivalent to discounting ε at the following rate[6]:

$$R'_W = -\left(1/t\right)\ln\left(\sum_i q_i^W e^{-r_i t}\right) \tag{12}$$

Alternatively, the "Gollier approach" consist of eliminating $V'_i(C_0)$ from (10) yielding:

$$\varepsilon \geq \delta \sum q^G{}_i\, exp(r_i\, t) \tag{13}$$

where $q^G{}_i = p_i\, V'_i(C_t) / \sum p_i\, V'_i(C_t)$

According to Gollier and Weitzman (2009) this is equivalent to discounting ε at the following rate:

$$R'_G = \left(1/t\right)\ln\left(\sum_i q_i^G e^{r_i t}\right) \tag{14}$$

Since both (12) and (14) were derived from (10), it must be true that $R'_W = R'_G$. The authors go on to state that "This means that the adjustment of the valuation for risk resolves the 'Weitzman-Gollier puzzle'," denote this the "risk adjusted discount rate" R_*, and go on to state that "qualitatively the properties of the efficient discount rate $R_*(t)$ resemble closely those of $R_W(t)$ recommended by Weitzman, with the only quantitative difference being the

[6] The prime symbol applied to the cetratinty equivalent rates R' does not denote a derivative. It is used to distinguish them from those derived from market yields in expressions (3) and (4).

substitution of "Weitzman-adjusted probabilities" $\{q^W{}_i\}$ for the unadjusted probabilities $\{p_i\}$."[7]

The pairwise morphological resemblance between the definitions of R'_W and R'_G, as defined in expressions (12) and (14) respectively, on the one hand, and those of R_W and R_G, as defined in expressions (3) and (4) respectively, on the other, does not make them all equal, however. This is why Gollier and Weitzman (2009) fails to resolve the puzzle. As $R_W \neq R_G$ and $R'_W = R'_G$, it is impossible for all four to be equal. Gollier and Weitzman (2009) uses the cited morphological similarities to suggest that R_W is right, but for the puzzle to really have been solved thereby, it would also have to show that $R_W = R_G$, which is contrary to fact. The equality between R'_W and R'_G neither makes $R_W = R_G$ nor provides an explanation for $R_W \neq R_G$.

It is also not clear why R'_W should be the same as R_W. The next section explores this relationship, with the aid of a simple numerical example.

5. A numerical example of the Gollier-Weitzman model

The model proposed in Gollier and Weitzman (2009) will be used in this Section. The utility function proposed by the authors is:

$$V(C) = \sum_t^{\infty} e^{-\rho t} U(C_t) \tag{15}$$

where $\rho > 0$ is the pure rate of time preference and $U(C_t)$ is a utility function that the authors did not specify, but that will be taken here to be of the constant-inter temporal-elasticity-of-substitution (CIES) type:

$$U(C) = \frac{C^{1-\sigma} - 1}{1-\sigma} \tag{16}$$

where consumption $C > 0$, and the elasticity of marginal utility with respect to consumption $\sigma > 0$ but not equal to 1. This is also the measure of the decision maker's constant proportional risk aversion.

The above utility function will be maximized subject to a budget constraint given by an original endowment of consumptions C_t for time periods 0 and t, and a constant interest rate r_i for each scenario i. Under these circumstances the welfare impact of investing at time 0 for a yield of $\exp(r\,t)$ at time t can be measured by the changes in the value of the utility function (15) used.

In the numerical example proposed, the following parameters are assumed:

Table 1

Parameters of the numerical example

Scenario 1 interest rate, r_1	1%
Scenario 2 interest rate, r_2	5%

[7] Gollier and Weitzman (2009:8)

Probability of scenario 1, p_1	0.5
Probability of scenario 2, p_2	0.5
Consumption at time = 0	\$2,000
Annual growth of consumption	0.75%
Constant proportional risk aversion, σ	0.8
Pure rate of time preference, ρ	0.5%
Time t in years	200

As the arguments of Gollier and Weitzman are predicated on the decision maker being on his optimal consumption trajectory, the first thing to do is to calculate it for each scenario. This means that the investor pondering whether to invest in the safe project of the Weitzman model at time 0^{-} must also prepare to take action to bring himself into optimality once the interest rate to prevail in an instant and forever thence is revealed. To this end, he must solve for the optimal capital market action by maximizing, subject to his budget constraint, the following expression for each interest rate scenario, where x is the amount to invest (or borrow if negative):

$$\max V(C) = \frac{(C_0 - x)^{1-\sigma} - 1}{1-\sigma} + \frac{(C_t + e^{rt}x)^{1-\sigma} - 1}{e^{\rho t}(1-\sigma)} \tag{17}$$

Differentiating[8] this with respect to x, and setting the result equal to 0, gives the first order condition of the optimization:

$$\frac{e^{rt-\rho t}}{(e^{rt}x + C_t)^{\sigma}} + \frac{1}{(C_0 - x)^{\sigma}} = 0 \tag{18}$$

The optimal market action to be taken can be obtained by solving for x in the above expression[9]:

$$x = \frac{e^{\frac{rt}{\sigma}}C_0 - e^{\frac{\rho t}{\sigma}}C_t}{e^{\frac{\rho t}{\sigma}+rt} + e^{\frac{rt}{\sigma}}} \tag{19}$$

With the data of our simple example, the investor would borrow \$182.24 in the low interest rate scenario and would invest \$1,554.51 in the high interest rate scenario. It will be on the basis of the resulting optimal consumption paths that he will ponder whether to invest in the safe project of the Weitzman (1998) model.

At this point we are in the position of computing R'_W. To this end expression (10) will be adapted to our simple example, as follows, keeping the notation of the previous Section:

$$\varepsilon\,(\,p_1\,V'_1(C_t) + p_2\,V'_2(C_t)\,) \geq \delta\,(\,p_1\,V'_1(C_0) + p_2\,V'_2(C_0)\,) \tag{20}$$

Employing the "Weitzman approach," which seeks to define a discount factor, this becomes:

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[8] Result courtesy of http://www.derivative-calculator.net

[9] Solution courtesy of http://www.wolframalpha.com

$$\frac{p_1 V_1^{'}(C_t) + p_2 V_2^{'}(C_t)}{p_1 V_1^{'}(C_0) + p_2 V_2^{'}(C_0)} \geq \frac{\delta}{\varepsilon} \tag{21}$$

If we convert the above relationship into a strict equality, and replace δ/ε by D, we can interpret D as the expected value of the Weitzman approach discount factor, from which we can compute:

$$R'_W(\mathrm{t}) = -\,(1/\mathrm{t})\ \ln(\ D\ ) \tag{22}$$

To follow the "Gollier approach" of finding a compound factor instead, all we have to do is invert expression (22), make it a strict equality, and replace ε/δ by F, which can be interpreted as the expected value of the Gollier approach compound factor, from which we can compute:

$$R'_G = (1/\mathrm{t})\ \ln(\ F\ ) \tag{23}$$

It is clear from this that $D = 1/F$, which is as it should be, as expected discount factors are always the inverses of expected compound factors, and therefore $R'_W = R'_G$. The implicit CER can be computed from either. Notice that the transformations defined by (11) through (14) are not needed for computational purposes. Gollier and Weitzman (2009) only performed those to show that expressions that look like (3) and (4) can be derived from (10).

Differentiating[10] (15) with respect to C_0 and C_t we obtain the marginal utilities[11]:

$$V_i^{'}(C_0) = \frac{1}{C_0^{\sigma}} \tag{24}$$

$$V_i^{'}(C_t) = \frac{1}{e^{r_i t} C_t^{\sigma}} \tag{25}$$

Using expressions (24) and (25) in expression (21) allows us to calculate discount factor D, and from that R'_W, which computes to 1.76% with the data assumed[12]. This is the annual monetary return of a small project that will leave the expected welfare of the decision maker unchanged, when it is defined as in (12), or in (22), its equivalent for our example.

Given that consumption has been optimized for both scenarios, the risk adjusted certainty equivalent market interest rate (CER) can be computed as well. The total utility value of the optimized scenario 1 case is 27.419, while that of scenario 2 is 69.196. Their[13] expected value is 48.3076. The certainty equivalent rate is obtained by finding out what

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[10] Results courtesy of http://www.derivative-calculator.net

[11] In scenario 1 C_0 = 2182.24, therefore $V'_1(C_0) = 1/(2182.24^{0.8}) = 0.002132$. In the same scenario C_t = 7,616.8, therefore $V'_1(C_t) = 1/(\exp(0.01\cdot200)\ 7616.8^{0.8}) = 0.28859$. In scenario 2 C_0 = 445.49, therefore $V'_2(C_0) = 1/(445.49^{0.8}) = 0.007602$. In the same scenario C_t = 34,249,304.8, therefore $V'_2(C_t) = 1/(\exp(0.05\cdot200)\ 34{,}249{,}304.8^{0.8}) = 3.45131\text{E-}07$.

[12] D = (0.5·0.002132+0.5·0.28859)/(0.5·0.007602+0.5·3.45131E-07) = 33.69025. From this R'_W = ln(33.69025)= 1.7586%

[13] In scenario 1: $(2182.24^{0.2}-1)/0.2+(7{,}616.8^{0.2}-1)/(0.2\cdot\exp(0.005\cdot200)) = 27.419$ and in scenario 2 $(445.49^{0.2}-1)/0.2+(34{,}249{,}304.8^{0.2}-1)\ /\ (0.2\cdot\exp(0.005\cdot200)) = 69.196$.

deterministic interest rate yields the same utility as the computed expected utility. In this calculation, done numerically using the goal seek function of Excel, consumption has to be continuously re-optimized, so that at the end the investor's consumption is also optimized at the computed CER. The risk adjusted CER that corresponds to the computed expected utility is 3.89%. The market action that turns out to be optimal at the risk adjusted CER is the investment of $1,333.02.

Table 2 shows the values obtained for R'_W, for risk adjusted market CERs, and for a number of other results of interest as a function of t. R_W, the certainty equivalent of time reversed negative compounding that Weitzman called certainty equivalent discount rate, is below 3% and declining. R'_W approaches this, but does not quite reach it. With the degree of risk aversion implicit in the numerical example, the risk adjusted market CER is above 3% and grows in the long term. The risk neutral CER is also above 3% and grows in the long term, as it should be for perfectly auto-correlated interest rates.

Table 2

Model results for selected time horizons

Years till time *t*	**50**	**100**	**200**	**300**	**400**	**500**
R_W, negative compounding CER	2.13%	1.67%	1.35%	1.23%	1.17%	1,14%
R'_W or R'_G	2.65%	2.18%	1.76%	1.62%	1.57%	1.54%
Risk adjusted market CER	3.81%	3.78%	3.89%	4.08%	4.23%	4.36%
Risk neutral market CER	3.87%	4.33%	4.65%	4.77%	4.83%	4.86%

Figure 4
Comparison of CER and *R'*$_W$

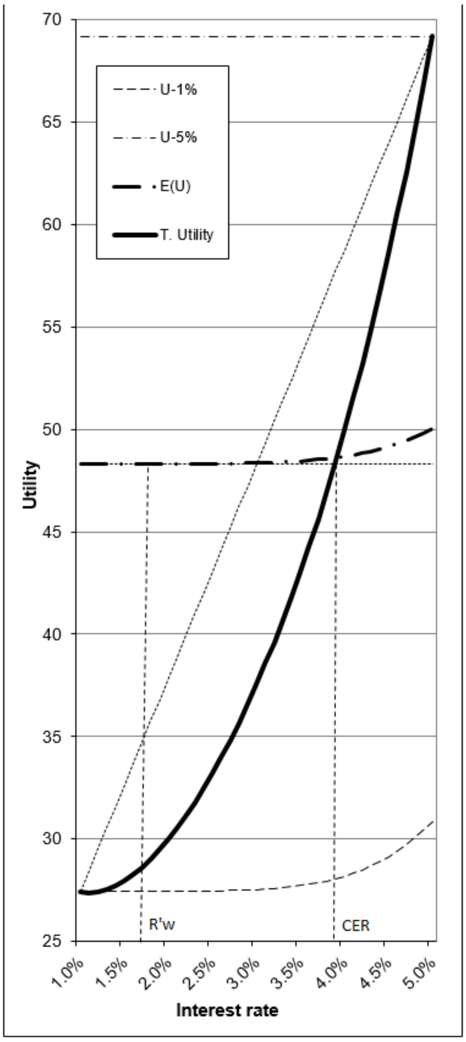

Figure 1 shows how the risk adjusted market CER and R'_W (or R'_G) are derived. The curve labeled “T. Utility” (total utility) plots the deterministic utility associated with all possible interest rates between the rates of the two scenarios of this case, 1% and 5%. The risk adjusted CER is computed in the usual way. The mid-point of the cord stretching between the 1% and 5% values of the total utility curve gives the expected utility, 48.3076 in this case. The dotted horizontal line through that point intersects the total utility curve at the risk adjusted CER, 3.89% in this case. For this calculation the horizontal axis measures market interest rates.

For calculating R'_W, the horizontal axis is interpreted as the annual yield of the safe project (its return is the same in both scenarios). The curve labeled “U-5%,” found at the top of the figure, gives total utility, as a function of project return, for the 5% interest rate scenario. Notice that this curve is virtually constant, because the small investment of $1 has little effect on the already high utility of the 5% scenario. The curve labeled “U-1%” does the same for the 1% project return scenario. In this case, however, higher small investment yields do make a difference. This curve begins to slope upwards perceptibly by the time that the project yield exceeds 3%. The curve labeled “E(U)” is the expected value of the preceding two. R'_W is found where this curve equals the expected utility of the two scenarios, which is the same as that of the risk adjusted CER. The reason why the total utility curve has a much higher slope than the average utility curve just described, is that the former reflects large consumption reallocations as a function of interest rate changes, whereas the latter was computed assuming a small investment of $1.

There are two CERs in this example, because there are two decisions. The market CER corresponds to the consumption path optimization decision, while R'_W corresponds to the small safe investment decision. If the model were modified to be more realistic, *i.e.*, to rule out the possibility of borrowing from the distant future, and to limit the fraction of current consumption that can be invested for the very long term, the two CERs would begin to converge, and would merge into one if only the return on the small project were at risk. As for some values of effective risk aversion the market CER is increasing with time, so will eventually R'_W in such cases. But none of this is of any consequence, for the arguments presented when discussing the declining CERs of Buchholz and Schumacher (2008) in Section 3 apply equally to R'_W or R'_G, even if they are always strictly declining, for they cannot be used to discount monetary project flows, and the fact that they are declining does not make the monetary opportunity cost of investments declining.

Because of the consumption path optimization that has taken place, the welfare value of a safe investment with a return of R'_W is the same as that of an investment of $1 in the market, which is by definition perfectly correlated with the market's own return. It is interesting to compute the value of investing $1 in a project that has the same returns as the market, but in a manner that is uncorrelated with it. The uncorrelated project's CER cannot be computed from expression (21), of course. It needs to be arrived at by numerical methods (the Goal Seek function of Excel will do), and can be computed from the following data:

Table 3
Utility of the uncorrelated market yield

		Market return 1%	Market return 5%
Project return 1% Yield in year 200 is $7.39	C_0	$2,181.2428	$444.4909
	C_{200}	$7,624.1648	$34,249,346.9708
	Total Utility	27.4189	69.1886
Project return 5% Yield in year 200 is $22,026.47	C_0	$2,181.2428	$444.4909
	C_{200}	$29,643.2416	$34,271,365.0475
	Total Utility	30.8492	69.1962

Table 3 contains four blocks corresponding to all combinations of market and project returns. In each, the consumption values C_0 and C_{200} are taken from the equilibrium path calculated with the Gollier Weitzman (2005) model, with the following adjustments: C_0 is always reduced by $1 indicating that an investment in that amount is being made, and C_{200} is augmented by the monetary yield of the project, as corresponds to the case (shown in the first column). The total utility is the third value in each cell, and corresponds to the consumption schedule specified in the cell. The expected value of the four utilities (each has a probability of 0.25 in our example) is 49.1633.

The project CER is found by constructing another table that is identical to the above in all respects, except that the project return is the same in both cells of column 1. That return is varied until the computed expected utility reaches 49.1633. That value is reached when the project yield is a certain $8,122.19. The project return is then 4.50%, which is the project CER.

We can also what happens when a project's return is the same as that of the market, but the correlation coefficient between the project's and market's returns is –1. Table 4 helps find the CER of that project, in the manner already described.

Table 4
Utility of the negatively correlated market return

		Market return 1%	Market return 5%
Project return 1% Yield in year 200 is $7.39	C_0		$444.49090
	C_{200}		$34,249,346.9708
	Total Utility		69.1886
Project return 5% Yield in year 200 is $22,026.47	C_0	$2,181.2428	
	C_{200}	$29,643.2416	
	Total Utility	30.8492	

The assumed correlation coefficient rules out the cases in which the market and project returns are the same. Therefore the expected utility is computed only over the remaining two possible cells. It is 50.0189. The project CER is found the same way as in the previous case. The certainty equivalent yield is $21,948.7490, and the corresponding CER is 5.00%.

Project with risks that are equal to the market's risks, but which are uncorrelated with them, are more valuable to risk averse investors than Divine IOUs. If the correlation is negative, then they are even more valuable. But this finding, as will be seen in the next section, does not affect the discount rate to be used.

Exploring the Gollier and Weitzman (2009) model has provided interesting insights and results, but the equality between R'_W and R_W is not one of them. Gollier and Weitzman (2009) has not resolved the Weitzman-Gollier Puzzle. It has defined $R'_W = R'_G$, the equality between which is due to the fact that rates derived from expected discount factors are the same as those derived from the corresponding expected compound factors, but it has not explained why R_W of Weitzman (1998) is not the same as R_G of Gollier (2003). The difference between these is not due to the fact that the former was derived from an expected discount factor while the latter was derived from an expected compound factor. R_W could also have been derived from its equivalent expected compound factor: $1 / \Sigma\, p_i \exp(-r_i t)$. Their difference is due, rather, to the fact that R_W was derived from the wrong expected discount factor.

6. Computing expected PVs under risk aversion

Risk neutral investors can compute EPVs simply by using expression (6), *i.e.*, discounting with the inverse expected compound factor. Examples were provided in Section 3 of the congruence between risk averse CER discounting and market monetary rate compounding. In this section, a variant of the Gollier-Weitzman (2005) model will be used to further explore the behavior of EPVs under risk aversion.

The cited model was first made somewhat more realistic though three changes: (1) there is no massive consumption realignment in the first eye-blink of the model's time frame. C_0 is the same in both scenarios; (2) borrowing from the distant future is ruled out; and (3) the fraction of C_0 that can be invested in the market is capped, in the case of the following examples to about 1% of C_0. Consequently C_0 is taken to be \$2,000, and the amount invested at the uncertain market rate is \$20[14]. All other assumptions of the numerical example used already are kept. The sample EPV calculations provided in Section 3 did not allow for the welfare consequence of making an initial investment. In the examples of this Section, it will be assumed that all projects entail an investment of \$1.

Because of the constraints on investment assumed in this modified model, the consumption path of the investor is not a global optimum. For this reason, in our example, the small investment return that leaves welfare unchanged, called sqCER (for *status quo* CER) in the Appendix[15], is not equal to mCER, the certainty equivalent of market rates. It is just the annual interest rate equivalent of the weighted average of the marginal rates of substitution between present and future consumptions under the two scenarios. Being purely a function of the utility function and of the investor's endowment, sqCER provides no opportunity cost of capital information and therefore cannot be used as a discount rate. Because of the global optimization that took place in the model of the previous section,

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[14] This is the optimal investment amount given the cited constraints, when $\sigma = 0.8$. C_t in the first scenrio becomes \$9,111.16 and in the second \$449,492.69.The calulations shown in this section are not dependent on the investor having optimized his consumption path.

[15] It is equal to 1.43% when $\sigma = 0.8$, and 2.37% when $\sigma = 2$.

sqCER and mCER were equal[16], because in it the investor adjusts his consumption path to market rates.

For a risk averse investor, the calculation of the EPV of a project's future yield can be undertaken in two steps[17]:

1. Compute the certainty equivalent yield of the project, *i.e.*, the EFV of an equally valuable safe project of the same investment cost;
2. Find the sum that, when invested in the market, will have the same certainty equivalent yield (EFV) as the project. This sum is the EPV.

The internal rate of return implicit in the EPV, EFV pair just defined, is the discount rate. It follows from this discussion that discount rates will vary, as they will be investor and project specific.

Table 5 shows the project CERs (pCER), EFVs, EPVs and the implicit certainty equivalent (CE) discount rates for projects with return structures equal to that of the market, and the correlations with the market's risk that are shown in the table, for two degrees of risk aversion. The pCER that corresponds to Correlation =1 is the market CER (that of the equivalent safe project), and is also exactly the discount rate to be used to discount project certainty equivalent yields, but only when Correlation = 1, as shown in Table 5.

Table 5

pCERs, EPVs and discount rates for equiprobable project rates $z_1 = 1\%$, $z_2 = 5\%$

	σ = 0.8				**σ = 2**			
Corre-lation	**pCER**	**EFV**	**EPV**	**CE discount rate**	**pCER**	**EFV**	**EPV**	**CE discount rate**
1	3.43%	$958.15	$1	3.43%	1.39%	$16.04	$1	1.39%
0	4.53%	$8,557.95	$7.97	3.49%	4.26%	$5,000.47	$621.02	1.04%
-1	4.95%	$19,795.40	$16.62	3.54%	5.00%	$21,925.91	$2,709.86	1.05%

The examples of Table 5 show that regardless of the degree of risk aversion, projects that have the same risk as the market ($z_1 = r_1$ and $z_2 = r_2$), but that is uncorrelated with that of the market, are more valuable to investors than safe projects (Divine IOUs). Having negative correlation is more valuable still, in which cases pCERs are higher even than the risk neutral market CER. We can also see that when Correlation $\neq 1$, the discount rate is not exactly the risk adjusted market CER. This is so because the EFVs of projects yielding uncorrelated market rates are higher than that of the "market project," and therefore involve different utility function ranges.

It is also interesting to note that the results of Table 5 are invariant to the value of pure rate of time preference assumed. Changing its value changes the levels of utility, but not the

[16] 1.76% for σ = 0.8

[17] The calculation formulas are given in the Appednix.

CERs. This is unsurprising, as the pure rate of time preference will affect the valuation of project and market yields equally, and therefore has no effect on their comparison.

The notion that CE discount rates can be used to discount raw monetary flows of projects, which is not the case, appears to be present in much of the discount rate literature. Only project pCERs can be discounted with mCERs, and even that is not exact, as Table 5 has shown, as the correct discount rates are project specific. None the less, and to further emphasize the difference, the monetary discount rates of the above project types will be calculated below.

The following Tables 6 and 7 present an analysis of the monetary CERs that projects with a risk structure similar to that of the market (and varying degrees of correlation with it) need to have to be accepted by investors of varying degrees of risk aversion. To this end, the stochastic annual returns of investments are defined as $zl = z\ (1 - 2/3)$ and $zh = z\ (1 + 2/3)$, where zl and zh are the low and high project rates of return, respectively, defined as a function of their expected value z. For instance, when $z = 3\%$, $zl = 1\%$ and $zh = 5\%$. Thus, reducing z will reduce the expected return of a project while keeping its risk structure intact (the probabilities of the scenarios are kept unchanged as well). Consequently, monetary CERs, which are also a function of t, will change as well.

Tables 6 and 7 show the monetary CERs of investments so defined that their pCER is the same as the investor's mCER. In other words, they possess the minimum monetary expected return needed to be feasible, given their risk structure and the investors' risk aversion and endowment. For the investment types in Table 5 that have EPV > \$1, the return structure has to be lowered, by lowering z until EPV = 1, or, equivalently, until pCER = mCER. Once the right zl and zh values have been found, the corresponding risk neutral or monetary CER was computed, taking the value of t into account. In Table 6, t = 200 years, while in Table 7, t = 300 years. The monetary CERs are actually the internal rates of return (IRRs) of these projects, because as their EPVs are equal to 1, their ENPVs equal zero. These are therefore the *hurdle monetary expected rates of return* for investments of the given type of risk.

Table 6

Monetary hurdle expected rates and CE discount rates for t =200 years

Risk aversion	σ = 0		σ = 0.8		σ = 2	
Correla-tion	**monetary CER**	**CE discount rate**	**hurdle monetary CER**	**CE discount**	**monetary CER**	**CE discount**
1	4.65%	4.65%	4.65%	3.43%	4.65%	1.39%
0	4.65%	4.65%	3.45%	3.43%	1.22%	1.39%
-1	4.65%	4.65%	3.11%	3.43%	1.09%	1.39%

Table 7

Monetary hurdle expected rates and CE discount rates for t =300 years

Risk aversion	σ = 0		σ = 0.8		σ = 2	
Correla-tion	**monetary CER**	**CE discount rate**	**hurdle monetary CER**	**CE discount**	**monetary CER**	**CE discount**
1	4.77%	4.77%	4.77%	2.86%	4.77%	1.00%
0	4.77%	4.77%	2.89%	2.86%	1.00%	1.00%
-1	4.77%	4.77%	2.63%	2.86%	0.80%	1.00%

The first rows of Tables 6 and 7 illustrate the conclusion of Section 3: declining risk averse CE discount rates correspond to growing monetary CERs. This means that investments with the same risk structure as the market, and whose risks are correlated with those of the market, must have monetary CERs equal to that of the market to be feasible. In this particular case, the EPV of the investment can equally be computed by discounting the CE EFV with the risk averse CE discount rate, or by discounting the monetary EFV by the risk neutral CE discount rate. This should make it clear that it is wrong to discount monetary yields with risk averse CE discount rates.

The first two columns of Tables 6 and 7 show that for risk neutral investors, which is likely to be the case of the public sector, all monetary CERs are all equal, as only expected returns matter, not correlations. The risk neutral market CER is always the discount rate, which is directly applicable to monetary flows of investments. (In these Tables even CE discount rates are the same in each column, because all project EPVs are equal to 1, hence touch the same utility function points.)

Investments with risks that are uncorrelated, or negatively correlated with the market are more valuable to risk averse investors that Divine IOUs or investments with the same risks as the market, because of the growing divergence of the values of the monetary compound factors of the two scenarios. This, which was already shown by their higher pCERs in Table 5, is now shown by the lower required expected monetary returns in Tables 6 and 7.

Comparing required monetary yields between Table 6 (200 years) and Table 7 (300 years), we can see that when investment and market risks are positively correlated, the monetary yield requirement is a growing function of time, regardless of degree of risk aversion.

However, for investments with the same risks as the market, but which are uncorrelated or negatively correlated to the risks of the market, the monetary yield requirements are a declining function of time for risk averse investors. This result is contingent, of course, on the assumption that the investor has already invested in the market for the same maturity *t*. This condition is met in this model, thanks to the fact that it is a two period model and that therefore the only investments possible are of matching maturities by definition. But in reality this condition cannot be met, for while it is possible to make investments that will yield benefits centuries from now, it is not possible to make balloon loans of matching maturity. When C_t is the same under both scenarios, the value of the correlation parameter makes no difference, and the monetary hurdle rates are those that are shown in the first row of Tables 6 and 7.

Buchholz and Schumacher (2008) show that risk adjusted CERs are declining function of time for certain degrees of risk aversion. This has been corroborated by the examples in this Section, which also show, however, that is this is not the same as saying that monetary CERs do, as Weitzman (1998) claimed.

Gollier and Weitzman (2009) concludes that "The bottom-line message that we wish for readers to take away from this paper is the following. When future discount rates are uncertain but have a permanent component, then the "effective" discount rate must decline over time toward its lowest possible value." The thought that the monetary flows of investments can be discounted with market CERs seems to be implicit in this conclusion, for otherwise the effect sought in Weitzman (1998) would not be present. But certainty equivalent discount rates can only be applied to certainty equivalent yields, and, as shown by the examples of this Section, monetary expected hurdle rates will be an increasing function of time for projects of a given risk profile.

7. Conclusions

The Weitzman-Gollier puzzle resulted from an inconsistency between alternative methods of computing expected market yields, in a model that assumes risk neutrality. This inconsistency cannot be resolved by adding risk aversion to the model, for an investor's risk aversion only affects his own valuations, not market yields. The resolution of the puzzle lies in the finding that in attempting to compute expected net present values, Weitzman (1998) uses an incorrect expression. Correcting it, the inconsistency disappears.

The conclusion that Weitzman (1998) derives from the model it presents, namely that the longer lived an investment project, the less efficient it would need to be in transferring resources from the present to the future, is inconsistent with the model's premises. The opposite follows from the assumption of perfect auto-correlation of interest rates.

The Weitzman (1998) model was framed in the context of risk neutrality. Most of the subsequent literature introduced utility functions that can describe varying types and degrees of risk aversion, and raises questions about investor behavior that are interesting to study. It is for this reason that the bulk of this paper was devoted to analyzing decisions to be taken by risk averse investors. The calculations presented in this paper derive from the assumptions specifically made in it, but the conclusions derived will probably hold for other parametrizations of the constant proportional risk aversion family of utility functions as well.

Absent a market in which interest rate risk can be hedged, investors in very long maturing projects must rely on the comparison of project CERs with market CERs to adequately measure the opportunity cost of their funds. In the case of risk neutral investors, the expected compound or discount factors can readily be used to compute EPVs and EFVs, because risk neutral CERs are the same as monetary CERs. Risk neutral investors are indifferent between sums that are certain (Divine IOUs) and risky yields of equivalent expected values. Risk neutral investors will agree on mCERs provided that they assess market return probabilities equally.

Risk averse investors will have different mCERs, however, even if they expect the same market returns, because their degrees of risk aversion and endowments differ. Their discount rates, applicable only to pCERs, will differ, but not just among investors, but also across projects, even if they are based on the same forecast of market rates.

Declining risk averse CE discount rates correspond to growing monetary CERs. This means that investments with the same risk structure as the market, must have monetary CERs equal to that of the market to be feasible.

It is not a good idea to offer policy advice based on results derived from theoretical models that are unrealistic in many respects. It should also be noted that cost benefit analysis is not based on the maximization of the type of utility function around which the discussion of this topic has centered. Nonetheless, the question of the term structure of interest rates raised in these models is relevant to the determination of the opportunity cost of capital. The conclusion of this paper is that to take this effect into account, monetary yields in the distant future would only not have to be discounted at increasing monetary discount rates (explicitly for risk neutral investors, such as the public sector, or implicitly for risk averse ones) if the assumption of perfectly auto-correlated interest rates failed to hold sufficiently.

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APPENDIX

This appendix contains the formulas used in the calculations performed to develop Tables 5-7. These formulas can be copied into Excel, the Goal seek function of which will reproduce the results of the calculations. The precision of the calculations should be set to at least 1E-10.

Assign values to the following variables by placing them in cells that are named with the given variable names:

Market data

Variable name	Concept	Value
mCER	risk adjusted market certainty equivalent rate	Solve for this
rl	low scenario market interest rate	1.00%
rh	high scenario market interest rate	5.00%
tp	pure rate of time preference	0.50%
mpl	probability of low market interest rate scenario	0.5
mph	probability of high market interest rate scenario	0.5
C0l	Consumption at time 0 in the low scenario	2000
C0h	Consumption at time 0 in the high scenario	2000
Ctl	Consumption at time t in the low scenario	2000*EXP(0.0075*t) +20*EXP(rl*t) = 9,111
Cth	Consumption at time t in the high scenario	2000*EXP(0.0075*t) +20*EXP(rh*t) = 449,493
t	time in years	200
ra	coefficient of risk aversion	0.8

With these values the expected utility investing $1 in the market is equal to:

mpl*(((C0l-1)^(1-ra)-1)/(1-ra)+((Ctl+EXP(rl*t))^(1-ra)-1)/((1-ra)*EXP(tp*t)))+
mph*(((C0h-1)^(1-ra)-1)/(1-ra)+((Cth+EXP(rh*t))^(1-ra)-1)/((1-ra)*EXP(tp*t)))

and the expected utility of investing $1 to receive a certain return of mCER is

mpl*(((C0l-1)^(1-ra)-1)/(1-ra)+((Ctl+EXP(mCER*t))^(1-ra)-1)/((1-ra)*EXP(tp*t)))+
mph*(((C0h-1)^(1-ra)-1)/(1-ra)+((Cth+EXP(mCER*t))^(1-ra)-1)/((1-ra)*EXP(tp*t)))

mCER can be found with the Goal seek function of Excel by changing the value of mCER until the difference between the above two expressions becomes zero.

Project data

Variable name	Concept	Value
pCER	project certainty equivalent rate	Solve for this
zl	low scenario project return	1.00%
zh	high scenario project return	5.00%
ppl	probability of the low scenario	0.5
pph	probability of the high scenario	0.5

corr	Correlation coefficient (-1, 0, or 1 only)	1

With these values the utility of investing $1 in a safe project yielding pCER is:

```
mpl*(((C0l-1)^(1-ra)-1)/(1-ra)+((Ctl+EXP(pCER*t))^(1-ra)-1)/((1-ra)*EXP(tp*t)))+
mph*(((C0h-1)^(1-ra)-1)/(1-ra)+((Cth+EXP(pCER*t))^(1-ra)-1)/((1-ra)*EXP(tp*t)))
```

while the utility of investing $1 in a project with parameters as described above is:

```
IF(corr<0,0,IF(corr=0,1,1/(ppl*mpl+pph*mph)))*ppl*mpl
*(((C0l-1)^(1-ra)-1)/(1-ra)+((Ctl+EXP(zl*t))^(1-ra)-1)/((1-ra)*EXP(tp*t)))
+IF(corr>0,0,IF(corr=0,1,1/(pph*mpl+ppl*mph)))*ppl*mph
*(((C0h-1)^(1-ra)-1)/(1-ra)+((Cth+EXP(zl*t))^(1-ra)-1)/((1-ra)*EXP(tp*t)))
+IF(corr>0,0,IF(corr=0,1,1/(pph*mpl+ppl*mph)))*pph*mpl
*(((C0l-1)^(1-ra)-1)/(1-ra)+((Ctl+EXP(zh*t))^(1-ra)-1)/((1-ra)*EXP(tp*t)))
+IF(corr<0,0,IF(corr=0,1,1/(ppl*mpl+pph*mph)))*pph*mph
*(((C0h-1)^(1-ra)-1)/(1-ra)+((Cth+EXP(zh*t))^(1-ra)-1)/((1-ra)*EXP(tp*t)))
```

Using Goal seek to set the difference between the above two expressions to zero by changing pCER will define the project's certainty equivalent rate. Notice that the above expression contains four terms, each corresponding to a combination of low and high market and project rates. The four IF functions ensure that combinations ruled out when corr $\neq 0$ will be multiplied by zero, and that conditional probabilities always add up to one.

Define the following additional variables:

Expected future and present values

Variable name	**Concept**	**Value**
pFV	project Future Value	= EXP(pCER*t)
pPV	project Present Value	solve for this
dscR	discount rate	= LN(pFV/pPV)/t

The certainty equivalent future value of the project being tested is computed as above from pCER. The project present value pPV can be solved for by setting to zero the difference between the following two expressions.

Expected utility of the future value of the project:

```
mpl*((C0l^(1-ra)-1)/(1-ra)+((Ctl+pFV)^(1-ra)-1)/((1-ra)*EXP(tp*t)))
+mph*((C0h^(1-ra)-1)/(1-ra)+((Cth+pFV)^(1-ra)-1)/((1-ra)*EXP(tp*t)))
```

Expected utility of investing amount pPV in the market:

```
mpl*(((C0l)^(1-ra)-1)/(1-ra)+((Ctl+pPV*EXP(rl*t))^(1-ra)-1)/((1-ra)*EXP(tp*t)))
+mph*(((C0h)^(1-ra)-1)/(1-ra)+((Cth+pPV*EXP(rh*t))^(1-ra)-1)/((1-ra)*EXP(tp*t)))
```

Once pPV has been obtained, the implicit discount rate can be computed as defined in the table above.

To compute the return of the small investment that will leave the investor's utility constant, define the following additional variable:

The *status quo* CER

Variable name	Concept	Value
sqCER	*status quo* CER	Solve for this

To solve for sqCER, change it until the difference between the two following expressions becomes zero:

Utility of an investment that returns sqCER:

mpl*(((C0l-1)^(1-ra)-1)/(1-ra)+((Ctl+EXP(sqCER*t))^(1-ra)-1)/((1-ra)*EXP(tp*t)))
+ mph*(((C0h-1)^(1-ra)-1)/(1-ra)+((Cth+EXP(sqCER*t))^(1-ra)-1)/((1-ra)*EXP(tp*t)))

Utility of the status *quo*, that is, the expected utility of the consumption path given, without additional investments:

mpl*((C0l^(1-ra)-1)/(1-ra)+(Ctl^(1-ra)-1)/((1-ra)*EXP(tp*t)))
+ mph*((C0h^(1-ra)-1)/(1-ra)+(Cth^(1-ra)-1)/((1-ra)*EXP(tp*t)))