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On Certain Indices for Ordinal Data with Unequally Weighted Classes

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Abstract. In this paper, some new indices for ordinal data are introduced. These indices have been developed so as to measure the degree of concentration on the "small" or the "large" values of a variable whose level of measurement is ordinal. Their advantage in relation to other approaches is that they ascribe unequal weights to each class of values. Although, they constitute a useful tool in various fields of applications, the focus here is on their use in sample surveys and specifically in situations where one is interested in taking into account the "distance" of the responses from the "neutral" category in a given question. The properties of these indices are examined and methods for constructing confidence intervals for their actual values are discussed. The performance of these methods is evaluated through an extensive simulation study.

1. Introduction

Various types of indices are widely used in real world applications. Some disciplines where the use of indices is widespread are index numbers (e.g., Allen, 1975), statistical quality control (e.g., Kotz and Lovelace, 1998; Perakis, 2002), accounting (e.g., Wild et al., 2000) and sample surveys (e.g., Bnerjee et al., 1999). An interesting discussion of the historical background and the present situation on the use of statistical indicators in various fields of applications is provided by De Vries (2001).

Recently, Maravelakis et al. (2003) developed some indices, which are similar in nature with the index suggested by Perakis and Xekalaki (2002) that have applications in statistical process control. The indices introduced by Maravelakis et al. (2003) can be used to measure the degree of concentration on the "large" or "small" values of a variable in ordinal scale. In that paper, the use of these indices in sample surveys is considered, where

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often one is faced with questions whose answers have a somewhat natural ordering. A common example is a question whose possible answers are "Very Good", "Good", "Moderate", "Bad" and "Very Bad". To our knowledge no other authors have dealt with such indices.

Different features of such types of data can be measured through various other indices such as Cohen's (1960) Kappa and its modifications (see e.g., Bnerjee et al., 1999; Doner, 1999) and the measure of nominal-ordinal association proposed by Agresti (1981) and Piccarreta (2001).

In this paper, the indices suggested by Maravelakis et al. (2003) are generalized so as to measure the observed concentration on the "small" or "large" values more objectively. Specifically, the indices proposed here, ascribe different weights to each value of the variable in question according to its rank. These indices, as those suggested by Maravelakis et al. (2003), can be used in connection with several types of ordinal data, as explained in Section 7. Nevertheless, in the sequel we focus on their use in sample surveys.

Section 2, describes the indices defined by Maravelakis et al. (2003) and gives the rationale that led to the definition of the indices proposed in this paper. In the third section, we introduce the new indices and investigate their basic properties. Section 4 is devoted to the derivation of the variances of their estimators. The construction of confidence intervals for their actual values, using three alternative bootstrap techniques, is discussed in Section 5. The performance of the three methods is tested through simulation. An illustrative example based on real data that clarifies their estimation is given in Section 6. Finally, some concluding remarks are provided in Section 7.

2. Motivation

Consider a discrete valued variable that takes a finite number of values from 1 to k, or a continuous variable with values grouped in k classes. Suppose that these k values or classes have a natural ordering starting from the "best" (value 1) to the "worst" (value k) and exhibit an inherent symmetry, i.e. the number of values characterized as "positive" coincides with that of the "negative" ones. Thus, the first $[k/2]$ values have a "positive" interpretation whereas the last $[k/2]$ have a "negative" one. If the value of k is odd, the $([k/2]+1)$st value does not belong to any of these two categories and in the sequel, it is termed "neutral".

In this paper, we consider the case where the variable under investigation consists of the answers to a question in a study which asks a person to choose one out of k possible categories. However, the analysis can be modified readily for any other variable with the above properties.

Let π_i (p_i), $i = 1, 2, \ldots, k$ denote the true (observed) proportion of answers for each of the k categories, where π_1 (p_1) refers to the "best" available answer, and π_k (p_k) to the "worst" one. Obviously, the "neutral" answer, if such an answer exists (i.e. if k is odd), is located at the point $[k/2]+1$. We should remark that among the k possible answers we include the "neutral" answer (if it exists), but we do not take into consideration answers of the type "No opinion/No answer". If such a type of answer exists, we should recalculate the observed proportions excluding this answer and we proceed using the theory developed in the sequel.

In Maravelakis et al. (2003), three alternative indices were defined for the assessment of the degree of concentration on "positive" answers. Index I_1 was defined as

$$I_1 = \frac{\sum_{i=1}^{[k/2]} \pi_i}{\pi_0},$$

where $\pi_0 = [k/2] \cdot (1/k)$.

Index I_2, when k is odd, is given by

$$I_2 = \frac{\sum_{i=1}^{[k/2]} \pi_i}{\sum_{i=[k/2]+2}^{k} \pi_i},$$

whereas, when k is even, it becomes

$$I_2 = \frac{\sum_{i=1}^{[k/2]} \pi_i}{\sum_{i=[k/2]+1}^{k} \pi_i}.$$

Finally, in situations where k is odd, I_3 is defined as

$$I_3 = \frac{\sum_{i=1}^{[k/2]+1} \pi_i}{\sum_{i=[k/2]+1}^{k} \pi_i}.$$

The actual values of these indices can be estimated by $\widehat{I_1}$, $\widehat{I_2}$, $\widehat{I_3}$, which are defined by the above formulae by substituting p_i for π_i, $i = 1, \ldots, k$.

A drawback of these indices is that they give equal importance to all categories. This fact may cause some vagueness in the results since fixed sums of "positive" or "negative" answers lead to the same values of the indices considered, without taking into account how these sums are composed. This statement is clarified through a simple artificial example given below.

Suppose that the obtained proportions of answers in two questions are as given in Table I. The resulting estimates of the indices for both questions are: $\widehat{I_1} = 2$, $\widehat{I_2} = 16$ and $\widehat{I_3} = 4.75$, even though there exist substantial differences in the proportions of the "positive" categories. Actually, in the first case, the "positive" answers are mainly comprised of the answer

Table I. An artificial example

Question	Very Good	Good	Moderate	Bad	Very Bad
1	0.02	0.78	0.15	0.02	0.03
2	0.78	0.02	0.15	0.02	0.03

"Good", while, in the second case, of the answer "Very Good". Thus, one would expect larger index values for the second question since, despite the fact that the sum of the proportions of the two "positive" categories coincides with the corresponding sum of question 1, its much larger proportion in the "best" category (i.e. "Very Good") provides evidence of a stronger tendency of the respondents to select the "positive" answers. The source of this deficiency of I_1, I_2 and I_3 is related to the fact that they assign common weights (equal to unity) to all the components of the "positive" or "negative" categories and hence their values do not reflect changes in the values of each component when the sums of "positive" and "negative" answers are fixed.

3. The New Indices

Our purpose is to define new indices in order to overcome the problem of equal weights for all the possible answers. First, we introduce the methodology of computing the appropriate weights and afterwards we propose the new indices.

3.1. The Weights

Let

$$w_j,\ j=1,\dots,[k/2]$$

denote the weight of the jth category of the "positive" ("negative") answers. The value $j=1$ corresponds to the "best positive" and the "worst negative" answer and $j=[k/2]$ corresponds to the "worst positive" and the "best negative" answer. An appropriate set of weights must satisfy the following conditions:

1. $\sum_{j=1}^{[k/2]} w_j=[k/2]$
2. If the weight of the $[k/2]$ category is equal to a positive constant c, then the weight of the jth category is defined as $([k/2]-j+1)\,c$. Hence, the weights for the categories 1 through $[k/2]$ are: $[k/2]c$, $([k/2]-1)\,c$, $\dots$, c. Obviously, these weights satisfy the property

$$w_1-w_2=w_2-w_3=\cdots=w_{[k/2]-1}-w_{[k/2]}=c$$

and ensure that $w_1 \geq w_2 \geq \cdots \geq w_{[k/2]}$.

The first condition is imposed so as to ensure the comparability of the values of the new indices to those of the indices proposed by Maravelakis et al. (2003). This arises from the fact that the sum of weights in both cases equals $[k/2]$. The second condition ensures that the difference between the weights that correspond to any pair of equidistant categories is fixed. By this definition, the weights reflect the strength of (positive or negative) views as expressed by the responses. Responses reflecting extreme situations should naturally carry more weight. The more "distant" from the "neutral" a category is, the greater its influence should be on the overall evaluation of the situation based on the totality of responses. This is indeed achieved by the suggested weights.

These conditions lead to the following system of $[k/2]+1$ equations with unknowns $w_1, w_2, \ldots, w_{[k/2]}$ and c

$$\underset{([k/2]+1)\times[k/2]}{\mathbf{A}} \times \underset{[k/2]\times 1}{\mathbf{w}} = \underset{([k/2]+1)\times 1}{\mathbf{c}}, \tag{1}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix},$$

$\mathbf{w} = \begin{bmatrix} w_1 & w_2 & \cdots & w_{[k/2]} \end{bmatrix}^{\mathsf{T}}$ and $\mathbf{c} = \begin{bmatrix} c & c & \cdots & c & [k/2] \end{bmatrix}^{\mathsf{T}}$.

The system of equations in (1) has a unique solution given by

$$w_j = 2\left(\frac{[k/2]-j+1}{[k/2]+1}\right), \quad j = 1, \ldots, [k/2] \tag{2}$$

and

$$c = w_{[k/2]} = \frac{2}{[k/2]+1}.$$

For example, if $k = 7$, the weights from the "best" or the "worst" category to the closest to the "neutral" category are 6/4, 4/4 and 2/4, respectively. In the sequel, some generalizations of the indices I_1, I_2 and I_3 are considered based on the weights defined in (2).

3.2. The Index I_1^*

Taking advantage of the weights defined in (2), the index I_1 can be generalized to the form

$$I_1^* = \frac{\sum_{i=1}^{[k/2]} w_i \pi_i}{\pi_0}.$$

The values that I_1^* can take are in the interval $[0,\ w_1/\pi_0]$. The index takes a value close to 0 when only a few of the given answers belong to the $[k/2]$ "positive" categories. On the contrary, values of I_1^* proximal to w_1/π_0, indicate that the respondents have a tendency to select the "positive" categories and, specifically, the "best" of them. For example, in the case $k=5$, the index I_1^* lies within the interval [0,10/3].

Obviously, I_1^* takes finite positive values. In addition, it is easy to compute confidence intervals for this index, not only via bootstrap, but also by using some methods for simultaneous confidence intervals for multinomial proportions (see Section 5). A drawback of this index is that it takes no account of the "negative" and "neutral" answers, thus ignoring the information provided by them.

The relationship between I_1^* and I_1 is determined through the sign of the quantity $r_1 = \sum_{i=1}^{[k/2]} \pi_i (1 - w_i)$. Specifically, if r_1 is positive (negative), then $I_1^* < I_1$ $\left(I_1^* > I_1\right)$. Finally, $I_1^* = I_1$ if $r_1 = 0$.

3.3. The Index I_2^*

A generalisation of the index I_2 can be obtained by

$$I_2^* = \frac{\sum_{i=1}^{[k/2]} w_i \pi_i}{\sum_{i=[k/2]+2}^{k} w_{k-i+1} \pi_i},$$

assuming that k is odd. If k is even, I_2^* is defined as

$$I_2^* = \frac{\sum_{i=1}^{[k/2]} w_i \pi_i}{\sum_{i=[k/2]+1}^{k} w_{k-i+1} \pi_i}.$$

Index I_2^* takes values between 0 and infinity. When none of the respondents have chosen any of the $[k/2]$ "positive" answers, I_2^* takes the value 0. On the other hand, the value of the index cannot be computed (it becomes infinite) when none of the respondents has selected any of the $[k/2]$ "negative" answers. It should be noted that, although this is an extreme case, it is a disadvantage for the index. Another drawback of I_2^* is that it excludes the "neutral" category. Furthermore, a difficulty with the use of this index is the fact that construction of confidence intervals is not possible without resorting to the method of bootstrap, since it requires knowledge of the distribution of the ratio of two weighted sums of multinomial proportions (see Section 5). However, I_2^* is more informative than I_1^* because it takes into account "negative" answers and, at the same time, its calculation is fairly easy. The mathematical formulation describing the relationship between I_1^* and I_2^* is established in the sequel.

Let $r_2 = \left(\sum_{i=[k/2]+2}^{k} w_{k-i+1}\pi_i\right)/\pi_0$. Then $I_1^* < I_2^*$, provided that $r_2 < 1$. On the other hand, in the case where $r_2 > 1$, it can be seen that $I_1^* > I_2^*$. Finally, the two indices take the same value if $r_2 = 1$. These relationships hold when k is odd and can be easily modified for even values of k.

3.4. The Index I_3^*

The index I_3^*, defined subsequently, can be used in situations where the total number of answers is odd. In this case, since the "neutral" category is involved in the computation, we have to assign a weight for it. In particular, the appropriate weights arise from (2) by substituting the value $[k/2]+1$ for $[k/2]$, i.e.

$$w_j^{'} = 2\left(\frac{[k/2]-j+2}{[k/2]+2}\right), \quad j = 1, \dots, [k/2]+1.$$

Therefore, the new index is defined as

$$I_3^* = \frac{\sum_{i=1}^{[k/2]+1} w_i^{'}\pi_i}{\sum_{i=[k/2]+1}^{k} w_{k-i+1}^{'}\pi_i}.$$

Index I_3^* takes values that lie between 0 and infinity. The value 0 arises when all of the respondents have opted for a "negative" answer. On the other hand, I_3^* approaches infinity as the number of "positive" respondents increases. The fact that its value may tend to infinity is a drawback. However, it should be noted that this is not a probable scenario. Another disadvantage of I_3^* is that it is difficult to obtain confidence limits for its true value analytically. This problem can be overcome by using the bootstrap method (see Section 5). Index I_3^* surpasses a drawback of the indices I_1^* and I_2^* since it takes into account the "neutral" category.

3.5. Estimation

The actual values of the indices I_1^*, I_2^*, I_3^* can be estimated by

$$\widehat{I}_1^* = \frac{\sum_{i=1}^{[k/2]} w_i p_i}{\pi_0},$$

$$\widehat{I}_2^* = \frac{\sum_{i=1}^{[k/2]} w_i p_i}{\sum_{i=[k/2]+2}^{k} w_{k-i+1} p_i}, \text{ when } k \text{ is odd,}$$

$$\widehat{I}_2^* = \frac{\sum_{i=1}^{[k/2]} w_i p_i}{\sum_{i=[k/2]+1}^{k} w_{k-i+1} p_i}, \text{ when } k \text{ is even,}$$

and

$$\widehat{I}_3^* = \frac{\sum_{i=1}^{[k/2]+1} w_i^{'} p_i}{\sum_{i=[k/2]+1}^{k} w_{k-i+1}^{'} p_i},$$

respectively. Obviously, these estimators arise by substituting p_i for π_i in the expressions of the indices.

3.6. An Example

Let us now reconsider the example of Section 2 so as to clarify the estimation of the new indices and illustrate their superiority over I_1, I_2 and I_3. In the first question, the estimates of the new indices are:

$$\widehat{I}_1^* = \frac{(4/3)0.02 + (2/3)0.78}{0.4} = 1.3667,$$

$$\widehat{I}_2^* = \frac{(4/3)0.02 + (2/3)0.78}{(4/3)0.03 + (2/3)0.02} = 10.25$$

and

$$\widehat{I}_3^* = \frac{(6/4)0.02 + (4/4)0.78 + (2/4)0.15}{(6/4)0.03 + (4/4)0.02 + (2/4)0.15} = 6.3214,$$

respectively. The corresponding values for the second question are

$$\widehat{I}_1^* = \frac{(4/3)0.78 + (2/3)0.02}{0.4} = 2.6333,$$

$$\widehat{I}_2^* = \frac{(4/3)0.78 + (2/3)0.02}{(4/3)0.03 + (2/3)0.02} = 19.75,$$

$$\widehat{I}_3^* = \frac{(6/4)0.78 + (4/4)0.02 + (2/4)0.15}{(6/4)0.03 + (4/4)0.02 + (2/4)0.15} = 9.0353$$

and reflect the stronger tendency of the respondents to select the “positive” answers in the second question.

4. The Variances of the Estimators

In this section, the variances of the estimators of the three indices are assessed. For all the indices the method of bootstrap is implemented. Especially for $\widehat{I_1^*}$, a formula for finding the exact value of its variance is derived.

In the case of the index I_1^*, the parametric calculation is as follows.

Let $\mathbf{p}=\left[\, p_1\ p_2 \ldots p_k \,\right]^{\mathsf{T}}$ denote the vector of observed proportions of the k answers and $\boldsymbol{\pi}=\left[\, \pi_1\ \pi_2 \ldots \pi_k \,\right]^{\mathsf{T}}$ represent the corresponding true proportions. The unrestricted unbiased maximum likelihood estimator of $\boldsymbol{\pi}$ is given by $\mathbf{p}$ (see e.g. May and Johnson (1997)) and the covariance matrix of $\mathbf{p}$ is

$$\boldsymbol{\Sigma}=\frac{1}{N}\begin{bmatrix} \pi_1(1-\pi_1) & -\pi_1\pi_2 & \ldots & -\pi_1\pi_k \\ -\pi_1\pi_2 & \pi_2(1-\pi_2) & \ldots & -\pi_2\pi_k \\ \vdots & \vdots & \ddots & \vdots \\ -\pi_1\pi_k & -\pi_2\pi_k & \ldots & \pi_k(1-\pi_k) \end{bmatrix},$$

where N is the number of available answers. The unrestricted maximum likelihood estimator of $\boldsymbol{\Sigma}$ is computed by replacing π_i with p_i and is denoted by $\mathbf{S}$. Then,

$$\operatorname{Var}(\widehat{I_1^*})=\operatorname{Var}\left(\frac{\mathbf{w}^{\mathsf{T}}\mathbf{p}^*}{\pi_0}\right)=\frac{1}{\pi_0^2}\mathbf{w}^{\mathsf{T}}\operatorname{Var}(\mathbf{p}^*)\mathbf{w}=\frac{1}{\pi_0^2}\mathbf{w}^{\mathsf{T}}\boldsymbol{\Sigma}^*\mathbf{w}, \tag{3}$$

where $\boldsymbol{\Sigma}^*$ is a partition of $\boldsymbol{\Sigma}$ containing the first $[k/2]$ rows and columns of $\boldsymbol{\Sigma}$ and $\mathbf{p}^*=\left[\, p_1\ p_2 \ldots p_{[k/2]} \,\right]^{\mathsf{T}}$. For instance, when $k=7$, the expression given in (3) simplifies to

$$\operatorname{Var}(\widehat{I_1^*})=-\frac{49}{36N}(-9\pi_1+9\pi_1^2+12\pi_1\pi_2+6\pi_1\pi_3-4\pi_2+4\pi_2^2 \\ +4\pi_3\pi_2-\pi_3+\pi_3^2).$$

An estimate of $\operatorname{Var}(\widehat{I_1^*})$ can be obtained by replacing π_i $(i=1,\ldots,k)$ by their sample counterparts.

The derivation of exact formulae for the variance of the estimators of the indices I_2^* and I_3^* is a difficult task since these are ratios of weighted sums of multinomial proportions. However, we may approximate the value of the variance for particular choices of $\boldsymbol{\pi}$ and N using the method of bootstrap. (A detailed description of this method and its applications can be found in Efron and Tibshirani (1993).

In the sequel, the method of bootstrap is implemented for the approximation of the variances of the estimators of the three new indices for various choices of $\boldsymbol{\pi}$ and N. Specifically, assuming that we have a question with k possible answers, N observations and proportions $\pi_1, \pi_2, \ldots, \pi_k$, we

Table II. The variances of the estimators of the three indices for $B = 1000$ and $N = 50$

							$\widehat{I_1^*}$		$\widehat{I_2^*}$	$\widehat{I_3^*}$
Proportions							E	A	A	A
0.03	0.02	0.05	0.45	0.05	0.20	0.20	0.0100	0.0093	0.0091	0.0093
0.06	0.06	0.08	0.40	0.10	0.15	0.15	0.0195	0.0187	0.0407	0.0265
0.10	0.10	0.10	0.35	0.15	0.10	0.10	0.0283	0.0287	0.1707	0.0680
0.15	0.15	0.10	0.30	0.10	0.10	0.10	0.0361	0.0376	0.4012	0.1484
0.20	0.15	0.15	0.20	0.10	0.10	0.10	0.0394	0.0408	0.8039	0.3135
0.20	0.20	0.20	0.10	0.10	0.10	0.10	0.0370	0.0378	0.7993	0.4863
0.25	0.25	0.20	0.10	0.10	0.05	0.05	0.0367	0.0372	5.7835	2.1036
0.25	0.25	0.30	0.10	0.05	0.03	0.02	0.0312	0.0326	–	19.148
0.30	0.30	0.30	0.05	0.02	0.02	0.01	0.0261	0.0261	–	–

generate a large number of multinomial samples, say $B = 1000$, via sampling with replacement. The B samples are termed bootstrap samples. For each bootstrap sample, the value of the index I^* is calculated. The general notation I^* is used here to denote any of the three new indices. An approximation of the variance for the estimator of each index ($S^2_{I^*}$) can be found through the formula

$$S^2_{I^*} = \frac{1}{B-1} \sum_{i=1}^{B} \left(I_i^* - \overline{I}^* \right)^2 ,$$

where I_i^* is the index value assessed on the basis of the ith bootstrap sample and $\overline{I}^*$ is the mean of the B bootstrap index values.

The obtained results, assuming $k = 7$, are summarized in Tables II–V. Each of these tables corresponds to a different sample size (N). Moreover, the proportions considered were selected to cover a wide range of cases, i.e. small, moderate or large index values.

In the case of I_1^*, the bootstrap approximations (A) can be compared with the exact ones (E) computed using formula (3).

From Tables II–V one may draw the following conclusions:

- As the sample size increases, the variance of all the estimators decreases
- The variance of $\widehat{I_1^*}$ appears to be generally smaller in comparison to the variances of the estimators of the other indices
- The variance of $\widehat{I_1^*}$ is not seriously affected by changes in the values of the proportions
- The variances of $\widehat{I_2^*}$ and $\widehat{I_3^*}$ increase as the degree of concentration on the "positive" answers increases

Table III. The variances of the estimators of the three indices for $B = 1000$ and $N = 100$

							$\widehat{I_1^*}$		$\widehat{I_2^*}$	$\widehat{I_3^*}$
Proportions							E	A	A	A
0.03	0.02	0.05	0.45	0.05	0.20	0.20	0.0050	0.0049	0.0045	0.0047
0.06	0.06	0.08	0.40	0.10	0.15	0.15	0.0097	0.0102	0.0190	0.0127
0.10	0.10	0.10	0.35	0.15	0.10	0.10	0.0142	0.0143	0.0777	0.0347
0.15	0.15	0.10	0.30	0.10	0.10	0.10	0.0181	0.0181	0.1683	0.0696
0.20	0.15	0.15	0.20	0.10	0.10	0.10	0.0197	0.0190	0.2182	0.1108
0.20	0.20	0.20	0.10	0.10	0.10	0.10	0.0185	0.0189	0.2717	0.1789
0.25	0.25	0.20	0.10	0.10	0.05	0.05	0.0183	0.0183	1.8565	0.8036
0.25	0.25	0.30	0.10	0.05	0.03	0.02	0.0156	0.0160	26.303	4.9743
0.30	0.30	0.30	0.05	0.02	0.02	0.01	0.0131	0.0135	–	76.386

Table IV. The variances of the estimators of the three indices for $B = 1000$ and $N = 250$

							$\widehat{I_1^*}$		$\widehat{I_2^*}$	$\widehat{I_3^*}$
Proportions							E	A	A	A
0.03	0.02	0.05	0.45	0.05	0.20	0.20	0.0020	0.0020	0.0017	0.0018
0.06	0.06	0.08	0.40	0.10	0.15	0.15	0.0039	0.0035	0.0066	0.0046
0.10	0.10	0.10	0.35	0.15	0.10	0.10	0.0057	0.0055	0.0261	0.0125
0.15	0.15	0.10	0.30	0.10	0.10	0.10	0.0072	0.0070	0.0578	0.0257
0.20	0.15	0.15	0.20	0.10	0.10	0.10	0.0079	0.0080	0.0868	0.0452
0.20	0.20	0.20	0.10	0.10	0.10	0.10	0.0074	0.0068	0.0951	0.0655
0.25	0.25	0.20	0.10	0.10	0.05	0.05	0.0073	0.0072	0.6194	0.2827
0.25	0.25	0.30	0.10	0.05	0.03	0.02	0.0063	0.0061	6.3119	1.5515
0.30	0.30	0.30	0.05	0.02	0.02	0.01	0.0052	0.0053	79.097	16.941

- The performance of the bootstrap method is fairly satisfactory as can be observed from the differences between the exact and the approximate (bootstrap) values of the variance of $\widehat{I_1^*}$
- For small sample sizes, the approximations of the variances of $\widehat{I_2^*}$ and $\widehat{I_3^*}$ cannot be obtained in situations where the proportions of the "positive" answers are very large (see the last rows of Tables II and III). This is a consequence of the fact that the values of these indices become infinite for some of the bootstrap samples.

Table V. The variances of the estimators of the three indices for $B = 1000$ and $N = 500$

							$\widehat{I}_1^*$		$\widehat{I}_2^*$	$\widehat{I}_3^*$
Proportions							*E*	*A*	*A*	*A*
0.03	0.02	0.05	0.45	0.05	0.20	0.20	0.0010	0.0010	0.0009	0.0010
0.06	0.06	0.08	0.40	0.10	0.15	0.15	0.0020	0.0019	0.0036	0.0025
0.10	0.10	0.10	0.35	0.15	0.10	0.10	0.0028	0.0027	0.0129	0.0062
0.15	0.15	0.10	0.30	0.10	0.10	0.10	0.0036	0.0037	0.0270	0.0124
0.20	0.15	0.15	0.20	0.10	0.10	0.10	0.0039	0.0039	0.0420	0.0225
0.20	0.20	0.20	0.10	0.10	0.10	0.10	0.0037	0.0038	0.0504	0.0348
0.25	0.25	0.20	0.10	0.10	0.05	0.05	0.0037	0.0040	0.2856	0.1391
0.25	0.25	0.30	0.10	0.05	0.03	0.02	0.0031	0.0031	2.7278	0.7130
0.30	0.30	0.30	0.05	0.02	0.02	0.01	0.0026	0.0026	27.230	7.1804

5. Confidence Intervals

This section is devoted to the construction of confidence intervals for the true values of the indices defined. Owing to the fact that these indices are functions of multinomial proportions, the construction of confidence intervals for them relates to the construction of simultaneous confidence limits for multinomial proportions. This is a problem dealt with by several authors (e.g. Quesenberry and Hurst, 1964; Goodman, 1965; Fitzpatrick and Scott, 1987; Sison and Glaz, 1995; Kwong, 1996, 1998; Ahmed, 2000). However, these confidence limits can be used only in the case of I_1^*. The construction of parametric confidence intervals for indices I_2^* and I_3^*, which are ratios of weighted sums of multinomial proportions, is much more complicated. For this reason, we resort to the method of bootstrap for obtaining confidence intervals for them. A $100(1-a)\%$ confidence interval of index I_1^* is given by

$$\left(\frac{\sum_{i=1}^{[k/2]} w_i p_L^{(i)}}{\pi_0}, \frac{\sum_{i=1}^{[k/2]} w_i p_U^{(i)}}{\pi_0} \right), \tag{4}$$

where $p_L^{(i)}$, $p_U^{(i)}$ are the lower and the upper simultaneous confidence limits for category i calculated using any of the suggested methods.

Alternatively, one may take advantage of the bootstrap method so as to assess confidence intervals for the actual values of the indices I_1^*, I_2^* and I_3^*. For simplicity, we adopt again the general notation I^* for any of these indices. For the calculation of bootstrap confidence intervals we order the

B index values, obtained following the procedure described in the previous section, in a non-descending order and we denote the ith of these values by

$$I^*_{(i)},\ i = 1, \dots, B.$$

We will now describe three alternative methods that one can apply in order to create bootstrap confidence intervals. These are the standard bootstrap, the percentile bootstrap and the bias-corrected percentile bootstrap.

According to the standard bootstrap method, a $100(1-\alpha)\%$ confidence interval for the index I^* is given by

$$\left(\widehat{I^*} - z_{1-\alpha/2} S_{I^*}, \widehat{I^*} + z_{1-\alpha/2} S_{I^*}\right),$$

where z_α denotes the 100α percentile of the standard normal distribution, S_{I^*} is the standard deviation of the B index values and $\widehat{I^*}$ is the index value that was assessed from the initial sample.

Following the percentile bootstrap technique, a $100(1-\alpha)\%$ confidence interval for the index I^* is given by

$$\left(I^*_{(B\alpha/2)},\ I^*_{(B(1-\alpha/2))}\right).$$

The bias-corrected percentile bootstrap method is similar to the percentile bootstrap, but involves a slight correction for the potential bias. According to this method, we firstly find the two successive values $I^*_{(i)}$ and $I^*_{(i+1)}$ between which the value of the index that was assessed from the initial sample ($\widehat{I^*}$) lies. Then, we derive the value for which the cumulative distribution function of the standard normal distribution (Φ) takes the value i/B. If we denote this value by z_0, then we calculate the probabilities p_l and p_u, which are defined as $p_l = \Phi(2z_0 + z_{\alpha/2})$ and $p_u = \Phi(2z_0 + z_{1-\alpha/2})$. Using these probabilities we end up with a $100(1-\alpha)\%$ confidence interval of the form

$$\left(I^*_{(B\cdot p_l)},\ I^*_{(B\cdot p_u)}\right).$$

The performance of the three bootstrap techniques is examined via a simulation study. The results obtained are provided in the Appendix. In this study, 10,000 random samples were generated from the multinomial distribution with parameters $N = 250$ and 500 and various combinations of $\pi_1, \pi_2, \dots, \pi_7$. These combinations are the same as those considered in Section 4. The number k of the selected categories was assumed to be 7, without loss of generality.

The number of bootstrap samples generated each time is $B = 1000$. For any case, the observed coverage (OC) and the mean range (MR) are computed. Tables X and XII refer to confidence level 0.90, whereas Tables XI and XIII refer to confidence level 0.95. The first entry of each cell corresponds to the standard bootstrap (SB) method, the second to the percentile bootstrap (PB) and the third to the bias-corrected percentile bootstrap (BB).

On the basis of these tables one may conclude that:

- The observed coverage is not seriously affected by the sample size. Hence, one may construct confidence intervals for the true values of the indices even when the number of available observations is not very large
- The mean range of the confidence intervals produced from all the techniques reduces as the sample size increases
- For the index I_1^*, all the methods appear to attain a coverage close to the nominal. Likewise, the mean range of the confidence intervals produced from the three methods is nearly the same
- In the case of index I_2^*, despite the fact that we do not observe substantial differences in the coverage of the three methods, BB seems to provide the confidence intervals with the best coverage. The mean range of the SB confidence intervals appears to exceed the ones of the other two methods
- For index I_3^*, method BB results in a coverage closer to the nominal in most of the examined cases. In addition, method SB gives the widest intervals

It should be remarked that the mean range of the confidence intervals I_2^* and I_3^* could not be computed in some cases because, for some of the generated bootstrap samples, the values of these indices become infinite.

6. An Illustrative Example

In the sequel, the data analyzed by Jensen (1986) are used in order to illustrate the advantages of the indices I_1^*, I_2^* and I_3^* proposed in this paper in comparison to the indices I_1, I_2 and I_3 suggested by Maravelakis et al. (2003). Jensen (1986) dealt with data acquired through a questionnaire that was given between 1973 and 1976 to 60% of the students of the only Catholic high school and its two neighboring public high schools in a southeastern city of the United States. For more details on this survey, the reader is referred to Jensen (1986). The questionnaires that were given to the students include several questions whose answers have a natural ordering. Therefore, one can take advantage of the theory developed in this paper so as to measure the observed degree of concentration on the "positive" categories in each question.

In Table IV of Jensen (1986), the obtained results for various questions associated with students's choices, moral evaluations and perceptions of risk are provided separately for the students of public and catholic schools. Some of these questions are:

Table VI. The proportions and the number of responses for questions 1 and 2 for Jensen's (1986) data

Question	DY	Y	U	NO	DN	N
1	0.473	0.327	0.086	0.065	0.049	1480
2	0.623	0.243	0.052	0.030	0.052	440

Table VII. Estimates of the values of the six indices for questions 1 and 2

Question	$\widehat{I_1}$	$\widehat{I_1^*}$	$\widehat{I_2}$	$\widehat{I_2^*}$	$\widehat{I_3}$	$\widehat{I_3^*}$
1	2	2.122	7.018	7.810	4.430	5.948
2	2.165	2.482	10.561	11.112	6.851	8.981

1. Suppose you and your friends were messing around one afternoon and they decided to steal something from a store just for kicks. Do you think it would be wrong to go along? (Public schools)
2. Suppose you and your friends were messing around one night and they decided to break into a place and steal some things. Would it be wrong to go along? (Catholic schools)

In both questions the possible answers were "Definitely Yes" (DY), "Yes" (Y), "Uncertain" (U), "No" (NO), "Definitely No" (DN).

The observed proportions and the number of responses for these questions are displayed in Table VI, while in Table VII the corresponding estimates of the values of the six indices are presented.

In both questions, the values of the six indices indicate a tendency of the respondents to prefer the "positive" answers. Likewise, this tendency seems to be stronger in question 2.

Suppose now that the obtained proportions of the two questions were as shown in Table VIII. Obviously, in this case the proportions of the first two categories are given in reverse order. However, as one may observe from Table IX, the indices I_1, I_2 and I_3 do not reflect these changes since their values remain unchanged. On the other hand, in both cases, the values of the new indices decreased as a consequence of the fact that even though the total proportion of the "positive" answers remains constant its distribution to the two categories has changed substantially.

7. Concluding Remarks

In this paper, some new indices were introduced and their properties were studied. These indices can be considered as generalizations of the indices proposed by Maravelakis et al. (2003). Their aim is to measure the degree

Table VIII. The proportions and the number of responses for questions 1* and 2*

Question	DY	Y	U	NO	DN	*N*
1*	0.327	0.473	0.086	0.065	0.049	1480
2*	0.243	0.623	0.052	0.030	0.052	440

Table IX. Estimates of the values of the six indices for questions 1* and 2*

Question	$\widehat{I}_1$	$\widehat{I}_1^*$	$\widehat{I}_2$	$\widehat{I}_2^*$	$\widehat{I}_3$	$\widehat{I}_3^*$
1*	2	1.878	7.018	6.914	4.430	5.545
2*	2.165	1.848	10.561	8.276	6.851	7.563

of concentration on the "small" or the "large" values of ordinal variables and have applications in various disciplines. The use of these indices was illustrated in connection with data obtained from sample surveys. Nevertheless, various other fields of applications where these indices may serve as a useful tool, exist. As an example, we refer to the educational field and especially the evaluation of different groups of students in situations where their grades are in ordinal scale.

As already mentioned, indices have been used in a number of fields. A natural question in using weighted indices is how the weights are chosen. In Statistical Process Control the Exponentially Weighted Moving Average (EWMA) control chart is a statistic (index) of the current level of a process. The weights in this statistic decrease geometrically, assigning the largest weight to the most recent observation and a continuously decreasing weight from the next most recent observation to the oldest (see, e.g. Montgomery, 2001). This selection of weights stems from the fact that the newer observation gives the best outlook of the process. In the education field different weights are assigned to the factors related to the quality of education. These weights are not based on a mathematical formulation but rather on a subjective selection (see Han, 1996). In the areas of classification and clustering different indices have been proposed with various types of weighting. The criteria for selecting these weights is based on the analyst (Cox and Cox, 2000). Therefore, one may conclude that the set of weights chosen in each subject depends on the nature of the problem and even for the same problem different weights may be assigned. In the problem studied, the set of weights used are based on mathematical relations with the aim to arrive at a logical selection.

Specifically, the first condition for selecting the appropriate weights in Section 3 is not a binding one because the sum of the weights could be any value. We choose the particular one for comparison purposes. On the other hand, the second condition is crucial on the selection of the weights. In the case of a questionnaire it seems natural that the weights of symmetrical classes be equal, although there may be cases where the researcher may decide otherwise.

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Appendix

Table X. The simulation study for $N = 250$ and $1 - \alpha = 0.9$

Proportions								I_1^*		I_2^*		I_3^*	
								OC	MR	OC	MR	OC	MR
							SB	0.8880	0.1459	0.8975	0.1400	0.9015	0.1435
0.03	0.02	0.05	0.45	0.05	0.20	0.20	PB	0.8928	0.1456	0.8918	0.1390	0.8962	0.1430
							BB	0.8889	0.1453	0.8923	0.1402	0.8968	0.1432
							SB	0.8938	0.2040	0.9012	0.2839	0.9000	0.2337
0.06	0.06	0.08	0.40	0.10	0.15	0.15	PB	0.9018	0.2038	0.8948	0.2817	0.8959	0.2327
							BB	0.9014	0.2036	0.8961	0.2827	0.8969	0.2330
							SB	0.9003	0.2466	0.9121	0.5440	0.9028	0.3703
0.10	0.10	0.10	0.35	0.15	0.10	0.10	PB	0.9070	0.2465	0.8991	0.5385	0.9005	0.3685
							BB	0.9062	0.2464	0.9021	0.5387	0.8998	0.3687
							SB	0.8972	0.2789	0.9176	0.8095	0.9102	0.5342
0.15	0.15	0.10	0.30	0.10	0.10	0.10	PB	0.8957	0.2789	0.9023	0.7995	0.9017	0.5308
							BB	0.8954	0.2789	0.9038	0.7982	0.9028	0.5307
							SB	0.9028	0.2912	0.9150	0.9616	0.9111	0.6932
0.20	0.15	0.15	0.20	0.10	0.10	0.10	PB	0.9026	0.2912	0.8989	0.9491	0.9003	0.6878
							BB	0.9023	0.2913	0.9005	0.9464	0.9024	0.6872
							SB	0.9011	0.2822	0.9147	1.0674	0.9111	0.8789
0.20	0.20	0.20	0.10	0.10	0.10	0.10	PB	0.9076	0.2822	0.8980	1.0537	0.8983	0.8706
							BB	0.9065	0.2823	0.9004	1.0500	0.9009	0.8687
							SB	0.9000	0.2812	0.9228	2.6657	0.9131	1.7879
0.25	0.25	0.20	0.10	0.10	0.05	0.05	PB	0.9005	0.2812	0.8927	2.6045	0.8921	1.7621
							BB	0.8991	0.2814	0.8941	2.5788	0.8927	1.7529
							SB	0.8977	0.2592	0.9399	9.0278	0.9271	4.2237
0.25	0.25	0.30	0.10	0.05	0.03	0.02	PB	0.8977	0.2592	0.8917	8.4496	0.8925	4.1226
							BB	0.8969	0.2594	0.8943	8.2090	0.8949	4.0723
							SB	0.8965	0.2372	0.8653	–	0.9385	14.703
0.30	0.30	0.30	0.05	0.02	0.02	0.01	PB	0.9028	0.2372	0.8784	–	0.8873	13.727
							BB	0.9036	0.2375	0.8871	–	0.8916	13.275

Table XI. The simulation study for $N = 250$ and $1 - \alpha = 0.95$

Proportions								I_1^*		I_2^*		I_3^*	
								OC	MR	OC	MR	OC	MR
							SB	0.9386	0.1739	0.9387	0.1668	0.9463	0.1710
0.03	0.02	0.05	0.45	0.05	0.20	0.20	PB	0.9431	0.1732	0.9432	0.1621	0.9445	0.1706
							BB	0.9420	0.1729	0.9473	0.1675	0.9452	0.1710
							SB	0.9414	0.2431	0.9464	0.3383	0.9468	0.2786
0.06	0.06	0.08	0.40	0.10	0.15	0.15	PB	0.9487	0.2426	0.9448	0.3372	0.9465	0.2779
							BB	0.9456	0.2424	0.9460	0.3384	0.9481	0.2782
							SB	0.9480	0.2939	0.9499	0.6483	0.9510	0.4413
0.10	0.10	0.10	0.35	0.15	0.10	0.10	PB	0.9504	0.2937	0.9488	0.6459	0.9496	0.4403
							BB	0.9494	0.2935	0.9492	0.6461	0.9493	0.4405
							SB	0.9462	0.3324	0.9584	0.9646	0.9551	0.6366
0.15	0.15	0.10	0.30	0.10	0.10	0.10	PB	0.9465	0.3321	0.9508	0.9605	0.9512	0.6348
							BB	0.9471	0.3320	0.9504	0.9590	0.9517	0.6346
							SB	0.9504	0.3470	0.9560	1.1458	0.9542	0.8261
0.20	0.15	0.15	0.20	0.10	0.10	0.10	PB	0.9508	0.3468	0.9502	1.1406	0.9493	0.8237
							BB	0.9508	0.3468	0.9499	1.1376	0.9495	0.8227
							SB	0.9482	0.3363	0.9559	1.2719	0.9540	1.0473
0.20	0.20	0.20	0.10	0.10	0.10	0.10	PB	0.9509	0.3362	0.9482	1.2665	0.9499	1.0438
							BB	0.9508	0.3363	0.9501	1.2621	0.9501	1.0416
							SB	0.9501	0.3350	0.9587	3.1764	0.9549	2.1304
0.25	0.25	0.20	0.10	0.10	0.05	0.05	PB	0.9495	0.3350	0.9442	3.1514	0.9457	2.1205
							BB	0.9478	0.3352	0.9447	3.1193	0.9460	2.1090
							SB	0.9467	0.3089	0.9624	10.757	0.9605	5.0329
0.25	0.25	0.30	0.10	0.05	0.03	0.02	PB	0.9469	0.3088	0.9397	10.461	0.9439	4.9899
							BB	0.9466	0.3090	0.9432	10.155	0.9456	4.9291
							SB	0.9484	0.2826	0.8870	–	0.9636	17.519
0.30	0.30	0.30	0.05	0.02	0.02	0.01	PB	0.9530	0.2825	0.9333	–	0.9378	17.024
							BB	0.9518	0.2828	0.9400	–	0.9421	16.442

Table XII. The simulation study for $N = 500$ and $1 - \alpha = 0.9$

Proportions								I_1^*		I_2^*		I_3^*	
								OC	MR	OC	MR	OC	MR
							SB	0.8962	0.1035	0.8990	0.0978	0.8979	0.1008
0.03	0.02	0.05	0.45	0.05	0.20	0.20	PB	0.8962	0.1034	0.8954	0.0975	0.8981	0.1006
							BB	0.8965	0.1033	0.8965	0.0979	0.8976	0.1007
							SB	0.9000	0.1448	0.9058	0.1981	0.9045	0.1644
0.06	0.06	0.08	0.40	0.10	0.15	0.15	PB	0.9045	0.1447	0.9016	0.1972	0.9024	0.1640
							BB	0.9030	0.1447	0.9017	0.1977	0.9012	0.1642
							SB	0.8998	0.1746	0.9018	0.3748	0.8991	0.2589
0.10	0.10	0.10	0.35	0.15	0.10	0.10	PB	0.9027	0.1746	0.8961	0.3728	0.8982	0.2582
							BB	0.9030	0.1746	0.8975	0.3730	0.8993	0.2584
							SB	0.8999	0.1974	0.9062	0.5555	0.9021	0.3728
0.15	0.15	0.10	0.30	0.10	0.10	0.10	PB	0.9029	0.1973	0.8983	0.5522	0.8995	0.3716
							BB	0.9003	0.1974	0.8996	0.5521	0.8998	0.3717
							SB	0.9021	0.2062	0.9085	0.6596	0.9044	0.4818
0.20	0.15	0.15	0.20	0.10	0.10	0.10	PB	0.9054	0.2063	0.9003	0.6553	0.8987	0.4799
							BB	0.9036	0.2063	0.8990	0.6548	0.8976	0.4798
							SB	0.8938	0.1998	0.9051	0.7333	0.9026	0.6086
0.20	0.20	0.20	0.10	0.10	0.10	0.10	PB	0.8989	0.1998	0.8947	0.7286	0.8958	0.6056
							BB	0.8992	0.2000	0.8961	0.7279	0.8976	0.6052
							SB	0.8983	0.1990	0.9067	1.7864	0.9012	1.2241
0.25	0.25	0.20	0.10	0.10	0.05	0.05	PB	0.9013	0.1989	0.8908	1.7672	0.8918	1.2157
							BB	0.9011	0.1991	0.8917	1.7601	0.8938	1.2132
							SB	0.8999	0.1836	0.9265	5.5162	0.9129	2.8152
0.25	0.25	0.30	0.10	0.05	0.03	0.02	PB	0.8988	0.1836	0.9002	5.3798	0.8991	2.7828
							BB	0.8982	0.1837	0.9015	5.3161	0.9003	2.7683
							SB	0.9013	0.1679	0.9371	19.016	0.9219	8.9700
0.30	0.30	0.30	0.05	0.02	0.02	0.01	PB	0.9048	0.1679	0.8841	17.789	0.8912	8.7384
							BB	0.9025	0.1681	0.8895	17.267	0.8947	8.6164

Table XIII. The simulation study for $N = 500$ and $1 - \alpha = 0.95$

Proportions								I_1^*		I_2^*		I_3^*	
								OC	MR	OC	MR	OC	MR
							SB	0.9414	0.1234	0.9473	0.1166	0.9507	0.1201
0.03	0.02	0.05	0.45	0.05	0.20	0.20	PB	0.9432	0.1231	0.9499	0.1163	0.9497	0.1200
							BB	0.9437	0.1230	0.9479	0.1168	0.9473	0.1201
							SB	0.9487	0.1725	0.9487	0.2361	0.9532	0.1959
0.06	0.06	0.08	0.40	0.10	0.15	0.15	PB	0.9510	0.1722	0.9493	0.2357	0.9510	0.1957
							BB	0.9513	0.1722	0.9505	0.2361	0.9510	0.1958
							SB	0.9484	0.2081	0.9486	0.4466	0.9494	0.3085
0.10	0.10	0.10	0.35	0.15	0.10	0.10	PB	0.9502	0.2079	0.9485	0.4457	0.9482	0.3081
							BB	0.9504	0.2079	0.9489	0.4459	0.9499	0.3083
							SB	0.9484	0.2353	0.9532	0.6620	0.9510	0.4442
0.15	0.15	0.10	0.30	0.10	0.10	0.10	PB	0.9512	0.2351	0.9491	0.6604	0.9483	0.4436
							BB	0.9500	0.2352	0.9481	0.6601	0.9466	0.4436
							SB	0.9493	0.2458	0.9537	0.7860	0.9519	0.5741
0.20	0.15	0.15	0.20	0.10	0.10	0.10	PB	0.9523	0.2456	0.9483	0.7840	0.9492	0.5731
							BB	0.9511	0.2457	0.9494	0.7832	0.9501	0.5729
							SB	0.9474	0.2382	0.9539	0.8737	0.9521	0.7252
0.20	0.20	0.20	0.10	0.10	0.10	0.10	PB	0.9506	0.2381	0.9466	0.8718	0.9472	0.7239
							BB	0.9483	0.2382	0.9456	0.8707	0.9463	0.7236
							SB	0.9458	0.2371	0.9545	2.1287	0.9522	1.4586
0.25	0.25	0.20	0.10	0.10	0.05	0.05	PB	0.9482	0.2370	0.9438	2.1205	0.9450	1.4547
							BB	0.9484	0.2371	0.9436	2.1118	0.9457	1.4517
							SB	0.9498	0.2188	0.9616	6.5728	0.9548	3.3545
0.25	0.25	0.30	0.10	0.05	0.03	0.02	PB	0.9489	0.2186	0.9481	6.5214	0.9484	3.3410
							BB	0.9486	0.2188	0.9489	6.4418	0.9495	3.3229
							SB	0.9517	0.2001	0.9604	22.658	0.9598	10.689
0.30	0.30	0.30	0.05	0.02	0.02	0.01	PB	0.9529	0.2000	0.9398	22.046	0.9410	10.597
							BB	0.9541	0.2002	0.9436	21.374	0.9425	10.448