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# Extreme Value Index Estimators and Smoothing Alternatives: Review and Simulation Comparison

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## Abstract

Extreme-value theory and corresponding analysis is an issue extensively applied in many different fields. The central point of this theory is the estimation of a parameter  $\gamma$ , known as extreme-value index. In this paper we review several extreme-value index estimators, ranging from the oldest ones to the most recent developments. Moreover, some smoothing and robustifying procedures of these estimators are presented. A simulation study is conducted in order to compare the behaviour of the estimators and their smoothed alternatives. Maybe, the most prominent result of this study is that no uniformly best estimator exist and that the behaviour of estimators depends on the value of the parameter  $\gamma$  itself.

**Keywords** : extreme value index; semi-parametric estimation; smoothing modification.

## 1 Introduction

Extreme value theory is an issue of major importance in many fields of application where extreme values may appear and have detrimental effects. Such fields range from hydrology (Smith (1989), Davison and Smith (1990), Coles and Tawn (1996), Barão and Tawn (1999)) to insurance (Beirlant et al. (1994), Mikosch (1997), McNeil (1997), Rootzen and Tajvidi (1997)) and finance (Danielsson and de Vries (1997), McNeil (1998 and 1999), Embrechts et al. (1998, 1999), Embrechts (1999)). Actually, extreme value theory is a blend of a variety of applications and sophisticated mathematical results on point processes and regular varying functions.

The cornerstone of extreme value theory is Fisher-Tippet's theorem for limit laws for maxima (Fisher and Tippet, 1928). According to this, if the maximum value of a distribution function (d.f.) tends (in distribution) to a non-degenerate d.f. then this limiting d.f. can *only* be the Generalized Extreme Value (GEV) distribution:

$$H_\theta(x) = H_{\gamma, \mu, \sigma}(x) = \exp\left\{-\left(1 + \gamma \frac{x - \mu}{\sigma}\right)^{-1/\gamma}\right\},$$

where  $1 + \gamma \frac{x - \mu}{\sigma} > 0$ , and  $\theta = (\gamma, \mu, \sigma) \in \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R}_+$ .

A comprehensive sketch of the proof can be found in Embrechts *et al.* (1997).

The random variable (r.v.)  $X$  (the d.f.  $F$  of  $X$ , or the distribution of  $X$ ) is said to belong to the maximum domain of attraction of the extreme value distribution  $H_\gamma$  if there exist constants  $c_n > 0$ ,  $d_n \in \mathfrak{R}$  such that  $c_n^{-1}(M_n - d_n) \xrightarrow{d} H_\gamma$  holds. We write  $X \in \text{MDA}(H_\gamma)$  (or  $F \in \text{MDA}(H_\gamma)$ ).

In this paper we deal with the estimation of the parameter (extreme-value index)  $\gamma$ . In section 2 a general theoretical background is provided, in section 3 several existing estimators for  $\gamma$  are presented, while in section 4 some smoothing methods on specific estimators are given and extended to other estimators, too. A simulation comparison of the presented extreme-value index estimators along with their smoothing alternatives is analytically described in section 5. Finally, concluding remarks are found in section 6.

## 2 Modelling Approaches

A starting point for modelling the extremes of a process is based on distributional models derived from asymptotic theory. The *parametric approach* to modelling extremes is based on the assumption that the data in hand ( $X_1, X_2, \dots, X_n$ ) form an i.i.d. sample from an exact GEV d.f. In this case, standard statistical methodology from parametric estimation theory can be utilised in order to derive estimates of the parameters  $\theta$ . In practice, this approach is adopted whenever our dataset is consisted of maxima of independent samples (e.g. in hydrology we have disjoint time periods). This method is often called method of *block maxima*. Such techniques are discussed in DuMouchel (1983), Hosking (1985), Hosking et al. (1985), Smith (1985), Scarf (1992), Embrechts et al. (1997) and Coles and Dixon (1999). However, this approach may seem restrictive and not very realistic since the grouping of data into epochs is sometimes rather arbitrary, while by using only the block maxima, we may loose important information (some blocks may contain several among the largest observations, while other blocks may contain

none). Moreover, in the case that we have few data, block maxima cannot be actually implemented.

In this paper we deal with another widely used approach, the so-called ‘Maximum Domain of Attraction Approach’ (Embrechts et al., 1997), or Non-Parametric. In the present context we prefer the term ‘*semi-parametric*’ since this term reflects the fact that we make only partly assumptions about the unknown d.f.  $F$ .

So, essentially, we are interested in the distribution of the maximum (or minimum) value. Here is the point where extreme-value theory gets involved. According to the Fisher-Tippett theorem, the limiting d.f. of the (normalized) maximum value (if it exists) is the GEV d.f.  $H_\theta = H_{\gamma, \mu, \sigma}$ . So, without making any assumptions about the unknown d.f.  $F$  (apart from some continuity conditions which ensure the existence of the limiting d.f.), extreme-value theory provides us with a fairly sufficient tool for describing the behaviour of extremes of the distribution that the data in hand stem from. The only issue that remains to be resolved is the estimation of the parameters of the GEV d.f.  $\theta = (\gamma, \mu, \sigma)$ . Of these parameters, the *shape parameter*  $\gamma$  (also called *tail index* or *extreme-value index*) is the one that attracts most of the attention, since this is the parameter that determines, in general terms, the behaviour of extremes.

According to extreme-value theory these are the parameters of the GEV d.f. that the maximum value follows asymptotically. Of course, in reality, we only have a finite sample and, in any case, we cannot use only the largest observation for inference. So, the procedure followed in practice is that we assume that the asymptotic approximation is achieved for the largest  $k$  observations (where  $k$  is large but not as large as the sample size  $n$ ), which we subsequently use for the estimation of the parameters. However, the choice of  $k$  is not an easy task. On the contrary, it is a very controversial issue. Many authors have suggested alternative methods for choosing  $k$ , but no method has been universally accepted.

### 3 Semi-Parametric Extreme-Value Index Estimators

In this section, we give the most prominent answers to the issue of parameter estimation. We mainly concentrate to the estimation of the shape parameter  $\gamma$  due to its (already stressed) importance. The setting on which we are working is :

Suppose that we have a sample of i.i.d r.v.'s  $X_1, X_2, \dots, X_n$  (where  $X_{1:n} \geq X_{2:n} \geq \dots \geq X_{n:n}$  are the corresponding descending order statistics) from an unknown *continuous* d.f.  $F$ . According to extreme-value theory, the normalized maximum of such a sample follows asymptotically a GEV d.f.  $H_{\gamma, \mu, \sigma}$ , i.e.  $F \in MDA(H_{\gamma, \mu, \sigma})$ .

In the remaining of this section, we give the most prominent answers to the above question of estimation of extreme-value index  $\gamma$ . Of course, it would be unrealistic to claim that we can cover the whole literature on these issues, since the literature is indeed vast. We describe the most known suggestions, ranging from the first contributions, of 1975, in the area to very recent modifications and new developments.

#### 3.1 Pickands Estimator

The Pickands estimator (Pickands, 1975), is the first suggested estimator for the parameter  $\gamma \in \mathfrak{R}$  of GEV d.f and is given by the formula

$$\hat{\gamma}_P = \frac{1}{\ln 2} \ln \left( \frac{X_{(k/4):n} - X_{(k/2):n}}{X_{(k/2):n} - X_{k:n}} \right).$$

The original justification of Pickands estimator was based on adopting a percentile estimation method for the differences among the upper-order statistics. A more formal justification is provided by Embrechts et al. (1997).

A particular characteristic of Pickands estimator is the fact that the largest observation is not explicitly used in the estimation. One can argue that this makes sense since the largest observation may add too much uncertainty.

The properties of Pickands estimator were mainly explored by Dekkers and de Haan (1989). They proved, under certain conditions, weak and strong consistency, as well as asymptotic normality. Consistency depends only on the behaviour of  $k$ , while asymptotic normality requires more delicate conditions (2<sup>nd</sup> order conditions) on the underlying d.f.  $F$ , which are difficult to verify in practice. Still, Dekkers and de Haan (1989) have shown

that these conditions hold for various known and widely-used d.f.'s (normal, gamma, GEV, exponential, uniform, Cauchy).

### 3.2 Hill Estimator

The most popular tail index estimator is the Hill estimator, (Hill, 1975), which, though, is restricted to the Fréchet case  $\gamma > 0$ . The the Hill estimator is provided by the formula

$$\hat{\gamma}_H = \frac{1}{k} \sum_{i=1}^k \ln X_{i:n} - \ln X_{k+1:n}.$$

The *original derivation* of the Hill estimator relied on the notion of conditional maximum likelihood estimation method.

The statistical behaviour and properties of the Hill estimator have been studied by many authors separately, and under diverse conditions. Weak and strong consistency as well as asymptotic normality of the Hill estimator hold under the assumption of i.i.d. data (Embrechts et al., 1997). Similar (or slightly modified) results have been derived for data with several types of dependence or some other specific structures (see for example Hsing, 1991 as well as Resnick and Stărică, 1995, 1996 and 1998).

Note that, the conditions on  $k$  and d.f.  $F$  that ensure the consistency and asymptotic normality of the Hill estimator are the same as those imposed for Pickands estimator. Such conditions have been discussed by many authors, such as Davis and Resnick (1984), Haeusler and Teugels (1985), de Haan and Resnick (1998).

Though the Hill estimator has the apparent disadvantage that is restricted to the case  $\gamma > 0$ , it has been widely used in practice and extensively studied by statisticians. Its popularity is partly due to its simplicity and partly due to the fact that in most of the cases where extreme-value analysis is called for, we have long-tailed d.f.'s (i.e.  $\gamma > 0$ ).

### 3.3 Adapted Hill Estimator

The popularity of the Hill estimator generated a tempting problem to try to extend the Hill estimator (with its simplicity and good properties) to the general case  $\gamma \in \mathfrak{R}$ . Such an attempt, led Beirlant et al. (1996) to the so-called adapted the Hill estimator, which is applicable for any  $\gamma$  in the range of real numbers :

$$\hat{\gamma}_{adH} = \frac{1}{k} \sum_{i=1}^k \ln(U_i) - \ln(U_{k+1}) , \text{ where } U_i = X_{(i+1):n} \left( \frac{1}{i} \sum_{j=1}^i \ln X_{j:n} - \ln X_{(i+1):n} \right).$$

### 3.4 Moment Estimator

Another estimator that can be considered as an adaptation of the Hill estimator, in order to obtain consistency for all  $\gamma \in \mathfrak{R}$ , has been proposed by Dekkers et al. (1989). This is the moment estimator, given by

$$\hat{\gamma}_M = M_1 + 1 - \frac{1}{2} \left( 1 - \frac{(M_1)^2}{M_2} \right)^{-1}, \text{ where } M_j \equiv \frac{1}{k} \sum_{i=1}^k (\ln X_{i:n} - \ln X_{(k+1):n})^j, j=1, 2.$$

Weak and strong consistency, as well as asymptotic normality of the moment estimator have been proven by its creators Dekkers et al. (1989).

### 3.5 QQ – Estimator

One of the approaches concerning Hill's derivation is the 'QQ-plot' approach (Beirlant et al., 1996). According to this, the Hill estimator is approximately the slope of the line fitted to the upper tail of Pareto QQ plot. A more precise estimator, under this approach, has been suggested by Kratz and Resnick (1996), who derived the following estimator of  $\gamma$ :

$$\hat{\gamma}_{qq} = \frac{\sum_{i=1}^k \ln \frac{i}{k+1} \left\{ \sum_{j=1}^k \ln X_{j:n} - k \ln X_{i:n} \right\}}{k \sum_{i=1}^k \left( \ln \frac{i}{k+1} \right)^2 - \left( \sum_{i=1}^k \ln \frac{i}{k+1} \right)^2}.$$

The authors proved weak consistency and asymptotic normality of qq-estimator (under conditions similar to the ones imposed for the Hill estimator). However, the asymptotic variance of qq-estimator is twice the asymptotic variance of the Hill estimator, while similar conclusions are drawn from simulations of small samples. On the other hand, one of the advantages of qq-estimator over the Hill estimator is that the residuals (of the Pareto plot) contain information which potentially can be utilised to confront the bias in the estimates when the approximation is not exactly valid.

### 3.6 Moments Ratio Estimator

Concentrating on cases where  $\gamma > 0$ , the main disadvantage of the Hill estimator is that it can be severely biased, depending on the 2<sup>nd</sup> order behaviour of the underlying d.f. F.

Based on an asymptotic 2<sup>nd</sup> order expansion of the d.f. F, from which one gets the bias of the Hill estimator, Danielsson et al. (1996) proposed the moments ratio estimator :

$$\hat{\gamma}_{MR} = \frac{1}{2} \cdot \frac{M_2}{M_1}.$$

They proved that  $\hat{\gamma}_{MR}$  has lower asymptotic square bias than the Hill estimator (when evaluated at the same threshold, i.e. for the same  $k$ ), though the convergence rates are the same.

### 3.7 Peng's and W estimators

An estimator related to the moment estimator  $\hat{\gamma}_M$  is Peng's estimator, suggested by Deheuvels et al. (1997):

$$\hat{\gamma}_L = \frac{M_2}{2M_1} + 1 - \frac{1}{2} \left( 1 - \frac{(M_1)^2}{M_2} \right)^{-1}.$$

This estimator has been designed to somewhat reduce the bias of the moment estimator.

Another related estimator suggested by the same authors is the W estimator

$$\hat{\gamma}_W = 1 - \frac{1}{2} \left( 1 - \frac{(L_1)^2}{L_2} \right)^{-1}, \text{ where } L_j \equiv \frac{1}{k} \sum_{i=1}^k (X_{i:n} - X_{(k+1):n})^j, j=1, 2.$$

As Deheuvels et al. (1997) mentioned,  $\hat{\gamma}_L$  is consistent for any  $\gamma \in \mathfrak{R}$  (under the usual conditions), while  $\hat{\gamma}_W$  is consistent only for  $\gamma < 1/2$ . Moreover, under appropriate conditions on  $F$  and  $k(n)$ ,  $\hat{\gamma}_L$  is asymptotically normal. Normality holds for  $\hat{\gamma}_W$  only for  $\gamma < 1/4$ .

### 3.8 Use of Mean Excess Plot

A graphical tool for assessing the behaviour of a d.f.  $F$  is the mean excess function (MEF). The limit behaviour of MEF of a distribution gives important information on the tail of that distribution function (Beirlant et al., 1995). MEF's and the corresponding mean excess plots (MEP's) are widely used in the first exploratory step of extreme-value analysis, while they also play an important role in the more systematic steps of tail index and large quantiles estimation. MEF is essentially the expected value of excesses over a threshold value  $u$ . The formal definition of MEF (Beirlant et al., 1996) is as follows:

Let  $X$  be a positive r.v.  $X$  with d.f.  $F$  and with finite first moment. Then MEF of  $X$  is

$$e(u) = E(X - u | X > u) = \frac{1}{\bar{F}(u)} \int_u^{x_F} \bar{F}(y) dy, \text{ for all } u > 0.$$

The corresponding MEP is the plot of points  $\{u, e(u), \text{ for all } u > 0\}$ .

The empirical counterpart of MEF based on sample  $(X_1, X_2, \dots, X_n)$ , is

$$\hat{e}(u) = \frac{\sum_{i=1}^n (X_i - u) \mathbf{1}_{(u, \infty)}(X_i)}{\sum_{i=1}^n \mathbf{1}_{(u, \infty)}(X_i)}, \text{ where } \mathbf{1}_{(u, \infty)}(x) = 1 \text{ if } x > u, 0 \text{ otherwise.}$$

Usually, the MEP is evaluated at the points. In that case, MEF takes the form

$$E_k = \hat{e}(X_{(k+1):n}) = \frac{1}{k} \sum_{i=1}^k X_{i:n} - X_{(k+1):n}, \quad k=1, \dots, n.$$

If  $X \in MDA(H_\gamma)$ ,  $\gamma > 0$ , then it is easy to show that

$$e_{\ln X}(\ln u) = E(\ln X - \ln u | X > u) \rightarrow \gamma, \text{ as } u \rightarrow \infty.$$

Intuitively, this suggests that if the MEF of the logarithmic-transformed data is ultimately constant, then  $X \in MDA(H_\gamma)$ ,  $\gamma > 0$  and the values of MEF converge to the true value  $\gamma$ .

Replacing  $u$ , in the above relation, by a high quantile  $Q\left(1 - \frac{k}{n}\right)$ , or empirically by  $X_{(k+1):n}$ , we find that the estimator  $e_{\ln X}(X_{(k+1):n})$  will be a consistent estimator of  $\gamma$  in case  $X \in MDA(H_\gamma)$ ,  $\gamma > 0$ . This holds when  $k/n \rightarrow 0$  as  $n \rightarrow \infty$ . Notice that the empirical counterpart of  $e_{\ln X}(X_{(k+1):n})$  is the well-known the Hill estimator.

### 3.9 Comparison of Estimators

The aforementioned estimators share some common desirable properties, such as weak consistency and asymptotic normality (though these properties may hold under slightly different, unverifiable in any case, conditions). On the other hand, simulation studies or applications on real data can end up in large differences among these estimators. In any case, there is no ‘uniformly best’ estimator. Of course, Hill, Pickands and Moment estimators are the most popular ones. This could be partly due to the fact that they are the oldest ones. The rest estimators have been introduced later. Actually, most of these have been introduced as alternatives to Hill, Pickands or Moment estimator and some of them

have been proven to be superior in some cases only. In the literature, there are several comparison studies of extreme-value index estimators (either theoretically or via Monte-Carlo methods), such as Deheuvels et al. (1997) and Rosen and Weissman (1996). Still, these studies are confined to a small number of estimators.

## **4 Smoothing and Robustifying Procedures for Semi-Parametric Extreme-Value Index Estimators**

### **4.1 Introduction**

In the previous section, we presented a series of (semi-parametric) estimators for the extreme value index  $\gamma$ . Still, one of the most serious objections one could raise against these methods is their sensitivity towards the choice of  $k$  (number of upper order statistics used in the estimation). The well-known phenomenon of bias-variance trade off turns out to be unresolved, and choosing  $k$  seems to be more of an art than a science.

Some refinements of these estimators have been proposed, in an effort to produce unbiased estimators even when a large number of upper order statistics is used in the estimation (see, for example, Peng, 1998, or Drees, 1996). From another standpoint, adaptive methods for choosing  $k$  were proposed for special classes of distributions (see Beirlant et al., 1996 and references in Resnick and Stărică, 1997). Still, no hard and fast rule seems to exist. Another common unfortunate feature of semi-parametric estimators of  $\gamma$  is that the optimal choice of  $k$  depends mainly on 2<sup>nd</sup> order assumptions of the underlying d.f.  $F$ , which are not verifiable in practice. Accordingly, it is not an exaggeration to say that choosing is the Achilles heel of semi-parametric estimation methods for extreme-value index  $\gamma$ . In this paper we present a different approach towards this issue. We go back to elementary notions of extreme-value theory, and statistical analysis in general, and try to explore methods to render (at least partially) this problem. The procedures we use are i) smoothing techniques and ii) robustifying techniques.

### **4.2 Smoothing Extreme-Value Index Estimators**

The essence of semi-parametric estimators of extreme-value index  $\gamma$ , is that we use information of only the most extreme observations in order to make inference about the behaviour of the maximum of a d.f. An exploratory way to subjectively choose the

number  $k$  is based on the plot of the estimator  $\hat{\gamma}(k)$  versus  $k$ . A stable region of the plot indicates a valid value for the estimator. The search for a stable region in the plot is a standard but problematic and ill-defined practice. The need for a stable region results from adapting theoretical limit theorems which are proved subject to the conditions  $k(n) \rightarrow \infty$  and  $k(n)/n \rightarrow 0$ .

But, since extreme events by definition are rare, there is only little data (few observations) that can be utilised and this inevitably involves an added large statistical uncertainty. Thus, sparseness of large observations and the unexpectedly large differences between them, lead to a high volatility of the part of the plot that we are interested in and makes the choice of  $k$  very difficult. That is, the practical use of the estimator on real data is hampered by the high volatility of the plot and bias problems and it is often the case that volatility of the plot prevents a clear choice of  $k$ . A possible solution would be to smooth ‘somehow’ the estimates with respect to the choice of  $k$  (i.e. make it more insensitive to the choice of  $k$ ), leading to a more stable plot and a more reliable estimate of  $\gamma$ . Such a method was proposed by Resnick and Stărică (1997, 1999) for smoothing Hill and Moment estimators, respectively.

#### 4.2.1 Smoothing Hill Estimator

Resnick and Stărică (1997) proposed a simple averaging technique that reduces the volatility of the Hill-plot. The smoothing procedure consists of averaging the Hill estimator values corresponding to different values of order statistics  $p$ . The formula of the proposed averaged-Hill estimator is :

$$av\hat{\gamma}_H(k) = \frac{1}{k - [ku]} \sum_{p=[ku]+1}^k \hat{\gamma}_H(p) ,$$

where  $u < 1$ , and  $[x]$  denotes the smallest integer greater than or equal to  $x$ .

The authors proved that through averaging (using the above formula), the variance of the Hill estimator can be considerably reduced and the volatility of the plot tamed. The smoothed graph has a narrower range over its stable regime, with less sensitivity to the value of  $k$ . This fact diminishes the importance of selecting the optimal  $k$ . The smoothing techniques make no (additional) unrealistic or uncheckable assumptions and are always

available to complement the Hill plot. Obviously, when considerable bias is present, the averaging technique offers no improvement.

Resnick and Stărică (1997) derived the adequacy (consistency and asymptotic normality) of the averaged-Hill estimator, as well as its improvement over the Hill estimator (smaller asymptotic variance). Since the asymptotic variance is a decreasing function of  $u$ , one would like to choose  $u$  as big as possible to ensure the maximum decrease of the variance. However, the choice of  $u$  is limited by the sample size. Due to the averaging, the larger the  $u$ , the fewer the points one gets on the plot of averaged Hill. Therefore, an equilibrium should be reached between variance reduction and a comfortable number of points on the plot. This is a problem similar to the variance-bias trade-off encountered in the simple extreme-value index estimators.

#### 4.2.2 Smoothing Moment Estimator

Resnick and Stărică (1999) also applied their idea of smoothing to the more general moment estimator  $\hat{\gamma}_M$ , essentially generalizing their reasoning of smoothing the Hill estimator.

The proposed smoothing technique consists of averaging the moment estimator values corresponding to different numbers of order statistics  $p$ . The formula of the proposed averaged-moment estimator, for given  $0 < u < 1$ , is :

$$av\hat{\gamma}_M(k) = \frac{1}{k - [ku]} \sum_{p=[ku]+1}^k \hat{\gamma}_M(p) .$$

In practice, the authors suggest to take  $u=0.3$  or  $u=0.5$  depending on the sample size (the smaller the sample size the larger  $u$  should be).

In this case the consequent reduction in asymptotic variance is not so profound. The authors actually showed that through averaging (using the above formula), the variance of the moment estimator can be considerably reduced only in the case  $\gamma < 0$ . In the case  $\gamma > 0$  the simple moment estimator turns out to be superior than the averaged-moment estimator. For  $\gamma \approx 0$  the two moment estimators (simple and averaged) are almost equivalent. These conclusions hold asymptotically, and have been verified via a graphical comparison, since the analytic formulas of variances are rather complicated to be

compared directly. Full treatment of this issue and proofs of the propositions can be found in Resnick and Stărică (1999).

### 4.3 Robust Estimators Based on Excess Plots

As we have previously mentioned the MEP constitutes a starting point for the estimation of extreme-value index. However, though theoretically the values of MEF consistently estimate parameter  $\gamma$ , in practice strong random fluctuations of the empirical MEF and the corresponding MEP are observed, especially in the right part of the plot (i.e. for large values of  $u$ ), since there we have fewer data. But this exactly is the part of plot that mostly concerns us, that is the part that theoretically informs us about the tail behaviour of the underlying d.f. Consequently, the calculation of the ‘ultimate’ value of MEF can be largely influenced by only a few extreme outliers, which may not even be representative of the general ‘trend’. It is striking the result of Drees and Reiss (1996) that the empirical MEF is an inaccurate estimate of the Pareto MEF, and that the shape of the empirical curve heavily depends on the maximum of the sample.

In an attempt to make the procedure more robust, that is less sensitive to the strong random fluctuations of the empirical MEF at the end of the range, the following adaptations of MEF have been considered (Beirlant et al., 1996):

- Generalized Median Excess Function  $M^{(p)}(k) = X_{([pk]+1):n} - X_{(k+1):n}$   
(for  $p=0.5$  we get the simple median excess function).
- Trimmed Mean Excess Function  $T^{(p)}(k) = \frac{1}{k - [pk]} \sum_{j=[pk]+1}^k X_{j:n} - X_{(k+1):n}$ .

The general motivations and procedures explained for the MEF and its contribution to the estimation of  $\gamma$  hold here as well. Thus, alternative estimators for  $\gamma > 0$  are :

- $\hat{\gamma}_{gen.med} = \frac{1}{\ln(1/p)} (\ln X_{([pk]+1):n} - \ln X_{(k+1):n})$

which for  $p=0.5$  gives  $\hat{\gamma}_{med} = \frac{1}{\ln(2)} (\ln X_{([k/2]+1):n} - \ln X_{(k+1):n})$

(the consistency of this estimator is proven by Beirlant et al., 1996).

- $\hat{\gamma}_{trim} = \frac{1}{k - [pk]} \sum_{j=[pk]+1}^k \ln X_{j:n} - \ln X_{(k+1):n}$

The performance of both of these families of alternative estimators is going to be explored (via simulation) in the next section.

## 5 Simulation Comparison of Extreme-Value Index Estimators

### 5.1 Details of Simulation Study

In this section, we try to investigate and compare, via Monte Carlo methods, the performance of the extreme-value index estimators introduced, as well as the performance of the modifications suggested previously. Apart from the standard form of estimators, we apply to all of them the averaging procedure presented in section 4. Resnick and Stărică (1997, 1999) suggested (and proved the adequacy and good properties of) this procedure only in the context of Hill and moment estimator. We apply the procedure to other extreme-value index estimators, so as to empirically evaluate its effect on these estimators. In addition, apart from these mean-averaged estimators, we apply analogously a median-averaging procedure to our estimators. Moreover, for  $\gamma > 0$ , we also examine estimators based on median excess plot as well as on trimmed mean excess plot. The following table 1 contains all the estimators that are included in the present simulation study. These estimators are compared with respect to the distributions of table 2.

**Table 1.** Estimators included in the simulation study

Estimators	Formula
<i>Standard Estimators</i>	
Pickands estimator (for $\gamma \in \mathfrak{R}$ )	$\hat{\gamma}_P = \frac{1}{\ln 2} \ln \left( \frac{X_{M:n} - X_{2M:n}}{X_{2M:n} - X_{4M:n}} \right)$
Hill estimator (for $\gamma > 0$ )	$\hat{\gamma}_H = \frac{1}{k} \sum_{i=1}^k \ln X_{i:n} - \ln X_{k+1:n}$
Adapted the Hill estimator (for $\gamma \in \mathfrak{R}$ )	$\hat{\gamma}_{adH} = \frac{1}{k} \sum_{i=1}^k \ln(UH_i) - \ln(UH_{k+1})$
Moment estimator (for $\gamma \in \mathfrak{R}$ )	$\hat{\gamma}_M = M_1 + 1 - \frac{1}{2} \left( 1 - \frac{(M_1)^2}{M_2} \right)^{-1}$

Moment-Ratio estimator (for $\gamma > 0$ )	$\hat{\gamma}_{MR} = \frac{1}{2} \cdot \frac{M_2}{M_1}$
QQ estimator (for $\gamma > 0$ )	$\hat{\gamma}_{qq} = \frac{\sum_{i=1}^k \ln \frac{i}{k+1} \left\{ \sum_{j=1}^k \ln X_{j:n} - k \ln X_{i:n} \right\}}{k \sum_{i=1}^k \left( \ln \frac{i}{k+1} \right)^2 - \left( \sum_{i=1}^k \ln \frac{i}{k+1} \right)^2}$
Peng's estimator (for $\gamma \in \mathfrak{R}$ )	$\hat{\gamma}_L = \frac{M_2}{2M_1} + 1 - \frac{1}{2} \left( 1 - \frac{(M_1)^2}{M_2} \right)^{-1}$
W estimator (for $\gamma < 1/2$ )	$\hat{\gamma}_w = 1 - \frac{1}{2} \left( 1 - \frac{(L_1)^2}{L_2} \right)^{-1}$
<i>Mean-Averaged Estimators (u=0.5)</i>	
Averaged X estimator (X stands for all the standard estimators)	$av\hat{\gamma}_X(k) = \frac{1}{k - [ku]} \sum_{p=[ks]+1}^k \hat{\gamma}_X(p)$
<i>Median-Averaged Estimators (u=0.5)</i>	
Median-Averaged X estimator (X stands for all the standard estimators)	$med.av\hat{\gamma}_X(k) = med\{\hat{\gamma}_X(p), p = [ku] + 1, \dots, k\}$
<i>Estimators based on Excess Plot</i>	
Estimator based on Median Excess Plot (for $\gamma > 0$ )	$\hat{\gamma}_{med} = \frac{1}{\ln(2)} (\ln X_{([k/2]+1):n} - \ln X_{(k+1):n})$
Estimators based on Trimmed Mean Excess Plot (for $\gamma > 0$ ) $p=0.01, 0.05, 0.10$	$\hat{\gamma}_{trim} = \frac{1}{k - [pk]} \sum_{j=[pk]+1}^k \ln X_{j:n} - \ln X_{(k+1):n}$

**Table 2.** Distributions used in the simulation study

Name	Extreme-value index $\gamma$	Other parameters
<b>Burr</b>	0.25, 0.55, 1, 2	$(\tau, \lambda) = (0.25, 1), (0.55, 1), (0.5, 2), (1, 0.5)$
<b>Fréchet</b>	0.25, 0.55, 1, 2	-
<b>Log-gamma</b>	0.25, 0.55, 1, 2	$\alpha=1, \lambda = 4, 1/0.55, 1, 0.5$
<b>Log-logistic</b>	0.25, 0.55, 1, 2	-
<b>Pareto</b>	0.25, 0.55, 1, 2	-
<b>Weibull</b>	0	$\lambda=1, \tau=0.5, 1.5$
<b>Exponential</b>	0	-
<b>Log-normal</b>	0	$\mu=100, \sigma=1$

<i>Normal</i>	0	$\mu=10, \sigma=1$
<i>Gamma</i>	0	$\alpha=1, \beta=0.5, 1.5$
<i>Beta</i>	-1/3, -2	$(\alpha, \beta) = (0.5, 3), (2, 3), (0.5, 0.5), (2, 0.5)$
<i>Uniform</i>	-1	$(\alpha, \beta) = (0,1), (5, 10)$

As seen from table 2, the distributions used in table 2, range from distributions with finite upper endpoint ( $\gamma < 0$ ) to long-tailed distributions ( $\gamma \geq 0$ ). From each of these distributions, 1000 samples were generated of moderate size ( $n=100$ ) and 500 samples of large size ( $n=1000$ ), based on which the performance of the estimators is examined. In our study, the performance of any estimator of  $\gamma$ , is evaluated in terms of the bias, standard error and root mean square error of the estimator based on  $k$  upper order statistics (where  $k$  ranges from 1 up to sample size). The root mean square error (rmse), being a combination of standard deviation and bias, is essentially the basis for comparisons of estimators. In the next section we summarize the main results of the simulation study.

## 5.2 Discussion of Simulation Results

Before proceeding to the discussion of the results, it should be noted that the performance of the estimators did not seem to remain stable for data stemming from different distributions. For this reason, in the sequel we provide the main findings of the simulation study distinguishing for each different class of distributions. Paragraphs with more general remarks are provided at the end of this section.

- **Burr Distribution**

In the case of Burr distribution, the estimation of extreme-value index, which ranges in the interval  $(0, +\infty)$ , seems to depend on the value of extreme-value index itself. So, for  $\gamma=0.25$  (i.e., more generally speaking, for  $\gamma < \frac{1}{2}$ ) the best estimator is the Hill estimator (it has the smallest rmse and in most cases the smallest bias). Moreover, in small samples ( $n=100$ ) Hill, also, has the smallest std, while for  $n=1000$  Moment-Ratio displays the smallest std. The mean averaging procedure improves the performance of Pickands estimator. The same general result holds for  $\gamma=0.55$  but only for small samples. For  $\gamma=0.55$  (large samples) and for  $\gamma=1$  Moment-Ratio outperforms the Hill estimator. Finally, for  $\gamma > 1$  (i.e. for  $\gamma=2$ ) Moment estimator is preferable for large sample sizes

( $n=1000$ ), while  $W$  is best when it comes to small sample sizes. As long as trimming effect is concerned, no clear effect exists.

- **Fréchet Distribution**

For samples of small size ( $n=100$ ), the Hill estimator, in general, displays the best performance. For Fréchet data with  $\gamma$  up to 1, the Hill estimator has the smallest rmse and, most of the times, the smallest bias and std. However, when  $\gamma$  exceeds 1 (in our case for  $\gamma=2$ ) the variability of Hill is much increased and  $W$  stands out as the best estimator. When we are dealing with large simulated data-sets ( $n=1000$ ), though the performance of Hill remains in the same level, Moment-Ratio estimator improves greatly (its bias and, consequently, its rmse is reduced significantly) and turns out to be the best estimator. Again this holds for values of  $\gamma$  smaller than or equal to 1, since for  $\gamma=2$ , Moment estimator has the smallest rmse. As long as the averaging procedures are concerned, the only improvement is observed in the case of mean-averaged Pickands estimator.

- **Log-Gamma Distribution**

For large sample sizes ( $n=1000$ ) and  $\gamma$  up to 1, the best estimator is, undeniably, the Moment-Ratio estimator. It has the smallest bias, std, as well as rmse. However, when  $\gamma$  exceeds 1, the behaviour of Moment-Ratio deteriorates (mainly due to the large increase of std) and though, still, Moment-Ratio has the smallest bias, it is Moment the estimator with the smallest rmse. A point that is worthy to be pointed out here is that while the behaviour of most estimators deteriorates as  $\gamma$  exceeds 1, the performance of  $W$  estimator is improved. Actually, now,  $W$  has the smallest std. The situation is quite different when we are dealing with small samples ( $n=100$ ). In those cases, it is the Hill the best estimator (smallest bias, std, rmse). Of course the behaviour of Moment-Ratio is not disappointing since it is the second best estimator (not differing much from the Hill). The only case where Hill is surpassed by another estimator is for  $\gamma=2$  and for small  $k$  ( $k=12, 20$ ), when  $W$  estimator displays the smallest std and rmse. No satisfactory improvement can be attributed to the averaging procedures.

- **Log-Logistic Distribution**

When we are dealing with large samples the situation is quite clear. The best estimator is the Moment-Ratio estimator, since it always has the smallest rmse (in many cases it also has minimum bias or std). However, the evaluation of estimators gets more complicated

for small sample sizes ( $n=100$ ). Generally speaking, we could say that among all estimators the estimator based on 10% trimmed mean excess plot is the best estimator (smallest rmse). If we confine our comparison among the standard estimators, then Hill and Moment-Ratio estimators are the most preferable. For  $\gamma < 1$  and for small and moderate choice of  $k$  ( $k=12, 20$ ) it is the Hill which has the smallest rmse (while the second smallest rmse is achieved by the Moment-Ratio), while when a larger portion of data is used in the estimation ( $k=40$ ) the Moment-Ratio stands out as the best estimator (followed by the Hill). Irrespectively of the choice of  $k$ , in the case of  $\gamma > 1$ , the best estimator is W.

- **Pareto Distribution**

As was the case for distributions previously discussed, the Moment-Ratio estimator tends to be, in general, preferable. For large samples and  $\gamma$  up to 1, Moment-Ratio has the smallest rmse and the smallest bias (for  $\gamma$  smaller than 1 it also has the smallest std). However, when  $\gamma$  exceeds 1, Moment estimator “outguns” Moment-Ratio. Even then, Moment-Ratio has the smallest bias, while W has the smallest std. Again, for small samples the situation is somewhat different. For  $\gamma$  up to 1, Hill is the best estimator (smallest rmse, bias and, usually, smallest std). But for  $\gamma=2$  and for  $k$  small ( $k=12, 20$ ) its variability is much increased while the behaviour of W is improved leading to a minimum rmse attributed to W. It is useful to add that Moment-Ratio is the second best estimator here.

- **Weibull Distribution**

Weibull d.f. belongs to the maximum domain of attraction of Gumbel d.f., i.e. the extreme value index equals zero. That means that not all of the estimators under examination should actually be applied to that type of data. Indeed, according to theoretical results, Hill, Moment-Ratio and QQ estimators are not consistent for  $\gamma \leq 0$ . However, in the context of our simulation study we applied all estimators even in cases that they are not applicable (for comparison reasons). The simulation results here do not suggest that any single estimator is uniformly best. So, for large samples and  $\tau=0.5$  Peng is the best estimator, while for  $\tau=1.5$  the result depends on the number of upper order statistics used ( $k$ ). For large  $k$ , again Peng is the best estimator. However for smaller  $k$  ( $k=12, 20$ ) QQ and Moment estimators display better performances (though even then the

performance of Peng is not bad). The situation is somehow similar for small samples ( $n=100$ ). That is, for  $\tau=0.5$  Peng is preferable but for  $\tau=1.5$ , it is outperformed by Moment and Moment-Ratio estimators.

- **Exponential Distribution**

For large sample sizes ( $n=1000$ ) Moment estimator has the best performance (for all choices of  $k$ ). Apart from having the smallest rmse, it also exhibits minimum bias. As long as standard error is concerned, Moment-Ratio has the smallest values (though Moment-Ratio is theoretically applicable only for  $\gamma > 0$ ). The mean averaging procedure improves the performance of Pickands estimator (though it still remains inferior to other estimators). Estimators based on trimmed mean excess plots have a better performance than simple the Hill estimator (the larger the proportion of trimming the better the performance of the estimator, while the median excess estimator is worse than Hill).

For small samples ( $n=100$ ) Moment estimators perform better only when a large portion of the sample is used in the estimation ( $k=40$ ). Even then when we are dealing with averaged estimators, Pickands outperforms the Moment. When a smaller  $k$  is used in the estimation of extreme-value index ( $k=12$  or  $20$ ) Moment-Ratio performs better (in terms of rmse) and has the smallest std, while Pickands has the smallest bias.

Generally speaking, for estimating the extreme-value index in the case of exponential d.f.s Moment estimator is most preferable for large samples, while for small samples we should opt for mean-averaged Pickands estimators.

- **Log-Normal Distributions**

For large samples the situation is quite clear, since Peng is the best estimator for all choices of  $k$ . However, things are not so simple for  $n=100$ . In that case, the choice of best estimator seems to depend on the choice of  $k$ . So, for large  $k$ , Peng is the best estimator, while for small  $k$  Moment and Moment-Ratio outperform it.

- **Normal Distribution**

In the case of normally distributed data the zero extreme-value index seems to be better estimated by the Moment-Ratio estimators. This holds for all choices of  $k$ , for small and for large sample sizes. The only problem seems to be the fact that Moment-Ratio estimator is not theoretically applicable when extreme-value index equals zero. Other estimators with satisfactory behaviour are Hill and Q-Q estimators (both of which are,

again, not applicable for zero extreme-value index). As long as averaging procedures are concerned, we should note that mean averaging greatly improves the performance of the standard Pickands estimator. However, mean averaged Pickands estimator is not one of the best estimators.

- **Gamma Distribution**

Here, for the first time the mean averaging procedure provides rather promising results. Though there is not a uniformly best estimator (for all choices of  $k$  and sample size  $n$ ), the mean averaged Pickands estimator displays the most satisfactory behaviour (in most cases it has the smallest rmse). It is really impressive its improvement over the simple, standard Pickands estimator. Other estimators with acceptable performance are Peng's, Moment, Moment-Ratio and the Hill estimator.

- **Beta Distribution**

Beta distribution belongs to the maximum domain of attraction of Weibull d.f, since it has finite upper end-point. That means that the extreme-value index that we are trying to estimate is negative and, as previously mentioned, not all estimators are applicable here. When we are dealing with large samples the situation is quite clear. Moment estimator is uniformly the best estimator (with the smallest rmse). Also, Peng's performance is almost as good as Moment's. Moreover, here it is evident the beneficial effect of the averaging procedures (the mean as well as the median averaging). Mean averaging procedure substantially improves the performance of Pickands estimator, while both averaging procedures improve the performance of Moment and Peng's estimator. So, in most cases it is the mean averaged Moment estimator that has the smallest rmse among all tested estimators. These results hold for different choices of  $k$  and parameters  $\alpha, \beta$  of the beta distribution (i.e. for different values of the extreme-value index). However, this is not so when it comes to small sample sizes. Actually, for small sample sizes ( $n=100$  in our case) the situation is much more complicated. The performance of the estimators depends not only on the value of  $k$  but also on the value of the extreme-value index  $\gamma$  itself. So, for values of  $\gamma$  close to 0 ( $-1/3$  in particular) mean averaged Pickands estimator seems to be the best choice, while among the standard estimators Moment-Ratio and Moment are more preferable. As the value of  $\gamma$  draws away from zero ( $\gamma=-2$ ) the performance of all estimators deteriorates (the values of rmse's are evidently larger). The behaviour of

Moment, Moment-Ratio and Peng's estimator is very disappointing. Pickands is the best estimator among standard estimators. Moreover, the effect of averaging procedures is much less significant. Anyhow, it is the median averaged Pickands and Moment estimators that exhibit the smallest rmse.

- **Uniform Distribution**

Uniform distribution is also bounded to the right and its tails are characterized by extreme-value index equal to -1. The situation here is somewhat similar to the case of Beta distribution. More specifically, for large sample sizes ( $n=1000$ ) Moment estimator is the most preferable estimator, followed by Peng's. Averaging procedures (mean and median) improve their performances, while mean averaging also improves the performance of Pickands estimator. Here it is the median averaging procedure that provides better results than the mean averaging. These results hold for small samples of uniformly distributed data. The only difference in this case is that for small  $k$  ( $k=12$ ) standard estimators display very large rmse's and the 'good' estimators previously mentioned are outperformed by estimators such as Hill and Moment-Ratio.

### **5.3 General Comments**

Though differences exist between different distribution, some common underlying behaviour of estimators, if we discern them according to the corresponding value of  $\gamma$ . More particularly:

- *For  $\gamma > 0$*

For large sample sizes Moment-Ratio seems to be the most preferable estimator. It is usually the best estimator, in terms of minimum rmse. Even in cases that other estimators outperform it, it is one of the bests, while in no case does it display very unsatisfactory performance. It is interesting to note that the W estimator tends to be appropriate for distributions with extreme-value index  $\gamma$  larger than 1, though for smaller values of  $\gamma$  its performance can be very unsatisfactory. So, it may be risky to use this estimator, since in real-life applications, the value of  $\gamma$  is unknown. For small samples (in our case  $n=100$ ), the Hill estimator turns out to be the best choice, while Moment-Ratio and Moment estimators can also be regarded as safe options. Among averaging procedures, only the mean averaging of Pickands estimator is effective. However, the improvement is not large enough to out-beat the other standard estimators. On the other hand, the trimming

procedure concerning the Hill estimator slightly improves the performance of Hill. This result combined with the fact that standard the Hill estimator is, in some cases, the best estimator lead to even better results. Still further exploration on this issue is required.

- *For  $\gamma=0$*

The maximum domain of attraction of Gumbel distribution contains a wide range of distributions differing a lot. Consequently, there is not a uniformly superior estimator. However, by examining more carefully the above results one could deduce that Peng's is the most preferable estimator of the extreme-value index. Moment and (surprisingly) Moment-Ratio also display an adequate behaviour. The usefulness of averaging procedures in these cases should also be stressed out. These procedures have an obvious profitable impact on Pickands estimator, so that mean-averaged Pickands estimator can also be regarded as an adequate estimator of zero  $\gamma$ .

- *For  $\gamma<0$*

This class contains upper-bounded distributions. Though the shape of distributions differs a lot from the distributions with  $\gamma=0$ , the behaviour of extreme-value index estimators in these two classes of d.f.'s displays great analogies. Here, Moment and Peng's estimators are undeniably the most preferable estimators and the beneficial effect of both mean and median averaging procedures is even more evident. Moreover, as we deviate from zero (and positive) values of  $\gamma$ , the inadequacy of estimators such as Hill, Moment-Ratio and so on, is more clear.

## **6 Discussion**

The comparison (via simulation) of semi-parametric estimation methods for the extreme-value index and some robustifying alternatives has been the central issue of this paper. The simulation study conducted led to some very interesting results. The first is that, as one could naturally expect, the performance of estimators on a specific data-set depends on the distribution of the data. So, there is not a uniformly best estimator. Nevertheless, by looking more carefully at the results, some general conclusions may be reached. More specifically, in cases of long-tailed data (with an infinite upper end-point) Moment and Moment-Ratio estimators seem to estimate more satisfactorily the non-negative extreme-value index  $\gamma$ . However, when it comes to upper-bounded distributions

(characterized by a negative value of  $\gamma$ ) Peng's and Moment estimators are more preferable. As far as the impact of smoothing (averaging) procedures is concerned, we deduce that it is effective (improving the performance of standard estimators) in cases where the true value of extreme-value index is non-positive. Particularly, mean-averaging procedures improve greatly the performance of Pickands estimator (in case of zero  $\gamma$ ), while median-averaging of Moment and Peng's also leads to improved estimators (for  $\gamma$  negative).

The dependence of estimators' performance on the distribution of data in hand can be alternatively seen as dependence on the true value of the index itself. So, before proceeding to the use of any estimation formula it would be useful if we could get an idea about the range of values where the true  $\gamma$  lies in. This can be achieved graphically via QQ and mean excess plots. Alternatively, there exist statistical tests which tests such hypothesis. See, for example, Hosking (1984), Hasofer and Wang (1992), Alves and Gomes (1996) and Marohn (1998).

Moreover, it should be pointed out that among the averaged estimators used in the simulation study only the mean-averaged Hill and Moment estimators have been theoretically explored by Resnick and Stărică (1997, 1999). As we have seen, the median averaging procedure has also displayed some interesting effectiveness, implying that it may be worthy to be also studied theoretically (with special emphasis on Moment and Peng's estimators). The same holds for the mean-averaged Pickands estimator.

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