"Luce problem" and discontinuity of Prelec's function at p = 1

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This short paper is devoted to two items: 1) An analysis of Prelec's weighting function at the probability $p = 1$ is highlighted (this analysis was performed by R. Duncan Luce in two articles with Ragnar Steingrimsson and János Aczél and here is referred to as the "Luce problem"). 2) The question of possible discontinuity of Prelec's weighting function at $p = 1$ is specially considered, as a manifestation of importance of the "Luce problem."

Contents

Introduction

The objectives of this short work are:

1) Highlighting and consideration of an undeservedly underestimated consequence of the articles: Steingrimsson and Luce (2007) and Aczél and Luce (2007).

2) An example of a development of this consequence.

The articles and this work deal with the probability weighting function $W(p)$ necessarily and widely used in prospect theories. Here, I shall usually refer it to as Prelec's weighting function (see Prelec, 1998) or shortly Prelec's function.

1. Background

1.1. Two articles

In 2007, R. Duncan Luce with Ragnar Steingrimsson and János Aczél published two articles:

Ragnar Steingrimsson and R. Duncan Luce, "Empirical evaluation of a model of global psychophysical judgments: IV. Forms for the weighting function."

János Aczél and R. Duncan Luce, "A behavioral condition for Prelec's weighting function on the positive line without assuming $W(1) = 1$."

The first article was essentially devoted to the analysis of weighting functions with $W(1) = 1$ and without $W(1) = 1$ (here $W(1)$ is the value of a weighting function at the probability $p = 1$). Two subchapters of the article are devoted to "function with $W(1) = 1$" and two subchapters of the article are devoted to "function without $W(1) = 1$." Moreover, even the title of the second article contains the item "without assuming $W(1) = 1$."

1.2. State of the art

Many years before the abovementioned two articles, at least the vast majority of authors in a lot of works assumed on default that Prelec's weighting function $W(p)$ is equal to 1 at $p = 1$ (one may agree that the assumption $W(1) = 1$ is, indeed, quite evident and natural).

For example, we see in Wakker (1994), page 9: "DEFINITION 1. Rank-dependent utility, (RDU) holds if there exist a strictly increasing continuous utility function $U'' \; [0,M] \rightarrow N$ and a strictly increasing probability transformation $\varphi:[0,1] \rightarrow [0,1]$ with $\varphi(0) = 0$ and $\varphi(1) = 1$,"

We see in Prelec (1998) that Prelec's formula "$w(p) = exp\{-\{-ln\, p\}^{\alpha}\}, 0<a<1$" in itself assumes only $w(1) = 1$.

ibid, page 515: "unique, nondecreasing weighting functions, satisfying $w(0) = 0, w(1) = 1$"

Note, there is no assumption of $W(1) \neq 1$ in these works.

2. The "Luce problem"

2.1. Against accepted view

In spite of the accepted view, R. Duncan Luce with Ragnar Steingrimsson and János Aczél had discovered a problem and question of a general mathematical and scientific nature. The essence of the problem and question was: "whether a well-known object is actually the same, as it seems to all people?" Generally, it can be compared with the question: "does the sun actually go round the earth?"

Namely, the problem was: a special analysis of Prelec's function at $p = 1$.

Prelec's function has been numerously analyzed in the middle of the probability scale, but an analysis at $p \approx 1$ is still undeservedly too rare event.

One can name this problem after R. Duncan Luce as the "Luce problem," or after Steingrimsson, Luce and Aczél as the "SLA problem," etc. Here I refer to this problem as the "Luce problem."

The question was: whether Prelec's weighting function $W(p)$ is actually equal to 1 at $p = 1$?[1]

Here, I refer to this question also as the "Luce question."

2.2. The importance of the problem and question

The above two articles are well cited. At 07 April 2015, Steingrimsson and Luce (2007) was cited by 23 and Aczél and Luce (2007) was cited by 8 articles.

Nevertheless, the "Luce problem" and "question" are still underestimated.

For example, we see in Diecidue, Schmidt, and Zank (2009), page 1105: "the weighting function $w: [0, 1] \rightarrow [0, 1]$ is strictly increasing and continuous with $w(0) = 0$ and $w(1) = 1$."

We see in Chechile and Barch (2013), page 16: "Assumption 2. If $p = 1$, then $w(p) = 1$"

We do not see an assumption like "If $p = 1$, then $w(p) \neq 1$" in these works.

We may note this problem and question were considered by R. Duncan Luce with his co-authors not once. They were considered twice. Therefore, they can have some featured consequences. For example:

1) The problem and question were not accident.

2) The question was not a purely quantitative one. That is, the question was not: "whether Prelec's weighting function $W(p)$ is a bit more or less than 1 at $p = 1$."

An example of consideration of possible importance of the "Luce problem" and "Luce question" is presented below.

[1] In Harin (2014), I named this question as the "Aczél–Luce question" or "Luce question" because at that time I knew about Steingrimsson and Luce (2007) but had not access to this article. Now this question may be named as the "Luce question" or "Steingrimsson–Luce–Aczél question" or "SLA question," etc.

3. Importance of the "Luce problem." Possible discontinuity of Prelec's function

3.1. A modification of the "Luce question"

The question of continuity or discontinuity of Prelec's function at $p = 1$ has been already considered among other questions (see, e.g., Wakker, 1994). Let us highlight it and make a special consideration of it.

There is a deal of evidence for the existence of a difference between subjects' treatment of the probabilities of uncertain (probable) and certain outcomes (see, e.g., Kahneman and Tversky, 1979; Halevy, 2008). Therefore, in the general case, one should distinguish between the values of the probability weighting function $W(p)$ of a certain outcome and the limit of the probability weighting function $W(p)$ of uncertain outcomes as the probability of uncertain outcomes tends to 1.

Let us specify a value $W_{Certain}$ of the probability weighting function $W(p)$ for a certain outcome. At that, $W_{Certain}$ may be assumed to be equal to 1. Otherwise, other values of $W(p)$ may be normalized by $W_{Certain}$.

Let us also specify a value $W(1)$ as the limit of the probability weighting function $W(p)$ for a probable (uncertain) outcome as p tends to 1

$$W(1) \equiv \lim_{p \to 1} W(p) \qquad (1).$$

If $W_{Impossible}$ is defined for the impossible case (for $p = 0$), then, similar to Aczél and Luce (2007), one can write

$$W(p) = \begin{cases} W_{Impossible} & p = 0 \\ W(p) & p \in]0,1[\\ W_{Certain} & p = 1 \end{cases} \qquad (2)$$

So, taking into account (1), one may modify the "Luce question" whether $W(1) = 1$ into the modified "Luce question"

$$W_{Certain} - W(1) = ? \qquad (3)$$

3.2. A possible discontinuity of Prelec's function at $p = 1$ and the importance of the "Luce problem"

The question $W_{Certain} - W(1) = ?$ or whether $W(1) = W_{Certain}$ is the question whether $W(p)$ is continuous at $p = 1$. If $W(1) = W_{Certain}$ then $W(p)$ is continuous (at $p = 1$). This is usually assumed by default. Nevertheless, this has not been proven for the general case. The answer

$$W_{Certain} - W(1) \neq 0$$

or

$$W(1) \neq W_{Certain}$$

to the modified "Luce question" means that the function $W(p)$ has a discontinuity at $p=1$.

A discontinuity is not a quantitative but a qualitative, moreover, a topological feature. Therefore, the possible discontinuity of Prelec's function can qualitatively change prospect theories, at least in their mathematical aspects.

So, the "Luce problem" can be of crucial importance for prospect theories.

4. Supports of possible discontinuity of Prelec's function at $p = 1$

4.1. A theoretical support

Purely mathematical theorems (see, e.g., Harin, 2012b) prove that the mean M and probability p cannot attain the border 1 of the interval $[A, B]$ under the condition of a non-zero minimal dispersion σ^2_{Min} of the data. Here is a very brief review of conclusions of the theorems:

Let us suppose, for example, a nonnegative random variable X takes values in a finite interval $[A, B]$. Let us write M for its mean. If there is a non-zero restriction on the dispersion $\sigma^2 \geq \sigma^2_{Min} > 0$, then

$$A < \left(A + \frac{\sigma^2_{Min}}{B - A} \right) \leq M \leq \left(B - \frac{\sigma^2_{Min}}{B - A} \right) < B \ .$$

That is,

$$\frac{\sigma^2_{Min}}{B - A} > 0$$

is the width of a non-zero "forbidden zone" for the mean near a border of the interval.

So, if there is a non-zero restriction on the dispersion, then a non-zero "forbidden zone" exists for the mean near a border of the interval.

A similar statement was proved as well for the probability p under the condition of non-zero minimal dispersion $\sigma^2 \geq \sigma^2_{Min} > 0$ of the data of the estimation of the probability on the unitary interval $[0, 1]$. It may be written as

$$0 < \sigma^2_{Min} \leq p \leq \left(1 - \sigma^2_{Min}\right) < 1 \ .$$

The theorems allow to explain (see, e.g., Harin, 2012a, b and in a form of hypotheses Harin 2007, 2005), at least partially, some well-known paradoxes and problems of utility and prospect theories.

The theorems have not been disproved.

4.2. An experimental support

A natural question is bound to arise: Why did not this discontinuity be discovered in numerous experiments?

This question is answered by Harin (2014). Here is a very brief review of this answer:

Unfortunately, in prevailing random–lottery incentive experiments, the choices of certain outcomes are stimulated by uncertain lotteries. This "certain–uncertain" inconsistency is evident, but only recently published. Because of it, conclusions from a random–lottery incentive experiment, that includes a certain outcome, cannot be unquestionably correct, especially near $p = 1$.

This "certain–uncertain" inconsistency has not been disproved also.

Moreover, the well-known experiment of Starmer and Sugden (1991) supports this "certain–uncertain" inconsistency.

5. Results and discussion

5.1. Results

This short paper leads to two interrelated main results:

1) The research of the "Luce problem" should be continued (in any case, the "Luce problem" should not be passed over in silence).

2) The importance of the "Luce problem" is manifested by means of the special consideration of possible discontinuity of Prelec's weighting function at the probability $p = 1$.

5.2. Discussion

Unfortunately, at present the "Luce question" is passed over in silence. In any case, $W(1) \neq 1$ should be manifestly mentioned as the alternative assumption among other general assumptions in appropriate articles.

The general "Luce problem" of special analysis of Prelec's weighting function at $p = 1$ is not attended also. Researches of Prelec's weighting function at probabilities $p = .99$, $p = .999$ and so on are needed.

The assumption of possible discontinuity of Prelec's weighting function merits to be mentioned more often as the alternative assumption in appropriate articles.

The question of possible discontinuity of Prelec's weighting function at $p = 1$ should be independently tested. In particular, both 1) the existence theorems for restrictions on the mean and probability and 2) the "certain-uncertain" inconsistency of the random-lottery experimental system should be analyzed by independent researchers. The experiment of Starmer and Sugden (1991) should be independently analyzed from the point of view of the "certain-uncertain" inconsistency and, advisable, be repeated at the probabilities $p = .99$, $p = .999$ and so on by independent groups of researchers.

One can make following statements and suppositions as well:

Chapter 3.2 shows that the possible discontinuity of Prelec's function can qualitatively change prospect theories.

Chapters 4.1–4.2 show that at present the theoretical and experimental supports of the possibility of existence of discontinuity of Prelec's weighting function at $p = 1$ have not been disproved. Therefore, a supposition can be made that at present one cannot exclude the possibility of existence of this discontinuity.

The research of the "Luce problem" is necessary either for the research of this important discontinuity or for the important proof of its absence. Therefore, a supposition can be made that in any case, at present one cannot deny the importance of the "Luce problem."

Conclusions

The "Luce problem" and "Luce question" are undeservedly underestimated. This short paper highlights them and their importance.

The possible discontinuity of Prelec's weighting function at the probability $p = 1$ is specially considered, as a manifestation of importance of the "Luce problem" and "Luce question."

Prelec's weighting function is represented as (2). The "Luce question," taking into account (1), is modified to (3), where the possible continuity or discontinuity of Prelec's function at $p = 1$ is manifested.

The possible discontinuity of Prelec's weighting function at $p = 1$ is supported both theoretically and experimentally.

A discontinuity is a topological feature. Therefore, the possible discontinuity of Prelec's function can qualitatively change the existing prospect theories.

Acknowledgments

This paper is written in Honor of R. Duncan Luce.

One cannot exclude, that, due to developments of the "Luce problem" (first by co-authors, colleagues, disciples and followers of R. Duncan Luce and then by all involved scholars), the existing prospect theories will be qualitatively changed.

In this case, these qualitative changes will be a memory and memorial in Honor of R. Duncan Luce.

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