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2001

Journal of Applied Statistical Science
Volume 10, No. 2 pp. 119-136

ISSN # 1067-5817

On Some Bivariate Extensions of the Folded Normal and the Folded T Distributions

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SUMMARY

In this paper two new bivariate distributions are defined and studied. They are the two - variate versions of the folded normal distribution (Leone et al. 1961) and the folded t distribution (Psarakis and Panaretos 1990).

They both arise in the context of evaluating the predictive behaviour of two competing linear models with the aim to select the one that leads to predictions closer to the actual value of the dependent variable.

Keywords and phrases: Folded normal, folded-t distribution, model selection.

1. INTRODUCTION

In this paper a practical problem in the area of validating forecasting models is mentioned that motivates the definition of bivariate extensions of folded normal and folded t distributions which are subsequently studied.

Xekalaki and Katti (1984) introduced a sequential scheme to evaluate the forecasting potential of models which amounted to scoring the forecasting performance of the model at each of a number of points in time and obtaining a final rating based on the scores. Next section describes the practical situation in which the bivariate extensions of folded normal and folded t distributions may be observed (section 2). In sections 3,4 and 5 the bivariate standard normal and the bivariate standard t distributions are studied, while in sections 6,7 and 8 the bivariate folded t and the bivariate standard folded t are introduced and studied. Finally, section 9 looks into the relationship between bivariate folded t and bivariate folded standard normal distributions.

2.MODEL SELECTION AND BIVARIATE FOLDING

Consider two linear models A and B of the form:

$$\mathbf{Y}_t = \mathbf{X}_t(M)\boldsymbol{\beta}_M + \boldsymbol{\varepsilon}_t(M)$$

where $\mathbf{Y}_t$ is an $\ell_t \times 1$ vector of observations on the dependent random variable, $\mathbf{X}_t(M)$ is an $\ell_t \times m_M$ matrix of known coefficients $\left(\ell_t \geq m_M, \left|\mathbf{X}_t'(M)\mathbf{X}_t(M)\right| \neq 0\right.$, $\boldsymbol{\beta}_M$ is an $m_M \times 1$ vector of regression coefficients and $\boldsymbol{\varepsilon}_t(M)$ is an $\ell_t \times 1$ vector of normal error random variables with

$$E(\boldsymbol{\varepsilon}_t(M)) = 0 \quad \text{and} \quad V(\boldsymbol{\varepsilon}_t(M)) = \sigma^2(M)\mathbf{I}_t \quad \left(\sigma^2(M) < \infty\right).$$

Here, $\mathbf{I}_t$ is the $\ell_t \times \ell_t$ identity matrix and M indexes the model (i.e. M=A or M=B). Therefore, a prediction for the value of the dependent random variable for time t+1 will be given by the statistic

$$\hat{Y}^0_{t+1}(M) = \mathbf{X}^0_{t+1}(M)\hat{\boldsymbol{\beta}}_t(M),$$

where $\hat{\boldsymbol{\beta}}_t(M)$ is the least squares estimator of $\boldsymbol{\beta}_M$ at time t, $\mathbf{X}^0_{t+1}(M)$ is a $1 \times m$ vector at time t+1 and Y^0_{t+1} is the observed actual value. Let Y^0_{t+1}, $\hat{Y}^0_{t+1}(M)$ are the observed and predicted values, respectively, for model M at time t+1. Then,

$$\hat{Y}^0_{t+1}(M) - Y^0_{t+1} = e_{t+1}(M)$$

will follow the $N(0, \sigma^2_M)$

where

$$\sigma^2_M = \sigma^2(M)\left[1 + \mathbf{X}^0_{t+1}(M)\left(\mathbf{X}_t'(M)\mathbf{X}_t(M)\right)^{-1}\mathbf{X}'^0_{t+1}(M)\right], \quad \text{(M=A or M=B)}$$

and so

$$\left|\hat{Y}^{0}_{t+1}(M)-Y^{0}_{t+1}\right|=\left|e_{t+1}(M)\right|$$

will follow the folded normal distribution with mean

$$\mu_f(M)=\sqrt{2/\pi}\,\sigma_M$$

and variance

$$\sigma_f^2(M)=\sigma_M^2(1-2/\pi)\quad \text{(Leone et al. 1961).}$$

In this way studying the distribution of the random variables $(|e_t(A)|,|e_t(B)|)$ when $|e_t(A)|$ and $|e_t(B)|$ have the folded normal distribution is in order, as this will facilitate inferences about the comparisons of the predictive performances of the two models. In the sequel we refer to this distribution as the bivariate folded normal distribution.

Similarly, proposing and studying bivariate extension of the folded t distribution (Psarakis and Panaretos 1990) is an interesting consideration from a theoretical and practical point of view.

3. FOLDING A BIVARIATE NORMAL DISTRIBUTION

If X,Y are two normal variables with means μ_1 , μ_2 and variances σ_1^2,σ_2^2, it is known that their joint distribution is the bivariate normal with probability density function given by:

$$f_{X,Y}(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2}-2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}+\frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right]$$

where $|\rho|\leq 1$.

Let $F_{X,Y}(x,y)$ denote the distribution function of this bivariate normal distribution. Then, folding the above distribution to obtain the joint distribution of the random variables $|X|$, $|Y|$ leads to:

$$F_{|X|,|Y|}(x,y) = P(|X| \le x, |Y| \le y) = P(-x \le X \le x, -y \le Y \le y) =$$

$$F_{X,Y}(x,y) + F_{X,Y}(-x,-y) - F_{X,Y}(-x,y) - F_{X,Y}(x,-y), \qquad x, y > 0.$$

This relationship, in terms of probability density functions, is equivalent to:

$$f_{|X|,|Y|}(x,y) = f_{X,Y}(x,y) + f_{X,Y}(-x,-y) - f_{X,Y}(-x,y) - f_{X,Y}(x,-y),$$

Definition: The joint distribution of two non-negatives real valued random variables X and Y will be said to be the <u>bivariate folded normal</u> distribution if its probability density function is given by:

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\left\{\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right)\right)\right.$$

$$+\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{(x+\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x+\mu_1)(y+\mu_2)}{\sigma_1\sigma_2} + \frac{(y+\mu_2)^2}{\sigma_2^2}\right)\right)$$

$$+\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{(x+\mu_1)^2}{\sigma_1^2} + 2\rho\frac{(x+\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right)\right)$$

$$\left.+\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} + 2\rho\frac{(x-\mu_1)(y+\mu_2)}{\sigma_1\sigma_2} + \frac{(y+\mu_2)^2}{\sigma_2^2}\right)\right)\right\} \qquad (3.1)$$

where $x > 0$, $y > 0$, $\sigma_i > 0$, $\mu_i \in R$, $i = 1,2$ and $|\rho| \le 1$

Letting $\mu_1=\mu_2=0$ in the above relationship yields the bivariate half normal distribution defined by Patil (1984).

Note also that the case $\mu_1=\mu_2=0$ and $\sigma_1=\sigma_2=1$ leads to the folded version of the bivariate standard normal distribution. In the sequel, we focus on the latter distribution and refer to it as the <u>bivariate folded standard normal distribution.</u>

Definition: Two non-negative real valued random variables X,Y will be said to follow the bivariate folded standard normal distribution if their joint probability density function is given by:

$$f_{X,Y}(x,y)=\frac{1}{\pi\sqrt{1-\rho^2}}\left\{\exp\left[-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}\right]+\exp\left[-\frac{x^2+y^2+2\rho xy}{2(1-\rho^2)}\right]\right\} \quad -1\le\rho\le 1 \quad x,y\ge 0 \tag{3.2}$$

It can be shown that:

$$E(X\,|\,Y)=|\rho|x$$
$$V(X\,|\,Y)=(1-\rho^2)$$

see e.g. Patil (1984).

Figures 1 and 2 in appendix give the distribution density function of the bivariate folded standard normal distribution for $\rho=0.2$ and $\rho=0.9$ respectively.

4. MARGINAL DISTRIBUTIONS OF THE BIVARIATE FOLDED STANDARD NORMAL DISTRIBUTION.

Let (X,Y) be a random vector with probability density function $f_{X,Y}(x,y)$ as given by (3.2). Then, the marginal distribution of X will be defined by the density function

$$f_X(x)=\int_0^\infty f_{X,Y}(x,y)dy=$$

$$\frac{1}{\pi\sqrt{1-\rho^2}}\left[\int_0^\infty \exp\left\{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)\right\}dy\right.$$

$$\left.+\int_0^\infty \exp\left\{-\frac{1}{2(1-\rho^2)}(x^2+2\rho xy+y^2)\right\}dy\right]$$

i.e.

$$f_X(x)=\frac{1}{\pi\sqrt{1-\rho^2}}\exp\left(-\frac{x^2}{2(1-\rho^2)}\right)\left[\int_0^\infty \exp\left\{-\frac{1}{2(1-\rho^2)}(-2\rho xy+y^2)\right\}dy+\int_0^\infty \exp\left\{-\frac{1}{2(1-\rho^2)}(2\rho xy+y^2)\right\}dy\right] \tag{4.1}$$

Let A and B denote the first and second integral on the right hand side of (4.1), respectively.

It is known (Gradshteyn (1980)) that

$$\int_0^{\infty} \exp\left(-\frac{x^2}{4\beta} - \gamma x\right)dx = \sqrt{\pi\beta}\exp\left(\beta\gamma^2\right)\left[1 - \Phi^*\left(\gamma\sqrt{\beta}\right)\right] \qquad [\operatorname{Re}\beta > 0] \tag{4.2}$$

$$\text{where } \Phi^*(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-t^2\right)dt \tag{4.3}$$

Then, setting $\beta = \dfrac{1-\rho^2}{2}$ and $\gamma = -\dfrac{\rho x}{1-\rho^2}$ we obtain

$$A = \sqrt{\pi\frac{1-\rho^2}{2}}\exp\left(\frac{\left(1-\rho^2\right)\rho^2 x^2}{2\left(1-\rho^2\right)^2}\right)\left[1-\Phi^*\left(-\frac{\rho x}{1-\rho^2}\sqrt{\frac{1-\rho^2}{2}}\right)\right].$$

Similarly, for $\beta = \dfrac{1-\rho^2}{2}$, $\gamma = \dfrac{\rho x}{1-\rho^2}$ we have for B, from (4.2)

$$B = \sqrt{\pi\frac{1-\rho^2}{2}}\exp\left(\frac{\left(1-\rho^2\right)\rho^2 x^2}{2\left(1-\rho^2\right)^2}\right)\left[1-\Phi^*\left(\frac{\rho x}{1-\rho^2}\sqrt{\frac{1-\rho^2}{2}}\right)\right].$$

So, the right hand side of (4.1) can be written as

$$\frac{1}{\pi\sqrt{1-\rho^2}}\exp\left(-\frac{x^2}{2\left(1-\rho^2\right)}\right)\sqrt{\pi\frac{1-\rho^2}{2}}\exp\left[\frac{\rho^2 x^2}{2\left(1-\rho^2\right)}\right]\times$$

$$\left[1-\Phi^*\left(-\frac{\rho x}{1-\rho^2}\sqrt{\frac{1-\rho^2}{2}}\right)+1-\Phi^*\left(\frac{\rho x}{1-\rho^2}\sqrt{\frac{1-\rho^2}{2}}\right)\right].$$

The above expression leads to:

$$f_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)\left[2-\Phi^*\left(-\frac{\rho x}{2\sqrt{1-\rho^2}}\right)-\Phi^*\left(\frac{\rho x}{2\sqrt{1-\rho^2}}\right)\right] \tag{4.4}$$

Since, from (4.3)

$$\Phi^{*}\left(-\frac{\rho\,x}{2\sqrt{1-\rho^{2}}}\right)=2\left[\Phi\left(-\frac{\rho\,x}{2\sqrt{2}\sqrt{1-\rho^{2}}}\right)-\Phi(0)\right]$$

and

$$\Phi^{*}\left(\frac{\rho x}{2\sqrt{1-\rho^{2}}}\right)=2\left[\Phi\left(\frac{\rho x}{2\sqrt{2}\sqrt{1-\rho^{2}}}\right)-\Phi(0)\right]$$

with $\Phi(\,.\,)$ denoting the distribution function of the standard normal distribution, (4.4) can be eventually written as

$$f_X(x)=\sqrt{\frac{2}{\pi}}\exp\left(-\frac{x^2}{2}\right) \tag{4.5}$$

But this is the probability density function of the folded standard normal distribution.

In a similar manner, the variable Y can be shown to follow the folded standard normal distribution with probability density function as given by (4.5).

So, both marginals are of the type (4.5) with

$$E(X)=E(Y)=\sqrt{2/\pi}$$

and

$$V(X)=V(Y)=1-2/\pi$$

Note: Following an analogous argument the above results on the bivariate folded standard normal distribution can be generalised to the case of the bivariate folded normal distribution to show that the marginal distributions of the bivariate folded normal distribution are the folded normal distributions.

5. THE MOMENT GENERATING FUNCTION OF THE BIVARIATE FOLDED STANDARD NORMAL DISTRIBUTION.

To find the moment generating function of the bivariate folded standard normal distribution we make use of the work of Tallis (1961), where the moment generating function (m.g.f.) of the truncated multi-normal distribution is derived. According to Tallis (1961) the m.g.f. of the truncated multi-normal distribution is given by the form:

$$m = a^{-1}\exp(T)\ \Phi_n(b_s, \mathbf{R}) \tag{5.1}$$

where $T = \frac{1}{2}\mathbf{t}'\mathbf{Rt}$, $\Phi_n(b_s, \mathbf{R})$ is the multivariate standard normal distribution with correlation matrix **R**, and

$$a = P(W_1>\alpha_1, W_2>\alpha_2, \ldots\ldots W_n>\alpha_n) = \Phi_n(\alpha_s, \mathbf{R}) \tag{5.2}$$

$b_s=\alpha_s-\zeta_s$, $\zeta_s=\Sigma\rho_{su}t_u$ $s=1,2,\ldots,n$

According to Tallis (1961), the m.g.f. of the truncated multi-normal distribution is

$$m = a^{-1}\ {}^{(n)}\!\int_{\alpha_s}^{\infty}\exp(\mathbf{t}'\mathbf{w})\varphi_n(w_s;\mathbf{R})dw_s = a^{-1}(2\pi)^{-n/2}|\mathbf{R}|^{-1/2}\ {}^{(n)}\!\int_{\alpha_s}^{\infty}\exp\left[-\frac{1}{2}\{\mathbf{w}'\mathbf{R}^{-1}\mathbf{w}-2\mathbf{t}'\mathbf{w}\}\right]dw_s, \tag{5.3}$$

where **t** is the column vector of the t_s and **w** is the column vector of w_s and

$${}^{(n)}\!\int_{\alpha_s}^{\infty}\exp\left[-\frac{1}{2}\{\mathbf{w}'\mathbf{R}^{-1}\mathbf{w}-2\mathbf{t}'\mathbf{w}\}\right]dw_s,$$

is the abbreviation of

$$\int_{\alpha_1}\int_{\alpha_2}\ldots\int_{\alpha_n}\exp\left[-\frac{1}{2}\{\mathbf{w}'\mathbf{R}^{-1}\mathbf{w}-2\mathbf{t}'\mathbf{w}\}\right]dw_1dw_2\ldots dw_n$$

Now from the identity

$$-\frac{1}{2}\{\mathbf{w}'\mathbf{R}^{-1}\mathbf{w}-2\mathbf{t}'\mathbf{w}\} = \frac{1}{2}\mathbf{t}'\mathbf{Rt} - \frac{1}{2}(\mathbf{w}-\zeta)'\mathbf{R}^{-1}(\mathbf{w}-\zeta) \tag{5.4}$$

for $\zeta=\mathbf{Rt}$ (5.3) becomes

$$m = a^{-1}(2\pi)^{-n/2}|\mathbf{R}|^{-1/2}\exp(T) \overset{(n)}{\int_{a_s}^{\infty}} \exp\left\{-\frac{1}{2}(\mathbf{w}-\zeta)'\mathbf{R}^{-1}(\mathbf{w}-\zeta)\right\}dw_s.$$

and letting $X_s=W_s-\zeta_s$ (5.1) follows.

If we apply the above argument in the case of the bivariate folded standard normal distribution (in which case $a_s = 0$) from (5.2) we have a=1 and the m.g.f. becomes:

$$m = 2\exp(T)\,\Phi_2(b_s;\mathbf{R}) + 2\exp(T')\Phi_2(b_s;\mathbf{R}')$$

where $T = t_1^2 - 2\rho t_1 t_2 + t_2^2$, $T' = t_1^2 + 2\rho t_1 t_2 + t_2^2$, $b_1 = -\zeta_1 = -t_1 - \rho t_2$

$$b_2 = -\zeta_2 = -\rho t_1 - t_2, \quad \mathbf{R} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \quad \mathbf{R}' = \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}.$$

6.THE DISTRIBUTIONS BIVARIATE FOLDED T AND BIVARIATE FOLDED STANDARD T

Let $\mathbf{R}=(\rho_{ij})$ i=1,2,..n; j=1,2,..n be an n×n symmetric matrix such that it is either positive - definite or positive semidefinite and $\rho_{ii}=1$, i=1,...n. Let $Z=(Z_1,...Z_n)'$ have an N(0, $\mathbf{R}$) distribution and let the random variable S be independent of Z and such that $vS^2 \sim X^2(v)$distribution. Then, a natural generalization of the Student's t variable, is (see Tong (1990), Dunnet and Sobel (1954)) the variable

$$\mathbf{t} = (t_1, t_2, ...t_n) = (Z_1/S, Z_2/S, ...Z_n/S).$$

For n=2 and by replacing t_1 , t_2 and ρ_{12} by x,y, ρ respectively, the probability density function of the bivariate t distribution with n degrees of freedom is obtained with p.d.f as given by (Dunnet and Sobel 1954):

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\left[1+\frac{x^2-2\rho\,xy+y^2}{n(1-\rho^2)}\right]^{-\frac{n+2}{2}}.$$

Folding the above will result in the joint distribution of the random variables $|X| = |Z_1|/S$, $|Y| = |Z_2|/S$, where $|Z_1|$, $|Z_2|$ have the folded normal distribution. For the distribution function of this distribution we have that :

$$F_{|X|,|Y|}(x,y) = P(|X| \le x, |Y| \le y) = P(-x \le X \le x, -y \le Y \le y) =$$

$$F_{X,Y}(x,y) + F_{X,Y}(-x,-y) - F_{X,Y}(-x,y) - F_{X,Y}(x,-y) \qquad x,y > 0.$$

In terms of probability density functions this is equivalent to

$$f_{|X|,|Y|}(x,y) = f_{X,,Y}(x,y) + f_{X,Y}(-x,-y) - f_{X,Y}(-x,y) - f_{X,Y}(x,-y)$$

This lead to the following definition:

Definition : The joint distribution of two non-negative real valued random variables $Z = |X|$ and $W = |Y|$ will be said to be **the bivariate folded t** distribution with n degrees of freedom if its probability density function is given by:

$$f_{Z,W}(x,y) = \frac{1}{\pi\sqrt{1-\rho^2}}\left[1+\frac{x^2-2\rho xy+y^2}{n(1-\rho^2)}\right]^{-\frac{n+2}{2}} + \frac{1}{\pi\sqrt{1-\rho^2}}\left[1+\frac{x^2+2\rho xy+y^2}{n(1-\rho^2)}\right]^{-\frac{n+2}{2}} \qquad x,y>0 \quad (6.1)$$

The **bivariate folded standard t** distribution with n degrees of freedom can also be defined by the probability density function:

$$f_{Z,W}(x,y) = \frac{4}{2\pi}\left(1+\frac{x^2+y^2}{n}\right)^{-\frac{n+2}{2}} \qquad x,y>0 \qquad (6.2)$$

by letting $\rho=0$ in (6.1).

Steffens (1969) proved that

$$\int_0^\infty \int_0^\infty \frac{1}{2\pi}\left(1+\frac{x^2+y^2}{n}\right)^{-\frac{n+2}{2}} = 1/4$$

so, it is easily deduced that (6.2) is a well defined density function.

Figures 3,4,5 and 6 in appendix provide graphs of the probability density function of the bivariate folded t distribution for (a) $n=4$ and $\rho=0.2$ (b) $n=4$ and $\rho=0.9$ (c) $n=20$ and $\rho=0.2$ and (d) $n=20$ and $\rho=0.9$.

Figures 7 and 8 in appendix provide graphs of the probability density function of the bivariate folded standard t distribution for $n=4$ and $n=20$.

7.RELATIONSHIPS OF THE BIVARIATE FOLDED T DISTRIBUTION WITH OTHER DISTRIBUTION PERTAINING TO COMPUTER GENERATION

The probability density function of the bivariate Cauchy distribution with parameter c is given by :

$$f_{X,Y}(x,y) = \frac{1}{2\pi} c\left[c^2 + x^2 + y^2\right]^{-3/2} \qquad -\infty < x, y < \infty$$

(see e.g. Patil 1984).

We may define the bivariate half Cauchy distribution as the distribution with probability density function given by:

$$f_{X,Y}(x,y) = \frac{2}{\pi} c\left(c^2 + x^2 + y^2\right)^{-3/2} \qquad x, y > 0$$

Then, it can be easily verified, that this distribution may be thought of as arising from the bivariate folded t distribution, for $n=1$ and $\rho=0$ in the case where $c=1$.

8.MARGINAL DISTRIBUTIONS OF THE BIVARIATE FOLDED T DISTRIBUTION

In this section we'll be concerned with the problem of determining the marginal distributions of the bivariate folded t distribution.

It is known that multivariate normal and multivariate t distributions belong to the family of elliptically contoured distributions. It is also known that the marginal distributions of the elliptically contoured distributions are also elliptically contoured distributions (Tong 1990).

It is interesting to note that the bivariate standard folded t distribution is an elliptically contoured distribution. This follows from the fact that its p.d.f., as given by (6.2), may take the form:

$$f_{Z,W}(x,y) = 4\frac{\Gamma\left(\frac{n+2}{2}\right)}{(n\pi)^{2/2}\Gamma(n/2)}\left[1+\frac{x^2+y^2}{n}\right]^{-\frac{n+2}{2}} \quad x, y > 0$$

So, following Tong's (1990) theory we have for the marginal distibutions of Z.

$$f_Z(x) = 2\frac{\Gamma\left(\frac{n+1}{2}\right)}{(n\pi)^{1/2}\Gamma(n/2)}\left[1+\frac{x^2}{n}\right]^{-\frac{n+1}{2}} \quad x>0$$

But, this is the density function of the folded t distribution. Hence, both marginal distributions of (6.2) are of the folded t form.

9.THE RELATIONSHIP OF THE BIVARIATE FOLDED T DISTRIBUTION TO THE BIVARIATE FOLDED STANDARD NORMAL DISTRIBUTION

It is known that the limiting case of the bivariate t distribution when $n \to \infty$ is the bivariate half - normal distribution. It would therefore be interesting, particularly from a practical point of view, to check whether the folded versions of these distributions exhibit relationships of similar nature. In the sequel, it is demonstrated that this is indeed the case.

It is well known that:

$$\lim_{n\to\infty}\left[1+\frac{x^2-2\rho x y+y^2}{n\left(1-\rho^2\right)}\right]^{-\frac{n+2}{2}} = \exp\left(\frac{x^2+y^2-2\rho x y}{2\left(1-\rho^2\right)}\right) \tag{9.1}$$

and

$$\lim_{n\to\infty}\left[1+\frac{x^2+2\rho x y+y^2}{n\left(1-\rho^2\right)}\right]^{-\frac{n+2}{2}} = \exp\left(\frac{x^2+y^2+2\rho x y}{2\left(1-\rho^2\right)}\right) \tag{9.2}$$

From the above relationships it is obvious that the limiting case of the bivariate folded t distribution when $n\to\infty$ is the bivariate folded standard normal distribution.

Also, since

$$\lim_{n\to\infty}\left[1+\frac{x^2+y^2}{n}\right]^{-\frac{n+2}{2}} = \exp\left(-\frac{x^2+y^2}{2}\right)$$

it follows that the probability density function of the bivariate folded standard t will tend to

$$\frac{2}{\pi}\exp\left(-\frac{x^2+y^2}{2}\right).$$

Hence, the folded standard normal distribution is a limiting case of the bivariate folded standard t distribution with ρ=0.

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Figure 1
Probability density function of the bivariate
folded standard normal distribution for ρ = 0.2

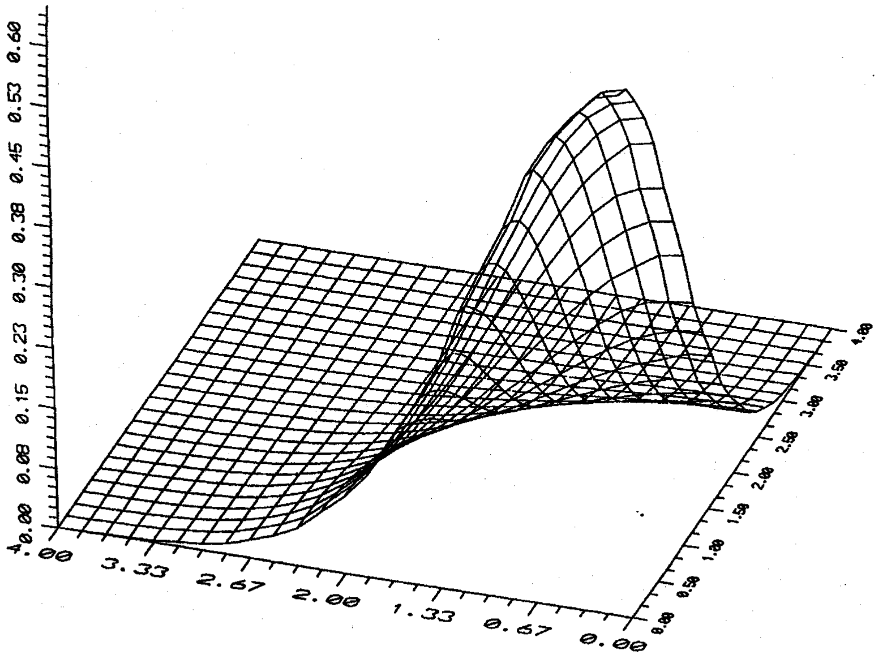

Figure 2
Probability density function of the bivariate
folded standard normal distribution for ρ = 0.9

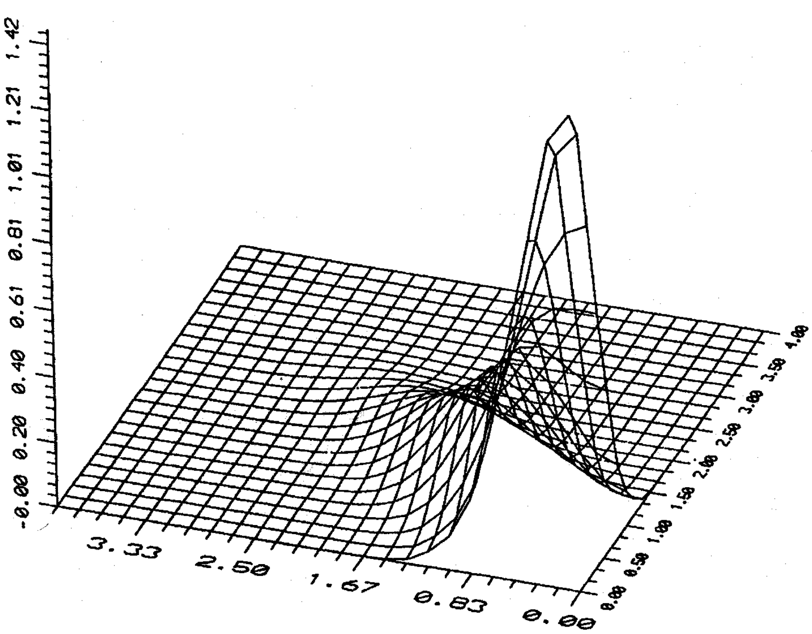

Figure 3
Probability density function of the bivariate folded t distribution for $n = 4$ and $\rho = 0.2$

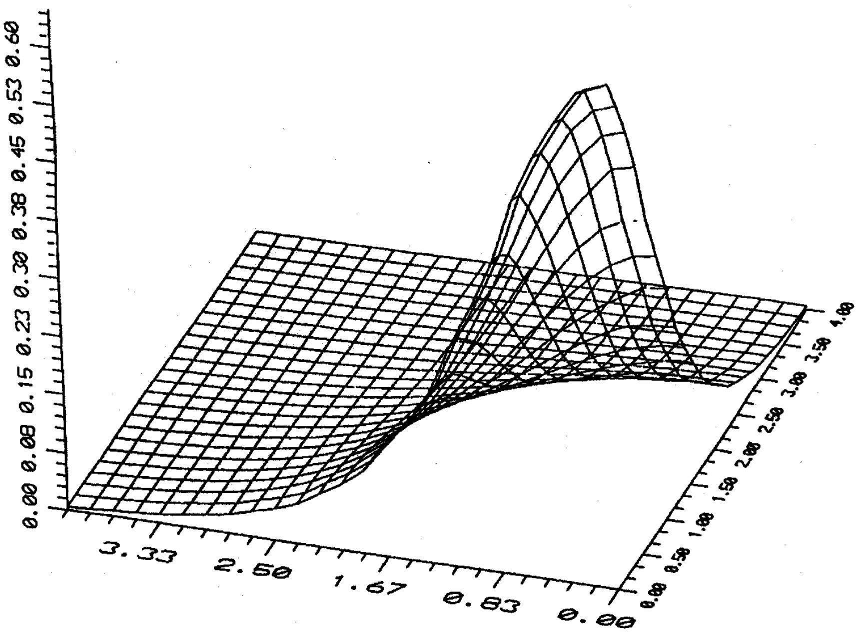

Figure 4
Probability density function of the bivariate folded t distribution for $n = 4$ and $\rho = 0.9$

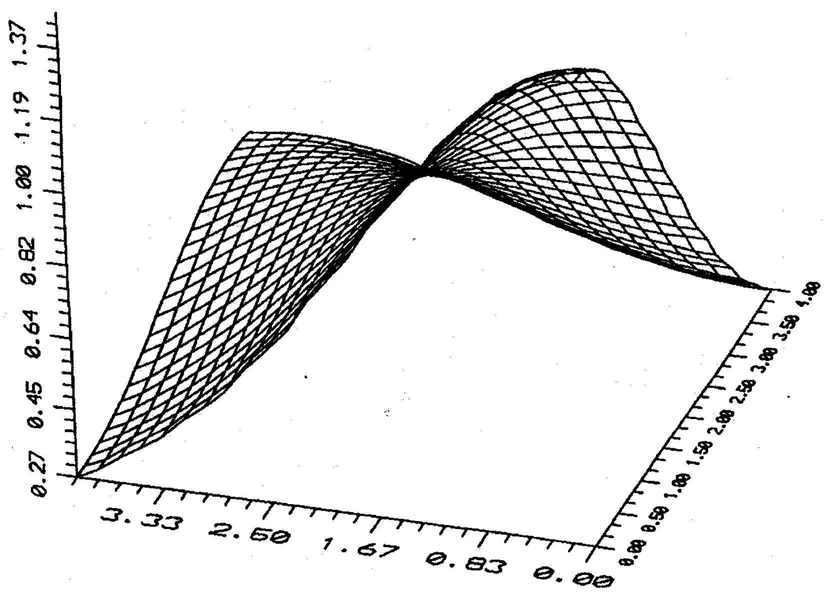

Figure 5
Probability density function of the bivariate folded t distribution for $n = 20$ and $\rho = 0.2$

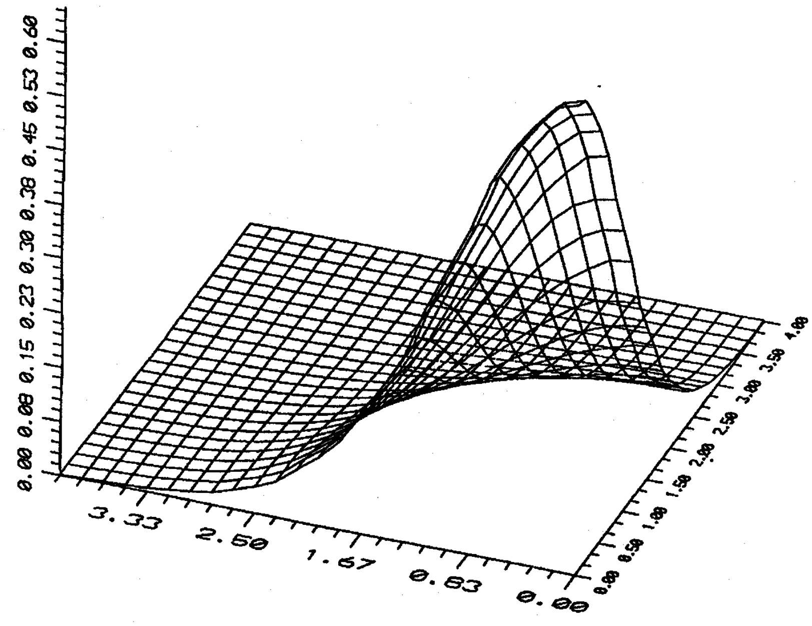

Figure 6
Probability density function of the bivariate folded t distribution for $n = 20$ and $\rho = 0.9$

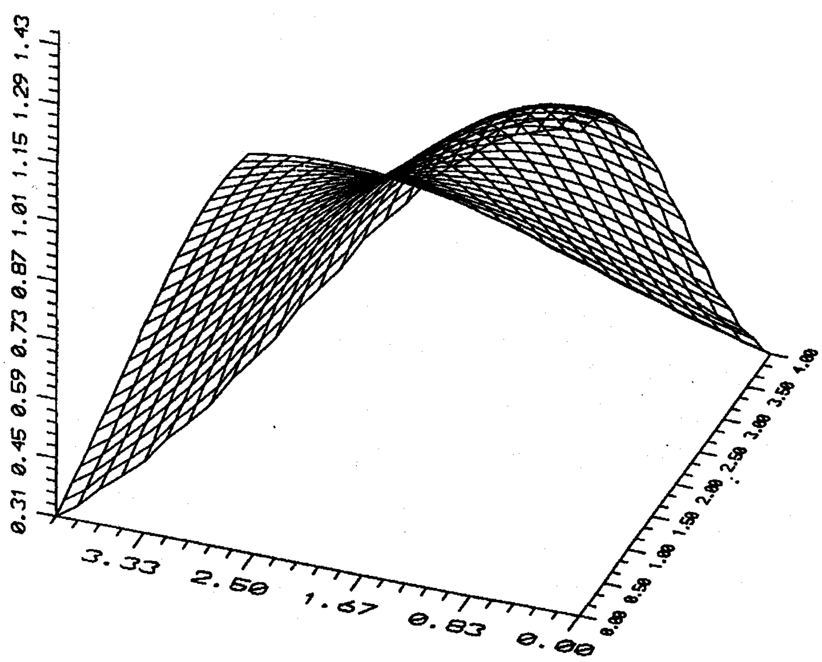

Figure 7
Probability density function of the
bivariate folded standard t distribution for n = 4

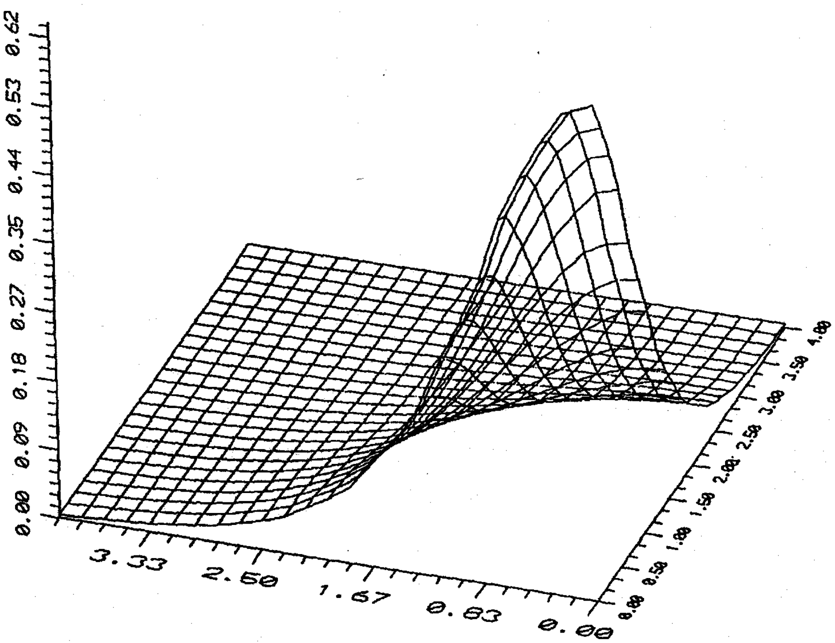

Figure 8
Probability density function of the
bivariate folded standard t distribution for n = 20