# Effect of Estimation on the Univariate Control Charts for Process Dispersion

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# Effect of estimation of the process parameters on the control limits of the univariate control charts for process dispersion

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***Abstract*— Control charts are extensively used in many real world appplications. Since process parameters are rarely known common practice is to estimate them. Then, the control limits are modified and become actually random variables. In this paper, we deal with the univariate control charts for dispersion for both rational subgroups and individual measurements. We study the effect of estimating the process parameters of this chart on the first two moments of the run length distribution. The results are used for proposing appropriate values of sample size and number of samples in order to make the estimated control limits perform as the theoretical ones.**

***Keywords*— Shewhart charts, ARL, S chart, X chart.**

## I. Introduction

Control charts are used for controlling and monitoring variables in any product or process. They have found considerable applications in industry for improving the quality of the products. The most known are the Shewhart type control charts for monitoring process mean and dispersion.

Quesenberry [7], was the first to examine the effect of estimation of the process mean and standard deviation on the control limits of the Shewhart chart for the mean for both rational subgroups and individual observations. Chen [1], extended this work by using three different estimators of the standard deviation in the $\overline{X}$ chart case. Nedumaran and Pigniatiello [5], investigated the estimation effect on the $T^2$ control charts. Woodall and Montgomery [12] emphasized the need for much more research in this area since it is proved that more data than usually recommended. are needed for the control charts to behave as expected from theory. In the same paper, Woodall and Montgomery state that much work has been done concerning the control of the process mean but not that much for the process dispersion. In an earlier paper Lowry et al. [4], examined the effect of run rules on the performance of control charts for detecting shifts in process standard deviation. Recently, Klein [3] proposed modified S-charts for keeping stable the process variability. In this paper, we examine the effect of estimation of the process parameters on the control limits of charts for process dispersion for both rational subgroups and individual observations.

The paper is organized as follows. In section 2, we present the classical $S$ chart with three sigma limits and extensive numerical calculations of the effect of estimating the process standard deviation on the values of average run length ($ARL$) and standard deviation of the run length ($SDRL$). Section 3, outlines the $S$ chart using probability limits and results of estimating the process standard deviation on the $ARL$ and $SDRL$ values again. The $X$ chart for individual observations is presented on section 4 and its use for process dispersion when we have estimated limits is suggested. Finally, in section 5 some conclusions are listed.

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## II. The S (three sigma) control chart

Assume that $X_{ij}$, $i = 1,..m$ and $j = 1,..,n$ are observations from a stable $N(\mu, \sigma^2)$ process comprising $m$ samples of size $n$ each. In this process we want to keep its variability in control. In order to develop control limits we need to know the value of the true standard deviation $\sigma$. If this value is known the control limits are

$$UCL = \left(c_4 + 3\sqrt{1-c_4^2}\right)\sigma \qquad (1)$$

$$LCL = \left(c_4 - 3\sqrt{1-c_4^2}\right)\sigma \qquad (2)$$

Usually, we do not know the value of $\sigma$ and therefore we have to estimate it from past data. The estimate used is

$$\bar{S} = \frac{1}{m}\sum_{i=1}^{m} S_i$$

where $m$ is the number of past samples used, $S_i^2 = \frac{1}{n-1}\sum_{j=1}^{n}\left(X_j - \bar{X}_i\right)^2$ is the unbiased estimator of $\sigma^2$ and $n$ is the sample size. However, we know that $\bar{S}$ is not an unbiased estimator of $\sigma$. It has been proved (see e.g. Ryan [10]), that an unbiased estimate of $\sigma$ is $\bar{S}/c_4$ where $c_4 = \left(\frac{2}{n-1}\right)^{1/2}\frac{\Gamma(n/2)}{\Gamma((n-1)/2)}$ and that the standard deviation of $S$ equals $\sigma\sqrt{1-c_4^2}$. The upper and lower control limits of the chart known as the $S$ chart are:

$$\widehat{UCL} = \left(1 + \frac{3}{c_4}\sqrt{1-c_4^2}\right)\bar{S} \qquad (3)$$

$$\widehat{LCL} = \left(1 - \frac{3}{c_4}\sqrt{1-c_4^2}\right)\bar{S} \qquad (4)$$

Approaches making use of these limits are known as the three sigma approaches based on the normal approximation proposed by Shewhart in the early thirties. However, it is easy to prove that this approximation is not satisfactory since, as is known

$$\frac{(n-1)S^2}{\sigma^2} \sim X^2_{n-1} \tag{5}$$

Although this approximation is not accurate, it is usually used as a first check.

Let $A_i$ denote the event that the $ith$ sample standard deviation $S_i$ exceeds $UCL$ or is exceeded by $LCL$. Then, since $S_i$ and $S_j$ are independent for $i \neq j$, the sequence of trials $A_i$ and $A_j$ are independent meaning that they constitute a sequence of Bernoulli trials. The mean and standard deviation of the run length distribution, $ARL$ and $SDRL$ respectively, of this process is that of a geometric distribution given by the following formulas

$$ARL = \frac{1}{1-\beta} \tag{6}$$

$$SDRL = \frac{\sqrt{\beta}}{1-\beta} \tag{7}$$

where $\beta = 1 - \Pr(A_i) = \Pr(LCL \leq S_i \leq UCL)$.

Assume now that we are in the case when the true value of the standard deviation is not known, which is the most usual case. Let $B_i$ denote the event that the $ith$ sample standard deviation $S_i$ exceeds $\widehat{UCL}$ or is exceeded by $\widehat{LCL}$. The formulas (6) and (7) for $ARL$ and $SDRL$ are not valid any more because the events $B_i$ and $B_j$ are not independent for $i \neq j$. We can prove that $E(\widehat{UCL}) = UCL$ and $Var(\widehat{UCL}) = \left(1 + \frac{3}{c_4}\sqrt{1-c_4^2}\right)^2 \sigma^2 \frac{(1-c_4^2)}{m}$ and using these relations we prove, after some calculations, that

$$Cov(S_i - \widehat{UCL}, S_j - \widehat{LCL}) = Var(\widehat{UCL})$$

$$= \left(1 + \frac{3}{c_4}\sqrt{1-c_4^2}\right)^2 \sigma^2 \frac{(1-c_4^2)}{m}$$

and

$$Var(S_i - \widehat{UCL}) = \left[1 + \frac{\left(1 + \frac{3}{c_4}\sqrt{1-c_4^2}\right)^2}{m}\right] \sigma^2 (1 - c_4^2)$$

Therefore, the correlation between the random variables $S_i - \widehat{UCL}$ and $S_j - \widehat{LCL}$ is

$$Corr(S_i - \widehat{UCL}, S_j - \widehat{LCL}) = \frac{Var(\widehat{UCL})}{Var(S_i - \widehat{UCL})} =$$

$$= \frac{\left(1 + \frac{3}{c_4}\sqrt{1-c_4^2}\right)^2}{m + \left(1 + \frac{3}{c_4}\sqrt{1-c_4^2}\right)^2} \tag{8}$$

It is obvious that the correlation is a function of $m$ and $n$ only. In table I we present values of the correlation for combinations of $m$ and $n$. From this table we see that as the sample size and the number of samples increases the correlation decreases. For small or moderate sample size ($n \leq 20$) we need 200 samples for the correlation to be negligible. However, for larger sample size we can afford $m = 50$.

In order to examine the values of the first two moments of the run length distribution, we performed a simulation study based on various numbers of samples and various sample sizes. In particular, the number of samples and samples sizes considered were $m = 5, 10, 20, 30, 50, 100, 200, 500, 1000$ and $n = 5, 10, 20$. For every combination of $m$ and $n$ we simulated $m$ samples of size $n$ from a $N(\mu, \sigma_0^2)$ distribution and computed $\widehat{UCL}$ and $\widehat{LCL}$. Then, we simulated samples from a $N(\mu, \sigma_1^2)$ distribution until we obtained a value above $\widehat{UCL}$ or below $\widehat{LCL}$. The number of samples simulated up to the one that led to a value outside the control limits constitutes one observation of the run length distribution. This procedure was repeated 10000 times in order to get an estimate of the values of $ARL$ and $SDRL$. Representative results for $n = 5, 10, 20$ are presented in tables II-XI.

From these tables certain conclusions are drawn. We see that we have results only for upward shifts when $n > 5$. This happens because for $n \leq 5$ the lower control limit is set to zero. Therefore, it can never be crossed by a simulation study, or in reality. For upward shifts, as $m$ increases, the $ARL$ and $SDRL$ values decrease and approach their theoretical values. For downward shifts as $m$ increases the same thing happens for $n = 50$. For $n = 10, 20$ the $ARL$ and $SDRL$ values do not follow a specific trend. In the in-control state we also do not have a clear pattern for either $ARL$ or $SDRL$ values. What we can say in every case is that $ARL$ and $SDRL$ values behave in the same way.

As $m$ increases the $ARL$ is getting closer to the theoretical value faster than the $SDRL$. Moreover, as $n$ increases the theoretical values, in the in-control state, approach the ones from a normal distribution which are $ARL = 370.4$ and $SDRL = 369.9$. The same, of course, happens for the out of control states.

If we use this type of chart for identifying shifts in process dispersion we have to use samples of size $n$ at least 20 for minimizing the effect of estimating $S$. If $n$ is less than this value the practitioner will face an increased number of false alarms. The effect of estimation is also severe for $m \leq 20$, especially in the in-control state and for small shifts. For values $30 \leq m \leq 50$ the effect is moderate and for values of 100 or larger the effect is small enough. A last point to make is that, when we have small downward shifts for $n \leq 20$, the $ARL$ and $SDRL$ values are larger than the corresponding in-control values. Consequently, in such cases special care must be taken and it is better to use control charts for small shifts like CUSUM and EWMA.

## III. The S (probability limits) control chart

A modification of the control limits (1), (2) and (3), (4) uses probability limits in place of the three sigma limits (see e.g. Ryan [10]). If the value of the standard deviation

$\sigma$ is known, the control limits (in place of (1) and (2)) are:

$$UCL = \sigma\sqrt{\frac{\chi^2_{0.999}}{n-1}}$$
$$LCL = \sigma\sqrt{\frac{\chi^2_{0.001}}{n-1}}$$

In these limits, if the process variability operates in control, the probability that the standard deviation of future subgroups will fall between them is 0.998, which is approximately equal to the 0.9973, the probability assumed when using the 3 sigma ones. If the true standard deviation is not known we use its unbiased estimate $\bar{S}/c_4$. The limits then become (in place of (3) and (4)):

$$\widehat{UCL} = \frac{\bar{S}}{c_4}\sqrt{\frac{\chi^2_{0.999}}{n-1}}$$
$$\widehat{LCL} = \frac{\bar{S}}{c_4}\sqrt{\frac{\chi^2_{0.001}}{n-1}}.$$

It is obvious that these limits are based on property (5). In the same way of thinking (as in the case of three sigma limits) we can prove that

$$Var(\widehat{UCL}) = [\sigma^2(1-c_4^2)\chi^2_{0.999}]/[(n-1)c_4^2 m]$$

and consequently

$$Cov(S_i - \widehat{UCL}, S_j - \widehat{LCL}) = Var(\widehat{UCL}) =$$

$$= \frac{\sigma^2(1-c_4^2)\chi^2_{0.999}}{(n-1)c_4^2 m}$$

Moreover,

$$Var(S_i - \widehat{UCL}) = \sigma^2(1-c_4^2)\left[1 + \frac{\chi^2_{0.999}}{(n-1)c_4^2 m}\right]$$

and finally

$$Corr(S_i - \widehat{UCL}, S_j - \widehat{LCL}) = \frac{Var(\widehat{UCL})}{Var(S_i - \widehat{UCL})} =$$

$$= \frac{\chi^2_{0.999}}{\chi^2_{0.999} + (n-1)c_4^2 m}.$$

As in the case of three sigma limits, this correlation depends only on $m$ and $n$. In table XII we calculated the correlation for various combinations of $m$ and $n$. From this table we conclude again that as the sample size and the number of samples increases the correlation decreases. The recommendation for sample sizes and number of samples is the same as in the previous section.

We computed the $ARL$ and $SDRL$ values for several values of $m$ and $n$ via simulation along the same lines as in the three sigma limits. The number of samples and samples sizes considered were $m$ = 5, 10, 20, 30, 50, 100, 200, 500, 1000 and $n$ = 5, 10, 20, 50. Representative results for $n = 5, 10$ are presented on tables XIII–XVI. From the results we deduce the following points. For upward shifts, as $m$ increases, the $ARL$ and $SDRL$ values generally decrease and approach their theoretical values. For downward shifts as $m$ increases the same thing happens for $n = 20, 50$. For $n = 5, 10$ the $ARL$ and $SDRL$ values do not follow a specific pattern. In the in-control state the $ARL$ and $SDRL$ values increase till they get close to their theoretical values. As an overall conclusion we can say that the $ARL$ and $SDRL$ values behave in the same way except that as $m$ increases the $ARL$ is getting closer to the theoretical value faster than the $SDRL$.

When we are in-control we need at least $m = 200$, otherwise the practitioner will face many false alarms whereas the value of $n$ is not equally important. In the out-of-control situations the value of $n$ is important for minimizing the effect of estimating $S$. Specifically, when $\sigma_1^2/\sigma_0^2 = 1.2$ the ARL values for $n = 5, 10, 20, 50$ are 239.29, 178.40, 117.98, 50.77, respectively. Therefore, we observe a dramatic reduction as $n$ becomes larger. A similar situation occurs for downward shifts. Consequently, large values of $n$, larger than 20, are recommended. The effect of estimation is severe for $m \leq 20$, especially for small shifts. For values $30 \leq m \leq 50$ the effect is moderate and for values of 100 or larger the effect is small enough. When we have small downward shifts for $n = 5$, and for $n = 10$ when $m \leq 10$, the $ARL$ and $SDRL$ values are larger than the corresponding in-control values. In such a situation it is better to use control charts for detecting small shifts like CUSUM and EWMA.

## IV. THE X CHART FOR MONITORING PROCESS DISPERSION

Let $X_i, i = 1, .., n$ denote independent and identically distributed observations from a $N(\mu, \sigma^2)$ process. If the parameters $\mu$ and $\sigma^2$ are known, the control limits are

$$UCL = \mu + 3\sigma$$
$$CL = \mu$$
$$LCL = \mu - 3\sigma$$

Usually, these parameters are not known and they have to be estimated. In this case, the variability is usually controlled using moving ranges. Nevertheless, Nelson [6], Roes et al. [9] and Rigdon et al. [8] have recommended either against the use of the moving range chart or its use together with the classical $X$ chart. Moreover, Sullivan and Woodall [11] showed that a moving range control chart does not contribute significantly to the identification of out of control situations. Therefore, the use of the $X$ control chart for monitoring the process standard deviation is recommended. The control limits of the $X$ control chart are

$$\widehat{UCL} = \bar{X} + 3\hat{\sigma}$$
$$\widehat{CL} = \bar{X}$$
$$\widehat{LCL} = \bar{X} - 3\hat{\sigma}$$

where $\bar{X}$ is an unbiased estimate of the mean of the process and $\hat{\sigma}$ is an estimate of the standard deviation $\sigma$ of the process. Usually, the estimate of the standard deviation used is $\overline{MR}/d_2$ where $\overline{MR}$ denotes the average of the moving ranges and $d_2$ is a constant used to make the estimator unbiased. However, Cryer and Ryan [2] showed that a preferable estimate of $\sigma$ is $s/c_4$ where $c_4$ is defined in the same way as in the case of rational subgroups and $s$ is the standard deviation of the observations.

In order to assess the effect of the number of observations on the control limits of the $X$ chart we performed a simulation study. The results are presented in table XVII. For each value in the table, we simulated $N$ values from a $N(\mu, \sigma_0^2)$ distribution, we computed the $\widehat{UCL}$ and $\widehat{LCL}$ and subsequently we generated values from a $N(\mu, \sigma_1^2)$ distribution until we obtained a value above $\widehat{UCL}$ or below $\widehat{LCL}$. The number of samples simulated up to the one that was outside the control limits constitutes one observation on the run length. This procedure was repeated 32000 times in order to get an estimate of the values of $ARL$ and $SDRL$.

From tables XVII and XVIII we see that we do not have results for downward shifts. This happens because a decreasing standard deviation will never cause a value below the lower control limit. The simulation reveals that the $ARL$ and $SDRL$ values decrease till they approach their theoretical values. We need at least 300 observations to minimize the effect of estimation in the control limits of the $X$ chart.

## V. Conclusions

In this paper, we examined the effect of estimation on the control limits for process dispersion on charts using rational subgroups and individual observations. Extensive numerical studies for several combinations of numbers of samples and of sample sizes in the case of rational subgroups and of numbers of observations in the case of individual control charts were presented. These values were used for proposing the $m$ and $n$ values that a practitioner should use in order to reduce the estimation effect on the univariate dispersion control charts.

In the rational subgroups case we propose larger $n$ values than usual and someone may argue that this is a problem. However, Woodall and Montgomery [12] remarked that in industry now there are large data sets available in contrast to the past. Therefore, such values for the sample size should not be a problem, generally. On the other hand, if for some special applications this still remains a problem, the practitioner should keep in mind the great influence on the estimated control chart performance displayed on the tables of this work.

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## APPENDIX

TABLE I

CORRELATION FOR SEVERAL VALUES OF M AND N.

| | $n$ | | | |
|---|---|---|---|---|
| $m$ | 5 | 10 | 20 | 50 |
| 5 | 0.16581 | 0.37055 | 0.30735 | 0.25370 |
| 10 | 0.30362 | 0.22741 | 0.18158 | 0.14528 |
| 20 | 0.17898 | 0.12829 | 0.09986 | 0.07833 |
| 30 | 0.12689 | 0.08935 | 0.06886 | 0.05362 |
| 50 | 0.08020 | 0.05560 | 0.04249 | 0.03288 |
| 100 | 0.04178 | 0.02859 | 0.02171 | 0.01671 |
| 200 | 0.02133 | 0.01450 | 0.01097 | 0.00843 |
| 500 | 0.00864 | 0.00585 | 0.00442 | 0.00339 |
| 1000 | 0.00434 | 0.00293 | 0.00221 | 0.00170 |

TABLE II

ARL VALUES FOR THE S (THREE SIGMA) CONTROL CHART WHEN N=5.

| | $\sigma_1^2/\sigma_0^2$ | | | | |
|---|---|---|---|---|---|
| | 1 | 1.2 | 1.4 | 1.6 | 1.8 |
| $m$ | $ARL$ | $ARL$ | $ARL$ | $ARL$ | $ARL$ |
| 5 | $4 \cdot 10^5$ | 2223.5 | 594.02 | 105.38 | 37.41 |
| 10 | 2200.1 | 310.65 | 86.99 | 39.29 | 21.06 |
| 20 | 551.16 | 139.42 | 54.88 | 27.47 | 16.48 |
| 30 | 415.06 | 112.55 | 48.74 | 25.79 | 15.52 |
| 50 | 346.68 | 101.62 | 43.32 | 23.36 | 14.73 |
| 100 | 298.59 | 91.09 | 40.75 | 22.35 | 14.20 |
| 200 | 276.08 | 85.28 | 39.28 | 21.55 | 13.93 |
| 500 | 262.29 | 85.20 | 38.55 | 21.75 | 13.94 |
| 1000 | 253.76 | 84.37 | 37.32 | 20.97 | 13.59 |
| $\infty$ | 249.31 | 82.44 | 37.72 | 21.22 | 13.69 |

TABLE III

SDRL VALUES FOR THE S(THREE SIGMA) CONTROL CHART WHEN N=5.

| | $\sigma_1^2/\sigma_0^2$ | | | | |
|---|---|---|---|---|---|
| | 1 | 1.2 | 1.4 | 1.6 | 1.8 |
| $m$ | $SDRL$ | $SDRL$ | $SDRL$ | $SDRL$ | $SDRL$ |
| 5 | $1 \cdot 10^5$ | $7 \cdot 10^4$ | $2 \cdot 10^4$ | 1353.3 | 143.16 |
| 10 | $3 \cdot 10^4$ | 2288.7 | 330.39 | 104.72 | 45.34 |
| 20 | 1699.7 | 297.08 | 95.14 | 41.75 | 21.70 |
| 30 | 840.14 | 182.48 | 74.75 | 34.47 | 18.54 |
| 50 | 545.72 | 134.96 | 55.43 | 27.19 | 16.19 |
| 100 | 407.05 | 106.99 | 44.42 | 23.56 | 14.55 |
| 200 | 318.09 | 93.97 | 41.12 | 22.18 | 13.92 |
| 500 | 275.14 | 88.07 | 39.24 | 21.79 | 13.52 |
| 1000 | 258.84 | 87.67 | 37.06 | 20.66 | 13.14 |
| $\infty$ | 248.81 | 81.94 | 37.21 | 20.71 | 13.18 |

TABLE IV

ARL VALUES FOR THE S (THREE SIGMA) CONTROL CHART WHEN N=10.

| | $\sigma_1^2/\sigma_0^2$ | | | | |
|---|---|---|---|---|---|
| | 1 | 1.2 | 1.4 | 1.6 | 1.8 |
| $m$ | $ARL$ | $ARL$ | $ARL$ | $ARL$ | $ARL$ |
| 5 | 606.61 | 236.14 | 78.31 | 29.43 | 14.39 |
| 10 | 538.65 | 145.57 | 45.83 | 19.21 | 10.17 |
| 20 | 461.44 | 106.92 | 33.86 | 15.85 | 9.04 |
| 30 | 430.50 | 95.59 | 32.34 | 15.02 | 8.59 |
| 50 | 389.91 | 88.05 | 30.29 | 14.38 | 8.37 |
| 100 | 359.35 | 80.89 | 28.79 | 13.85 | 8.26 |
| 200 | 344.38 | 78.19 | 28.46 | 13.43 | 7.97 |
| 500 | 334.53 | 76.10 | 27.45 | 13.52 | 8.06 |
| 1000 | 334.56 | 75.88 | 27.31 | 13.50 | 7.98 |
| $\infty$ | 331.17 | 75.66 | 27.52 | 13.47 | 8.00 |

TABLE V

SDRL VALUES FOR THE S (THREE SIGMA) CONTROL CHART WHEN N=10.

| | $\sigma_1^2/\sigma_0^2$ | | | | |
|---|---|---|---|---|---|
| | 1 | 1.2 | 1.4 | 1.6 | 1.8 |
| $m$ | $SDRL$ | $SDRL$ | $SDRL$ | $SDRL$ | $SDRL$ |
| 5 | 1064.81 | 634.06 | 263.87 | 112.67 | 41.18 |
| 10 | 919.10 | 329.89 | 99.37 | 31.64 | 13.87 |
| 20 | 725.80 | 175.82 | 48.22 | 19.75 | 10.34 |
| 30 | 626.79 | 137.72 | 40.07 | 17.08 | 9.14 |
| 50 | 510.09 | 106.54 | 33.98 | 15.65 | 8.58 |
| 100 | 411.69 | 88.11 | 30.81 | 14.24 | 8.16 |
| 200 | 367.08 | 82.25 | 28.96 | 13.31 | 7.58 |
| 500 | 340.97 | 76.60 | 27.27 | 13.14 | 7.65 |
| 1000 | 337.96 | 75.93 | 27.01 | 13.04 | 7.48 |
| $\infty$ | 330.67 | 75.16 | 27.01 | 12.96 | 7.48 |

TABLE VI

(CONTINUED) ARL VALUES FOR THE S (THREE SIGMA) CONTROL CHART WHEN N=10.

| | $\sigma_1^2/\sigma_0^2$ | | | |
|---|---|---|---|---|
| | 0.2 | 0.4 | 0.6 | 0.8 |
| $m$ | $ARL$ | $ARL$ | $ARL$ | $ARL$ |
| 5 | 24.01 | 306.28 | 1019.6 | 1136.2 |
| 10 | 21.04 | 254.28 | 1071.7 | 1316.9 |
| 20 | 19.77 | 230.60 | 1079.4 | 1472.6 |
| 30 | 19.15 | 223.33 | 1056.5 | 1569.2 |
| 50 | 18.47 | 218.20 | 1047.3 | 1644.7 |
| 100 | 18.21 | 210.57 | 1037.5 | 1696.2 |
| 200 | 17.95 | 205.32 | 1023.1 | 1744.5 |
| 500 | 18.20 | 205.59 | 1009.1 | 1773.9 |
| 1000 | 17.53 | 206.96 | 1006.7 | 1768.3 |
| $\infty$ | 17.90 | 206.06 | 1011.7 | 1777.2 |

TABLE VII

(CONTINUED) SDRL VALUES FOR THE S (THREE SIGMA) CONTROL CHART WHEN N=10.

| | $\sigma_1^2/\sigma_0^2$ | | | |
|---|---|---|---|---|
| | 0.2 | 0.4 | 0.6 | 0.8 |
| $m$ | $SDRL$ | $SDRL$ | $SDRL$ | $SDRL$ |
| 5 | 37.43 | 520.37 | 1274.5 | 1433.6 |
| 10 | 25.88 | 377.34 | 1253.2 | 1514.7 |
| 20 | 22.62 | 275.74 | 1207.5 | 1603.4 |
| 30 | 20.62 | 249.71 | 1155.1 | 1656.9 |
| 50 | 19.15 | 229.50 | 1106.2 | 1686.9 |
| 100 | 18.19 | 215.40 | 1061.3 | 1729.9 |
| 200 | 17.93 | 205.49 | 1027.8 | 1746.7 |
| 500 | 17.79 | 203.14 | 1022.0 | 1785.0 |
| 1000 | 17.28 | 204.95 | 1007.9 | 1773.9 |
| $\infty$ | 17.39 | 205.56 | 1011.2 | 1776.7 |

TABLE VIII

ARL VALUES FOR THE S (THREE SIGMA) CONTROL CHART WHEN N=20.

| | $\sigma_1^2/\sigma_0^2$ | | | | |
|---|---|---|---|---|---|
| | 1 | 1.2 | 1.4 | 1.6 | 1.8 |
| $m$ | $ARL$ | $ARL$ | $ARL$ | $ARL$ | $ARL$ |
| 5 | 332.72 | 121.96 | 32.29 | 11.46 | 5.54 |
| 10 | 362.96 | 92.99 | 23.32 | 8.71 | 4.62 |
| 20 | 371.24 | 75 | 19.63 | 7.87 | 4.32 |
| 30 | 372.32 | 68.53 | 18.23 | 7.67 | 4.29 |
| 50 | 362.66 | 63.80 | 17.52 | 7.49 | 4.24 |
| 100 | 364.01 | 60.17 | 17.01 | 7.36 | 4.11 |
| 200 | 359 | 59.56 | 16.61 | 7.15 | 4.11 |
| 500 | 355.18 | 59.11 | 16.36 | 7.13 | 4.09 |
| 1000 | 353.23 | 57.59 | 16.26 | 7.15 | 4.08 |
| $\infty$ | 356.50 | 57.37 | 16.39 | 7.15 | 4.07 |

TABLE IX

SDRL VALUES FOR THE S (THREE SIGMA) CONTROL CHART WHEN N=20.

| | $\sigma_1^2/\sigma_0^2$ | | | | |
|---|---|---|---|---|---|
| | 1 | 1.2 | 1.4 | 1.6 | 1.8 |
| $m$ | $SDRL$ | $SDRL$ | $SDRL$ | $SDRL$ | $SDRL$ |
| 5 | 444.02 | 244.63 | 78.86 | 27.03 | 10.33 |
| 10 | 457.01 | 166.56 | 45.79 | 11.78 | 5.39 |
| 20 | 439.25 | 115.39 | 25.36 | 8.77 | 4.20 |
| 30 | 430.13 | 86.90 | 21.84 | 8.17 | 4.12 |
| 50 | 403.51 | 76.74 | 18.73 | 7.60 | 3.97 |
| 100 | 393.80 | 65.57 | 17.34 | 7.15 | 3.58 |
| 200 | 374.30 | 60.31 | 16.45 | 6.82 | 3.59 |
| 500 | 358.14 | 59.61 | 16.15 | 6.70 | 3.56 |
| 1000 | 353.28 | 57.23 | 15.79 | 6.66 | 3.59 |
| $\infty$ | 356.00 | 56.87 | 15.88 | 6.63 | 3.53 |

TABLE X

(CONTINUED) ARL VALUES FOR THE S (THREE SIGMA) CONTROL CHART WHEN N=20.

| | $\sigma_1^2/\sigma_0^2$ | | | |
|---|---|---|---|---|
| | 0.2 | 0.4 | 0.6 | 0.8 |
| $m$ | $ARL$ | $ARL$ | $ARL$ | $ARL$ |
| 5 | 1.32 | 11.92 | 111.01 | 383.43 |
| 10 | 1.28 | 10.03 | 90.20 | 423.04 |
| 20 | 1.26 | 9.21 | 80.28 | 442.70 |
| 30 | 1.26 | 8.97 | 78.03 | 444.96 |
| 50 | 1.24 | 8.90 | 75.70 | 451.20 |
| 100 | 1.25 | 8.68 | 73.42 | 450.90 |
| 200 | 1.23 | 8.70 | 73.69 | 447.50 |
| 500 | 1.24 | 8.55 | 73.62 | 441.19 |
| 1000 | 1.24 | 8.54 | 72.08 | 445.81 |
| $\infty$ | 1.24 | 8.56 | 72.91 | 449.79 |

TABLE XI

(CONTINUED) SDRL VALUES FOR THE S (THREE SIGMA) CONTROL CHART WHEN N=20.

| | $\sigma_1^2/\sigma_0^2$ | | | |
|---|---|---|---|---|
| | 0.2 | 0.4 | 0.6 | 0.8 |
| $m$ | $SDRL$ | $SDRL$ | $SDRL$ | $SDRL$ |
| 5 | 0.75 | 18.83 | 190.51 | 457.13 |
| 10 | 0.64 | 12.23 | 127.11 | 463.05 |
| 20 | 0.60 | 9.96 | 94.68 | 473.45 |
| 30 | 0.58 | 9.20 | 88.22 | 471.63 |
| 50 | 0.54 | 8.59 | 80.02 | 469.84 |
| 100 | 0.57 | 8.20 | 75.19 | 455.92 |
| 200 | 0.54 | 8.27 | 74.61 | 446.43 |
| 500 | 0.55 | 8.05 | 73.39 | 441.96 |
| 1000 | 0.56 | 8.14 | 71.77 | 448.49 |
| $\infty$ | 0.54 | 8.04 | 72.41 | 449.29 |

TABLE XII

CORRELATION FOR SEVERAL VALUES OF M AND N.

| | $n$ | | | |
|---|---|---|---|---|
| $m$ | 5 | 10 | 20 | 50 |
| 5 | 0.51095 | 0.39568 | 0.32137 | 0.26032 |
| 10 | 0.34314 | 0.24663 | 0.19144 | 0.14964 |
| 20 | 0.20710 | 0.14066 | 0.10585 | 0.08087 |
| 30 | 0.14831 | 0.09839 | 0.07315 | 0.05541 |
| 50 | 0.09460 | 0.06145 | 0.04521 | 0.03400 |
| 100 | 0.04965 | 0.03170 | 0.02313 | 0.01729 |
| 200 | 0.02545 | 0.01611 | 0.01170 | 0.00872 |
| 500 | 0.01034 | 0.00650 | 0.00471 | 0.00351 |
| 1000 | 0.00520 | 0.00326 | 0.00236 | 0.00176 |

TABLE XIII

ARL VALUES FOR THE S (PROBABILITY LIMITS) CONTROL CHART WHEN N=5.

| | $\sigma_1^2/\sigma_0^2$ | | | | |
|---|---|---|---|---|---|
| | 1 | 1.2 | 1.4 | 1.6 | 1.8 |
| $m$ | $ARL$ | $ARL$ | $ARL$ | $ARL$ | $ARL$ |
| 5 | 359.97 | 267.35 | 173.40 | 111.11 | 71.17 |
| 10 | 401.46 | 268.52 | 154.68 | 83.88 | 47.93 |
| 20 | 441.09 | 254.39 | 127.40 | 64.92 | 36.69 |
| 30 | 462.04 | 247.68 | 115.34 | 58.02 | 33.35 |
| 50 | 472.24 | 239.29 | 108.19 | 52.48 | 30.29 |
| 100 | 489.90 | 229.28 | 99.08 | 49.79 | 28.81 |
| 200 | 498.35 | 221.61 | 94.66 | 48.20 | 27.67 |
| 500 | 500.93 | 216.74 | 93.45 | 46.06 | 28.00 |
| 1000 | 497.73 | 213.01 | 92.29 | 47.12 | 27.31 |
| $\infty$ | 500.02 | 214.74 | 91.78 | 46.51 | 27.33 |

TABLE XIV

SDRL VALUES FOR THE S (PROBABILITY LIMITS) CONTROL CHART WHEN N=5.

| | $\sigma_1^2/\sigma_0^2$ | | | | |
|---|---|---|---|---|---|
| | 1 | 1.2 | 1.4 | 1.6 | 1.8 |
| $m$ | $SDRL$ | $SDRL$ | $SDRL$ | $SDRL$ | $SDRL$ |
| 5 | 463.12 | 405.54 | 312.06 | 231.57 | 173.0 |
| 10 | 491.51 | 395.19 | 263.77 | 161.01 | 102.77 |
| 20 | 495.15 | 350.22 | 199.04 | 106.92 | 58.40 |
| 30 | 509.78 | 320.05 | 164.84 | 84.90 | 49.45 |
| 50 | 504.56 | 295.97 | 137.80 | 65.23 | 35.47 |
| 100 | 512.64 | 262.50 | 115.37 | 54.68 | 31.03 |
| 200 | 505.20 | 240.21 | 102.45 | 50.97 | 29.10 |
| 500 | 505.59 | 223.58 | 95.24 | 46.00 | 27.60 |
| 1000 | 503.09 | 217.36 | 94.50 | 47.08 | 26.70 |
| $\infty$ | 499.52 | 214.24 | 91.28 | 46.01 | 26.82 |

TABLE XV

ARL VALUES FOR THE S (PROBABILITY LIMITS) CONTROL CHART WHEN N=10.

| | $\sigma_1^2/\sigma_0^2$ | | | | |
|---|---|---|---|---|---|
| | 1 | 1.2 | 1.4 | 1.6 | 1.8 |
| $m$ | $ARL$ | $ARL$ | $ARL$ | $ARL$ | $ARL$ |
| 5 | 341.44 | 217.14 | 110.54 | 52.43 | 25.60 |
| 10 | 391.03 | 208.08 | 86.61 | 36.61 | 17.72 |
| 20 | 428.95 | 194.55 | 70.14 | 28.50 | 14.96 |
| 30 | 448.41 | 187.90 | 65.03 | 27.33 | 14.20 |
| 50 | 464.28 | 178.40 | 60.27 | 25.81 | 13.61 |
| 100 | 479.05 | 169.77 | 56.35 | 24.52 | 13.16 |
| 200 | 484.86 | 166.70 | 54.73 | 24.26 | 12.81 |
| 500 | 490.54 | 161.32 | 52.91 | 24.02 | 13.11 |
| 1000 | 492.16 | 161.60 | 53.81 | 23.60 | 12.70 |
| $\infty$ | 500.05 | 161.99 | 53.44 | 23.46 | 12.74 |

TABLE XVI

SDRL VALUES FOR THE S (PROBABILITY LIMITS) CONTROL CHART WHEN N=10.

| | $\sigma_1^2/\sigma_0^2$ | | | | |
|---|---|---|---|---|---|
| | 1 | 1.2 | 1.4 | 1.6 | 1.8 |
| $m$ | $SDRL$ | $SDRL$ | $SDRL$ | $SDRL$ | $SDRL$ |
| 5 | 422.99 | 329.46 | 218.04 | 130.38 | 62.05 |
| 10 | 456.05 | 307.21 | 155.83 | 67.22 | 28.51 |
| 20 | 469.37 | 257.65 | 106.07 | 39.48 | 18.23 |
| 30 | 480.41 | 234.75 | 88.79 | 33.58 | 16.15 |
| 50 | 481.37 | 209.85 | 72.62 | 28.64 | 14.78 |
| 100 | 488.20 | 184.28 | 61.10 | 25.62 | 13.69 |
| 200 | 493.03 | 176.09 | 56.70 | 24.50 | 12.66 |
| 500 | 489.97 | 164.74 | 53.41 | 24.08 | 12.82 |
| 1000 | 480.65 | 161.87 | 53.11 | 23.23 | 12.38 |
| $\infty$ | 499.55 | 161.48 | 52.94 | 22.95 | 12.23 |

TABLE XVII

ARL VALUES FOR THE X CONTROL CHART.

| | $\sigma_1^2/\sigma_0^2$ | | | | |
|---|---|---|---|---|---|
| | 1 | 1.2 | 1.4 | 1.6 | 1.8 |
| $N$ | $ARL$ | $ARL$ | $ARL$ | $ARL$ | $ARL$ |
| 30 | 986.31 | 315.36 | 147.93 | 84.36 | 53.74 |
| 50 | 614.94 | 229.95 | 116.69 | 69.61 | 47.23 |
| 75 | 503.75 | 202.02 | 105.18 | 64.51 | 43.99 |
| 100 | 467.07 | 190.53 | 100.73 | 61.98 | 42.78 |
| 200 | 413.88 | 173.68 | 93.86 | 58.63 | 40.67 |
| 300 | 398.94 | 167.79 | 92.76 | 57.93 | 41.26 |
| 500 | 387.38 | 167.90 | 90.34 | 56.80 | 39.69 |
| 1000 | 379.32 | 162.96 | 89.12 | 57.03 | 39.90 |
| 2000 | 372.64 | 162.70 | 89.45 | 56.35 | 39.62 |
| $\infty$ | 370.40 | 162.08 | 89.05 | 56.48 | 39.45 |

TABLE XVIII

SDRL VALUES FOR THE X CONTROL CHART.

| | $\sigma_1^2/\sigma_0^2$ | | | | |
|---|---|---|---|---|---|
| | 1 | 1.2 | 1.4 | 1.6 | 1.8 |
| $N$ | $SDRL$ | $SDRL$ | $SDRL$ | $SDRL$ | $SDRL$ |
| 30 | 5024.83 | 1058.44 | 439.79 | 187.50 | 98.54 |
| 50 | 1565.0 | 476.60 | 200.50 | 107.23 | 66.81 |
| 75 | 948.78 | 318.54 | 150.77 | 84.15 | 54.87 |
| 100 | 770.60 | 274.54 | 131.39 | 75.26 | 50.48 |
| 200 | 518.65 | 205.96 | 105.77 | 63.56 | 42.81 |
| 300 | 476.34 | 187.69 | 100.47 | 61.37 | 42.29 |
| 500 | 429.45 | 179.39 | 93.58 | 58.96 | 40.54 |
| 1000 | 401.55 | 168.50 | 91.10 | 57.78 | 39.85 |
| 2000 | 383.71 | 166.87 | 89.41 | 55.82 | 39.17 |
| $\infty$ | 369.90 | 161.58 | 88.55 | 55.98 | 38.95 |