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# INVARIANCE OF THE DISTRIBUTION OF THE MAXIMUM

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**ABSTRACT.** Many models in economics involve probabilistic choices where each decision-maker selects the best alternative from a finite set. Viewing the value of each alternative as a random variable, the analyst is then interested in the choice probabilities, that is, the probability for an alternative to give the maximum value. Much analytical power can be gained, both for positive and normative analysis, if the maximum value is statistically independent of which alternative obtains the highest value. This note synthesizes and generalizes previous results on this invariance property. We provide characterizations of the property within a wide class of distributions that comprises the McFadden GEV class, show implications in several directions, and establish connections with copulas. We illustrate the usefulness of the invariance property by way of a few examples.

**Keywords:** Choice, random utility, extreme value, leader-maximum, invariance, independence.

**JEL codes:** C10, C25, D01.

This note characterizes, within certain much-used function classes, those multivariate probability distributions that have a certain remarkable and useful invariance property that can be informally described as follows. Consider a population of consumers who face a finite set of alternatives, say, lunch restaurants. At each time a choice is made, the consumer chooses exactly one of the restaurants; he or she is well informed and chooses the one that she finds best. Her choice of restaurant will result in an experienced utility. Let us now compare the experienced utility distributions at the restaurants. The invariance property holds if these distributions are the same for all restaurants. In a statistical sense then, people are thus just as satisfied in one restaurant as in another. The reason why such invariance may even be a possibility is that consumers make voluntary and well-informed choices. The randomness is only in the eyes of the outside observer, who does not know every individual's preferences,

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only the population distribution of preferences. Imagine that one restaurant improves its food quality or services and/or reduces its prices, and assume that all consumers are informed of this change. Some consumers will then switch to this restaurant from the one they used to go to. The ones who are most eager to do this are those who were least happy at their original restaurant, which they before found to be the best, but no longer so. Since the least happy left, average experienced utility of those who remain in their usual restaurants goes up. If the invariance property holds, the new utility distributions, among clients in the restaurants after one improved its services, will again be identical across restaurants, but now at a higher level.

The invariance property leads to analytical simplification in many applications including discrete choice with an outside option, rent seeking, innovation contests, patent races, and auctions, see Section 3. First, however, Section 1 provides general definitions and some preliminaries. Our main results are presented in Section 2. Section 4 discusses a number of earlier contributions.

### 1. DEFINITIONS AND PRELIMINARIES

Let  $\mathbb{N}$  be the positive integers,  $\mathbb{R}$  be the reals and  $\mathbb{R}_+$  the non-negative reals. Denote by  $\mathcal{F}$  the class of cumulative distribution functions  $F : \mathbb{R}^n \rightarrow [0, 1]$  that are twice continuously differentiable and have positive continuous density on some product set  $D^n \subseteq \mathbb{R}^n$ , where  $n \in \mathbb{N}$  and  $D$  is an open (bounded or unbounded) interval in  $\mathbb{R}$ . Let  $\mathbf{X} = (X_1, \dots, X_n)$  be a random vector distributed according to some such  $F$ . (We write vectors in bold-face.) Let  $\hat{X} = \max_j X_j$  be the maximum of the random vector and let  $\hat{F}$  denote its c.d.f.. We write  $F_j$  for the partial derivative of  $F$  with respect to its  $j^{\text{th}}$  variable and  $F^{(j)}$  for the  $j^{\text{th}}$  marginal distribution of the multivariate distribution  $F$ . Define the selection  $\xi \in J = \{1, \dots, n\}$  by  $\xi = \arg \max_j X_j$ , where the latter set with probability one is a singleton. Let  $p_j = P(\xi = j)$ . One may also consider the distribution of the maximum conditional on the selection of a particular alternative  $j \in J$ :  $\hat{F}^{(j)}(t) = P(\hat{X} \leq t \mid \xi = j)$ . In a discrete-choice setting,  $\hat{F}^{(j)}$  is the distribution of achieved (or experienced) utility, conditional on the choice of alternative  $j \in J$ , and  $p_j$  is its choice probability.

It is relatively straight-forward to prove the following three equalities:<sup>1</sup>

$$\begin{aligned} p_j &= \int F_j(s, \dots, s) ds \quad \forall j \in J \\ \hat{F}^{(j)}(t) &= p_j^{-1} \cdot \int_{-\infty}^t F_j(s, \dots, s) ds \quad \forall j \in J, t \in \mathbb{R} \\ \hat{F}(t) &= F(t, \dots, t) \quad \forall t \in \mathbb{R}. \end{aligned} \tag{1}$$

We note that  $p_j > 0$  for all  $j \in J$ , and that the quantities in (1) only depend on how the cumulative distribution function (c.d.f.)  $F \in \mathcal{F}$  behaves near the diagonal of its domain. By ‘‘invariance’’ we mean that the conditional distributions  $\hat{F}^{(j)}$  are identical across alternatives  $j \in J$ :

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<sup>1</sup>See e.g. Lindberg, Eriksson and Mattsson (1995) and Lindberg (2012).

**Definition 1.** A multivariate distribution  $F$  has the **Invariance Property** if  $\hat{F}^{(j)} = \hat{F}$  for all  $j \in J$ . A random vector  $\mathbf{X}$  has the Invariance Property if its distribution has the Invariance Property.

The following proposition, stated without proof, provides some immediate conclusions regarding the Invariance Property. We use the notational convention that a univariate function applied to a vector is applied to each component of the vector.

**Proposition 1.** Consider a random vector  $\mathbf{X}$  with c.d.f.  $F \in \mathcal{F}$ .

- (i) If  $X_j$  are i.i.d. then  $\mathbf{X}$  has the Invariance Property.
- (ii) If  $\mathbf{X}$  has the Invariance Property and  $T : \mathbb{R} \rightarrow \mathbb{R}$  is a strictly increasing function, then  $T(\mathbf{X})$  has the Invariance Property.
- (iii)  $\mathbf{X}$  has the Invariance Property if and only if the selection  $\xi$  and the maximum  $\hat{X}$  are independent.

We will express results in terms of so-called copulas. These are functions on the unit cubes in Euclidean spaces, defined as follows (see Nelsen, 2006, for an excellent introduction):

**Definition 2.** A copula is any function  $C : [0, 1]^n \rightarrow [0, 1]$  such that

- (i)  $C(\mathbf{x}) = 0$  if  $\prod_{j \in J} x_j = 0$ ,
- (ii)  $C(\mathbf{x}) = x_k$  if  $\prod_{j \in J \setminus \{k\}} x_j = 1$ ,
- (iii) If  $\mathbf{x}, \mathbf{y} \in [0, 1]^n$  and  $\mathbf{x} \leq \mathbf{y}$ , then  $V_C([\mathbf{x}, \mathbf{y}]) \geq 0$ .

Here  $[\mathbf{x}, \mathbf{y}]$  denotes the box  $\times_{j=1}^n [x_j, y_j] \subseteq [0, 1]^n$  and  $V_C([\mathbf{x}, \mathbf{y}])$  is the  $C$ -volume of this box, defined as the signed sum of the values  $C(\mathbf{v})$  at all vertices  $\mathbf{v}$  of  $[\mathbf{x}, \mathbf{y}]$ , where the sign is positive (negative) if  $v_j = x_j$  for an even (odd) number of coordinates  $j \in J$ . Condition (iii) ensures that the copula assigns non-negative probability mass to any box. By construction, copulas are then c.d.f.s on the unit cube that have uniform marginal distributions.

By Sklar's theorem (e.g. Theorem 2.10.9 in Nelsen, 2006), every multivariate distribution  $F : \mathbb{R}^n \rightarrow [0, 1]$  can be written in terms of its marginal distributions  $F^{(j)}$  and a copula  $C$ , so that  $F(\mathbf{x}) = C(F^{(1)}(x_1), \dots, F^{(n)}(x_n))$  for all  $\mathbf{x} \in \mathbb{R}^n$ . The copula associated with any c.d.f. thus captures the statistical dependence structure of the multivariate distribution in question.

In order to state our main result we define the class of copulas that are associated with multivariate extreme-value (MEV) distributions, to be called *MEV copulas*. A *multivariate extreme-value (MEV) distribution* is any multivariate distribution  $H$  with non-degenerate margins for which there exists an i.i.d. sequence of random vectors,  $(\mathbf{X}_m)_{m \in \mathbb{N}}$ , all distributed according to some multivariate c.d.f.  $F$ , and normalizing vectors  $\mathbf{a}_m > 0$  and  $\mathbf{b}_m$  for each  $m \in \mathbb{N}$  such that, with component-wise maximization, multiplication and division,

$$\lim_{m \rightarrow \infty} P\left(\frac{\max_{1 \leq l \leq m} \mathbf{X}_l - \mathbf{b}_m}{\mathbf{a}_m} \leq \mathbf{x}\right) = \lim_{m \rightarrow \infty} [F(\mathbf{a}_m \mathbf{x} + \mathbf{b}_m)]^m = H(\mathbf{x}),$$

(see Joe, 1997).<sup>2</sup> MEV copulas can be shown to be exactly those copulas that satisfy the following homogeneity property

$$C(x_1^\alpha, \dots, x_n^\alpha) = [C(x_1, \dots, x_n)]^\alpha \quad \forall \mathbf{x} \in (0, 1)^n, \forall \alpha > 0.$$

We prove this claim in Lemma 1 in the Appendix, thereby completing an argument given in Joe (1997).

## 2. RESULTS

The main result of this note is the following theorem, which generalizes previous results, see Section 4. All proofs are given in the Appendix.

**Theorem 1.** *Consider any  $F \in \mathcal{F}$  such that  $F(\mathbf{x}) = C(F^{(1)}(x_1), \dots, F^{(n)}(x_n)) \forall \mathbf{x} \in \mathbb{R}^n$  for some MEV copula  $C$  that is twice continuously differentiable on the unit cube. Then  $F$  and all the multivariate marginal distributions of  $F$  have the Invariance Property if and only if for each  $j \in J$  there exists a  $\alpha_j > 0$  such that  $F^{(j)} = [F^{(1)}]^{\alpha_j}$ .*

The condition that the marginal distributions are positive powers of each other has special significance when the marginal distribution  $F^{(1)}$  is standard extreme-value type 1 (Gumbel). Then  $F^{(j)}(t) = \exp(-\exp[-t + \ln \alpha_j])$ , meaning that the marginal distributions are identical up to a location shift. Conversely, under the hypothesis of Theorem 1 it is straightforward to establish that if the Invariance Property is preserved under all location shifts of  $\mathbf{X}$  then the marginal distributions must be extreme-value type 1. An MEV distribution with type 1 margins is the basis of the well-known logit family of models, pioneered by McFadden (1974, 1978, 1981).

Theorem 1 characterizes the Invariance Property for those distributions in  $\mathcal{F}$  that have twice continuously differentiable MEV copulas. Could it be that invariance of a probability distribution is *equivalent* to the distribution having an MEV copula and marginal distributions that are powers of each other? That conjecture is false because, as noted above, invariance depends only on the properties of the distribution on the diagonal  $L = \{\mathbf{x} \in D^n : x_1 = \dots = x_n\}$  of its domain:

**Proposition 2.** *Consider any distributions  $F, \tilde{F} \in \mathcal{F}$  on the same domain  $D^n$ . If  $F$  has the Invariance Property and  $\tilde{F} = F$  on an open neighborhood of  $L$ , then also  $\tilde{F}$  has the Invariance Property.*

The next proposition shows that the Invariance Property is preserved under aggregation of components to blocks represented by their maximal member.

**Proposition 3.** *Suppose that  $\mathbf{X} = (X_1, \dots, X_n)$  has the Invariance Property, and consider any partition of  $J$  into  $k$  subsets  $J_i$ . For each subset  $J_i$  let  $Y_i = \max_{j \in J_i} (X_j)$ . Then  $\mathbf{Y} = (Y_1, \dots, Y_k)$  has the Invariance Property.*

<sup>2</sup>More precisely, if  $\mathbf{X}_m = (X_{m1}, \dots, X_{mn})$ , then  $\max_{1 \leq l \leq m} \mathbf{X}_m$  is the vector with  $j^{\text{th}}$  component  $\max_{1 \leq l \leq m} X_{lj}$ , and  $\mathbf{a}_m \mathbf{x}$  is the vector with  $j^{\text{th}}$  component  $a_{mj} x_j$ .

## 3. APPLICATIONS

This section discusses a number of economics contexts in which the Invariance Property is useful. Throughout we rely on the equivalence of the Invariance Property to the independence of the selection  $\xi$  and the maximum  $\hat{X}$ .

**3.1. Discrete choice with outside option.** The probabilistic basis of a discrete choice model is a random vector  $\mathbf{X}$  of (indirect) utilities, each associated with an alternative  $j \in J$ . The decision-maker, or, as we will here say, the consumer, chooses the alternative with the highest utility. Such a discrete choice model is also embedded in models of monopolistic competition (Sattinger, 1984; Perloff and Salop, 1985). In many applications, consumers also have an outside option. The utility associated with the outside option acts as a threshold such that the consumer only chooses one of the alternatives in  $J$  if the utility of that alternative exceeds that of the outside option. We shall see that such situations are easily treated when the random utilities have the Invariance Property. This seems not to have been observed before.

Suppose then that  $\mathbf{X}$  has the Invariance Property, and that alternative  $j$  is chosen if  $X_j = \hat{X}$  and  $\hat{X} > X_0$ , where  $X_0$  is the random utility of the outside option, which we take to be statistically independent from  $\mathbf{X}$ . Let  $F_0$  be the c.d.f. of  $X_0$ . We still use  $\xi$  to denote the alternative in  $J = \{1, \dots, n\}$  with maximum utility. The outside option is labelled  $j = 0$ , and  $\eta \in \{0\} \cup J$  denotes the chosen option among all options. The probability for the outside option is

$$P(\eta = 0) = P(\hat{X} \leq X_0) = \int F(s, \dots, s) dF_0(s),$$

a probability that can be calculated from the primitives of the model.

It follows from Proposition 1 (iii) that  $P(\xi = j \mid \hat{X} > s) = p_j$  for any  $j \in J$  and  $s \in \mathbb{R}$ . Thus,

$$\begin{aligned} P(\eta = j) &= P(\xi = j \wedge \hat{X} > X_0) = \int P(\xi = j \mid \hat{X} > s) \cdot P(\hat{X} > s) dF_0(s) \\ &= p_j \cdot [1 - P(\eta = 0)]. \end{aligned}$$

The presence of an outside option does not affect the probabilities of the inside options, conditional on the outside option not being chosen.

**3.2. Rent-seeking, innovation contests and patent races.** Baye and Hoppe (2003) establish the strategic equivalence between wide classes of rent-seeking games, innovation contests and patent-races. In their innovation-contest game,  $n$  firms compete by employing a finite and positive number of scientists, where each scientist costs  $c > 0$  and independently produces an innovation of a random value with c.d.f.  $H$  on the unit interval. All firms pay the costs of their scientists, and the firm with the best idea among all firms wins the value of its best idea. The other firms win nothing.

Let  $X_j$  be the value of firm  $j$ 's best idea and let  $\mathbf{X} = (X_1, \dots, X_n)$ . Then  $\mathbf{X}$  has the joint c.d.f.  $F(x_1, \dots, x_n) = \prod_{j=1}^n F^{(j)}(x_j)$ , where  $F^{(j)}(x_j) = [H(x_j)]^{m_j}$  and  $m_j$

is the number of scientists at firm  $j$ . Clearly  $F$  has the invariance property, since  $F^{(j)} = (F^{(1)})^{m_j/m_1}$  for all  $j \in J = \{1, \dots, n\}$ . By Proposition 1 (iii), the identity  $\xi \in J$  of the firm that wins the contest and the value  $\hat{X}$  of the best idea are statistically independent. Hence, the expected profit to firm  $j$  from hiring  $m_j \in \mathbb{N}$  scientists can be expressed as

$$\pi_j(m_1, \dots, m_n) = P(\xi = j) \cdot E(\hat{X}) - c \cdot m_j$$

where  $P(\xi = j) = m_j/M$  for  $M = m_1 + \dots + m_n$ , and

$$E(\hat{X}) = \int_0^1 P(\hat{X} > s) ds = 1 - \int_0^1 \hat{F}(s) ds = 1 - \int_0^1 [H(s)]^M ds.$$

Hence,

$$\pi_j(m_1, \dots, m_n) = \left[ \frac{1}{M} \left( 1 - \int_0^1 [H(s)]^M ds \right) - c \right] \cdot m_j$$

This setting can be generalized in several ways within our framework. The R&D inputs  $m_j$  may be any positive real numbers, and the c.d.f.  $H$  may be defined on any interval. Moreover, the marginal c.d.f.  $F^{(j)}$  of the value of the best idea within any firm  $j$  may be of the power form  $H^{\varphi_j(m_j)}$  for any positive increasing function  $\varphi_j$  of its R&D input  $m_j$ . For example, an S-shaped such function could represent the "synergy" or "critical mass" effect that the marginal return (in terms of innovations or discoveries) from an additional scientist may be highest at some intermediate size of the research unit. Moreover, by using MEV copulas, one may allow for statistical dependence among the values of the best ideas in the different firms. For instance, using the linearly homogenous function  $G$  in Lemma 1, one obtains

$$P(\xi = j) = \frac{\varphi_j(m_j) \cdot G_j(\varphi_1(m_1), \dots, \varphi_n(m_n))}{G(\varphi_1(m_1), \dots, \varphi_n(m_n))}$$

from Theorem 1 in Mattsson, Weibull and Lindberg (2014), and  $E(\hat{X})$  can be derived from the c.d.f.  $\hat{F} = H^{G(\varphi_1(m_1), \dots, \varphi_n(m_n))}$  of  $\hat{X}$ . We conjecture that the results in Baye and Hoppe (2003) can be generalized, by use of the invariance property, in many important directions for a wide variety of innovation contests, patent races and rent-seeking games.

**3.3. Auctions.** Following Milgrom and Weber (1982) and Krishna (2002) we briefly consider auctions for an indivisible item in a situation where each bidder  $j \in J$  has private information about the item for sale in the form of a random signal  $X_j$ . The signal vector  $\mathbf{X} = (X_1, \dots, X_n)$  has positive dependence among the components, formalized as affiliation (see Krishna, 2002). More precisely, assume that all signals take values in some interval  $D \subseteq \mathbb{R}_+$ , and let  $F : D^n \rightarrow [0, 1]$  be the joint c.d.f. of

the signal vector  $\mathbf{X}$  with density  $f : D^n \rightarrow \mathbb{R}_+$ . The signals are (positively) *affiliated* (Milgrom and Weber, 1982) if

$$f(\mathbf{x}) \cdot f(\mathbf{y}) \leq f(\mathbf{x} \vee \mathbf{y}) \cdot f(\mathbf{x} \wedge \mathbf{y})$$

for all  $\mathbf{x}, \mathbf{y} \in D^n$ .<sup>3</sup> We now add the assumption that the distribution  $F$  of  $\mathbf{X}$  takes the form  $F(\mathbf{x}) = C(\Phi(x_1)^{\alpha_1}, \dots, \Phi(x_n)^{\alpha_n})$  for some MEV copula  $C$ , univariate c.d.f.  $\Phi$ , and constants  $\alpha_j > 0, j \in J$ . To the best of our knowledge, this assumption is new to the auction literature.<sup>4</sup> We proceed to show that it is useful.

A *strategy* for bidder  $j$  is any measurable function  $\beta_j : D \rightarrow \mathbb{R}_+$  that maps bidder  $j$ 's private signal  $X_j$  to a bid  $\beta_j(X_j)$ . We proceed to calculate the distribution and expectation of the highest and second-highest bids for the case when all bidders use the same strictly increasing strategy  $\beta$ .<sup>5</sup> The bidder with the highest signal,  $\hat{X}$ , then makes the highest bid,  $\beta(\hat{X})$ , and wins the auction. The distribution of the highest bid is easy to obtain. Writing  $\Psi(t)$  for  $\Phi[\beta^{-1}(t)]$ ,  $c$  for  $C(e^{-\alpha_1}, \dots, e^{-\alpha_n})$ , and recalling that  $C$  is a MEV copula, we find that the distribution of the maximum bid is

$$\begin{aligned} P\left[\beta(\hat{X}) \leq t\right] &= C(\Psi(t)^{\alpha_1}, \dots, \Psi(t)^{\alpha_n}) = C(e^{-\alpha_1(-\ln \Psi(t))}, \dots, e^{-\alpha_n(-\ln \Psi(t))}) \\ &= c^{-\ln \Psi(t)} = \exp\left(\ln\left[\Psi(t)^{-\ln c}\right]\right) = \Psi(t)^{-\ln c}. \end{aligned}$$

The inverse of the c.d.f. of the highest bid  $\beta(\hat{X})$  is the mapping  $s \rightarrow \Psi^{-1}\left(s^{-\frac{1}{\ln c}}\right)$ . Hence, the expected maximum bid is

$$E\left[\beta(\hat{X})\right] = \int_0^1 \Psi^{-1}\left(s^{-\frac{1}{\ln c}}\right) ds = \int_0^1 \beta\left(\Phi^{-1}\left(s^{-\frac{1}{\ln c}}\right)\right) ds.$$

Writing  $c_j$  for  $C(e^{-\alpha_1}, \dots, 1, \dots, e^{-\alpha_n})$ , where  $e^{-\alpha_j}$  in position  $j$  is replaced by 1, it is also straightforward to derive the distribution of the second highest bid,  $\hat{X}_{(2)}$  (the actual payment in a second-price auction):

$$\begin{aligned} P\left[\beta(\hat{X}_{(2)}) \leq t\right] &= P\left[\hat{X} \leq \beta^{-1}(t)\right] + \\ &\quad + \sum_{j=1}^n [C(\Psi(t)^{\alpha_1}, \dots, 1, \dots, \Psi(t)^{\alpha_n}) - C(\Psi(t)^{\alpha_1}, \dots, \Psi(t)^{\alpha_n})] \\ &= \sum_{j=1}^n \Psi(t)^{-\ln c_j} - (n-1) \Psi(t)^{-\ln c}. \end{aligned}$$

<sup>3</sup>This property is known in the statistics literature as multivariate total positivity ( $MTP_2$ ) of the density, see e.g. Karlin and Rinott (1980). See also Krishna (2002).

<sup>4</sup>Joe (1997) collects some results showing the relationship between affiliation and MEV copula. If a density is  $MTP_2$  then all bivariate margins of the c.d.f. are max-infinitely divisible (Thm. 2.3, 2.5 and 2.6). Any MEV copula is max-infinitely divisible. Thus affiliation and MEV copula are related concepts.

<sup>5</sup>Such monotonicity is known to hold in symmetric equilibria in sealed-bid first-price and second-price auctions, see e.g. Krishna (2002).



The expectation of the second highest bid is then

$$\begin{aligned}
E \left[ \beta \left( \hat{X}_{(2)} \right) \right] &= \int_0^\infty \left( 1 - \sum_{j=1}^n \Psi(t)^{-\ln c_j} + (n-1) \Psi(t)^{-\ln c} \right) dt \\
&= \sum_{j=1}^n \int_0^\infty \left[ 1 - \Psi(t)^{-\ln c_j} \right] dt - (n-1) \int_0^\infty \left[ 1 - \Psi(t)^{-\ln c} \right] dt \\
&= \sum_{j=1}^n \int_0^1 \beta \left( \Phi^{-1} \left( s^{-\frac{1}{\ln c_j}} \right) \right) ds - (n-1) E \left[ \beta \left( \hat{X} \right) \right].
\end{aligned}$$

We thus have explicit expressions for the distribution and expectation of the highest and second highest bids. By the Invariance Property, the probability distributions of the highest and second-highest bids are independent of which bidder actually won. We finally note that these derivations were made without any symmetry assumption concerning the distribution of the signal vector  $X$ . In particular, asymmetric statistical dependency among bidders' signals is permitted.

#### 4. PREVIOUS LITERATURE

Researchers in economics and in probability theory have returned to the topic of invariance a number of times. This section presents a summary and highlights how our results are related to earlier results.

Strauss (1979, Theorem 5) claimed that if the components of a random vector  $\mathbf{X}$  are of the additive form  $X_j = \nu_j + Z_j$ , where the  $\nu_j$ 's are scalars and  $\mathbf{Z}$  is a random vector with c.d.f.  $F$ , then the Invariance Property holds if and only if  $F(\mathbf{z}) = \phi[G(e^{-\mathbf{z}})]$  for some positive homogenous function  $G$  (with positive degree) and some function  $\phi: \mathbb{R}_+ \rightarrow [0, 1]$  such that  $F$  is a c.d.f.<sup>6</sup> It is straightforward to show that the subset of such functions  $G$  that are linearly homogenous corresponds exactly to the set of MEV copula (see Lemma 1). Strauss (1979, Theorem 4) further showed that the multinomial logit model is the only additive random utility model with independent error terms  $Z_j$  for which the Invariance Property holds. Anas and Feng (1988) later showed that the multinomial logit model has the weaker Invariance Property  $E[\hat{X}_j] = E[\hat{X}]$  for all  $j \in J$ . This follows, however, directly from Strauss (1979).

de Palma and Kilani (2007) also considered the additive random utility case with independent error terms and reproved Theorem 4 by Strauss (1979). They also proved the stronger claim that for error terms that are i.i.d. with finite expectation, the above Invariance Property of the conditional expected maximum utility for any choice of the  $\nu_j$ 's, is equivalent to the error terms being extreme-value type 1 (Gumbel) distributed. The if-part is the same as the previous claim by Anas and Feng (1988).

Also Train and Wilson (2008) derived the Invariance Property for Gumbel distributed random variables. They used this property for maximum likelihood estimation in

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<sup>6</sup>The same claim was essentially proved by Robertson and Strauss (1981), with Lindberg, Eriksson and Mattsson (1995) completing the argument.

a class of combined stated- and revealed-preference experiments, where each respondent considered the same alternatives in both experiments and where the attributes of the alternatives in the stated-preference experiments were varied on the basis of the respondent's revealed-preference choice.

Our Theorem 1 states that, for any multivariate distribution with MEV copula, the Invariance Property holds for all multivariate marginal distributions if and only if the univariate marginal distributions are positive powers of each other. A special case of this result was given by Resnick and Roy (1990b) who showed that MEV distributions with Gumbel marginals have the Invariance Property. Mattsson, Weibull and Lindberg (2014) generalized this to the "if" claim in the present Theorem 1. Resnick and Roy (1990a) proved a special case of the "only if" claim in our Theorem 1, namely that the Invariance Property for two independent non-negative random variables implies that their c.d.f.s are positive powers of each other. Mattsson, Weibull and Lindberg (2014) generalized this to an arbitrary finite number of independent random variables.

## 5. APPENDIX

The following Lemma is essentially in Joe (1997) but the proof given there is incomplete.

**Lemma 1.** *A copula  $C$  is a copula of an MEV distribution if and only if it is of the form*

$$C(\mathbf{x}) = \exp(-G(-\ln \mathbf{x})) \quad (2)$$

for some linearly homogenous function  $G$ .

**Proof.** Necessity: Note with Joe (1997, p. 173-174) that an MEV copula  $C$  must satisfy  $C(\mathbf{x}^\alpha) = C(\mathbf{x})^\alpha$  for all  $\mathbf{x} \in (0, 1)^n$  and  $\alpha > 0$ . Then  $G(\mathbf{y}) \equiv -\ln C(e^{-\mathbf{y}})$  is linearly homogenous.

Sufficiency: Suppose  $C$  has the form (2). Let  $F$  be a c.d.f. with copula  $C$  and marginals that are unit exponential on the negative half-axis, i.e.  $F_i(x) = e^x$  on  $(-\infty, 0]$ . Then  $F(\mathbf{x}) = C(e^{\mathbf{x}})$ . Let  $\mathbf{X}_j = (X_{j1}, \dots, X_{jn})$  be i.i.d. with c.d.f.  $F$ , and let  $M_i^k = \max_{j \leq k} X_{ji}$ . Let  $\mathbf{Y}^k = (kM_1^k, \dots, kM_n^k)$ , then  $\mathbf{Y}^k$  has the c.d.f.

$$\begin{aligned} P(\mathbf{Y}^k \leq \mathbf{x}) &= P[(kM_1^k, \dots, kM_n^k) \leq (x_1, \dots, x_n)] \\ &= P\left[\left(\max_{j \leq k} X_{j1}, \dots, \max_{j \leq k} X_{jn}\right) \leq \left(\frac{x_1}{k}, \dots, \frac{x_n}{k}\right)\right] \\ &= \prod_{j \leq k} P\left[X_{j1}, \dots, X_{jn} \leq \left(\frac{x_1}{k}, \dots, \frac{x_n}{k}\right)\right] \\ &= C(e^{x_1/k}, \dots, e^{x_n/k})^k = C(e^{\mathbf{x}}). \end{aligned}$$

Thus all  $\mathbf{Y}^k$  have the same distribution, and hence they converge in distribution to  $\mathbf{Y}^1$ , say, with the same c.d.f.  $C(e^{\mathbf{x}})$ . Therefore  $C(e^{\mathbf{x}})$  is the c.d.f. of an MEV distribution, and  $C$  is an MEV copula. ■

**Proof of Theorem 1.** Sufficiency: This follows from Theorem 1 in Mattsson, Weibull and Lindberg (2014), combined with the observation that the multivariate marginal distributions of MEV copula are also MEV copula, and that twice differentiability is inherited.

Necessity: Consider the c.d.f.  $P(X_1 \leq x_1, X_2 \leq x_2) = F(x_1, x_2, \infty, \dots, \infty) = C(F^{(1)}(x_1), F^{(2)}(x_2), 1, \dots, 1)$ . By the assumptions of the theorem, this c.d.f. has the Invariance Property. Furthermore, since  $F$  and  $C$  are twice continuously differentiable then also  $F^{(i)}$  and  $P(X_1 \leq x_1, X_2 \leq x_2)$  are twice continuously differentiable. The copula for  $(X_1, X_2)$  inherits the MEV property from  $C$  by continuity. We may thus ignore the last but two dimensions of  $F$  and  $C$  and assume that  $J = 2$  at no loss of generality. Hence, let  $\mathbf{X} = (X_1, X_2)$  have multivariate c.d.f.  $F \in \mathcal{F}$  on  $D^2$  for some open interval  $D \subseteq \mathbb{R}$ . Then

$$F(\mathbf{x}) = C(F^{(1)}(x_1), F^{(2)}(x_2)),$$

where  $C$  is an MEV copula that by assumption is twice continuously differentiable and  $F$  has the Invariance Property. We have to show that  $F^{(2)} = [F^{(1)}]^\alpha$  for some  $\alpha > 0$ . By Lemma 1, we may equivalently write

$$F(\mathbf{x}) = \exp(-G(-\ln F^{(1)}(x_1), -\ln F^{(2)}(x_2))), \quad (3)$$

where  $G$  is linearly homogenous and satisfies the properties necessary for  $F$  to be a c.d.f. We have assumed that  $C$  is twice continuously differentiable and then so is  $G$ .

By Proposition 1, the Invariance Property remains under any strictly increasing transformation of the components. Hence, it is no loss of generality to apply such a transformation so that  $F^{(1)}(x_1) = e^{x_1}$  on  $D = (-\infty, 0)$ . By (1) we then have, for all  $t \in D$ ,

$$\begin{aligned} \hat{F}^{(1)}(t) &= \frac{1}{p_1} \int_{-\infty}^t F(s, s) G_1(-s, -\ln F^{(2)}(s)) ds = \\ \hat{F}^{(2)}(t) &= \frac{1}{p_2} \int_{-\infty}^t F(s, s) G_2(-s, -\ln F^{(2)}(s)) \frac{f^{(2)}(s)}{F^{(2)}(s)} ds, \end{aligned}$$

where  $f^{(2)}$  is the density of  $F^{(2)}$  and  $p_i$  is defined in (1), for  $i = 1, 2$ . Differentiation with respect to  $t \in D$  gives

$$\frac{1}{p_1} F(t, t) G_1(-t, -\ln F^{(2)}(t)) = \frac{1}{p_2} F(t, t) G_2(-t, -\ln F^{(2)}(t)) \cdot \frac{f^{(2)}(t)}{F^{(2)}(t)}$$

or

$$f^{(2)}(t) = \frac{p_2 G_1(-t, -\ln F^{(2)}(t))}{p_1 G_2(-t, -\ln F^{(2)}(t))} \cdot F^{(2)}(t) \quad \forall t \in D. \quad (4)$$

Given  $p_1, p_2$ , let  $\alpha$  solve  $G_1(1, \alpha)/G(1, \alpha) = p_1$ ; such  $\alpha$  always exists since the mapping from  $\alpha > 0$  to  $(0, 1)$  is surjective (Norets and Takahashi, 2013). Then it is

easily verified that  $F^{(2)}(\cdot) = [F^{(1)}(\cdot)]^\alpha = \exp(\alpha(\cdot))$  is a solution to (4) and that  $p_2 = P(X_2 > X_1)$  if  $(X_1, X_2)$  is distributed according to  $C(F^{(1)}, F^{(2)})$ .

It remains to prove that no other solution to (4) exists that leads to the same value of  $p_1, p_2$ . According to the Picard-Lindelöf Theorem (see e.g. Theorem 3.1 in Hale (1969)), such an ordinary differential equation has a unique (local) solution through any given point  $(t_0, F^{(2)}(t_0)) \in D \times (0, 1)$  if  $\Psi(t, x) = \frac{p_2 G_1(-t, -\ln x)}{p_1 G_2(-t, -\ln x)} \cdot x$  defines a continuous function on  $D \times (0, 1)$  that is locally Lipschitz continuous in  $x$ . By assumption on  $C$ ,  $G_{12}$  and  $G_{22}$  are continuous, and we also have  $G_2 > 0$ , so  $\Psi$  is continuously differentiable, and hence also locally Lipschitz continuous, in  $x$ . Any local solution can be uniquely extended to a maximal solution on a subinterval  $D_0 \subseteq D$  (see e.g. pages 17-18 in Hale (1969)). Either  $D_0 = D$  or the solution attains the value 0 at the lower bound of  $D_0$  or the value 1 at its upper bound. In either case,  $f^{(2)}(t) = 0$  at such a point. Then the solution extends uniquely to all of  $D$  by letting  $F^{(2)}$  be 0 below  $D_0$  and 1 above. Moreover, by the Picard-Lindelöf theorem, an alternative solution,  $\tilde{F}^{(2)}$ , if it exists, cannot intersect the solution,  $F^{(2)}$ , anywhere in  $D_0$ . Hence,  $\tilde{F}^{(2)}(t)$  either lies above or below our solution for all  $t \in D_0$ . Assume it always lies above;  $F^{(2)} < \tilde{F}^{(2)}$  (the opposite case can be treated in the same way). Define the random vector  $\mathbf{Z} = (Z_1, Z_2)$  such that its c.d.f. is the above copula  $C$  (on  $[0, 1]^2$ ). Let  $F^{-i}$  be the inverse of  $F^{(i)}$ , for  $i = 1, 2$ , and let  $\tilde{F}^{-2}$  be the inverse of  $\tilde{F}^{(2)}$ .<sup>7</sup> Clearly  $\tilde{F}^{-2} < F^{-2}$ .

Let the random vector  $\mathbf{Y} = (Y_1, Y_2) = (F^{-1}(Z_1), F^{-2}(Z_2))$ . Then  $\mathbf{Y}$  has the c.d.f.

$$F_{\mathbf{Y}}(\mathbf{y}) = P[Z_1 \leq F^{(1)}(y_1) \wedge Z_2 \leq F^{(2)}(y_2)] = C(F^{(1)}(y_1), F^{(2)}(y_2)).$$

By the choice of  $\alpha$ ,  $p_2 = P(Y_2 \geq Y_1) = P[F^{-2}(Z_2) \geq F^{-1}(Z_1)]$ . Similarly, the random vector  $\tilde{\mathbf{Y}} = (Y_1, \tilde{Y}_2) = (F^{-1}(Z_1), \tilde{F}^{-2}(Z_2))$  has the c.d.f.  $\tilde{F}(y_1, y_2) = C(F^{(1)}(y_1), \tilde{F}^{(2)}(y_2))$ . We assume that  $\tilde{F}^{(2)}$  also satisfies the boundary condition  $p_2 = P(\tilde{Y}_2 \geq Y_1) = P[\tilde{F}^{-2}(Z_2) \geq F^{-1}(Z_1)]$ . But since  $\tilde{F}^{-2} < F^{-2}$ ,

$$p_2 = P[F^{-2}(Z_2) \geq F^{-1}(Z_1)] > P[\tilde{F}^{-2}(Z_2) \geq F^{-1}(Z_1)] = p_2,$$

a contradiction. Then  $F^{(2)}$  is the unique solution to (4) and the boundary condition  $p_2 = P[F^{-2}(Z_2) \geq F^{-1}(Z_1)]$ . ■

**Proof of Proposition 3.** Let  $\mathbf{X}$  have the invariance property, i.e.,  $\hat{F}^{(j)} = \hat{F}$  for all  $j$ . It is sufficient to establish the proposition for the case when two alternatives, say 1 and 2, are merged via the maximum operation. Define  $Y = X_1 \vee X_2$ . It is sufficient

<sup>7</sup>These inverse functions are well-defined since  $F$  has positive density on  $D^2$ .

to show that  $P(Y \leq t \mid \xi \in \{1, 2\}) = \hat{F}(t)$ . But

$$\begin{aligned}
 P(Y \leq t \mid \xi \in \{1, 2\}) &= \frac{P([Y \leq t] \wedge [\xi \in \{1, 2\}])}{P(\xi \in \{1, 2\})} \\
 &= \frac{P([Y \leq t] \wedge [\xi = 1]) + P([Y \leq t] \wedge [\xi = 2])}{p_1 + p_2} \\
 &= \frac{P(Y \leq t \mid \xi = 1) P(\xi = 1) + P(Y \leq t \mid \xi = 2) P(\xi = 2)}{p_1 + p_2} \\
 &= \frac{p_1}{p_1 + p_2} \hat{F}^1(t) + \frac{p_2}{p_1 + p_2} \hat{F}^2(t) = \hat{F}(t). \blacksquare
 \end{aligned}$$

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