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Abstract

We consider the role of the nonlinear commuting cost function in determination of the equilibrium commuting pattern where all agents are mobile. Previous literature has considered only linear commuting cost, where in equilibrium, all workers are indifferent about their workplace location. We show that this no longer holds for nonlinear commuting cost. The equilibrium commuting pattern is completely determined by the concavity or convexity of commuting cost as a function of distance. We show that a monocentric equilibrium exists when the ratio of the firm agglomeration externality to commuting cost is sufficiently high. Finally, we find empirical evidence of both long and short commutes in equilibrium, implying that the commuting cost function is likely concave.

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1 Introduction

How does the local labor market interact with commuting cost to produce the equilibrium commuting pattern? In a model with identical commuters, can cross-commuting, where one commuter's path to their job strictly contains another commuter's path, occur in equilibrium? In fact, Fujita and Thisse (2013, p. 201) state that "in any spatial equilibrium configuration, cross-commuting does not occur." We show in this paper that when commuting cost is strictly concave in distance, cross-commuting is an equilibrium phenomenon.

For an empirical viewpoint, consider the 242 municipalities in Tokyo Metropolitan Area consisting of Saitama, Chiba, Tokyo, and Kanagawa prefectures. If we regress the municipal yearly wage in the manufacturing sector in 2012 on a quadratic function of distance from Tokyo Station, we have:

$$\begin{aligned} \text{wage} \quad = \quad & -0.0294 * \text{distance}^2 + 1.49 * \text{distance} + 429 \\ & (-2.26) \qquad\qquad\qquad (1.29) \qquad\qquad\qquad (18.8) \\ & (t\text{-statistics in parentheses}) \end{aligned}$$

That is, the wage is decreasing and concave in the distance. This is in accord with the findings of Timothy and Wheaton (2001), where the wage is increasing and concave in commuting time in Boston and Minneapolis St. Paul in 1990. We will attempt to explain these phenomena using a simple theoretical model.

In general, the urban labor market interacts with commuting in interesting ways. For instance, the wage arbitrage condition says that no one can gain by changing her workplace location, given her residence location. In the literature, this statement is further specialized to mean that each worker is indifferent to her workplace. It is known that in order to satisfy this condition, one has to assume a linear commuting cost, as in the previous literature such as Ogawa and Fujita (1980), Fujita and Ogawa (1982), and Lucas and Rossi-Hansberg (2002).[1] If the commuting cost is nonlinear, the wage arbitrage condition does not hold, and hence, given her residence location, each worker strictly prefers the one particular location of their workplace that max-

[1] Although Lucas and Rossi-Hansberg use a time cost of commuting that is exponential in commuting distance, by taking logarithms of their equations, for example (3.3) and (3.4), we obtain a linear wage no-arbitrage equality.

imizes her utility relative to other workplaces. In this case, there is no guarantee that each worker can find a workplace so that the urban spatial labor market clears. Nevertheless, we show in this paper that when the commuting cost is either strictly convex or strictly concave in distance, the urban spatial labor market clears. Our paper extends the literature in that we allow the commuting cost to be nonlinear in distance and derive the equilibrium commuting pattern explicitly.

To be precise, our theoretical conclusions are as follows. If the commuting cost function is increasing and convex in distance, then the equilibrium commuting pattern is exclusively *parallel*, where every worker commutes the same distance[2] and there is no cross commuting. If the commuting cost function is increasing and concave in distance, then the equilibrium commuting pattern is exclusively *cross commuting*.

Our next task in this work is to investigate whether cross commuting occurs in the real world. More precisely, using commuter flow data from Tokyo, we will empirically test whether the actual commuting pattern is cross commuting or parallel commuting. We find that cross commuting is prevalent.

Our main purpose here is to investigate commuting patterns as a function of commuting cost, accounting for interactions with the labor market. Hence we simplify other aspects of our urban model. We leave to future work non-monocentric equilibrium urban patterns as well as the investigation of equilibrium under commuting cost functions that are neither convex nor concave. Analyses of these issues involve solving complex systems of equations, and likely require computer simulations rather than an analytical solution for tractability reasons.

The remainder of the paper is organized as follows. Section 2 gives our model, Section 3 presents our theoretical results, whereas Section 4 presents the empirics. An Appendix contains proofs of the results.

[2] For simplicity of this statement, we implicitly assume that the numbers of firms and consumers are the same and their quantity of land use is the same. Our model is more general than this.

2 The model

2.1 Preliminaries

We build on Ogawa and Fujita (1980) by extending the model to nonlinear commuting cost.[3] In particular, we consider a city on a line. The equilibrium configuration is endogenous, but we give conditions sufficient for it to be monocentric.[4] The locations of firms and workers are denoted by y and x, respectively. The exogenous mass of workers is N, whereas the exogenous mass of firms is M. There is one unit of land available at every location, so this is an example of a linear city. Let composite consumption good be denoted by z and let s denote quantity of land, s_w for workers and s_f for firms. The wage paid by a firm at location y is denoted $w(y)$. The rent paid per unit by a consumer at location x is denoted by $r(x)$, whereas the rent paid per unit by a firm at location y is denoted by $r(y)$.

The commuting cost for a consumer living at x but working at y is $T(|x-y|) = t \cdot \delta(|x-y|)$, where t is the commuting cost parameter, $\delta(0) = 0$ and $\delta(\cdot)$ is twice continuously differentiable with $\delta' > 0$. It represents the generalized cost of commuting between x and y, consisting of both the pecuniary and time costs of commuting. On the one hand, the pecuniary cost of commuting involves train fares and gasoline prices and is normally increasing and concave in distance due to the presence of fixed costs, for example the cost of a car or the cost of getting to a train station. On the other hand, the time cost of commuting involves the opportunity cost of time and fatigue from a long commute.[5] Thus, from the perspective of non-pecuniary costs, the commuting cost function is increasing and convex, especially when commuting time is prohibitively long. Therefore, the functional form of the generalized cost of commuting is unknown and subject to empirics.

To keep the model tractable and our focus on commuting cost, as is common in

[3] Although Ogawa and Fujita (1980) allow endogenous lot sizes for firms in a technical sense, since output of each firm is assumed to be constant and there is a fixed coefficient production technology in land and labor, in fact the land and labor demand of firms are fixed.

[4] If firms and workers are spatially integrated, then the only possible spatial equilibrium is where each worker commutes distance 0. In that case, the equilibrium and model are not very interesting for the purpose of analyzing commuting patterns.

[5] An exception is Fujita-sensei, who does most of his research on trains.

this literature, we shall assume that factors are inelastic in supply and demand. This will be made precise below.

2.2 Consumers

Each worker supplies 1 unit of labor inelastically, and will demand s_w units of land inelastically. To obtain this from primitives, we assume that the utility function has the following form:

$$U(z,s) = \begin{cases} z \text{ if } s \geq s_w \\ -\infty \text{ otherwise} \end{cases}$$

The budget constraint faced by a consumer who lives at x but works at y is:

$$w(y) = z + r(x)s + T(|x-y|)$$

Therefore, if the price of land is positive, the quantity of land consumed by a worker at x is s_w, whereas the consumption of composite good as well as the utility level of a worker living at x but working at y is $z(x,y)$:

$$z(x,y) = w(y) - r(x)s_w - T(|x-y|)$$

A worker residing at x and working at y has indirect utility

$$V(x,y) = z(x,y) = w(y) - r(x)s_w - T(|x-y|) \tag{1}$$

2.3 Producers

Turning to the production side, let Q be firm output and let l be labor input. Recall that s_f is firm land use. The function $A(y)$ is the firm externality, which is increasing in access to all other firms in the business district. It is specified as

$$A(y) = \beta - \gamma \int_{-\infty}^{\infty} h\left(y' - y\right) m(y')dy' \tag{2}$$

where $h \geq 0$ is continuous and increasing, $m(y')$ is the endogenous measure of firms at location y', γ is an exogenous parameter representing the strength of the firm agglomeration externality, and β will be the maximum amount that can be produced. Firms will demand $\frac{N}{M}$ units of labor inelastically and s_f units of land inelastically.

To derive this from primitives, the production function of a firm located at y is given by:

$$Q(l, s; y) = \begin{cases} A(y) \text{ if } l \geq \frac{N}{M} \text{ and } s \geq s_f \\ 0 \text{ otherwise} \end{cases}$$

The profit of a firm locating at y is therefore given by

$$\pi(y) = A(y) - w(y)l - r(y)s \tag{3}$$

Therefore, if wages and rents are positive, firms will demand labor $l = \frac{N}{M}$ and land $s = s_f$. Then the indirect profit of a firm locating at y is given by

$$\Pi(y) = A(y) - \frac{N}{M}w(y) - r(y)s_f = \overline{\Pi} \tag{4}$$

where $\overline{\Pi} \geq 0$ represents the equilibrium profit of every firm, since firms are free to choose their locations.

In what follows, we determine the endogenous variables: wages, land use, land rent, and the commuting pattern.

2.4 Commuting pattern functions and spatial equilibrium

If x and y in the commuting cost function are additively separable (conditional on the sign of the difference), for example $T(x - y) = t \cdot |x - y|$ as in the previous literature, then the wage arbitrage condition is met, namely $V(x, y)$ is constant in y for each x. Hence, it is not difficult to show existence of a spatial equilibrium. However, once the transport cost function is slightly different from linear, which is very likely in the real world, then we can no longer rely on the wage arbitrage condition. In this paper, we explore spatial equilibrium when the transport cost is nonlinear in distance. In our framework, indirect utility $V(x, y)$ is given in equation (1). For reference, we shall call the equilibrium indirect utility level $\overline{V}$. Let $y = f(x)$ be the commuting pattern function indicating that all workers living at x commute to firms locating at y.[6]

At equilibrium, we do not require that indirect utility V is constant for all x and y, but rather:

$$\overline{V} = \frac{M}{N}\left[A(f(x)) - \overline{\Pi} - r(f(x))s_f\right] - r(x)s_w - T(|x - f(x)|), \qquad \forall x \in \mathbb{R}$$

[6] Implicit in this definition is the idea that all consumers at one location commute to the same work location. In fact, this will hold in equilibrium.

and

$$\overline{V} \geq w(y) - r(x)s_w - T(|x-y|), \qquad \forall x, y \in \mathbb{R}$$

The densities of firms and consumers with respect to location will be given by $m(y)$ and $n(x)$, respectively. The transfer to absentee landlords, who are endowed with all of the land but obtain utility only from consumption commodity is denoted by $R \geq 0$.

Definition 1 *An **allocation** is a list of five measurable functions and a scalar:*

$\{n(x), m(y), z(x), f(x), A(y); R\}$, where the first three functions all have domain $\mathbb{R}$ and range $\mathbb{R}_+$, the fourth and fifth functions have domain $\mathbb{R}$ and range $\mathbb{R}$, whereas $R \geq 0$.

Definition 2 *An allocation $\{n(x), m(y), z(x), f(x), A(y); R\}$ is called **feasible** if:*

(i) $\int_{-\infty}^{\infty} m(y)dy = M, \qquad \int_{-\infty}^{\infty} n(x)dx = N$

(ii) $m(y)s_f + n(y)s_w \leq 1$ *a.s.* $y \in \mathbb{R}$

(iii) $\int_{-\infty}^{\infty} A(y)m(y)dy = \int_{-\infty}^{\infty} [z(x) + T(|x - f(x)|)]\, n(x)dx + R$

(iv) $\frac{N}{M}\int_C m(y)dy = \int_{\{x\in\mathbb{R}|f(x)\in C\}} n(x)dx$ *for all Lebesgue measurable* $C \subseteq \mathbb{R}$

(v) $A(y) = \beta - \gamma \int_{-\infty}^{\infty} h\left(|y' - y|\right) m(y')dy'$ *a.s.* $y \in \mathbb{R}$

The first condition represents population balance, the second condition represents material balance in land, the third condition represents material balance in consumption good, the fourth condition represents material balance in labor, whereas the last condition says that the externality is correct given the distribution of firms.

Definition 3 *A **spatial equilibrium** is a feasible allocation*

$\{n(x), m(y), z(x), f(x), A(y); R\}$ and a price system $\{w(y), r(x)\}$, where w and r are both measurable functions from $\mathbb{R}$ to $\mathbb{R}_+$, satisfying the following conditions:

(vi) *For almost every $x \in \mathbb{R}$ where $n(x) > 0$, $f(x)$ solves:*

$$z(x) = \max_y w(y) - r(x)s_w - T(|x-y|) + \frac{1}{N}\int_{-\infty}^{\infty} \Pi(y')m(y')dy'$$

(vii) $n(x) > 0$ *if and only if* $z(x) = \sup_{x'\in\mathbb{R}} z(x') \geq 0$

(viii) $m(y) > 0$ *if and only if* $\Pi(y) = A(y) - w(y)\frac{N}{M} - r(y)s_f = \sup_{y'\in\mathbb{R}} \Pi(y') \geq 0$

(ix) $\int_{-\infty}^{\infty} r(x)dx = R \geq 0$

The first condition represents consumer optimization over workplaces, the second represents consumer optimization over residential location, the third condition represents producer optimization over locations, whereas the last condition represents absentee landlord rent collection. For tractability and determinacy reasons, we assume that the consumers own the firms.

Definition 4 *We say that a spatial equilibrium* $\{n(x), m(y), z(x), f(x), A(y), R\}$, $\{w(y), r(x)\}$ *is* ***symmetric*** *if all of its component functions (excluding* R*) are symmetric around* 0*. We say that a symmetric spatial equilibrium* $\{n(x), m(y), z(x), f(x), A(y)\}$, $\{w(y), r(x)\}$ *is* ***monocentric*** *if there exists* $b \in \mathbb{R}_+$ *such that* $m(y) > 0$ *only if* $|y| \leq b$*, and* $n(x) > 0$ *only if* $|x| > b$*.*

Assuming a symmetric monocentric configuration $m(y) = 1/s_f$, we compute the derivatives of A for future use:

$$\begin{aligned} A'(y) &= -\frac{\gamma}{s_f}\left[h\left(b+y\right) - h\left(b-y\right)\right] \lesseqgtr 0 \text{ for } y \gtreqless 0 \\ A''(y) &= -\frac{\gamma}{s_f}\left[h'\left(b+y\right) + h'\left(b-y\right)\right] < 0 \end{aligned}$$

Definition 5 *We say that a symmetric monocentric spatial equilibrium* $\{n(x), m(y), z(x), f(x), A(y), R\}$, $\{w(y), r(x)\}$ *features* ***parallel commuting*** *if for almost all* $x, x' \in \mathbb{R}_+$ *with* $n(x), n(x') > 0$ *and* $x' > x$*, then* $f(x') > f(x)$*. We say that a symmetric monocentric spatial equilibrium* $\{n(x), m(y), z(x), f(x), A(y), R\}$, $\{w(y), r(x)\}$ *features* ***cross commuting*** *if for almost all* $x, x' \in \mathbb{R}_+$ *with* $n(x), n(x') > 0$ *and* $x' > x$*,* $f(x) > f(x')$*.*

3 Analysis of equilibrium

We distinguish two cases: commuting cost is convex and commuting cost is concave. First, we perform preliminary calculations that are common to our analysis of both cases to reduce repetition.

3.1 Preliminaries

Define $R = \int_{-\infty}^{\infty} r(x)dx$. From the material balance condition for consumption good, let

$$z(x) = \overline{z} = \frac{1}{N}\left[\frac{2}{s_f}\int_0^b A(y)dy - \frac{2}{s_w}\int_b^B T(x - f(x))dx - R\right] \tag{5}$$

Eventually, we must and shall specify $\overline{z}$ and R in terms only of parameters. Next, we provide informal intuition for the correlation between the second derivative of the commuting cost function and the commuting pattern.

Each worker residing at x chooses the best location $y(\leq x)$ of a firm. The consumer optimization problem is specified as:

$$\overline{z} = \max_y w(y) - r(x)s_w - T(x - y) + \frac{1}{N}\int_{-\infty}^{\infty} \Pi(y')m(y')dy' \tag{6}$$

with the first order condition:

$$w'(y) = -T'(x - y) = -T'(f^{-1}(y) - y) \tag{7}$$

Since the RHS is negative, the wage gradient $w'(y)$ is always negative. That is, the wage monotonically decreases from the city center $y = 0$. The higher wage near the city center offsets longer commute.

Proposition 6 *If the transport cost function is increasing and convex, then parallel commuting is an equilibrium commuting pattern. If the transport cost function is increasing and concave, then cross commuting is an equilibrium commuting pattern.*

Proof. Since

$$\begin{aligned} w''(y) - T''(f^{-1}(y) - y) &= -T''(f^{-1}(y) - y)(f^{-1\prime}(y) - 1) - T''(f^{-1}(y) - y) \\ &= -T''(f^{-1}(y) - y)f^{-1\prime}(y) \end{aligned}$$

the second-order condition is

$$-T''(f^{-1}(y) - y)f^{-1\prime}(y) < 0 \tag{8}$$

If (8) is met for all y, $y = f(x)$ is the global maximizer for every worker at residence x.

If $f^{-1\prime}(y) > 0$, then the second-order condition (8) is satisfied only if the transport cost function is strictly convex in distance. On the other hand, if $f^{-1\prime}(y) < 0$, then (8) is satisfied only if the transport cost is strictly concave. ■

Intuition for Proposition 6 is as follows. The horizontal axis is the location of the worker's job, y, whereas the vertical axis is in dollars. Define

$$z(y; x) = w(y) - r(x)s_w - T(x - y) + \frac{1}{N}\int_{-\infty}^{\infty} \Pi(y')m(y')dy'$$

so that

$$\frac{\partial z(y; x)}{\partial y} = w'(y) + T'(x - y)$$

We have graphed $\frac{\partial z(y;x)}{\partial y}$ in Figure 1. This curve must be downward sloping according to the second order condition. The first order condition (7) tells us that where this curve crosses the horizontal axis at y^* is the solution to the worker's optimization problem for given residence x. In the case where T is convex, when we increase x, for each given y, T' rises. Hence the $\frac{\partial z(y;x)}{\partial y}$ curve shifts up, represented by the red curve, and the new optimal job location y^*_+ must be to the right of y^*. In other words, if we look at workers more distant from the firms, they will pay higher commuting cost. The marginal commuting cost is increasing in distance, so they will choose a job location closer to the residences even though wages are lower. In the case where T is concave, when we increase x, for each given y, T' falls. Hence the $\frac{\partial z(y;x)}{\partial y}$ curve shifts down, represented by the green curve, and the new optimal job location y^*_- must be to the left of y^*. In other words, if we look at workers more distant from the firms, they will pay higher commuting cost. The marginal commuting cost is decreasing in distance, so they will choose a job location farther from the residences where wages are higher.

Due to the inelastic demand for land, the commuting pattern function will be linear in both cases described above.

Set $b = \frac{Ms_f}{2}$, $B = \frac{Ms_f}{2} + \frac{Ns_w}{2}$, $m(y) = \frac{1}{s_f}$ for $y \leq b$, $m(y) = 0$ for $y > b$, $n(x) = 0$ for $x \leq b$, $n(x) = \frac{1}{s_w}$ for $b < x \leq B$, $n(x) = 0$ for $x > B$. $A(y)$ is defined by equation (2) for this density m.

3.2 Convex commuting cost

A monocentric equilibrium exists if the ratio $\frac{\gamma}{t}$ of the firm agglomeration externality to the commuting cost is sufficiently large:

Proposition 7 *If $T'' > 0$, for sufficiently large $\frac{\gamma}{t}$, there exists a symmetric monocentric equilibrium with only parallel commuting. Moreover, this is the only symmetric monocentric equilibrium allocation.*

The proof is in the Appendix. We can see from the proof that the sign of $Ms_f - Ns_w$ coincides with the signs of the second derivatives of $w(y)$ and $r(x)$ in the case of $T'' > 0$. Since the business district is much smaller than the residential district in reality, $Ns_w > Ms_f$ holds, and hence the wage and the residential land rent are decreasing and concave in the distance from the city center. Otherwise, they are decreasing and convex. On the other hand, business land rent is not necessarily decreasing in distance from the city center.

In order to understand these conditions further, we specify functional forms as follows: $A(y) = \beta - \gamma \int_{-b}^{b} (y - y_1)^2 \, dy$ and $T(|x-y|) = t(x-y)^2$. Then, specializing the calculations in the Appendix to this example, one sufficient condition for monocentric spatial equilibrium can be written as

$$\Delta w(x) = (B-x)^2 t - (B-b)^2 t - \frac{b(x-b)(Bx - 2bx + bB)t}{(B-b)^2} + \frac{M^2(x^2-b^2)\gamma}{N} \geq 0$$

Since $\Delta w''(x) > 0$, $\Delta w(x)$ is convex. Because $\Delta w(b) = 0$, this condition is $\Delta w'(b) \geq 0$, namely

$$t \leq \frac{b(B-b)M^2\gamma}{(B^2 - 2bB + 2b^2)N} \tag{9}$$

The second sufficient condition for equilibrium can be written as

$$\Delta z(y) \equiv -N\left[b\left(B^2 + y^2\right) - b^2(B + 2y) - By^2 + b^3\right]t + M^2 b^2 (B-b)\gamma \geq 0$$

Because $\Delta z''(y) > 0$, $\Delta z(y)$ is convex. Since the minimizer of $\Delta z(y)$ is negative, this condition is $\Delta z(0) \geq 0$, namely

$$t \leq \frac{b(B-b)M^2\gamma}{(B^2 - bB + b^2)N} \tag{10}$$

The RHS of (9) is larger than that of (10). In fact, we have shown that the condition sufficient for a monocentric spatial equilibrium is given by (10). It exists when the commuting cost t is small and the face-to-face communication cost γ is large. Substituting $b = \frac{Ms_f}{2}$ and $B = \frac{Ms_f + Ns_w}{2}$ into the RHS of (10), it can shown that the RHS is increasing in M and decreasing in N. Thus, a monocentric spatial equilibrium exists if the number of firms is large and the number of workers is small. Whereas the former acts as an agglomeration force for firms, the latter acts as a dispersion force for workers.

3.3 Concave commuting cost

Proposition 8 *If $T'' < 0$, for sufficiently large $\frac{\gamma}{t}$, there exists a symmetric monocentric equilibrium with only cross commuting. Moreover, this is the only symmetric monocentric equilibrium allocation.*

The proof is in the Appendix. The wage is decreasing and concave in the distance from the city center and the residential land rent is decreasing and convex in the distance from the city center. On the other hand, business land rent is not necessarily decreasing in distance from the city center.

3.4 Land rent

Proposition 9 *The equilibrium residential land rent is concave under convex transport cost, whereas it is convex under concave transport cost.*

According to the empirical literature such as McMillen (1996), the actual residential land rent is decreasing and convex in the distance from the city center. Therefore, we conjecture that *generalized transport costs of commuting are concave in distance and the commuting pattern is cross commuting.* This is to be tested statistically in section 4.

We know that $A'(y) < 0$ since the access to all firms decreases from the city center and that $w'(y) < 0$ from (7). Hence, $r'_f(y)$ can be positive or negative. That is, while the residential land rent monotonically decreases in the distance from the city center, the business land rent can be non-monotonic.

4 Empirics

In order to test whether parallel commuting or cross commuting is dominant, we employ commuter flows from the 2010 Population Census in Japan. We extract an origin-destination matrix of 12 municipalities located along the Chuo Line commuter railroad out of 62 total municipalities in Tokyo prefecture. In Figure 2, the 3 light brown municipalities are in the central city whereas the 9 light green municipalities are suburbs of the central city.

Since we are interested in commuting flows from the suburbs to the central city, distinguish between the 9 suburbs (S) and 3 central city municipalities (C). We have constructed a $S \times C = 9 \times 3$ matrix of commuter flows, whose elements are denoted by c_{ij}. The 9 suburban municipalities and the 3 central city municipalities are sorted according to the distance from Tokyo station. Roughly speaking, parallel (respectively, cross) commuting is dominant if the elements near the diagonal, for example c_{11}, c_{21}, c_{83} and c_{93}, are larger (respectively, smaller) than those far from the diagonal, for example c_{13}, c_{23}, c_{81} and c_{91}. In order to examine this, we conduct a test of Kendall's τ_c statistic; see Kendall and Gibbons (1990). The test statistic is:

$$\tau_c = \frac{q\,(P-Q)}{(q-1)\,W^2}$$

and the average standard error is

$$ASE = \frac{2q}{(q-1)\,W^2}\left[\sum_{i=1}^{S}\sum_{j=1}^{C} c_{ij}\,(C_{ij}-D_{ij})^2 - \frac{1}{W}\,(P-Q)^2\right]^{\frac{1}{2}}$$

where

$$\begin{aligned}
q &= \min\{S, C\}, \qquad W = \sum_{i=1}^{S}\sum_{j=1}^{C} c_{ij} \\
P &= \sum_{i=1}^{S}\sum_{j=1}^{C} c_{ij}C_{ij}, \qquad Q = \sum_{i=1}^{S}\sum_{j=1}^{C} c_{ij}D_{ij} \\
C_{ij} &= \sum_{h<i}^{S}\sum_{k<j}^{C} c_{hk} + \sum_{h>i}^{S}\sum_{k>j}^{C} c_{hk}, \qquad D_{ij} = \sum_{h<i}^{S}\sum_{k>j}^{C} c_{hk} + \sum_{h>i}^{S}\sum_{k>j}^{C} c_{hk}
\end{aligned}$$

It is known that τ_c/ASE is standard normally distributed (Götaş and İşçi, 2011; Kendall and Gibbons, 1990).

Using the commuter flow data, we find that $\tau_c = -0.0136$ and $ASE = 0.00259$. Since $\tau_c/ASE = -5.26$, τ_c is negative and significantly different from zero, implying that *cross commuting is dominant* in the Chuo Line.[7] That is, there are many short-distance commutes between nearby municipalities and many long-distance commutes between distant municipalities as compared to intermediate-distance commutes. This test also suggests that the transport cost in commuting is increasing and concave in distance, possibly due to the presence of a fixed cost for infrastructure.

5 Conclusion

We have examined equilibrium of a rather standard model, similar to the previous literature with the exception that we allow general convex or concave commuting cost as a function of distance. We have shown that a monocentric equilibrium exists if the ratio of the firm agglomeration externality to the commuting cost is sufficiently large. Moreover, when commuting cost is convex, we have the following properties of equilibrium: residential land rent and business district wages are decreasing and concave in distance to the CBD, and there is exclusively parallel commuting. When commuting cost is concave, we have the following: residential land rent is decreasing and convex in distance to the CBD, business district wages are decreasing and concave in distance to the CBD, and there is exclusively cross commuting. The empirical evidence is broadly consistent with a concave commuting cost function and cross commuting, in contrast with the statement of Fujita and Thisse cited in the introduction.

Further work should focus on analyzing the model with even more general commuting cost functions, especially those that are neither globally convex nor concave. As a preview, consider a more realistic commuting cost function that is concave for shorter distances but convex for longer distances, with exactly one inflection point at an intermediate distance. Our conjecture is that there will be cross commuting for shorter distances but parallel commuting for longer distances, with a discontinuity in the commuting pattern function at the inflection point, but rents and wages are

[7] Using data from 1995, we have obtained almost the same result: $\tau_c = -0.0149$, $ASE = 0.00228$, and $\tau_c/ASE = -6.54$.

continuous everywhere but not necessarily differentiable everywhere.

Appendix

Proof of Proposition 7

We focus on locations in $\mathbb{R}_+$; the prices and allocations for the other locations are defined symmetrically. The technique of proof that equilibrium exists is guess and verify. Set $b = \frac{Ms_f}{2}$, $B = \frac{Ms_f}{2} + \frac{Ns_w}{2}$, $m(y) = \frac{1}{s_f}$ for $y \leq b$, $m(y) = 0$ for $y > b$, $n(x) = 0$ for $x \leq b$, $n(x) = \frac{1}{s_w}$ for $b < x \leq B$, $n(x) = 0$ for $x > B$. $A(y)$ is defined by equation (2) for this density m. For $b < x \leq B$, define

$$f(x) = \frac{Ms_f}{Ns_w}(x - b) \tag{11}$$

The function f is arbitrary otherwise. Hence, for $0 \leq y \leq b$,

$$f^{-1}(y) = \frac{Ns_w}{Ms_f}y + b \tag{12}$$

For $x > B$, define $r(x) = 0$.

There are two cases to consider:

(i) $Ns_w \neq Ms_f$. We shall find an explicit expression for equilibrium rent on the portion of the city where consumers live. This involves integrating (7), plugging back into equation (6) to eliminate $w(y)$, and solving for $r(x)$ using the fact that rent must be 0 at the boundary of the city. The details are as follows:

Integrating (7), we obtain

$$\begin{aligned} w(y) &= -\int T'(f^{-1}(y) - y)dy \\ &= -\frac{Ms_f}{Ns_w - Ms_f}T\left(\frac{Ns_w - Ms_f}{Ms_f}y + b\right) + C_a \\ &= -\frac{b}{B - 2b}T\left(f^{-1}(y) - y\right) + C_a \end{aligned} \tag{13}$$

where C_a is the constant of integration.

From (6) and (13),

$$r(x) = \frac{1}{s_w}\left[w(y) - T(x-y) - \overline{z}\right]$$

$$= \frac{1}{s_w}\left[-\frac{b}{B-2b}T\left(f^{-1}(y)-y\right) + C_a - T(x-y) - \overline{z}\right]$$

$$= \frac{1}{s_w}\left[-\frac{B-b}{B-2b}T\left(x-f(x)\right) + C_a - \overline{z}\right]$$

$$\text{Setting } C_b = \frac{1}{s_w}\left(C_a - \overline{z}\right),$$

$$= -\frac{N}{2\left(B-2b\right)}T\left(x-f(x)\right) + C_b \tag{14}$$

Since $r(B) = 0$, (14) leads to

$$C_b = \frac{N}{2\left(B-2b\right)}T\left(B-b\right)$$

Plugging C_b into (14) yields our ultimate expression for rent (17) below.

(ii) $Ns_w = Ms_f$. We have

$$\begin{aligned} w(y) &= -\int T'(f^{-1}(y)-y)dy \\ &= -\int T'\left(\frac{Ns_w}{Ms_f}y + b - y\right)dy \\ &= -\int T'\left(b\right)dy \\ &= -T'\left(b\right)y + C_c \end{aligned} \tag{15}$$

where C_c is again a constant of integration. From (6) and (15),

$$\begin{aligned} r(x) &= \frac{1}{s_w}\left[w(y) - T(x-y) - \overline{z}\right] \\ &= \frac{1}{s_w}\left[-T'\left(b\right)y + C_c - T(x-y) - \overline{z}\right] \\ &= \frac{1}{s_w}\left[-T'\left(b\right)\frac{Ms_f}{Ns_w}\left(x-b\right) + C_c - T(B-b) - \overline{z}\right] \\ &= \frac{1}{s_w}\left[-T'\left(b\right)x + C_d\right] \end{aligned}$$

where $C_d = bT'\left(b\right) + C_c - T(B-b) - \overline{z}$. Since $r(B) = 0$, we obtain

$$r(x) = \frac{1}{s_w}T'\left(b\right)\left(B-x\right) \tag{16}$$

Having analyzed both cases, we can summarize as follows:

$$
\begin{aligned}
\text{For } b &< x \le B, \text{ define} \\
r(x) &= \begin{cases} \frac{N}{2(B-2b)} \left[T(B-b) - T(x - f(x))\right] & \text{if } Ns_w \neq Ms_f \\ \frac{N}{2(B-b)} T'(b)(B-x) & \text{if } Ns_w = Ms_f \end{cases}
\end{aligned} \tag{17}
$$

$$
\text{Let } R_w = 2\int_b^B r(x)dx
$$

$$
\begin{aligned}
\text{For } 0 &\le y \le b, \text{ define} \\
w(y) &= \bar{z} + r(f^{-1}(y))s_w + T(f^{-1}(y) - y)
\end{aligned} \tag{18}
$$

Using the profit function (4) and the fact that the rent for consumers and producers must be equal at b,

$$
\begin{aligned}
\text{For } 0 &\le y \le b, \text{ define} \\
r(y) &= \frac{1}{s_f}\left\{A(y) - A(b) + \frac{N}{M}\left[w(b) - w(y)\right]\right\} + C_1
\end{aligned} \tag{19}
$$

where

$$
C_1 \equiv \begin{cases} \frac{N}{2(B-2b)} \left[T(B-b) - T(b)\right] & \text{if } Ns_w \neq Ms_f \\ \frac{N}{2} T'(b) & \text{if } Ns_w = Ms_f \end{cases} \tag{20}
$$

is a function of only exogenous parameters.

(i) If we substitute (13) into (19), we have

$$
\begin{aligned}
r(y) &= \frac{1}{s_f}\left\{A(y) - A(b) + \frac{N}{M}\left[w(b) + \frac{b}{B-2b}T\left(f^{-1}(y) - y\right) - C_a\right]\right\} + C_1 \\
&= \frac{1}{s_f}\left[A(y) + \frac{N}{M}\frac{b}{B-2b}T\left(f^{-1}(y) - y\right)\right] + C_2
\end{aligned}
$$

Using the fact that $r(y) = r(x)$ evaluated at $y = x = b$,

$$
C_2 \equiv -\frac{N}{2(B-2b)}T(b) - \frac{M}{2b}A(b)
$$

(ii) If we substitute (15) into (19), we get

$$
\begin{aligned}
r(y) &= \frac{1}{s_f}\left\{A(y) - A(b) + \frac{N}{M}\left[w(b) + T'(b)y - C_c\right]\right\} + C_1 \\
&= \frac{M}{2b}\left[A(y) + \frac{N}{M}T'(b)y\right] + C_2
\end{aligned}
$$

Using the fact that $r(y) = r(x)$ evaluated at $y = x = b$,

$$C_2 \equiv -\frac{M}{2b} A(b)$$

Hence,

$$r(y) = \begin{cases} \frac{M}{2b}\left[A(y) - A(b) + \frac{N}{M}\frac{b}{B-2b}\left[T\left(f^{-1}(y) - y\right) - T(b)\right]\right] & \text{if } Ns_w \neq Ms_f \\ \frac{M}{2b}\left[A(y) - A(b) + \frac{N}{M}T'(b)\, y\right] & \text{if } Ns_w = Ms_f \end{cases} \quad (21)$$

Let

$$R_f = 2\int_0^b r(y)dy$$

Then $R = R_f + R_w$, a function of only exogenous parameters. So $\overline{z}$ can be found as only a function of exogenous parameters by plugging R into (5). In addition, $w(y)$ $(0 \leq y \leq b)$ is a function of only exogenous variables by using (18).

Notice that (4) and (19) imply that profits are constant on $0 \leq y \leq b$, namely

$$\Pi(y) = \overline{\Pi} = A(b) - \frac{N}{M} w(b) - s_f C_1 \quad (22)$$

Hence,

$$\begin{aligned} \text{For } 0 &\leq y \leq b, \text{ define} \\ w(y) &= \frac{M}{N}\left[A(y) - \overline{\Pi} - r(y)s_f\right] \end{aligned}$$

To show that this represents an equilibrium, we must verify that $\overline{\Pi} \geq 0$, $\overline{z} \geq 0$, that no consumer wishes to move to $[0, b]$, and that no firm wants to move to (b, ∞).

In the case of $T'' > 0$ and $Ns_w \neq Ms_f$,[8] the total land rent is

$$\begin{aligned} R &= R_f + R_w = 2\int_0^b r(y)dy + 2\int_b^B r(x)dx \\ &= \frac{M}{b}\int_0^b A(y) - A(b) + \frac{N}{M}\frac{b}{B-2b}\left[T\left(f^{-1}(y) - y\right) - T(b)\right] dy \\ &\quad + \frac{N}{B-2b}\int_b^B \left[T(B-b) - T(x - f(x))\right] dx \\ &= \frac{M}{b}\int_0^b A(y) - A(b)\, dy + \frac{N}{B-2b}\int_b^B \left[T(x - f(x)) - T(b)\right]\frac{b}{B-b} dx \\ &\quad + \frac{N}{B-2b}\int_b^B \left[T(B-b) - T(x - f(x))\right] dx \\ &= \frac{M}{b}\int_0^b A(y) - A(b)\, dy - \frac{N}{B-b}\int_b^B T(x - f(x))\, dx + Ng \end{aligned}$$

[8] The case $Ns_w = Ms_f$ is similar.

where $dy = \frac{Ms_f}{Ns_w}dx = \frac{b}{B-b}dx$ and

$$g \equiv \frac{1}{B-2b}\left[(B-b)\,T\,(B-b) - bT\,(b)\right]$$

which is positive for all $b \neq B/2$. So (5) can be rewritten as

$$\begin{aligned}
\overline{z} &= \frac{1}{N}\left[\frac{M}{b}\int_0^b A(y)dy - \frac{N}{B-b}\int_b^B T(x-f(x))dx - R\right] \\
&= \frac{1}{N}\left[\begin{array}{c} \frac{M}{b}\int_0^b A(y)dy - \frac{N}{B-b}\int_b^B T(x-f(x))dx - \frac{M}{b}\int_0^b A(y) - A\,(b)\,dy \\ +\frac{N}{B-b}\int_b^B T\,(x-f(x))\,dx - Ng \end{array}\right] \\
&= \frac{M}{N}A\,(b) - g
\end{aligned}$$

The condition for spatial equilibrium is $\overline{z} \geq 0$ or

$$MA\,(b) \geq Ng \tag{23}$$

(i) First, we show that no firm will want to move to the residential area. Suppose a firm deviates from the business district $y \in [0,b]$ to the residential district $x \in (b,B]$. We compute the wage $w(x)$ that makes a worker indifferent if she resides at $x_1 \in [b,B]$ but shifts her workplace from $y \in [0,b]$ to $x \in [b,B]$. We focus on the case: $x \leq x_1$. The case $x > x_1$ can be ruled out because a worker residing at $x_1 = x - \delta$ must pay higher land rent than a worker residing at $x_1 = x + \delta$ for all $\delta > 0$ due to negative rent gradient $r(x)$ in the residential district. That is, the latter worker always achieves a higher utility level, since compared to the former, their wages are the same, their land rent is lower, and their commuting cost is the same. So if a consumer residing at $x_1 = x - \delta$ is happier, so is a consumer residing at $x_1 = x + \delta$. Hence we focus only on the case: $x_1 \in [x,B]$.

If she works at $y = f(x_1) \in [0,b]$, in equilibrium her consumption of composite good is the same as the consumer who lives at B:

$$\begin{aligned}
\overline{z}_b &= w(f(x_1)) - r(x_1)s_w - T\,(x_1 - f(x_1)) + \frac{1}{N}\int_{-\infty}^{\infty}\Pi(y')m(y')dy' \\
&= w(b) - T\,(B-b) + \frac{1}{N}\int_{-\infty}^{\infty}\Pi(y')m(y')dy'
\end{aligned}$$

from (6). On the other hand, if she works at $x \in [b,B]$, her consumption of composite good is

$$\overline{z}_a = w(x) - r(x_1)s_w - T\,(x_1 - x) + \frac{1}{N}\int_{-\infty}^{\infty}\Pi(y')m(y')dy'$$

In order to guarantee equal utility, these two consumptions should be the same $\overline{z}_b = \overline{z}_a$. That is, the wage offered by a firm at location x for a worker at location x_1 should be

$$w(x) = w(b) - T\left(B - b\right) + r(x_1)s_w + T\left(x_1 - x\right)$$

Then, the profit of a firm relocating from y to x and paying this wage $w(x)$ for a worker living at x_1 is

$$\begin{aligned} \Pi(x, x_1) &= A(x) - \frac{N}{M} w(x) - r(x)s_f \\ &= A(x) - \frac{N}{M}\left[w(b) - T\left(B - b\right) + r(x_1)s_w + T\left(x_1 - x\right)\right] - r(x)s_f \end{aligned} \tag{24}$$

For equilibrium, this profit does not exceed $\overline{\Pi}$, which was obtained before relocation:

$$\Pi(x, x_1) \leq \overline{\Pi}, \qquad \forall b \leq x \leq x_1 \leq B \tag{25}$$

We have

$$\frac{\partial \Pi(x, x_1)}{\partial x_1} = \frac{N}{M}\left[T'\left(x_1 - f(x_1)\right) - T'\left(x_1 - x\right)\right] > 0$$

because $x_1 - f(x_1) > x_1 - x$ and $T''\left(x\right) > 0$. This implies $x_1 = B$ is the maximizer of $\Pi(x, x_1)$. Hence, the no-deviation condition (25) is replaced with

$$\Pi(x, B) \leq \overline{\Pi}, \qquad \forall x \in (b, B] \tag{26}$$

which can be rewritten as

$$\begin{array}{ll} \frac{M}{N}\left[A(b) - A\left(x\right)\right] + \left[T(B - x) - T\left(B - b\right)\right] + \frac{b}{B-2b}\left[T(b) - T\left(x - f(x)\right)\right] \geq 0 & \text{if } Ns_w \neq Ms_f \\ \frac{M}{N}\left[A(b) - A(x)\right] + \left[T\left(B - x\right) - T\left(B - b\right)\right] + \frac{b}{B-b} T'\left(b\right)\left(b - x\right) \geq 0 & \text{if } Ns_w = Ms_f \end{array} \tag{27}$$

for all $x \in (b, B]$ by using (17) and (24). Observe that the term in the first brackets is positive, whereas those in the second and third brackets are negative from $\operatorname{sgn}\left[T(b) - T\left(x - f(x)\right)\right] = -\operatorname{sgn}\left(B - 2b\right)$. Condition (27) is rewritten as

$$\frac{\gamma}{t} \geq \max_{y,\, y_1} F_1(x, s_f, s_w, M, N), \text{ where}$$

$$F_1(x, s_f, s_w, M, N) \equiv \begin{cases} \dfrac{Ns_f}{M} \dfrac{\frac{Ms_f}{Ns_w - Ms_f}\left[\delta(x - f(x)) - \delta\left(\frac{Ms_f}{2}\right)\right] - \delta\left(\frac{Ns_w}{2} + \frac{Ms_f}{2} - x\right) + \delta\left(\frac{Ns_w}{2}\right)}{\int_{-\frac{Ms_f}{2}}^{\frac{Ms_f}{2}} h(x - y') - h\left(\frac{Ms_f}{2} - y'\right) dy'} & \text{if } Ns_w \neq Ms_f \\ \dfrac{Ns_f}{M} \dfrac{\frac{Ms_f}{Ns_w}\left[-T'\left(\frac{Ms_f}{2}\right)\left(\frac{Ms_f}{2} - x\right)\right] - \delta\left(\frac{Ns_w}{2} + \frac{Ms_f}{2} - x\right) + \delta\left(\frac{Ns_w}{2}\right)}{\int_{-\frac{Ms_f}{2}}^{\frac{Ms_f}{2}} h(x - y') - h\left(\frac{Ms_f}{2} - y'\right) dy'} & \text{if } Ns_w = Ms_f \end{cases}$$

and G_1 is finite and differentiable with respect to $x \in (b, B]$. Condition (27) is stricter than condition (23) because plugging $x = B$ into (27) yields

$$M\left[A(b) - A(B)\right] \geq Ng$$

where $A(B) > 0$.

(ii) Second, we consider no deviation condition of a worker. Suppose a worker deviates to from the residential district $x \in [b, B]$ to the business district $y \in [0, b)$. The consumption of composite good before deviation was

$$\overline{z}_b(y) = w(y) - r(f^{-1}(y))s_w - T\left(f^{-1}(y) - y\right) + \frac{1}{N}\int_{-\infty}^{\infty} \Pi(y')m(y')dy'$$

On the other hand, the consumption of composite good after deviation is

$$\overline{z}_a(y, y_1) = w(y) - r(y_1)s_w - T\left(|y - y_1|\right) + \frac{1}{N}\int_{-\infty}^{\infty} \Pi(y')m(y')dy'$$

Let $y_1 = y_1^*$ be the maximizer of $\overline{z}_a(y, y_1)$. For a symmetric monocentric spatial equilibrium, the consumption of composite good before deviation is not smaller than that after deviation. That is,

$$\overline{z}_b(y) \geq \overline{z}_a(y, y_1^*), \qquad \forall y \in [0, b) \tag{28}$$

or

$$\min_{y_1}\left[r(y_1)s_w + T\left(|y - y_1|\right)\right] - r(f^{-1}(y))s_w - T\left(f^{-1}(y) - y\right) \tag{29}$$

for all $y \in [0, b)$. Condition (29) is rewritten as

$$\frac{\gamma}{t} \geq \max_{y,\, y_1} F_2(y, y_1, s_f, s_w, M, N),$$

where $F_2(y, y_1, s_f, s_w, M, N)$

$$\equiv \begin{cases} \frac{s_f^2}{s_w}\frac{\frac{Ns_w}{Ns_w - Ms_f}\left[\delta\left(\frac{Ms_f}{2}\right) + \delta\left(\frac{Ns_w}{2}\right) - \delta\left(f^{-1}(y_1) - y_1\right)\right] - \delta(|y - y_1|) - \frac{Ms_f}{Ns_w - Ms_f}\delta\left(f^{-1}(y) - y\right)}{\int_{-\frac{Ms_f}{2}}^{\frac{Ms_f}{2}} h\left(\frac{Ms_f}{2} - y'\right) - h(|y' - y_1|)dy'} & \text{if } Ns_w \neq Ms_f \\ \frac{s_f}{s_w}\frac{-\frac{Ns_w}{Ms_f}\delta\left(\frac{Ms_f}{2}\right)y_1 + \delta'\left(\frac{Ms_f}{2}\right)\left(\frac{Ms_f}{2} + \frac{Ns_w}{2} - f^{-1}(y)\right) - \delta(|y - y_1|) + \delta\left(f^{-1}(y) - y\right)}{\int_{-\frac{Ms_f}{2}}^{\frac{Ms_f}{2}} h\left(\frac{Ms_f}{2} - y'\right) - h(|y' - y_1|)dy'} & \text{if } Ns_w = Ms_f \end{cases}$$

Hence, the sufficient conditions for a symmetric monocentric spatial equilibrium is given by (27) and (29).

Next, we show that there is no cross commuting in any symmetric, monocentric equilibrium. Thus, the equilibrium specified above is the only one.

In equilibrium, for a worker residing at x to weakly prefer commuting to y instead of $\widetilde{y}$, the following should hold:

$$w(y) - T(|x - y|) \geq w(\widetilde{y}) - T(|x - \widetilde{y}|)$$

In equilibrium, for a worker residing at $\widetilde{x}$ to weakly prefer commuting to $\widetilde{y}$ instead of y, the following should hold:

$$w(y) - T(|\widetilde{x} - y|) \leq w(\widetilde{y}) - T(|\widetilde{x} - \widetilde{y}|)$$

Hence,

$$\varphi(x) = T(|\widetilde{x} - y|) + T(|x - \widetilde{y}|) - T(|x - y|) - T(|\widetilde{x} - \widetilde{y}|) \geq 0 \tag{30}$$

should hold for all x.

Suppose that there is cross commuting. Then for some $\widetilde{y} < y \leq x < \widetilde{x}$, we have

$$\varphi'(x) = T'(x - \widetilde{y}) - T'(x - y)$$

This is positive because $x - \widetilde{y} > x - y$ and $T''(x) > 0$. We also get $\varphi(\widetilde{x}) = 0$, and thus $\varphi(x) < 0$ for all $x < \widetilde{x}$, which contradicts $\varphi(x) \geq 0$.

Hence there is only parallel commuting at a symmetric, monocentric equilibrium. Thus, we conclude that the only such equilibrium is the one we have specified.

Proof of Proposition 8

Similar to the previous proof, we focus on locations in $\mathbb{R}_+$. The technique of proof that equilibrium exists is guess and verify. For $b < x \leq B$, define

$$f(x) = \frac{Ms_f}{Ns_w}(B - x)$$

The function f is arbitrary otherwise. Hence, for $0 \leq y \leq b$,

$$f^{-1}(y) = B - \frac{Ns_w}{Ms_f}y$$

For $x > B$, define $r(x) = 0$.

Integrating (7), we obtain:

$$\begin{aligned} w(y) &= -\int T'(f^{-1}(y)-y)dy \\ &= -\int T'\left(B-\frac{B}{b}y\right)dy \\ &= \frac{b}{B}T\left(f^{-1}(y)-y\right)+C_a \end{aligned} \tag{31}$$

where once again C_a is a constant of integration.

From (6) and (31),

$$r(x) = \frac{1}{s_w}\left[w(y)-T(x-y)-\overline{z}\right]$$

$$= \frac{1}{s_w}\left[-\frac{B-b}{B}T\left(x-f(x)\right)+C_a-\overline{z}\right]$$

$$\text{Setting } C_b = \frac{1}{s_w}\left(C_a-\overline{z}\right)$$

$$= -\frac{N}{2B}T\left(x-f(x)\right)+C_b \tag{32}$$

Observe that the residential land rent is decreasing and convex in x.

Since $r(B)=0$ in (32), we get

$$C_b = \frac{N}{2B}T\left(B\right)$$

Plugging C_b into (32) yields our final expression for rent (33):

$$\begin{aligned} \text{For } b &< x \leq B, \text{ define} \\ r(x) &= \frac{N}{2B}\left[T\left(B\right)-T\left(x-f(x)\right)\right] \end{aligned} \tag{33}$$

Using the profit function and the fact that the rent for consumers and producers must be equal at b,

$$\begin{aligned} \text{For } 0 &\leq y \leq b, \text{ define} \\ r(y) &= \frac{1}{s_f}\left\{A(y)-A\left(b\right)+\frac{N}{M}\left[w\left(b\right)-w(y)\right]\right\}+\frac{N}{2B}T\left(B\right) \end{aligned} \tag{34}$$

If we substitute (31) into (34), we obtain

$$r(y) = \frac{1}{s_f}\left\{A(y)-A\left(b\right)+\frac{N}{M}\left[w\left(b\right)-\frac{b}{B}T\left(f^{-1}(y)-y\right)-C_a\right]\right\}+\frac{N}{2B}T\left(B\right)$$

Using $r(y) = r(x)$ evaluated at $y = x = b$, we obtain

$$r(y) = \frac{M}{2b}\left[A(y) - A(b)\right] - \frac{N}{2B}\left[T\left(f^{-1}(y) - y\right) - T(B)\right] \tag{35}$$

Then $R = R_f + R_w$, a function of only exogenous parameters. So $\overline{z}$ can be found as only a function of exogenous parameters by plugging R into (5). In addition, $w(y)$ $(0 \leq y \leq b)$ is a function of only exogenous variables by using (18).

Notice that (4) and (35) imply that profits are constant on $0 \leq y \leq b$, namely

$$\Pi(y) = \overline{\Pi} = A(b) - \frac{N}{M}w(b) - \frac{Nb}{MB}T(B)$$

Hence,

$$\begin{aligned} \text{For } 0 &\leq y \leq b, \text{ define} \\ w(y) &= \frac{M}{N}\left[A(y) - \overline{\Pi} - r(y)s_f\right] \end{aligned}$$

To show that this represents an equilibrium, we must verify that $\overline{\Pi} \geq 0$, $\overline{z} \geq 0$, that no consumer wishes to move to $[0, b]$, and that no firm wants to move to (b, ∞).

As in the previous proof, we verify that what we have constructed is an equilibrium. In the case of $T'' < 0$, the total land rent is

$$\begin{aligned} R_f + R_w &= 2\int_0^b r(y)dy + 2\int_b^B r(x)dx \\ &= \int_0^b \frac{M}{b}\left[A(y) - A(b)\right]dy + \frac{N}{B-b}\int_b^B \left[T(B) - T(x - f(x))\right]dx \end{aligned}$$

So (5) can be rewritten as

$$\begin{aligned} \overline{z} &= \frac{1}{N}\left[\frac{M}{b}\int_0^b A(y)dy - \frac{N}{B-b}\int_b^B T(x - f(x))dx - R\right] \\ &= \frac{M}{N}A(b) - T(B) \end{aligned}$$

The condition for spatial equilibrium is then given by

$$MA(b) \geq NT(B) \tag{36}$$

which is similar to (23).

(i) First, we seek no deviation condition of a firm. Suppose a firm deviates to from $y \in [0, b]$ to $x \in (b, B]$. We compute the wage $w(x)$ that makes a worker indifferent if she residing at $x_1 \in [b, B]$ shifts her workplace from $y \in [0, b]$ to $x \in [x_1, B]$. As before, we can focus on the interval of $x_1 \in [x, B]$.

If she works at $y = f(x_1)$, her consumption of composite good is

$$\overline{z}_b = w(f(x_1)) - r(x_1)s_w - T(x_1 - f(x_1)) = w(0) - T(B) + \frac{1}{N}\int_{-\infty}^{\infty} \Pi(y')m(y')dy'$$

On the other hand ,if she works at x, her consumption of composite good is

$$\overline{z}_a = w(x) - r(x_1)s_w - T(x_1 - x) + \frac{1}{N}\int_{-\infty}^{\infty} \Pi(y')m(y')dy'$$

Because $\overline{z}_b = \overline{z}_a$ for equal utility, the wage offered by a firm at location x for a worker at location x_1 should satisfy

$$w(x, x_1) = w(0) - T(B) + r(x_1)s_w + T(x_1 - x)$$

The profit of a firm relocating from y to x is

$$\begin{aligned}\Pi(x, x_1) &= A(x) - \frac{N}{M}w(x, x_1) - r(x)s_f \\ &= A(x) - \frac{N}{M}\left[w(0) - T(B) + r(x_1)s_w + T(x_1 - x)\right] - r(x)s_f\end{aligned}$$

The no-deviation condition is given by

$$\Pi(x, x_1) \leq \overline{\Pi}, \qquad \forall b \leq x, x_1 \leq B$$

We have

$$\frac{\partial \Pi(x, x_1)}{\partial x_1} = \frac{N}{M}\left[T'(x_1 - f(x_1)) - T'(x_1 - x)\right] > 0$$

because $x_1 - f(x_1) > x_1 - x$ and $T''(x) > 0$. This implies $x_1 = B$ is the minimizer of $\Pi(x, x_1)$. Hence, the no-deviation condition (25) is replaced with

$$\Pi(x, B) \leq \overline{\Pi}, \qquad \forall x \in (b, B] \tag{37}$$

We have

$$\begin{aligned}\Pi(x, B) &= A(x) - \frac{N}{M}\left[w(0) - T(B) + T(B - x)\right] - r(x)s_f \\ &= A(x) - A(0) + \frac{Nb}{MB}\left[T(x - f(x)) - T(B)\right] + \overline{\Pi} + \frac{N}{M}\left[T(B) - T(B - x)\right]\end{aligned}$$

Then, (37) can be rewritten as

$$\frac{M}{N}\left[A(0)-A\left(x\right)\right]+\left[T(B-x)-T\left(B\right)\right]+\frac{b}{B}\left[T(B)-T\left(x-f(x)\right)\right]\geq 0 \qquad (38)$$

for all $x \in (b, B]$. This is rewritten as

$$\frac{\gamma}{t} \geq \max_{y,\, y_1} G_1(x, s_f, s_w, M, N)$$

where $G_1(x, s_f, s_w, M, N)$

$$\equiv \quad \frac{Ns_f}{M}\frac{\frac{Ns_w}{Ns_w+Ms_f}\delta\left(\frac{Ns_w}{2}+\frac{Ms_f}{2}\right)+\frac{Ms_f}{Ns_w+Ms_f}\delta\left(x-f(x)\right)-\delta\left(\frac{Ns_w}{2}+\frac{Ms_f}{2}-x\right)}{\int_{-Ms_f/2}^{Ms_f/2}-h\left(|y'|\right)+h\left(x-y'\right)dy'}$$

and G_1 is finite and differentiable with respect to $x \in (b, B]$.

If we plug $x = B$ into (38), we get

$$M\left[A(0)-A\left(B\right)\right]-NT\left(B\right)\geq 0$$

which is stricter than the previous condition (36).

(ii) Second, we consider no deviation condition of a worker. Suppose a worker deviates to from $x \in [b, B]$ to $y_1 \in [0, b)$. The consumption of composite good before deviation was

$$\overline{z}_b(y) = w(y) - r(f^{-1}(y))s_w - T\left(f^{-1}(y) - y\right)$$

On the other hand, the consumption of composite good after deviation is

$$\overline{z}_a(y, y_1) = w(y) - r(y_1)s_w - T\left(|y - y_1|\right)$$

Let $y_1 = y_1^*$ be the maximizer of $\overline{z}_a(y, y_1)$. For a symmetric monocentric spatial equilibrium, the consumption of composite good before deviation is not smaller than that after deviation. That is,

$$\overline{z}_b(y) \geq \overline{z}_a(y, y_1^*), \qquad \forall y \in [0, b)$$

or

$$\min_{y_1}\left[r(y_1)s_w + T\left(|y-y_1|\right)\right] - r(f^{-1}(y))s_w - T\left(f^{-1}(y)-y\right) \geq 0 \qquad (39)$$

for all $y \in [0, b)$. This is rewritten as

$$\frac{\gamma}{t} \geq \max_{y,\, y_1} G_2(y, y_1, s_f, s_w, M, N)$$

where $G_2(y, y_1, s_f, s_w, M, N)$

$$\equiv \quad \frac{s_f^2}{s_w}\frac{\frac{Ms_f}{Ns_w+Ms_f}\delta\left(f^{-1}(y)-y\right)+\frac{Ns_w}{Ns_w+Ms_f}\delta\left(f^{-1}(y_1)-y_1\right)-\delta\left(|y-y_1|\right)}{\int_{-Ms_f/2}^{Ms_f/2}-h\left(|y'-y_1|\right)+h\left(Ms_f/2-y'\right)dy'}$$

and G_2 is finite and differentiable with respect to y, $y_1 \in [0, b)$.

Hence, the conditions for a symmetric monocentric spatial equilibrium is given by (38) and (39).

Next, we show that there is no cross commuting in any symmetric, monocentric equilibrium. Thus, the equilibrium specified above is the only one.

In equilibrium, for a worker residing at x to prefer commuting to y instead of $\widetilde{y}$, the following should hold:

$$w(y) - T(|x - y|) \geq w(\widetilde{y}) - T(|x - \widetilde{y}|)$$

In equilibrium, for a worker residing at $\widetilde{x}$ to prefer commuting to $\widetilde{y}$ instead of y, the following should hold:

$$w(y) - T(|\widetilde{x} - y|) \leq w(\widetilde{y}) - T(|\widetilde{x} - \widetilde{y}|)$$

Hence,

$$\varphi(x) = T(|\widetilde{x} - y|) + T(|x - \widetilde{y}|) - T(|x - y|) - T(|\widetilde{x} - \widetilde{y}|) \geq 0 \tag{40}$$

should hold for all x.

Suppose that there is parallel commuting. Then for some $y < \widetilde{y} \leq x < \widetilde{x}$, we have

$$\varphi'(x) = T'(x - \widetilde{y}) - T'(x - y) > 0$$

because $x - \widetilde{y} < x - y$ and $T''(x) < 0$. We also have $\varphi(\widetilde{x}) = 0$, and thus $\varphi(x) < 0$ for all $x < \widetilde{x}$, which contradicts $\varphi(x) \geq 0$.

Hence there is only cross commuting at a symmetric, monocentric equilibrium. Thus, we conclude that the only such equilibrium is the one we have specified.

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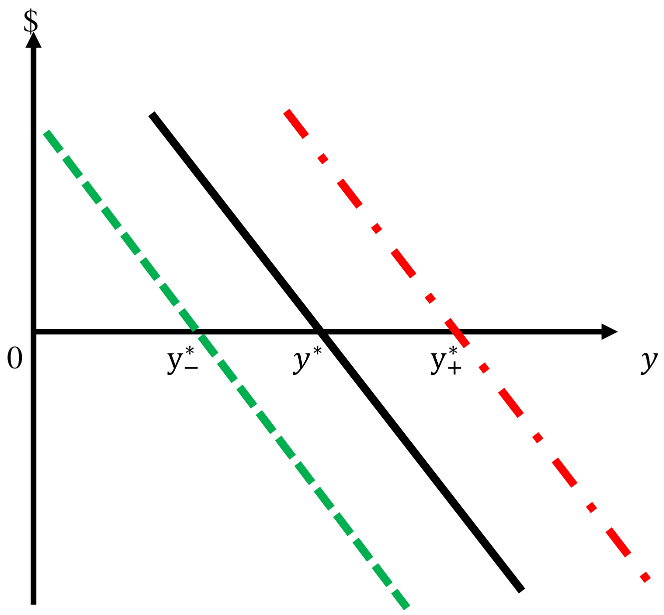

Figure 1: Determination of the commuting pattern

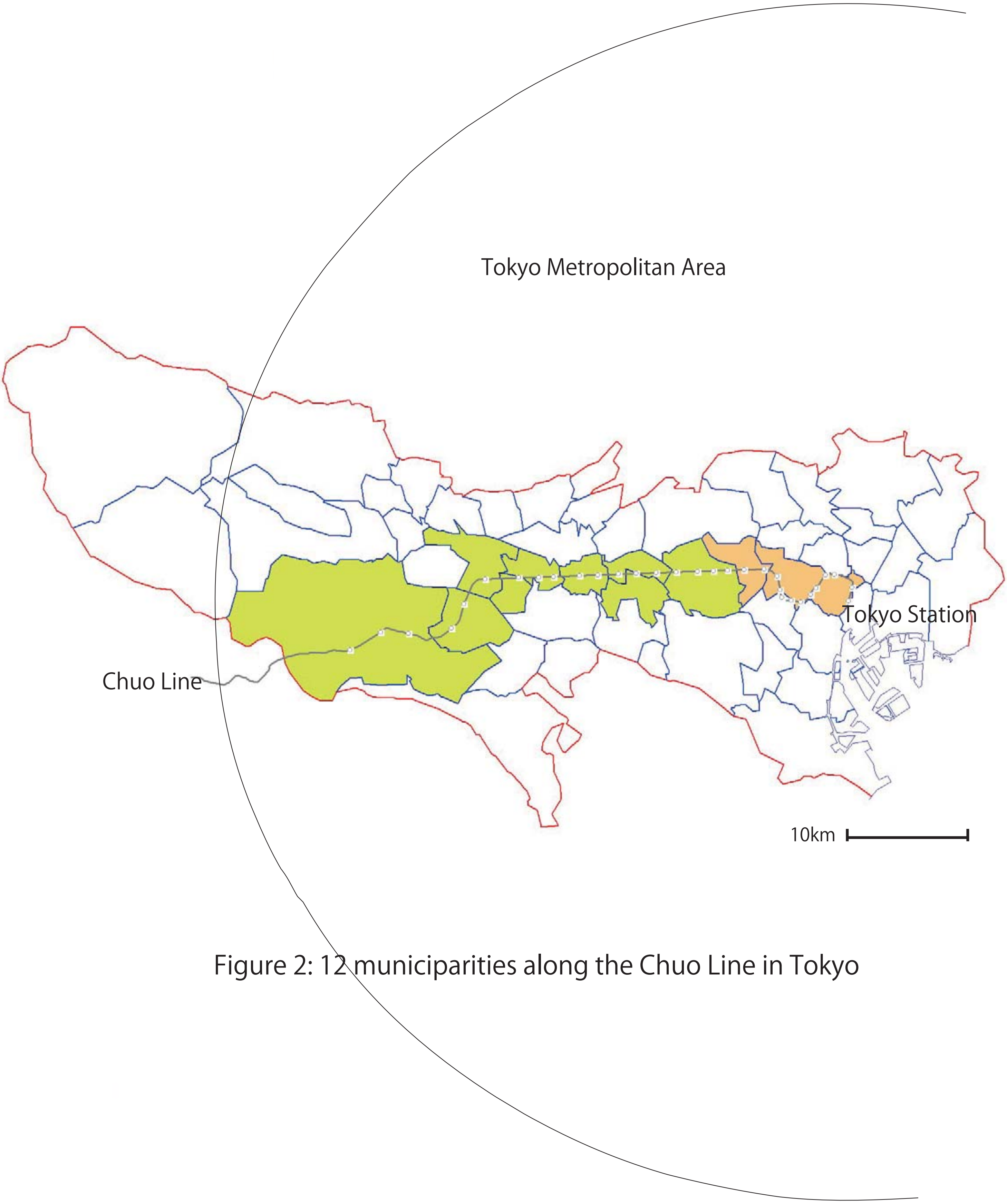

Figure 2: 12 municiparities along the Chuo Line in Tokyo