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Abstract

This paper provides a tractable continuous-time constant-absolute-risk averse (CARA)-Gaussian framework to quantitatively explore how the preference for robustness (RB) affects the interest rate, the dynamics of consumption and income, and the welfare costs of model uncertainty in general equilibrium. We show that RB significantly reduces the equilibrium interest rate, and reduces (increases) the relative volatility of consumption growth to income growth when the income process is stationary (non-stationary). Furthermore, we find that the welfare costs of model uncertainty are nontrivial for plausibly estimated income processes and calibrated RB parameter values. Finally, we extend the benchmark model to consider the separation of risk aversion and intertemporal substitution, incomplete information about income, and regime-switching in income growth.

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1. Introduction

Hansen and Sargent (1995) first formally introduced the preference for robustness (RB, a concern for model misspecification) into linear-quadratic-Gaussian (LQG) economic models.[1] In robust control problems, agents are concerned about the possibility that their true model is misspecified in a manner that is difficult to detect statistically; consequently, they make their optimal decisions as if the subjective distribution over shocks is chosen by an evil agent in order to minimize their expected lifetime utility.[2] As showed in Hansen, Sargent, and Tallarini (HST, 1999) and Luo and Young (2010), robustness models can produce precautionary savings even within the class of discrete-time LQG models, which leads to analytical simplicity. Specifically, using the explicit consumption-saving rules, they explored how RB affects consumption and saving decisions and found that the preference for robustness and the discount factor are observationally equivalent in the sense that they lead to the identical consumption and saving decisions within the discrete-time representative-agent LQG setting. However, if we consider problems outside the LQG setting (e.g., when the utility function is constant-absolute-risk-averse, i.e., CARA, or constant-relative-risk-averse, i.e., CRRA), RB-induced worst-case distributions are generally non-Gaussian, which greatly complicates the computational task.[3]

The permanent income hypothesis (PIH) of Friedman states that the individual consumer's optimal consumption is determined by permanent income that equals the annuity value of his total resources: the sum of (i) financial wealth and (ii) human wealth defined as the discounted present value of the current and expected future labor income using the exogenously given risk-free rate. Hall (1978) showed that when some restrictions are imposed (e.g., quadratic utility and the equality between the interest rate and the discount rate), the PIH emerges and changes in consumption are unpredictable. Consequently, the PIH consumer saves only when he anticipates that their future labor income will decline. This saving motive is called the demand for "savings for a rainy day". In contrast, Caballero (1990) examined a precautionary saving motive due to the interaction of risk aversion and unpredictable future income uncertainty when the consumer has CARA utility. The Caballero model leads to a constant precautionary savings demand and a constant dissavings term due to relative impatience. Wang (2003) showed in a Bewley-Caballero-Huggett equilibrium model that the precautionary saving demand and the impatience dissavings term cancel out in a general equilibrium and the PIH reemerges.

[1]See Hansen and Sargent (2007) for a textbook treatment on robustness.

[2]The solution to a robust decision-maker's problem can be regarded as the equilibrium of a max-min game between the decision-maker and the evil agent.

[3]See Chapter 1 of Hansen and Sargent (2007) for discussions on the computational difficulties in solving non-LQG RB models, and Colacito, Hansen, and Sargent (2007), Bidder and Smith (2012), and Young (2012) for using numerical methods to compute the worst-case distributions.

The main goal of this paper is to construct a tractable continuous-time CARA-Gaussian heterogenous-agent dynamic stochastic general equilibrium (DSGE) model to link the two research lines discussed above and explore how robustness affects the interest rate, the cross-sectional distributions of consumption and income, and welfare costs of model uncertainty in the presence of uninsurable labor income.[4] As the first contribution of this paper, we show that this continuous-time DSGE model featuring incomplete markets and the separation of risk aversion and robustness can be solved explicitly. Using the explicit consumption-saving rules, we find that risk aversion is more important than robustness in determining the precautionary savings demand.[5] In addition, we establish the observational equivalence results between risk aversion, robustness, and discounting in our continuous-time model.

Second, using the explicit decision rules, we show that a general equilibrium under RB can be constructed in the vein of Bewley (1986) and Huggett (1993).[6] In the general equilibrium, we find that the interest rate decreases with the degree of RB. The intuition is that the stronger the preference for RB, the greater the amount of model uncertainty determined by the interaction of risk aversion, RB, and labor income uncertainty, and the less the interest rate. In addition, we show that the relative volatility of consumption growth to income growth is determined by the interaction of the equilibrium interest rate and the persistence coefficient of the income process. Specifically, this relative volatility decreases (increases) with RB when the income process is stationary (non-stationary).

Third, after calibrating the RB parameter using the detection error probabilities (DEP), we find that RB has significant impacts on the equilibrium interest rate and consumption volatility. In the U.S. economy the average real risk-free interest rate is only about 1 percent between 1985 and 2014. The full-information rational expectations model requires the coefficient of risk aversion parameter to be 10 to match this rate.[7] In contrast, when consumers take into account model uncertainty, the model can generate an equilibrium interest rate of 1 percent with much lower values of the coefficient of risk aversion.[8] In addition, we find that when income uncertainty

[4]See Cagetti, Hansen, Sargent, and Williams (2002), Anderson, Hansen, and Sargent (2003), Maenhout (2004), and Kasa (2006) for the applications of robustness in continuous-time models.

[5]Within the discrete-time LQG setting, Luo, Nie, and Young (2012) showed that although both RB and CARA preferences increase the precautionary savings demand via the intercept terms in the consumption functions, they have distinct implications for the marginal propensity to consume out of permanent income (MPC).

[6]Wang (2003) constructed a general equilibrium under full-information rational expectations (FI-RE) in the same Bewley-Huggett type model economy with the CARA utility.

[7]Note that since we set the mean income level to be 1, the coefficient of relative risk aversion (CRRA) evaluated at this level is equal to the coefficient of absolute risk aversion (CARA).

[8]Barillas, Hansen, and Sargent (2009) showed that most of the observed high market price of risk in the U.S. can be reinterpreted as a market price of model uncertainty and we can thus reinterpret the risk-aversion parameter as measuring the representative agent's doubts about the model specification.

increases, the relative volatility decreases for any values of ϑ.[9] Using the Lucas elimiation-of-risk method, we find that the welfare costs due to model uncertainty are non-trivial. For plausibly parameter values, they could be as high as 10% of the typical consumer's permanent income.

Finally, we consider three extensions. In the first extension, we assume that consumers have recursive utility and have distinct preferences for risk and intertemporal substitution. In the second extension, we follow Pischke (1995) and Wang (2004) and assume that consumers can observe the total income but cannot distinguish the individual income components. In the final extension, we consider regime-switching in income growth and explore how it interacts with RB and affects the equilibrium interest rate and consumption volatility.

This paper is organized as follows. Section 2 presents a robustness version of the Caballero–Bewley-Huggett type model with incomplete markets and precautionary savings. Section 3 discusses the general equilibrium implications of RB for the interest rate and consumption and wealth dynamics. Section 4 present our quantitative results after estimating the income process and calibrating the RB parameter. Section 5 discusses how RB help explain the observed low interest rate in the U.S. Section 6 considers three extensions. Section 7 concludes.

2. A Continuous-time Heterogeneous-Agent Economy with Robustness

2.1. The Full-information Rational Expectations Model with Precautionary Savings

Following Wang (2003, 2009), we first formulate a continuous-time full-information rational expectations (FI-RE) Caballero-type model with precautionary savings. Specifically, we assume that there is only one risk-free asset in the model economy and there are a continuum of consumers who face uninsurable labor income and make optimal consumption-saving decisions. Uninsurable labor income (y_t) is assumed to follow an Ornstein-Uhlenbeck process:

$$dy_t = \rho \left(\frac{\mu}{\rho} - y_t \right) dt + \sigma_y dB_t, \tag{1}$$

where the unconditional mean and variance of income are $\overline{y} = \mu / \rho$ and $\sigma_y^2 / (2\rho)$, respectively, the persistence coefficient ρ governs the speed of convergence or divergence from the steady state,[10] B_t is a standard Brownian motion on the real line $\mathcal{R}$, and σ_y is the unconditional volatility of the

[9] This theoretical result might provide a potential explanation for the empirical evidence documented in Blundell, Pistaferri, and Preston (2008) that income and consumption inequality diverged over the sampling period they study.

[10] If $\rho > 0$, the income process is stationary and deviations of income from the steady state are temporary; if $\rho \leq 0$, income is non-stationary. The last case catches the flavor of Hall and Mishkin (1982)'s the specification of individual income that includes a non-stationary component. The $\rho = 0$ case corresponds to a simple Brownian motion without drift. The larger ρ is, the less y tends to drift away from $\overline{y}$. As ρ goes to ∞, the variance of y goes to 0, which means that y can never deviate from $\overline{y}$.

income change over an incremental unit of time. The typical consumer is assumed to maximize the following expected lifetime utility:

$$J_0 = E_0 \left[\int_{t=0}^{\infty} \exp\left(-\delta t\right) u(c_t) dt \right], \tag{2}$$

subject to the evolution of financial wealth (w_t):

$$dw_t = \left(rw_t + y_t - c_t\right) dt, \tag{3}$$

and a transversality condition, $\lim_{t\to\infty} E \left|\exp\left(-\delta t\right) J_t\right| = 0$, where r is the return to the risk-free asset, c is consumption, and the utility function takes the CARA form: $u(c_t) = -\exp\left(-\gamma c_t\right)/\gamma$, where $\gamma > 0$ is the coefficient of absolute risk aversion.[11] To present the model more compact, we define a new state variable, s_t:

$$s_t \equiv w_t + h_t,$$

where h_t is human wealth at time t and is defined as the expected present value of current and future labor income discounted at the risk-free interest rate r:

$$h_t \equiv E_t \left[\int_{t}^{\infty} \exp\left(-r\left(s-t\right)\right) y_s ds \right].$$

For the given the income process, (1), $h_t = y_t/\left(r+\rho\right) + \mu/\left(r\left(r+\rho\right)\right)$.[12] Following the state-space-reduction approach proposed in Luo (2008) and using the new state variable s, we can rewrite (3) as

$$ds_t = \left(rs_t - c_t\right) dt + \sigma_s dB_t, \tag{4}$$

where $\sigma_s = \sigma_y/\left(r+\rho\right)$ is the unconditional variance of the innovation to s_t.[13] It is not difficult to show that the above model with the univariate income process, (1), can be easily extended to the model with distinguishable multiple income components that have differencing persistence and volatility coefficients. In this more complicated case, we can still apply the state-space-reduction approach to simplify the model. To make our benchmark model tractable, we focus on the univariate income specification.

In this benchmark full-information rational expectations (FI-RE) model, we assume that the

[11]It is well-known that the CARA utility specification is tractable for deriving optimal policies in different settings. See Caballero (1990), Wang (2003, 2004), and Angeletos and La'O (2010).

[12]Here we need to impose the restriction that $r > -\rho$ to guarantee below the finiteness of human wealth.

[13]In the next section, we will introduce robustness directly into this "reduced" precautionary savings model. It is not difficult to show that the reduced univariate model and the original multivariate model are equivalent in the sense that they lead to the same consumption and saving functions. The detailed proof is available from online appendix.

consumer trusts the model and observes the state perfectly, i.e., no model uncertainty and no state uncertainty. Denoting the value function by $J(s_t)$. The Hamilton-Jacobi-Bellman (HJB) equation for this optimizing problem can be written as:

$$0 = \sup_{c_t} \left[-\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(s_t) + \mathcal{D} J(s_t) \right],$$

where

$$\mathcal{D} J(s_t) = J_s (r s_t - c_t) + \frac{1}{2} J_{ss} \sigma_s^2. \tag{5}$$

Solving the above HJB subject to (4) leads to the following consumption function:

$$c_t = r s_t + \Psi - \Gamma, \tag{6}$$

where $\Psi = (\delta - r) / (r\gamma)$ and

$$\Gamma \equiv \frac{1}{2} r \gamma \sigma_s^2, \tag{7}$$

is the consumer's precautionary saving demand. Following the literature of precautionary savings, we measure the demand for precautionary saving as the amount of saving due to the interaction of the degree of risk aversion and uninsurable labor income risk. From (7), it can see that the precautionary saving demand is larger for a larger value of the coefficient of absolute risk aversion (γ), a more volatile income innovation (σ_y), and a larger persistence coefficient (ρ).[14]

2.2. Incorporating Model Uncertainty due to Robustness

Robustness (robust control or robust filtering) emerged in the engineering literature in the 1970s and was introduced into economics and further developed by Hansen, Sargent, and others. A simple version of robustness considers the question of how to make optimal decisions when the decision maker does not know the true probability model that generates the data. The main goal of introducing robustness is to design optimal policies that not only work well when the reference (or approximating) model governing the evolution of the state variables is the true model, but also perform reasonably well when there is some type of model misspecification. To introduce robustness into our model proposed above, we follow the continuous-time methodology proposed by Anderson, Hansen, and Sargent (2003) (henceforth, AHS) and adopted in Maenhout (2004) to assume that consumers are concerned about the model misspecifications and take Equation (4) as the approximating model.[15] The corresponding distorting model can thus be obtained by adding

[14] As argued in Caballero (1990) and Wang (2004), a more persistent income shock takes a longer time to wear off and thus induces a stronger precautionary saving demand of a prudent forward-looking consumer.

[15] As argued in Hansen and Sargent (2007), the agent's committment technology is irrelevant under RB if the evolution of the state is backward-looking. We therefore do not specifiy the committment technology of the consumer in the RB

an endogenous distortion $v(s_t)$ to (4):

$$ds_t = (rs_t - c_t)\,dt + \sigma_s\left(\sigma_s v(s_t)\,dt + dB_t\right). \tag{8}$$

As shown in AHS (2003), the objective $\mathcal{D}J$ defined in (5) can be thought of as $E[dJ]/dt$ and plays a key role in introducing robustness. A key insight of AHS (2003) is that this differential expectations operator reflects a particular underlying model for the state variable because this operator is determined by the stochastic differential equations of the state variables. Consumers accept the approximating model, (4), as the best approximating model, but is still concerned that it is misspecified. They therefore want to consider a range of models (i.e., the distorted model, (8)) surrounding the approximating model when computing the continuation payoff. A preference for robustness is then achieved by having the agent guard against the distorting model that is reasonably close to the approximating model. The drift adjustment $v(s_t)$ is chosen to minimize the sum of (i) the expected continuation payoff adjusted to reflect the additional drift component in (8) and (ii) an entropy penalty:

$$\inf_{v}\left[\mathcal{D}J + v(s_t)\,\sigma_s^2 J_s + \frac{1}{2\vartheta_t}v^2(s_t)\,\sigma_s^2\right], \tag{9}$$

where the first two terms are the expected continuation payoff when the state variable follows (8), i.e., the alternative model based on drift distortion $v(s_t)$.[16] ϑ_t is fixed and state independent in AHS (2003), whereas it is state-dependent in Maenhout (2004). The key reason of using a state-dependent counterpart ϑ_t in Maenhout (2004) is to assure the homotheticity or scale invariance of the decision problem with the CRRA utility function.[17] Note that the evil agent's minimization problem, (9), becomes invariant to the scale of total resource s_t when using the state-dependent specification of ϑ_t. In this paper, we also specify that ϑ_t is state-dependent ($\vartheta(s_t)$) in the CARA-Gaussian setting. The main reason for this specification is to guarantee the homotheticity, which makes robustness not wear off as the value of the total wealth increases.[18]

Applying these results in the above model yields the following HJB equation under robustness:

$$\sup_{c_t}\inf_{v_t}\left[-\frac{1}{\gamma}\exp(-\gamma c_t) - \delta J(s_t) + \mathcal{D}J(s_t) + v(s_t)\,\sigma_s^2 J_s + \frac{1}{2\vartheta(s_t)}v^2(s_t)\,\sigma_s^2\right], \tag{10}$$

where the last term in the HJB above is due to the agent's preference for robustness and reflects a concern about the quadratic variation in the partial derivative of the value function weighted by

models of this paper.

[16]Note that the $\vartheta = 0$ case corresponds to the standard expected utility case.

[17]See Maenhout (2004) for detailed discussions on the appealing features of "homothetic robustness".

[18]In the detailed procedure of solving the robust HJB proposed in Appendix 8.2, it is clear that the impact of robustness wears off if we assume that ϑ_t is constant.

$\vartheta(s_t)$.[19] Solving first for the infimization part of (10) yields:

$$v(s_t)^* = -\vartheta(s_t) J_s,$$

where $\vartheta(s_t) = -\vartheta / J(s_t) > 0$. (See Appendix 8.2 for the derivation.) Following Uppal and Wang (2003) and Liu, Pan, and Wang (2005), here we can also define "$1/J(s_t)$" in the $\vartheta(s_t)$ specification as a *normalization* factor that is introduced to convert the relative entropy (i.e., the distance between the approximating model and the distorted model) to units of uility so that it is consistent with the units of the expected future value function evaluated with the distorted model. It is worth noting that adopting a slightly more general specification, $\vartheta(s_t) = -\varphi\vartheta / J(s_t)$ where φ is a constant, does not affect the main results of the paper. The reason is as follows. We can just define a new constant, $\widetilde{\vartheta} = \varphi\vartheta$, and $\widetilde{\vartheta}$, rather than ϑ, will enter the decision rules. Using a given detection error probability, we can easily calibrate the corresponding value of $\widetilde{\vartheta}$ that affects the optimal consumption-portfolio rules.[20]

Since $\vartheta(s_t) > 0$, the perturbation adds a negative drift term to the state transition equation because $J_s > 0$. Substituting for v^* in (10) gives:

$$\sup_{c_t} \left[-\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(s_t) + (r s_t - c_t) J_s + \frac{1}{2} \sigma_s^2 J_{ss} - \frac{1}{2} \vartheta(s_t) \sigma_s^2 J_s^2 \right]. \tag{11}$$

2.3. The Robust Consumption Function and Precautionary Saving

Following the standard procedure, we can then solve (11) and obtain the optimal consumption-portfolio rules under robustness. The following proposition summarizes the solution:

Proposition 1. *Under robustness, the consumption function and the saving function are*

$$c_t^* = r s_t + \Psi - \Gamma, \tag{12}$$

and

$$d_t^* = f_t + \Gamma - \Psi, \tag{13}$$

respectively, where $f_t = \rho(y_t - \overline{y}) / (r + \rho)$ *is the demand for savings "for a rainy day",* $\Psi(r) = (\delta - r) / (r\gamma)$ *captures the dissavings effect of relative impatience,*

$$\Gamma \equiv \frac{1}{2} r \widetilde{\gamma} \sigma_s^2 \tag{14}$$

[19] See AHS (2003) and Maenhout (2004) for detailed discussions.

[20] See Section 4.2 for the detailed procedure to calibrate the value of ϑ using the detection error probabilities.

is the demand for precautionary savings due to the interaction of income uncertainty, risk aversion, and uncertainty aversion, and $\widetilde{\gamma} \equiv (1+\vartheta)\gamma$ *is the effective coefficient of absolute risk aversion. Finally, the worst possible distortion is*

$$v^* = r\gamma\vartheta. \tag{15}$$

Proof. See Appendix 8.2. ■

From (12), it is clear that robustness does not change the marginal propensity to consume out of permanent income (MPC), but affects the amount of precautionary savings (Γ). In other words, in the continuous-time setting, consumption is not sensitive to unanticipated income shocks. This conclusion is different from that obtained in the discrete-time robust LQG-PIH model of Hansen, Sargent, and Tallarini (1999) (henceforth, HST) in which the MPC increases with model uncertainty determined by the interaction between RB and income uncertainty.[21] It is worth noting that this univariate RB model unique state variable s leads to the same consumption and saving functions as the corresponding multivariate RB model in which the state variables are w and y. The intuition behind this result is that the level of financial wealth w evolves deterministically over time, so that the evil agent cannot influence it.[22] Adopting the univariate setting here can significantly help solve the model explicitly when we consider state uncertainty into the RB model.

Expression (14) shows that the precautionary savings demand is increasing with the degree of robustness (ϑ) via increasing the value of $\widetilde{\gamma}$ and interacting with the fundamental uncertainty: labor income uncertainty (σ_s^2). An interesting question here is the relative importance of RB (ϑ) and CARA (γ) in determining the precautionary savings demand, holding other parameters constant. We can use the elasticities of precautionary saving as a measure of their importance. Specifically, using (14), we have the following proposition:

Proposition 2. *The relative sensitivity of precautionary saving to robustness (RB, ϑ) and CARA (γ) can be measured by:*

$$\mu_{\gamma\vartheta} \equiv \frac{e_\gamma}{e_\vartheta} = \frac{1+\vartheta}{\vartheta} > 1, \tag{16}$$

where $e_\vartheta \equiv \frac{\partial\Gamma/\Gamma}{\partial\vartheta/\vartheta}$ *and* $e_\gamma \equiv \frac{\partial\Gamma/\Gamma}{\partial\gamma/\gamma}$ *are the elasticities of the precautionary saving demand to RB and CARA, respectively. (16) means that the precautionary savings demand is more sensitive to the coefficient of (absolute) risk aversion measured by γ than RB measured by ϑ.*

Proof. The proof is straightforward. ■

[21] Consequently, consumption is more sensitive to unanticipated shocks. See HST (1999) for a detailed discussion on how RB affects consumption and precautionary savings within the discrete-time LQG setting.

[22] The proof of the equivalence between the univaritae and multivariate RB models is available from the corresponding author by request.

HST (1999) showed that the discount factor and the concern about robustness are observationally equivalent in the sense that they lead to the same consumption and investment decisions in a discrete-time LQG representative-agent permanent income model. The reason for this result is that introducing a concern about robustness increases savings in the same way as increasing the discount factor, so that the discount factor can be changed to offset the effect of a change in RB on consumption and investment.[23] In contrast, for our continuous-time CARA-Gaussian model discussed above, we have a more general observational equivalence result between δ, γ, and ϑ:

Proposition 3. *Let*

$$\gamma^{fi} = \gamma\left(1+\vartheta\right),$$

where γ^{fi} is the coefficient of absolute risk aversion in the FI-RE model, consumption and savings are identical in the FI-RE and RB models, holding other parameter values constant. Furthermore, let $\delta = r$ in the RB model, and

$$\delta^{fi} = r - \frac{1}{2}\vartheta\left(r\gamma\right)^2\sigma_s^2,$$

where δ^{fi} is the discount rate in the FI-RE model, consumption and savings are identical in the FI-RE and RB models, ceteris paribus.

Proof. Using (12) and (14), the proof is straightforward. ■

3. General Equilibrium Implications of RB

3.1. Definition of the General Equilibrium

As in Huggett (1993) and Wang (2003), we assume that the economy is populated by a continuum of *ex ante* identical, but *ex post* heterogeneous agents, with each agent having the saving function, (14). In addition, we also assume that the risk-free asset in our model economy is a pure-consumption loan and is in zero net supply. The initial cross-sectional distribution of income is assumed to be its stationary distribution $\Phi\left(\cdot\right)$. By the law of large numbers in Sun (2006), provided that the spaces of agents and the probability space are constructed appropriately, aggregate income and the cross-sectional distribution of permanent income $\Phi\left(\cdot\right)$ are constant over time.

Proposition 4. *The total savings demand "for a rainy day" in the precautionary savings model with RB equals zero for any positive interest rate. That is, $F_t\left(r\right) = \int_{y_t} f_t\left(r\right) d\Phi\left(y_t\right) = 0$, for $r > 0$.*

[23] As shown in HST (1999), the two models have different implications for asset prices because continuation valuations would alter as one alters the values of the discount factor and the robustness parameter within the observational equivalence set.

Proof. Given that labor income is a stationary process, the LLN can be directly applied and the proof is the same as that in Wang (2003). ■

This proposition states that the total savings "for a rainy day" is zero, at any positive interest rate. Therefore, from (13), for $r > 0$, the expression for total savings under RB in the economy at time t can be written as:

$$D(\vartheta, r) \equiv \Gamma(\vartheta, r) - \Psi(r). \tag{17}$$

We can now define the equilibrium in our model as follows:

Definition 1. *Given (17), a general equilibrium under RB is defined by an interest rate r^* satisfying:*

$$D(\vartheta, r^*) = 0. \tag{18}$$

3.2. Theoretical Results

The following proposition shows the existence of the equilibrium and the PIH holds in the RB general equilibrium:

Proposition 5. *There exists at least one equilibrium interest rate $r^* \in (0, \delta)$ in the precautionary-savings model with RB; if $\delta < \rho$ the equilibrium interest rate is unique on $(0, \delta)$. In equilibrium, each consumer's optimal consumption is described by the PIH, in that*

$$c_t^* = r^* s_t. \tag{19}$$

Furthermore, the evolution equations of wealth and consumption are

$$dw_t^* = f_t dt, \tag{20}$$

$$dc_t^* = \frac{r^*}{r^* + \rho}\sigma_y dB_t, \tag{21}$$

respectively, where $f_t = \rho(y_t - \overline{y}) / (r^* + \rho)$. Finally, the relative volatility of consumption growth to income growth is

$$\mu \equiv \frac{\mathrm{sd}(dc_t^*)}{\mathrm{sd}(dy_t)} = \frac{r^*}{r^* + \rho}. \tag{22}$$

Proof. If $r > \delta$, both $\Gamma(\vartheta, r)$ and $\Psi(r)$ in the expression for total savings $D(\vartheta, r^*)$ are positive, which contradicts the equilibrium condition: $D(\vartheta, r^*) = 0$. Since $\Gamma(\vartheta, r) - \Psi(r) < 0$ (> 0) when $r = 0$ ($r = \delta$), the continuity of the expression for total savings implies that there exists at least one interest rate $r^* \in (0, \delta)$ such that $D(\vartheta, r^*) = 0$. To prove this equilibrium is unique, note that

$$\frac{\partial D(\vartheta, r)}{\partial r} = (1+\vartheta)\gamma \frac{\sigma^2}{(r+\rho)^2}\left(\frac{1}{2} - \frac{r}{r+\rho}\right) + \frac{\delta}{r^2\gamma}.$$

Let $r > 0$; the derivative is positive if

$$\rho > r.$$

Therefore, if $\rho > \delta$ there is only one equilibrium in $(0, \delta)$. From Expression (12), we can obtain the individual's optimal consumption rule under RB in general equilibrium as $c_t^* = r^* s_t$. Therefore, there exists a unique equilibrium in this aggregate economy. Substituting (72) into (3) yields (20). Using (4) and (72), we can obtain (21). ■

The intuition behind this proposition is similar to that in Wang (2003). With an individual's constant total precautionary savings demand $\Gamma(\vartheta, r)$, for any $r > 0$, the equilibrium interest rate r^* must be at a level with the property that individual's dissavings demand due to impatience is exactly balanced by their total precautionary-savings demand, $\Gamma(\vartheta, r^*) = \Psi(r^*)$. Following Caballero (1991) and Wang (2003, 2009), we set that $\gamma = 1.5$, $\sigma_y = 0.309$, and $\rho = 0.128$.[24] Figure 1 shows that the aggregate saving function $D(\vartheta, r)$ is increasing with the interest rate, and there exists a unique interest rate r^* for every given ϑ such that $D(\vartheta, r^*) = 0$.[25]

Given (12), (14), and (18), it is clear that even though precautionary saving at the individual level increases with the degree of concerns about model misspecifications, the level of aggregate savings is equal to zero in the general equilibrium. That is, RB does not affect the level of aggregate wealth in the economy. Figure 1 shows how RB (ϑ) affects the equilibrium interest rate (r^*). It is clear from the figure that the stronger the preference for robustness, the less the equilibrium interest rate. From (22), we can see that RB can affect the volatility of consumption by reducing the equilibrium interest rate. The following proposition summarizes the results about how the persistence coefficient of income affects the impact of RB on the relative volatility:

Proposition 6. *Using (22), we have:*

$$\frac{\partial \mu}{\partial \vartheta} = \frac{\rho}{(r^*+\rho)^2}\frac{\partial r^*}{\partial \vartheta} \lesseqgtr 0 \text{ iff } \rho \gtreqless 0,$$

because $\partial r^* / \partial \vartheta < 0$.

[24] In Section 4.1, we will provide more details about how to estimate the income process using the U.S. panel data. The main result here is robust to the choices of these parameter values.

[25] We ignore negative interest rate equilibria because the resulting consumption function does not make economic sense. It is easy to see that D has the same zeroes as a cubic function, so that there exist conditions under which the equilibrium is globally unique, but these conditions are not amenable to analysis.

Proof. The proof is straightforward. ∎

In the next section, we will fully explore how RB affects the equilibrium interest rate and the equilibrium dynamics of consumption after estimating the income process and calibrating the RB parameter ϑ.

4. Quantitative Analysis

In this section, we first describe how we estimate the income process and calibrate the robustness parameter. We then present quantitative results on how RB affects the equilibrium interest rate and relative volatility of consumption to income.

4.1. Estimation of the Income Process

To implement the quantitative analysis, we need to first estimate the income process. That is, we need to estimate ρ and σ_y in the income process specification (1). We use micro data from the Panel Study of Income Dynamics (PSID). Following Blundell, Pistaferri, and Preston (2008), we define the household income as total household income (including wage, financial, and transfer income of head, wife, and all others in household) minus financial income (defined as the sum of annual dividend income, interest income, rental income, trust fund income, and income from royalties for the head of the household only) minus the tax liability of non-financial income. This tax liability is defined as the total tax liability multiplied by the non-financial share of total income. Tax liabilities after 1992 are not reported in the PSID and so we estimate them using the TAXSIM program from the NBER. Details on sample selection are reported in Appendix 8.1.

Following Floden and Lindé (2001), we normalize household income measures as ratios of the mean for that year. We then exclude all household values in years in which the income is less than 10% of the mean for that year or more than ten times the mean. To eliminate possible heteroskedasticity in the income measures, we follow Floden and Lindé (2001) to regress each on a series of demographic variables to remove variation caused by differences in age and education. We next subtract these fitted values from each measure to create a panel of income residuals. We then use this panel to estimate the household income process as specified by an stationary AR(1) process by running panel regressions on lagged income. Specifically, we specify the AR(1) process with Gaussian innovations as follows:

$$y_t = \phi_0 + \phi_1 y_{t-1} + \sigma\varepsilon_t,\ t \geq 1,\ \ |\phi_1| < 1, \tag{23}$$

where $\varepsilon_t \sim N(0,1)$, $\phi_0 = (1-\phi_1)\,\overline{y}$, $\overline{y}$ is the mean of y_t, and the initial level of labor income y_0

are given. Once we have estimates of ϕ_1 and σ, we can recover the drift and diffusion coefficients in the Ornstein-Uhlenbeck process specified in (1). This can be done by rewriting (23) in the time interval of $[t, t+\Delta t]$:[26]

$$y_{t+\Delta t} = \phi_0 + \phi_1 y_t + \sigma\sqrt{\Delta t}\varepsilon_{t+\Delta t}, \tag{24}$$

where $\phi_0 = \mu\left(1 - \exp\left(-\rho\Delta t\right)\right) / \left(\rho\Delta t\right)$, $\phi_1 = \exp\left(-\rho\Delta t\right)$, $\sigma = \sigma_y\sqrt{\left(1 - \exp\left(-2\rho\Delta t\right)\right) / \left(2\rho\Delta t\right)}$, and $\varepsilon_{t+\Delta t}$ is the time-$(t+\Delta t)$ standard normal distributed innovation to income. The estimation results are reported in Table 1.

4.2. Calibration of the Robustness Parameter

To fully explore how RB affects the dynamics of consumption and labor income, we adopt the calibration procedure outlined in HSW (2002) and AHS (2003) to calibrate the value of the RB parameter (ϑ) that governs the degree of robustness. Specifically, we calibrate ϑ by using the method of detection error probabilities (DEP) that is based on a statistical theory of model selection. We can then infer what values of ϑ imply reasonable fears of model misspecification for empirically-plausible approximating models. The model detection error probability denoted by p is a measure of how far the distorted model can deviate from the approximating model without being discarded; low values for this probability mean that agents are unwilling to discard many models, implying that the cloud of models surrounding the approximating model is large. In this case, it is easier for the consumer to distinguish the two models. The value of p is determined by the following procedure. Let model P denote the approximating model, (4) and model Q be the distorted model, (8). Define p_P as

$$p_P = \text{Prob}\left(\ln\left(\frac{L_Q}{L_P}\right) > 0 \middle| P\right), \tag{25}$$

where $\ln\left(\frac{L_Q}{L_P}\right)$ is the log-likelihood ratio. When model P generates the data, p_P measures the probability that a likelihood ratio test selects model Q. In this case, we call p_P the probability of the model detection error. Similarly, when model Q generates the data, we can define p_Q as

$$p_Q = \text{Prob}\left(\ln\left(\frac{L_P}{L_Q}\right) > 0 \middle| Q\right). \tag{26}$$

Given initial priors of 0.5 on each model and the length of the sample is N, the detection error probability, p, can be written as:

$$p\left(\vartheta; N\right) = \frac{1}{2}\left(p_P + p_Q\right), \tag{27}$$

[26] Note that here we use the fact that $\Delta B_t = \varepsilon_t\sqrt{\Delta t}$, where ΔB_t represents the increment of a Wiener process.

where ϑ is the robustness parameter used to generate model Q. Given this definition, we can see that $1 - p$ measures the probability that econometricians can distinguish the approximating model from the distorted model.

The general idea of the calibration procedure is to find a value of ϑ such that $p(\vartheta; N)$ equals a given value (for example, 20%) after simulating model P, (4), and model Q, (8).[27] In the continuous-time model with the iid Gaussian specification, $p(\vartheta; N)$ can be easily computed. Since both models P and Q are arithmetic Brownian motions with constant drift and diffusion coefficients, the log-likelihood ratios are Brownian motions and are normally distributed random variables. Specifically, the logarithm of the Radon-Nikodym derivative of the distorted model (Q) with respect to the approximating model (P) can be written as

$$\ln\left(\frac{L_Q}{L_P}\right) = -\int_0^N \overline{v} dB_s - \frac{1}{2}\int_0^N \overline{v}^2 ds, \tag{28}$$

where

$$\overline{v} \equiv v^* \sigma_s = r^* \vartheta \gamma \sigma_s. \tag{29}$$

Similarly, the logarithm of the Radon-Nikodym derivative of the approximating model (P) with respect to the distorted model (Q) is

$$\ln\left(\frac{L_P}{L_Q}\right) = \int_0^N \overline{v} dB_s + \frac{1}{2}\int_0^N \overline{v}^2 ds. \tag{30}$$

Using (25)-(30), it is straightforward to derive $p(\vartheta; N)$:

$$p(\vartheta; N) = \Pr\left(x < -\frac{\overline{v}}{2}\sqrt{N}\right), \tag{31}$$

where x follows a standard normal distribution. From the expressions of $\overline{v}$, (29), and $p(\vartheta; N)$, (31), it is clear that the value of p is decreasing with the value of ϑ.

We first explore the relationship between the DEP (p) and the value of the RB parameter, ϑ. A general finding is a negative relationship between these two variables. The left panel of Figure 2 illustrates how DEP (p) varies with the value of ϑ for different values of CARA (γ).[28] We can see from the figure that the stronger the preference for robustness (higher ϑ), the less the DEP (p)

[27] The number of periods used in the calculation, N, is set to be the actual length of the data.

[28] Based on the estimation results, we set $\overline{y} = 1$, $\sigma_y = 0.309$, and $\rho = 0.128$. It is worth noting that the implied coefficient of relative risk aversion (CRRA) in our CARA utility specification can be written as: γc or γy. Given that the value of the CRRA is very stable and $\overline{v}$ can be expressed as $r\vartheta\gamma\sigma_y / (r+\rho)$, proportional changes in the mean and standard deviation of y do not change our calibration results because their impacts on γ and σ_y are just cancelled out. For example, if both $\overline{y}$ and σ_y are doubled, γ is reduced to half such that the product of γ and σ_y remains unchanged.

is. For example, let $\gamma = 1.5$, then $p = 0.22$ and $r^* = 2.79\%$ when $\vartheta = 2.5$, while $p = 0.31$ and $r^* = 3.02\%$ when $\vartheta = 1.5$.[29] Both values of p are reasonable as argued in AHS (2002), HSW (2002), Maenhout (2004), and Hansen and Sargent (Chapter 9, 2007). In other words, a value of ϑ between 1.5 and 2.5 is reasonable.[30] Using (16), we have $\mu_{\gamma\vartheta} = 1.4$ and 1.67 when we set $p = 0.22$ and 0.31, respectively. That is, risk aversion is relatively more important than RB in determining the precautionary savings demand given plausibly calibrated values of ϑ.

The right panel of Figure 2 illustrates how DEP (p) varies with ϑ for different values of σ_y when γ equals 1.5.[31] It also shows that the higher the value of ϑ, the less the DEP (p). In addition, to calibrate the same value of p, less values of σ_y (i.e., more volatile labor income processes) leads to higher values of ϑ.[32] The intuition behind this result is that σ_s and ϑ have opposite effects on $\overline{v}$. (It is clear from (29).) To keep the same value of p, if one parameter increases, the other one must reduce to offset its effect on $\overline{v}$.

An important comment follows these calibration results. As emphasized in Hansen and Sargent (2007), in the robustness model, p can be used to measure the amount of model uncertainty, whereas ϑ is used to measure the degree of the agent's preference for RB. If we keep p constant when recalibrating ϑ for different values of γ, ρ, or σ_y, the amount of model uncertainty is held constant, i.e., the set of distorted models with which we surround the approximating model does not change. In contrast, if we keep ϑ constant, p will change accordingly when the values of γ, ρ, or σ_y change. That is, the amount of model uncertainty is "elastic" and will change accordingly when the fundamental factors change.

4.3. Effects of RB on the Equilibrium Interest Rate and Consumption Volatility

As shown in the theoretic results, the equilibrium interest rate and relative volatility of consumption to income are jointly determined by the degree of robustness, the risk aversion, and the income process. To better see how RB affects the equilibrium interest rate and the relative volatility, we present two quantitative exercises here. The first exercise fixes the parameters of the income process at the estimated values and allows the risk aversion parameter to change, while the second exercise fixes risk aversion parameter and allows the key income process parameter to vary.

Figure 3 shows that the equilibrium interest rate and the equilibrium relative volatility decrease with the calibrated value of ϑ for different values of γ when $\sigma_y = 0.309$, and $\rho = 0.128$. For

[29]Caballero (1990) and Wang (2009) also consider the $\gamma = 2$ case.

[30]As shown by Figure 2, when DEP declines, ϑ increases monotonically.

[31]Since $\sigma_s = \sigma_y / (r + \rho)$, both changes in the persistence coefficient (ρ) and changes in volatility coefficient (σ_y) will change the value of σ_s.

[32]It is straightforward to show that a reduction in ρ has similar impacts on the calibrated value of ϑ as an increase in σ_y.

example, when ϑ is increased from 1.5 to 2 (i.e., when p decreases from 0.313 to 0.223), r^* is reduced from 3.02% to 2.79%, and μ is reduced from 0.191 to 0.179 when $\gamma = 1.5$. In addition, the figure also shows that the interest rate and the relative volatility decrease with γ for different values of ϑ.

Figure 4 shows that the equilibrium interest rate and the equilibrium relative volatility decrease with the value of ϑ for different values of σ_y when $\gamma = 1.5$ and $\rho = 0.128$. The pattern of this figure is similar to that of Figure 3. In addition, the figure also shows that the interest rate and the relative volatility decrease with σ_y for different values of ϑ. For example, when σ_y is doubled from 0.2 to 0.4, r^* is reduced from 3.48% to 2.66% and μ is reduced from 0.214 to 0.172 when $\gamma = 1.5$ and $\vartheta = 1.5$.

Using the same constructed panel of household income and consumption described in the previous subsection, Figure 5 shows the relative dispersion of consumption, defined as the ratio of the standard deviation of the consumption change to the standard deviation of the income change between 1980 and 2000.[33] From the figure, the average empirical value of the relative volatility (μ) is 0.209, and the minimum and maximum values of the empirical relative volatility are 0.159 and 0.285, respectively. Comparing these results with Figures 3 and 4, we can see that our model, with plausibly estimated and calibrated parameter values, can match the empirical evidence on the relative volatility of consumption to income.

4.4. The Welfare Cost of Model Uncertainty

We can also quantify the effects of RB on the welfare cost of volatility in the general equilibrium using the Lucas elimination-of-risk method. (See Lucas 1987, 2000; Tallarini 2000).[34] It is worth noting that although we do not discuss the welfare costs of business cycles in our heterogeneous-agent economy without aggregate uncertainty, we can still use the Lucas approach to explore the welfare cost of model uncertainty due to RB.[35] Specifically, following the literature, we define the total welfare cost of volatility as the percentage of permanent income the consumer is ready to give

[33]Details on how the panel data are constructed are described in Appendix 8.1.

[34]Tallarini (2000) found that the welfare costs of aggregate fluctuations are non-trivial when the representative agent has a recursive utility that breaks the link between risk aversion and intertemporal substitution. However, in Tallarini's model, high welfare costs also require the agent to have implausibly high levels of risk aversion. In contrast, Barillas, Hansen, and Sargent (2009) showed that the high coefficients of risk aversion in Tallarini (2000) may not only reflect the agent's risk attitudes but also reflect his concerns about model misspecification. They found that market prices of model uncertainty contain information about the benefits of removing model uncertainty, not the consumption fluctuations that Lucas (1987, 2000) studied.

[35]Ellison and Sargent (2014) found that idiosyncratic consumption risk has a greater impact on the cost of business cycles when they fear model misspecification. In addition, they showed that endowing agents with fears about misspecification leads to greater welfare costs that the exisiting idiosyncratic consumption risk.

up at the initial period to be as well off in the FI-RE economy as he is in the RB economy:[36]

$$\widetilde{J}\left(s_0\left(1-\Delta\right)\right) = J\left(s_0\right), \tag{32}$$

where

$$\widetilde{J}\left(s_0\left(1-\Delta\right)\right) = -\frac{1}{\widetilde{\alpha}_1}\exp\left(-\widetilde{\alpha}_0-\widetilde{\alpha}_1 s_0\left(1-\Delta\right)\right) \text{ and } J\left(s_0\right) = -\frac{1}{\alpha_1}\exp\left(-\alpha_0-\alpha_1 s_0\right)$$

are the value functions under FI-RE and RB, respectively, Δ is the compensating amount measured by the percentage of s_0, $\alpha_1 = r^*\gamma$, $\widetilde{\alpha}_1 = \widetilde{r}^*\gamma$, $\alpha_0 = \delta/r^* - 1 - (1+\vartheta)\, r^*\gamma^2\sigma_s^2/2$, $\widetilde{\alpha}_0 = \delta/\widetilde{r}^* - 1 - \widetilde{r}^*\gamma^2\widetilde{\sigma}_s^2/2$, and r^* and $\widetilde{r}^*$ are the equilibrium interest rates in the RB and FI-RE economies, respectively.[37] The following proposition summarizes the result about how RB affects the welfare costs in general equilibrium:

Proposition 7. *When the equilibrium condition, (18), holds, the welfare costs due to model uncertainty can be written as:*

$$\Delta = \frac{s_0\left(\widetilde{\alpha}_1-\alpha_1\right)-\ln\left(\widetilde{\alpha}_1/\alpha_1\right)}{\widetilde{\alpha}_1 s_0} = \left(1-\frac{r^*}{\widetilde{r}^*}\right) - \frac{1}{\widetilde{r}^*\gamma s_0}\ln\left(\frac{\widetilde{r}^*}{r^*}\right), \tag{33}$$

which implies that

$$\frac{\partial\Delta}{\partial\vartheta} = \frac{\partial\Delta}{\partial r^*}\frac{\partial r^*}{\partial\vartheta} > 0$$

because $\partial r^*/\partial\vartheta < 0$, *and* $\partial\Delta/\partial r^* = -1/\widetilde{r}^*\left[1-1/\left(r^*\gamma s_0\right)\right] < 0$ *for plausible parameter values.*

Proof. Substituting (18) into the expressions of α_0 and $\widetilde{\alpha}_0$ in the value functions under FI-RE and RB, we obtain that $\alpha_0 = \widetilde{\alpha}_0 = 0$. Combining these results with (32) yields (33). ∎

To do quantitative welfare analysis, we need to know the initial level of s, s_0. We assume that $s_0 = E[s]$ and the ratio of the initial level of financial wealth (w_0) to mean income $(y_0 \equiv E[y_t])$ is 5, that is, $w_0/y_0 = 5$.[38] Given that $y_0 = 1$, $\gamma = 1.5$, and $\rho = 0.128$, we can easily calculate that $s_0 = w_0 + y_0/r$.[39] Figure 6 illustrates how the welfare cost of model uncertainty varies with ϑ for different values of γ and σ_y.[40] We can see from this figure that the welfare costs of model

[36] This approach is also used in Epaulard and Pommeret (2003) to examine the welfare cost of volatility in a representative-agent model with recursive utility. In their model, the total welfare cost of volatility is defined as the percentage of capital the representative agent is ready to give up at the initial period to be as well off in a certain economy as he is in a stochastic one.

[37] See Appendix 8.2 for the derivation of the value functions. Note that $\Delta = 0$ when $\vartheta = 0$.

[38] This number varies largely for different individuals, from 2 to 20. 5 is the average wealth/income ratio in the Survey of Consumer Finances 2001. We find that changing the value of this ratio does not change our conclusion about the welfare implication of RB.

[39] Note that here we use the definition of s_t: $s_t = w_t + y_t/\left(r+\rho\right) + \rho\overline{y}/\left[r\left(r+\rho\right)\right]$.

[40] When generating the left and right panels of this figure, we set $\sigma_y = 0.309$ and $\gamma = 1.5$, respectively.

uncertainty are nontrivial, and increase with γ and σ_y. The intuition behind this result is that higher income uncertainty leads to higher the induced model uncertainty. For example, when $\gamma = 1.5$ and $\vartheta = 1.5$, the welfare cost of model uncertainty Δ is 5.37%. When ϑ increases from 1.5 to 2.5, Δ increases from 5.37% to 13.64%. Furthermore, the figure also shows that an increase income volatility can significantly increase the welfare cost of model uncertainty. For example, when $\gamma = 1.5$, $\vartheta = 1.5$, and income volatility (σ_y) is reduced from 0.4 to 0.2, Δ decreases from 6.7% to 3.13%. We can thus learn from this result that macroeconomic policies that aim to reduce income inequality are beneficial in an economy in which agents have a fear of model misspecification.

5. Further Discussion on the Impact of RB on the Interest Rate

Our theoretical and quantitative results obtained in the previous sections have implications for explaining the observed low real interest rate as well as the declines in the equilibrium real interest rate (or the natural rate of interest) in the U.S. economy. We discuss them in this section.

5.1. The Observed Low Interest Rate

Our theoretical results show that a larger concern about model uncertainty lowers the equilibrium real interest rate. In the U.S. the average real risk-free interest rate is about 1.15 percent between 1985 and 2014.[41] The full-information model without RB requires the coefficient of risk aversion parameter to be 10 to match this rate.[42] This value of CRRA might be too high to be plausible for ordinary consumers. In contrast, when consumers take into account model uncertainty, the model can generate an equilibrium interest rate of 1.15 percent with much lower values of the coefficient of risk aversion.[43] Figure 7 shows the relationship between γ and ϑ for the given real interest rate 1.15%.[44] For example, when $\gamma = 5$ and $\vartheta = 3$, the RB model leads to the same interest rate as in the FI model with $\gamma = 10$. Using the same calibration procedure discussed in Section 4.1, we find that the corresponding DEP is $p = 0.16$. In other words, agents have 16% probability that they cannot distinguish the distorted model from the approximating model. As argued in Hansen and Sargent

[41] Following Campbell (2003), we calculate the average of the real 3-month Treasury yields. The averages from the beginning of 1985 to the end of 2014 are 0.88% using core CPI inflation and 1.37% using core PCE inflation. Therefore, depending on what inflation index is used, the risk-free rate is between 0.9 and 1.4. (In our following discussion, we set the risk free rate to be 1.15 which is the average of the two real interest rates under CPI and PCE.) We choose this period because it is more consistent with our sample period in estimating the income process. The average 3-month real treasury yields over the 1949 − 2014 period is 0.79% using headline CPI inflation. Notice that the core CPI inflation became available only starting from 1958.

[42] Note that since we set the mean income level to be 1, the coefficient of relative risk aversion (CRRA) evaluated at this level is equal to the coefficient of absolute risk aversion (CARA).

[43] This result is comparable to that obtained in Barillas, Hansen, and Sargent (2009). They found that most of the observed high market price of uncertainty in the U.S. can be reinterpreted as a market price of model uncertainty rather than the traditional market price of risk.

[44] The pattern is robust for different values of the equilibrium interest rate.

(2007) and in Section 4.2, this value seems reasonable in the literature. In summary, incorporating model uncertainty due to RB can relax the restriction on CRRA imposed by the model and thus has the potential to explain the low interest rate we observed in the U.S. economy.

5.2. Declines in the Equilibrium Real Interest Rate in the U.S.

Recent studies on monetary policy suggest a possible decline in the U.S. equilibrium real interest rate (Hamilton et al., 2015). In the monetary policy literature, this equilibrium real interest rate is also called the natural rate of interest or the neutral rate of interest, which simply refers to the equilibrium interest rate that is consistent with full employment and stable inflation.[45] Within the context of a New Keynesian Dynamic Stochastic General Equilibrium (DSGE) model, it is the equilibrium rate when the economy has no wage and price rigidities and no shocks to wage markups, price markups, or technology. This concept is important because it helps to determine the level at which policymakers should set the interest rate to be given the current inflation and economic conditions. In general, when the equilibrium interest rate is lower, policymakers should also lower the nominal interest rate (i.e., the federal funds rate in the U.S.) to keep the economy to stay at or move back to a full employment level with stable inflation (i.e, an inflation level of 2 percent in the U.S.).

The equilibrium real interest rate is unobserved because the real economy consists of distortions such as price and wage rigidities as mentioned above. However, many researchers have applied statistical methods to estimate the equilibrium real interest rate and show it has been lower , especially following the financial crisis (Laubach and Williams 2003, 2014 and Hamilton et al., 2015). Figure 8 plots Laubach and Williams' estimates. It clearly shows the equilibrium real interest rate became significantly lower after the 2007-09 financial crisis.[46]

Our results provide an explanation for a lower equilibrium interest rate by showing an increase in model uncertainty (i.e., an increase in ϑ) could contribute to a decline in the equilibrium real interest rate. First, the comparison between a model without model uncertainty (the FI-RE model) with a model taking into account model uncertainty (the RB model) shows agents' concern about model misspecification will increase aggregate savings and thus drives down the equilibrium interest rate. Second, within the RB framework, we show an increase in the degree of model uncertainty will further reduce the equilibrium interest rate through increasing precautionary savings.

[45] In a Taylor rule (Taylor 1993, 1999), it is the r^* in the rule: $i_t = r^* + \pi_t + \alpha_\pi(\pi_t - \pi^*) + \alpha_y(y_t - y_t^*)$, where π^* is the inflation target and y^* is potential output. Policymakers thus set the nominal interest rate (i) based on the equilibrium real interest rate (r^*), inflation (π), inflation gap $(\pi_t - \pi^*)$, and output gap $(y_t - y_t^*)$.

[46] It worth noting that in a standard Taylor rule which prescribes the monetary policy, the equilibrium interest rate is set to be a constant. In other words, a change in this equilibrium interest rate could be interpreted as a change in fundamentals in the economy.

The explanation that agents have become more concerned about model misspecification after the $2007-09$ financial crisis is not unreasonable given the long and deep recession which generated skepticism about whether the standard macro models can fully capture how the economy is working. Actually, as Figure 8 shows, most large declines in the equilibrium rate followed recessions, which is consistent with the view that recessions may have caused people to be more concerned about how the economy is truly working. Consider the following numerical example. Before the financial crisis, the equilibrium real interest rate is 2% when $\gamma = 3$ and $\vartheta = 2$. After the financial crisis, consumers became more concerned about the true model governing the macroeconomy and ϑ is thus increased from 2 to 6; consequently, the equilibrium interest rate reduces to 1.4%.

It is worth noting that the explanation of a lower equilibrium real interest rate due to higher savings is not new. Summers (2014) and Blanchard et al. (2014) also argue that increases in global savings could be a reason for a lower equilibrium real interest rate in the U.S. and other advanced economies. However, their explanations for higher savings usually rely on demographic trends (such as an aging population) and capital flows from emerging economies to advanced economies, while our explanation for increases in savings comes purely from agents' concern about model uncertainty. In addition, neither of these papers provides a structural model to quantify the effects, while we explicitly solve a stochastic general equilibrium model to show both the channel and the effect.

6. Extensions

In this section, we consider three extensions. In the first extension, we assume that consumers have recursive utility and thus risk aversion and intertemporal substitution are separated in their preferences. In the second extension, we consider incomplete information about individual income (IC). Specifically, in this extension we assume that consumers only observe total income and cannot perfect distinguish individual income components (see Muth 1960, Pischke 1995 and Wang 2003). In the third extension, we incorporate regime-swtiching in income process into the benchmark model and discuss how regime-switching in income growth affects individual consumption and savings decisions and the equilibrium interest rate.

6.1. Separation of Risk Aversion and Intertemporal Substitution

In the previous sections, we discussed how the interaction of risk aversion and robustness affects the equilibrium interest rate, consumption volatility, and welfare costs of model uncertainty. However, given the time-separable utility setting, we cannot examine how intertemporal substitution affects the equilibrium outcomes. In this section, we consider a continuous-time recursive utility

(RU) model with iso-elastic intertemporal substitution and exponential risk aversion. This recursive utility specification is proposed in Weil (1993) in a discrete-time consumption-saving model. In our continuous-time setting, the Bellman equation for the optimization problem can be written as:

$$J(s_t)^{1-1/\varepsilon} = \max_{c_t} \left\{ \left(1 - e^{-\delta dt}\right) c_t^{1-1/\varepsilon} + e^{-\delta dt} \mathcal{C}E_t^{1-1/\varepsilon} \right\} \tag{34}$$

subject to (4), where ε is the intertemporal elasticity of substitution, δ is the discount rate, γ is the coefficient of absolute risk aversion, and

$$\mathcal{C}E_t \equiv -\frac{1}{\gamma} \ln\left(E_t\left[\exp\left(-\gamma J\left(s_{t+dt}\right)\right)\right]\right)$$

denotes the certainty equivalent in terms of period-t consumption of the uncertain total utility in the future periods. Furthermore, (34) can be reduced to

$$0 = \max_{c_t} \left\{ \delta c_t^{1-1/\varepsilon} - \delta \widetilde{J}(s_t) + \left(rs_t - c_t - \frac{1}{2}\gamma A \sigma_s^2 \right) \widetilde{J}_s(s_t) \right\}, \tag{35}$$

where $\widetilde{J}(s_t) = J(s_t)^{1-1/\varepsilon} = (As_t + A_0)^{1-1/\varepsilon}$, and A and A_0 are undetermined coefficients.[47] (See Online Appendix for the derivation.)

If the consumer trusts the model represented by (4), we can solve for the consumption function and the corresponding value function as follows:

$$c_t^* = \left[r + (\delta - r)\varepsilon\right] s_t - \frac{1}{2}\gamma A \left[1 + \left(\frac{\delta}{r} - 1\right)\varepsilon\right] \sigma_s^2 \tag{36}$$

and $J(s_t) = As_t + A_0$, where $A = \left[\frac{r+(\delta-r)\varepsilon}{\delta^\varepsilon}\right]^{1/(1-\varepsilon)}$ and $A_0 = -\gamma A^2 \sigma^2 / (2r)$.[48] Here we need to impose that $r + (\delta - r)\varepsilon > 0$ to guarantee the existence of an optimal plan. In addition, as in Weil (1993), we also need to assume that the initial financial wealth level, w_0, is sufficiently high and the share of risky human wealth is sufficiently low in total wealth to guarantee that consumption would not become negative in finite time with positive probability.

[47] Note that here we use the fact that the log-exponenial operator can be simplified to:

$$\ln\left(E_t\left[\exp\left(-\gamma J\left(s_{t+dt}\right)\right)\right]\right) = -\gamma A s_t - \gamma A_0 - \gamma A\left(rs_t - c_t\right) dt + \frac{1}{2}\gamma^2 A^2 \sigma_s^2 dt.$$

[48] Note that when $\delta = r$, i.e., the discount rate equals the interest rate, the consumption rule reduces to: $c_t^* = rs_t - \frac{1}{2}\gamma\sigma_s^2$, which means that consumption is independent of intertemporal substitution in this special case.

6.1.1. Consumption and Saving Rules under RB

To introduce robustness into the above recursive utility model, we follow the same procedure as in the previous section and write the distorting model by adding an endogenous distortion $v(s_t)$ to the law of motion of the state variable s_t,

$$ds_t = (rs_t - c_t)\,dt + \sigma_s\left(\sigma_s v(s_t)\,dt + dB_t\right). \tag{37}$$

The drift adjustment $v(s_t)$ is chosen to minimize the sum of the expected continuation payoff, but adjusted to reflect the additional drift component in (37), and of an entropy penalty:

$$0 = \sup_{c_t}\inf_{v_t}\left\{\delta c_t^{1-1/\varepsilon} - \delta\widetilde{J}(s_t) + \left(rs_t - c_t - \frac{1}{2}A\alpha\sigma_s^2\right)\widetilde{J}_s(s_t) + \sigma_s^2 v_t \widetilde{J}_s(s_t) + \frac{1}{2\vartheta_t}\sigma_s^2 v_t^2\right\},$$

where $\widetilde{J}(s_t) = (As_t + A_0)^{1-1/\varepsilon}$ and $\widetilde{J}_s(s_t) = (1-1/\varepsilon)\,A\,(As_t + A_0)^{-1/\varepsilon}$. The following proposition summarizes the solution to this RB problem:

Proposition 8. *Given ϑ, the optimal consumption and saving functions under robustness are*

$$c_t^* = rs_t + \Psi_t - \Gamma, \tag{38}$$

$$d_t^* = f_t - \Psi_t + \Gamma, \tag{39}$$

respectively, where $f_t = \rho(y_t - \overline{y})\,/\,(r+\rho)$ is the demand for savings "for a rainy day",

$$\Psi_t \equiv (\delta - r)\,\varepsilon s_t \tag{40}$$

captures the dissavings effect of relative impatience,

$$\Gamma \equiv \frac{1}{2}\frac{A}{r}\left(A^{1-\varepsilon}\delta^{\epsilon}\right)\widetilde{\gamma}\sigma_s^2 \tag{41}$$

is the precautionary savings demand, $\widetilde{\gamma} \equiv \gamma + \vartheta$ is the effective coefficient of absolute risk aversion, and

$$A = \left[\frac{r + (\delta - r)\,\varepsilon}{\delta^{\varepsilon}}\right]^{1/(1-\varepsilon)} \geq r, \tag{42}$$

Proof. See Online Appendix. ∎

When $\delta = r$, $A = r$ and this RU model is reduced to the benchmark model. The reason is that when the interest rate equals the discount rate, the effect of EIS on consumption growth and saving disappears. When $\delta \neq r$, A is increasing in ε. (We can see this from Figure 9.)[49] From (38), (40), and (41), we can see that EIS affects both the MPC out of s_t and Γ when $\delta \neq r$. Specifically, both MPC and the precautionary saving demand increases with ε when $\delta > r$. It is worth noting that the OE between the discount factor and a concern about robustness established in HST (1999) also no longer holds in this RU model. It is clear from (38) to (41) that δ affect the MPC, $r + (\delta - r)\,\varepsilon$, whereas ϑ does not appear in the MPC.

The saving function, (39), can be decomposed as follows:

$$d_t^* = f_t - \Psi_{1,t} - \Psi_2 + \Gamma, \tag{43}$$

where

$$\Psi_{1,t} \equiv (\delta - r)\,\varepsilon\,(s_t - \bar{s}) \text{ and } \Psi_2 \equiv (\delta - r)\,\varepsilon\bar{s}.$$

The term, $\Psi_t = \Psi_{1,t} + \Psi_{2,t}$, captures the dissaving effect due to relative impatience, which is affine in the value of total source, the sum of financial wealth and human wealth. Furthermore, $\Psi_{1,t}$ is a mean reverting process and Ψ_2 is a constant term. It is worth noting that this part of saving measures consumers' intertemporal consumption smoothing motive, and is independent of the degree of risk aversion and labor income uncertainty. Unlike the benchmark model with the time-additive utility, in the RU case the Ψ_t term increases with the value of total wealth (s_t) when the consumers are relatively more impatient, i.e., $\delta > r$. This result is consistent with that obtained in Wang (2006) in which the dissaving effect is generated by the endogenous discount factor. In addition, the Ψ_t term can also capture the intuition that richer consumers are more impatient and thus dissave more in the long run used to model the endogenous discount factor.

6.1.2. General Equilibrium Implications

Using the individual saving function (43) and following the same aggregation procedure used in the previous sections, we have the following result on the total saving demand:

Proposition 9. *Both the total demand of savings "for a rainy day" and the total demand for the estimation-risk-induced savings in the RB model with IC equal zero for any positive interest rate. That is,* $F_t(r) = \int_{y_t} f_t(r)\, d\Phi(y_t) = 0$ *and* $H_t(r) = \int_{s_t} \Psi_{1,t} d\Phi_s(s_t) = 0$*, for* $r > 0$.

Proof. The proof uses the LLN and is the same as that in Wang (2003). ■

[49] Empirical studies using aggregate data usually find the EIS to be close to zero, whereas calibrated RBC models usually require it to be close to one. For example, Hall (1988) found in the expected utility setting that the value of ε is close to 0.1. Guvenen (2006) allowed heterogeneity and estimated that the true value of ε is 0.47 in an economy with both stockholders who have high EIS and non-stockholders who have low EIS. Although theoretically we cannot rule out the $\varepsilon > 1$ case, we follow the literature and assume that $\varepsilon \leq 1$ in this paper.

This proposition states that the total savings "for a rainy day" is zero, at any positive interest rate. Therefore, from (43), after aggregating across all consumers, the expression for total savings in this RU model can be written as:

$$D(r) \equiv \Gamma(r) - \Psi_2(r), \tag{44}$$

where the first term measures the amount of precautionary savings due to risk aversion and uncertainty aversion, and the second term captures the steady state dissavings effects of impatience. As in the benchmark model, we define the equilibrium in our model as: $D(r^*) = 0$. The following proposition shows the existence of the equilibrium and the PIH holds in the general equilibrium:

Proposition 10. *There exists at least one equilibrium with an interest rate $r^* \in (0, \delta)$ in the RB model with IC. In any such equilibrium, each consumer's optimal consumption is described by the PIH, in that*

$$c_t^* = [r^* + (\delta - r)\varepsilon] s_t - (\delta - r)\varepsilon\bar{s}. \tag{45}$$

Furthermore, in this equilibrium, the evolution equations of wealth and consumption are

$$dw_t^* = (f_t - \Psi_{1,t})\, dt, \tag{46}$$

$$dc_t^* = [r^* + (\delta - r^*)\varepsilon]\, ds_t, \tag{47}$$

respectively. Finally, the relative volatility of consumption growth to income growth is

$$\mu \equiv \frac{\mathrm{sd}(dc_t^*)}{\mathrm{sd}(dy_t)} = \frac{r^* + (\delta - r^*)\varepsilon}{r^* + \rho}. \tag{48}$$

Proof. If $r > \delta$, $D(\vartheta, r^*) > 0$ because $\Gamma > 0$ and $\Psi_2 < 0$, which contradicts the equilibrium condition: $D(\vartheta, r^*) = 0$. When $r = \delta$, it is straightforward to show that $\Gamma > 0$ and $\Psi_2 = 0$, which implies that Since $\Gamma - \Psi_2 > 0$. When r converges to 0, $\Psi_2 > 0$ and Γ converges to 0 because the value of A/r converges to 1, which implies that $\Gamma - \Psi_2 < 0$. The continuity of the expression for total savings thus implies that there exists at least one interest rate $r^* \in (0, \delta)$ such that $D(r^*) = \Gamma - \Psi_2 = 0$. ■

We can establish that uniqueness obtains on $(0, \delta)$ under a restriction that households are sufficiently close to expected utility.

Proposition 11. *The equilibrium is unique if $\varepsilon > 0$ is small enough.*

Proof. We have
$$\frac{\partial D(\vartheta, r)}{\partial r} > 0$$
if
$$(2\varepsilon - 1) r^2 + (\rho - 3\delta\varepsilon) r - \delta\varepsilon\rho > 0.$$
There are no real roots of this quadratic if the discriminant is negative:
$$\Delta = (\rho - 3\delta\varepsilon)^2 + 4(2\varepsilon - 1)\delta\varepsilon\rho.$$
A necessary condition for $\Delta < 0$ is $0 < \varepsilon < \frac{1}{2}$; thus, necessary conditions for uniqueness are
$$\varepsilon < \frac{1}{2}$$
$$(2\varepsilon - 1) r^2 + (\rho - 3\delta\varepsilon) r - \delta\varepsilon\rho > 0.$$
At $\varepsilon = 0$ the second condition reduces to
$$\rho > r,$$
which holds as before if $\rho > \delta$. By continuity these conditions continue to be satisfied for ε close enough to zero, so that D is monotonic on $(0, \delta)$. ■

Following the same calibration procedure adopted in Section 4.2, we can easily calibrate the value of ϑ using the DEP. Specifically, given that $v^* = \vartheta A$, the DEP for this RU case, $p(\vartheta; N)$, can be expressed as:
$$p(\vartheta; N) = \Pr\left(x < -\frac{\overline{v}}{2}\sqrt{N}\right), \tag{49}$$
where $\overline{v} \equiv v^*\sigma_s = \vartheta A\sigma_s$. Since A increases with ε, (49) clearly shows that p decreases with ε for given values of ϑ. For example, when $\vartheta = 2.5$ and $\gamma = 2$, p decreases from $p = 0.3465$ to 0.3373 when ε increases from 0.5 to 0.9. That is, EIS does not have significant impacts on the amount of model uncertainty facing the consumer if we fix ϑ and allow for elastic model uncertainty. This result is not surprising because ε does not influence A significantly. (We can see this from Figure 9.)

Figure 10 shows that the aggregate saving function $D(r)$ is increasing with the interest rate, and there exists a unique interest rate r^* for different values of ε such that $D(r^*) = 0$.[50] From this figure, it is clear that that the equilibrium interest rate (r^*) increases with ε. That is, the larger the elasticity of intertemporal substitution, the larger the equilibrium interest rate. Furthermore, the

[50] As in the benchmark mode, here we also set that $\gamma = 2$ and $\vartheta = 1.5$

impact of ε on r^* is significant. For example, r^* increases from 0.88% to 1.87% as ε increases from 0.1 to 0.4. In addition, the impact of ε on μ is also significant. For example, μ decreases from 0.1173 to 0.2150 as ε increases from 0.1 to 0.4.

6.2. Incomplete Information about Individual Income Components

In this section, we consider a more realistic and interesting income specification. Following Wang (2004), we assume that labor income has two distinct components:

$$y_t = y_{1,t} + y_{2,t},$$

where

$$dy_{1,t} = (\mu_1 - \rho_1 y_{1,t})\, dt + \sigma_1 dB_{1,t}, \tag{50}$$

$$dy_{2,t} = (\mu_2 - \rho_2 y_{2,t})\, dt + \rho_{12}\sigma_2 dB_{1,t} + \sqrt{1-\rho_{12}}\sigma_2 dB_{2,t}, \tag{51}$$

and ρ_{12} is the instantaneous correlation between the two individual components, $y_{1,t}$ and $y_{2,t}$.[51] All the other notations are similar to that we used in our benchmark model. Without loss of generality, we assume that $\rho_1 < \rho_2$. That is, the first income component is a unit root and the second component is mean-reverting. It is straightforward to show that if both components in the income process are observable, this model is essentially the same as our benchmark model. We therefore consider a more interesting case in which consumers only observe total income but cannot observe the two individual components. In this incomplete-information case, we need to use the filtering technique to obtain the best estimates of the unobservable income components first and then solve the optimization problem given the estimated income components. Following the same technique adopted in Wang (2004), in the steady state in which the conditional variance-covariance matrix is constant, we can obtain the following updating equations for the conditional means of $(y_{1,t}, y_{2,t})$:

$$d\begin{pmatrix} \widehat{y}_{1,t} \\ \widehat{y}_{2,t} \end{pmatrix} = \begin{pmatrix} \mu_1 - \rho_1 \widehat{y}_{1,t} \\ \mu_2 - \rho_2 \widehat{y}_{2,t} \end{pmatrix} dt + \begin{pmatrix} \widehat{\sigma}_1 \\ \widehat{\sigma}_2 \end{pmatrix} dZ_t, \tag{52}$$

where $\widehat{y}_{i,t} = E_t\left[y_{i,t}\right]$ and Z_t is a standard Brownian motion. The standard deviations of $d\widehat{y}_{1,t}$ and $d\widehat{y}_{2,t}$ are:

$$\widehat{\sigma}_1 = \frac{1}{\sigma}\left[(\rho_2 - \rho_1)\Sigma_{11} + \sigma_1^2 + \sigma_{12}\right] \text{ and } \widehat{\sigma}_2 = \frac{1}{\sigma}\left[-(\rho_2 - \rho_1)\overline{\Sigma}_{11} + \sigma_2^2 + \sigma_{12}\right],$$

[51] Pischke (1995) considers a similar two-component income specification in a discrete-time setting.

respectively, where

$$\Sigma_{11} = \frac{1}{(\rho_2 - \rho_1)^2}\left(\sqrt{\Theta^2 + (1-\rho_{12}^2)\,\sigma_1^2\sigma_2^2\,(\rho_2-\rho_1)^2} - \Theta\right) \tag{53}$$

is the steady state conditional variance of $y_{1,t}$, $\sigma = \sqrt{\sigma_1^2 + 2\sigma_{12} + \sigma_2^2}$, $\Theta = \rho_1\sigma_2^2 + \rho_2\sigma_1^2 + (\rho_1+\rho_2)\,\sigma_{12}$, and $\sigma_{12} = \rho_{12}\sigma_1\sigma_2$.[52] It is worth noting that for this bi-variate Gaussian income specification, Σ_{11} can fully characterize the estimation risk induced by partially observed income. Figure 11 illustrates how Σ_{11} varies with ρ_2 and σ_2/σ_1.[53] It clearly shows that given the persistence and volatility coefficients of $y_{1,t}$, the estimation risk increases with the persistence and volatility of $y_{2,t}$ (i.e., the less ρ_2 and the higher σ_2/σ_1).

Following the same procedure used in the benchmark model, we can introduce RB into this incomplete-information (IC) model by assuming that the consumers take (52) as the approximating model. The corresponding distorted model can thus be written as:

$$d\widehat{y}_{1,t} = (\mu_1 - \rho_1\widehat{y}_{1,t})\,dt + \widehat{\sigma}_1\,(\widehat{\sigma}_1 v_{1,t}dt + dZ_t)\,, \tag{54}$$

$$d\widehat{y}_{2,t} = (\mu_2 - \rho_2\widehat{y}_{1,t})\,dt + \widehat{\sigma}_2\,(\widehat{\sigma}_2 v_{2,t}dt + dZ_t)\,, \tag{55}$$

where we denote $v_t \equiv \begin{bmatrix} v_{1,t} & v_{2,t} \end{bmatrix}^T$ the distortion vector chosen by the evil agent. Following the same procedure we use in the preceding sections, we can solve this IC model with RB. The following proposition summarizes the solution to this IC model:

Proposition 12. *Given ϑ, the consumption and saving functions under RB and IC are:*

$$c_t^* = r\left[w_t + \frac{1}{r+\rho_1}\left(\widehat{y}_{1,t} + \frac{\mu_1}{r}\right) + \frac{1}{r+\rho_2}\left(\widehat{y}_{2,t} + \frac{\mu_2}{r}\right)\right] + \Psi - \Gamma, \tag{56}$$

$$d_t^* = f_t + h_t + \Gamma - \Psi, \tag{57}$$

respectively, where $f_t = \rho_1\,(y_{1,t} - \overline{y}_1)\,/\,(r+\rho_1) + \rho_2\,(y_{2,t} - \overline{y}_2)\,/\,(r+\rho_2)$ captures the consumer's demand for savings "for a rainy day", $h_t = rx_t$ is the estimation-risk induced saving,

$$x_t = \frac{1}{r+\rho_1}\,(y_{1,t} - \widehat{y}_{1,t}) + \frac{1}{r+\rho_2}\,(y_{2,t} - \widehat{y}_{2,t})$$

[52]The detailed derivation of these equations is similar to that in Wang (2004) and is available from the corresponding author by request.

[53]Here we set $\rho_1 = 0$, $\sigma_1 = 0.05$, and $\rho_{12} = 0$. That is, the first income component is a unit root and the two components are independent. The pattern of the figure does not change if these parameters change. The only exception is the $\rho_{12} = \pm 1$ case. In this specifical, $\Sigma_{11} = 0$ because the two components are perfectly correlted and the bivariate income specificaiton is essentially the same as the univariate income specification.

is the estimation risk, $\Psi = (\delta - r) / (r\gamma)$ *captures the dissavings effect of relative impatience, and*

$$\Gamma = \frac{1}{2} r \widetilde{\gamma} \left(\frac{\widehat{\sigma}_1}{r + \rho_1} + \frac{\widehat{\sigma}_2}{r + \rho_2} \right)^2 \tag{58}$$

is the precautionary savings demand, where $\widetilde{\gamma} \equiv (1 + \vartheta) \gamma$.

Proof. See Online Appendix. ■

Expression (58) shows that for given ϑ, IC increases precautionary savings by increasing the variance of perceived permanent income from $\left(\frac{\sigma_1}{r+\rho_1}\right)^2 + 2\frac{\rho_{12}\sigma_1\sigma_2}{(r+\rho_1)(r+\rho_2)} + \left(\frac{\sigma_2}{r+\rho_2}\right)^2$ to $\left(\frac{\widehat{\sigma}_1}{r+\rho_1} + \frac{\widehat{\sigma}_2}{r+\rho_2}\right)^2$.[54] In other words, both RB and IC lead to higher precautionary savings. Using the individual saving function (57) and following the same aggregation procedure used in the previous sections, we have the following result on the total saving demand:

Proposition 13. *Both the total demand of savings "for a rainy day" and the total demand for the estimation-risk-induced savings in the RB model with IC equal zero for any positive interest rate. That is,* $F_t(r) = \int_{y_t} f_t(r)\, d\Phi(y_t) = 0$ *and* $H_t(r) = \int_{x_t} h_t(r)\, d\Phi_x(x_t) = 0$*, for* $r > 0$.

Proof. The proof uses the LLN and is the same as that in Wang (2003). ■

This proposition states that the total savings "for a rainy day" and for the estimation risk is zero, at any positive interest rate. Therefore, from (57), after aggregating across all consumers, we define the equilibrium in this RB model with SU as: $D(r^*) \equiv \Gamma(r^*) - \Psi(r^*) = 0$, where $D(r^*)$ measures aggregate savings in equilibrium. The following proposition shows the existence of the equilibrium and the PIH holds in the general equilibrium:

Proposition 14. *There exists at least one interest rate* $r^* \in (0, \delta)$ *in the RB model with IC. In any such equilibrium, each consumer's optimal consumption is described by the PIH, in that*

$$c_t^* = r^* \left[w_t + \frac{1}{r^* + \rho_1} \left(\widehat{y}_{1,t} + \frac{\mu_1}{r^*} \right) + \frac{1}{r^* + \rho_2} \left(\widehat{y}_{2,t} + \frac{\mu_2}{r^*} \right) \right]. \tag{59}$$

[54] Note that

$$\begin{aligned} \Delta &\equiv \left(\frac{\widehat{\sigma}_1}{r+\rho_1} + \frac{\widehat{\sigma}_2}{r+\rho_2} \right)^2 - \left[\left(\frac{\sigma_1}{r+\rho_1} \right)^2 + \frac{2\rho_{12}\sigma_1\sigma_2}{(r+\rho_1)(r+\rho_2)} + \left(\frac{\sigma_2}{r+\rho_2} \right)^2 \right] \\ &= 2r\Sigma_{11} \left[\frac{(\rho_2 - \rho_1)}{(r+\rho_1)(r+\rho_2)} \right]^2 > 0. \end{aligned}$$

Furthermore, in this equilibrium, the evolution equations of wealth and consumption are

$$dw_t^* = (f_t + h_t)\,dt, \tag{60}$$

$$dc_t^* = \left(\frac{r^*}{r^*+\rho_1}\sigma_{\widehat{y}_1} + \frac{r^*}{r^*+\rho_2}\sigma_{\widehat{y}_2}\right) dZ_t, \tag{61}$$

respectively, where $f_t = \rho_1 (y_{1,t} - \overline{y}_1) / (r^* + \rho_1) + \rho_2 (y_{2,t} - \overline{y}_2) / (r^* + \rho_2)$ *and* $h_t = r^* (y_{1,t} - \widehat{y}_{1,t}) / (r^* + \rho_1) + r^* (y_{2,t} - \widehat{y}_{2,t}) / (r^* + \rho_2)$. *Finally, the relative volatility of consumption growth to income growth is*

$$\mu \equiv \frac{\mathrm{sd}\,(dc_t^*)}{\mathrm{sd}\,(dy_t)} = \left[\left(\frac{r^*}{r^*+\rho_1}\widehat{\sigma}_1\right)^2 + \left(\frac{r^*}{r^*+\rho_2}\widehat{\sigma}_2\right)^2\right] / \sqrt{\sigma_1^2 + 2\sigma_{12} + \sigma_2^2}. \tag{62}$$

Proof. The proof is the same as that in the benchmark model in Section 3. ■

The sufficient conditions for uniqueness on $(0, \delta)$ are much more cumbersome than in the benchmark case and we omit them (they involve the discriminant of a cubic). Continuity will imply that the uniqueness result will still obtain if one component has small enough variance or the correlation of the two shocks is close enough to 1 or -1, since these special cases reduce to the univariate environment.

As in the above numerical analysis, we still set that $\gamma = 2$ and $\vartheta = 1.5$ when examining how IC interacts with RB in this model. In addition, we assume that $\rho_{12} = 0$ and $\rho_1 = 0$. That is, the two individual income components are uncorrelated and the first component is a unit root.[55] The upper panel of Figure 12 shows that the aggregate saving function $D(r)$ is increasing with the interest rate, and there exists a unique interest rate r^* for different values of σ_2/σ_1 such that $D(r^*) = 0$. From this figure, it is clear that that the equilibrium interest rate (r^*) decreases with σ_2/σ_1. That is, the larger the standard deviation of the transitory income innovation, the less the equilibrium interest rate. However, the impact of σ_2/σ_1 on r^* is not significant. For example, r^* decreases from 1.25% to 1.18% as σ_2/σ_1 increases from 0.2 to 2. In contrast, the impact of σ_2/σ_1 on μ is significant. For example, μ decreases from 0.9809 to 0.4473 as σ_2/σ_1 increases from 0.2 to 2. The lower panel of Figure 12 shows that the aggregate saving function $D(r)$ is increasing with the interest rate, and there exists a unique interest rate r^* for different values of ρ_2 such that $D(r^*) = 0$. This panel also shows that the equilibrium interest rate (r^*) increases with ρ_2. That is, the less the persistence of the transitory income component, the larger the equilibrium interest rate. In addition, we find that the impact of ρ_2 on r^* and μ are not very significant. For example, r^* and μ increase from 1.14% to 1.24% and decreases from 0.7108 to 0.7073, respectively, as ρ_2 increases from 0.1 to 0.5.

[55]Changing the values of ρ_{12} and ρ_1 does not change our main results here. By setting them to be 0, we can use ρ_2 and σ_2/σ_1 to characterize the degree of IC. In addition, we set σ_1 to be 0.05. Recall that this IC model can be reduced to the case with perfect information about income when $\rho_2 = \rho_1$.

6.3. Regime-Switching in Mean Income Growth

In this subsection, we consider aggregate uncertainty due to regime-switching. Specifically, we assume that the mean income growth parameter, μ, is no longer constant and is governed by a two-state continuous-time regime-switching process.[56] For simplicity, here we assume that there are two states for the macroeconomic condition: low-income growth (0) and high-income growth (1). Specifically, let $X_t = \{0, 1\}$ denote the regime for the economy's income growth $\mu_t = \{\mu_0, \mu_1\}$ at t. For a small time period Δt, the state of μ_t jumps from 1 to 0 with the transition probability $\pi_1 \Delta t$ and jumps from 0 to 1 with the transition probability $\pi_0 \Delta t$. The transition densities, π_1 and π_0, measure the persistence of the Markov chain:[57]

$$\pi\left(X_t\right) = \begin{cases} \pi_1, \ X_t = 1, \\ \pi_0, \ X_t = 0. \end{cases} \tag{63}$$

Under RB, the HJB can be written as

$$\delta J^1\left(w_t, y_t\right) = \max_{c_t} \left\{ u\left(c_t\right) + \left(rw_t + y_t - c_t\right) J^1_w + \left(\mu^1 - \rho y_t\right) J^1_y + \frac{1}{2}\sigma_y^2 J^1_{yy} - \frac{1}{2}\vartheta_t^1 \sigma_y^2 \left(J^1_y\right)^2 + \pi_1 \left(J^0 - J^1\right) \right\}, \tag{64}$$

$$\delta J^0\left(w_t, y_t\right) = \max_{c_t} \left\{ u\left(c_t\right) + \left(rw_t + y_t - c_t\right) J^0_w + \left(\mu^0 - \rho y_t\right) J^0_y + \frac{1}{2}\sigma_y^2 J^0_{yy} - \frac{1}{2}\vartheta_t^0 \sigma_y^2 \left(J^0_y\right)^2 + \pi_0 \left(J^1 - J^0\right) \right\}, \tag{65}$$

subject to the distorted equation:

$$dy_t = \rho\left(\frac{\mu^i}{\rho} - y_t\right) dt + \sigma_y \left(\sigma_y v_t^i + dB_t\right), \tag{66}$$

for $i = 0, 1$, where $J^i\left(w_t, y_t\right)$ is the value function when the macro state is i, and the last terms in (64) and (65) measure how regime-switching affects the expected change in the value function.

Following the same procedure as in the benchmark model, we can solve for robust consumption-portfolio rules under regime-switching. The following proposition summarizes the solution:

Proposition 15. *In the RB model with regime-switching, the consumption function and the saving function are*

$$c_t^i = rs_t^i + \Psi\left(r\right) - \Gamma^i\left(\vartheta, r\right), \tag{67}$$

[56] See Honda (2003) and Wang (2009) for studying optimal consumption and portfolio problem in a model with regime-switching.

[57] The steady state distribution of the good and bad states in this regime-switching model are thus $\frac{\pi_0}{\pi_1+\pi_0}$ and $\frac{\pi_1}{\pi_1+\pi_0}$, respectively.

and

$$d_t^i = f_t + \Gamma^i(\vartheta, r) - \Psi(r), \tag{68}$$

where $s_t^i = w_t + \frac{y_t}{r+\rho} + \frac{\mu^i}{r(r+\rho)}$, $f_t = \rho(y_t - \overline{y}) / (r+\rho)$, $\Psi(r) = \frac{\delta - r}{r\gamma}$,

$$\Gamma^i(\vartheta, r) \equiv \frac{r\widetilde{\gamma}\sigma_y^2}{2(r+\rho)^2} + \frac{\pi_i}{r\gamma}(\exp(r\gamma x) - 1), \tag{69}$$

where $\widetilde{\gamma} \equiv (1+\vartheta)\gamma$, *and* $x > 0$ *is determined by*

$$rx = \frac{\phi}{r+\rho} + \frac{\pi_1}{r\gamma}(1 - \exp(r\gamma x)) - \frac{\pi_0}{r\gamma}(1 - \exp(-r\gamma x)), \tag{70}$$

for $i = 1, 0$, *where* $\phi = \mu^1 - \mu^0$.[58]

Proof. See Online Appendix. ■

Expression (69) measures the precautionary saving demand in the regime-swtiching case. The first term in (69) is the same as the expression for the precautionary saving demand in the benchmark model. The second term is the precautionary saving demand induced by the stochastic regime-switching process.[59]

As in the benchmark model, we define the equilibrium in our model as: $D^i(r^*) = \Gamma^i(\vartheta, r^*) - \Psi(r^*) = 0$, i.e.,

$$\frac{\delta - r}{r\gamma} = \frac{r\gamma(1+\vartheta)\sigma_y^2}{2(r+\rho)^2} + \frac{\pi_i}{r\gamma}(\exp(r\gamma x) - 1), \tag{71}$$

where $x \cong \frac{\phi}{(r+\rho)(r+\pi_0+\pi_1)}$.[60]

The following proposition shows the existence of the equilibrium and the PIH holds in the RB general equilibrium:

Proposition 16. *There exists at least one equilibrium interest rate* $r^* \in (0, \delta)$ *in the RB model with regime-switching. In any such equilibrium, each consumer's optimal consumption is described by the PIH, in that*

$$c_t^{i,*} = r^{i,*} s_t^i, \tag{72}$$

[58] If $\pi_0 = \pi_1 = \pi$, (70) reduces to

$$rx = \frac{\phi}{r+\kappa} + \frac{\pi}{r\gamma}(\exp(-r\gamma x) - \exp(r\gamma x)).$$

If $\phi = 0$ (i.e., $\mu^1 = \mu^0$), $x = 0$.

[59] Note that when $x = 0$ or $\pi = 0$, this term reduces to 0.

[60] This approximation is accurate because the value of $r\gamma x$ is a small value in equilibrium.

where $s_t^i = w_t + \frac{y_t}{r+\kappa} + \frac{\mu^i}{r(r+\kappa)}$.

Proof. If $r > \delta$, both $\Gamma^i(\vartheta, r)$ and $-\Psi(r)$ are positive, which contradicts the equilibrium condition: $D(\vartheta, r^*) = 0$. Since $\Gamma^i(\vartheta, r) - \Psi(r) < 0$ (> 0) when $r = 0$ $(r = \delta)$, the continuity of the expression for total savings implies that there exists at least one interest rate $r^* \in (0, \delta)$ such that $D(\vartheta, r^*) = 0$. From (12), we can obtain the individual's optimal consumption rule under RB in general equilibrium as $c_t^{i,*} = r^{i,*} s_t^i$ for $i = 1, 0$. ■

As with the case above, the conditions that establish uniqueness are impenetrable and are therefore omitted. Again, we can use continuity to argue that if the regimes are sufficiently similar uniqueness still obtains on $(0, \delta)$.

To explore how regime-switching affects the equilibrium interest rate via interacting with robustness. We first set $\gamma = \vartheta = 1.5$ and $\pi_1 = \pi_0 = 0.1$.[61] Figure 13 shows how the gap between high- and low-income growth rates affects the equilibrium interest rate (r^*). It is clear from the figure that for given values of RB, the larger the value of ϕ $(= \mu^1 - \mu^0)$, the larger the value of x, and the less the equilibrium interest rate. We then study how the transition probability between the two states affects the equilibrium interest rate. Figure 14 shows the interest rate decreases with the transition probability π when $\pi_0 = \pi_1 = \pi$.[62] In this economy with RB, the precautionary saving demand due to regime-switching measured by ϕ and π further drives down the equilibrium interest rate.

However, if we assume that $\pi_0 > \pi_1$, the equilibrium interest rate is different across good and bad regimes.[63] When $\pi_1 = 10\%$ and $\pi_0 = 15\%$, the interest rate is 2.74% in the bad state, while it is 2.83% in the good state.[64] Under RB, if consumers are more concerned about model misspecification in a recession (i.e., the value of ϑ is higher in a recession), they choose to save more and thus further reduce the equilibrium interest rate.

7. Conclusions

This paper has developed a tractable continuous-time CARA-Gaussian framework to explore how model uncertainty due to robustness affects the interest rate and the dynamics of consumption and wealth in a general equilibrium heterogenous-agent economy. Using the explicit consumption-

[61] Wang (2009) considers two values, 0 and 3%, for the transition probabilities, π. In Krusell and Smith (1998), they set $\pi = 20\%$ such that the average duration of a boom or a recession is five years.

[62] Here we also set $\gamma = \vartheta = 1.5$ and $\phi = 0.04$.

[63] In post-WWII U.S. data, we observe that the average length of a boom is longer than that of a recession.

[64] Here we set $\gamma = \vartheta = 1.5$ and $\varphi = 3\%$. The results are robust for higher values of π_0. For example, holding all the other parameters fixed, if π_0 is increased from 0.15 to 0.2, the interest rate becomes 2.70% in the bad state and 2.86% in the good state.

saving rules, we explored the relative importance of robustness and risk aversion in determining precautionary savings. Furthermore, we evaluated the quantitative effects of model uncertainty measured by the interaction of labor income uncertainty and calibrated values of the RB parameter on the general equilibrium interest rate, consumption volatility, and the welfare costs of model uncertainty. Finally, we studied how RB interacts with recursive utility, incomplete information about income, and stochastic volatility in income, and affect the equilibrium interest rate and consumption volatility.

8. Appendix

8.1. Data and Sample Selections

This appendix provides details on how we select the sample and construct a panel of both household income and consumption for our empirical analysis.

The PSID does not include enough consumption expenditure data to create full picture of household nondurable consumption. Such detailed expenditures are found, though, in the Consumer Expenditure Survey (CEX) from the Bureau of Labor Statistics. But households in this study are only interviewed for four consecutive quarters and thus do not form a panel. To create a panel of consumption to match the PSID income measures, we use an estimated demand function for imputing nondurable consumption created by Guvenen and Smith (2014). Using an IV regression, they estimate a demand function for nondurable consumption that fits the detailed data in the CEX. The demand function uses demographic information and food consumption which can be found in both the CEX and PSID. Thus, we use this demand function of food consumption and demographic information (including age, family size, inflation measures, race, and education) to estimate nondurable consumption for PSID households, creating a consumption panel.

Our household sample selection closely follows that of Blundell et al. (2008) as well.[65] We exclude households in the PSID low-income and Latino samples. We exclude household incomes in years of family composition change, divorce or remarriage, and female headship. We also exclude incomes in years where the head or wife is under 30 or over 65, or is missing education, region, or income responses. We also exclude household incomes where non-financial income is less than \$1000, where year-over-year income change is greater than \$90,000, and where year-over-year consumption change is greater than \$50,000. Our final panel contains 7,220 unique households with 54,901 yearly income responses and 50,422 imputed nondurable consumption values.[66]

[65]They create a new panel series of consumption that combines information from PSID and CEX, focusing on the period when some of the largest changes in income inequality occurred.

[66]There are more household incomes than imputed consumption values because food consumption - the main input

In order to estimate the income process, we narrow the sample period to the years $1980-1996$, due to the PSID survey changing to a biennial schedule after 1996. To further restrict the sample to exclude households with dramatic year-over-year income and consumption changes, we eliminate household observations in years where either income or consumption has increased more than 200% or decreased more than 80% from the previous year.

8.2. Solving the Benchmark RB Model

The Bellman equation associated with the optimization problem is

$$J(s_t) = \sup_{c_t}\left[-\frac{1}{\gamma}\exp(-\gamma c_t) + \exp(-\delta dt) J(s_{t+dt})\right],$$

subject to (8), where $J(s_t)$ is the value function. The Hamilton-Jacobi-Bellman (HJB) equation for this problem is then

$$0 = \sup_{c_t}\left[-\frac{1}{\gamma}\exp(-\gamma c_t) - \delta J(s_t) + \mathcal{D}J(s_t)\right],$$

where $\mathcal{D}J(s_t) = J_s(rs_t - c_t) + \frac{1}{2}J_{ss}\sigma_s^2$. Under RB, the HJB can be written as

$$\sup_{c_t}\inf_{v_t}\left[-\frac{1}{\gamma}\exp(-\gamma c_t) - \delta J(s_t) + \mathcal{D}J(s_t) + v(s_t)\sigma_s^2 J_s + \frac{1}{2\vartheta(s_t)}v(s_t)^2\sigma_s^2\right]$$

subject to the distorting equation, (8). Solving first for the infimization part of the problem yields

$$v^*(s_t) = -\vartheta(s_t) J_s.$$

Given that $\vartheta(s_t) > 0$, the perturbation adds a negative drift term to the state transition equation because $J_s > 0$. Substituting for v^* in the robust HJB equation gives:

$$\sup_{c_t}\left[-\frac{1}{\gamma}\exp(-\gamma c_t) - \delta J(s_t) + (rs_t - c_t)J_s + \frac{1}{2}\sigma_s^2 J_{ss} - \frac{1}{2}\vartheta(s_t)\sigma_s^2 J_s^2\right]. \quad (73)$$

Performing the indicated optimization yields the first-order condition for c_t:

$$c_t = -\frac{1}{\gamma}\ln(J_s). \quad (74)$$

variable in Guvenen and Smith's nondurable demand function - is not reported in the PSID for the years 1987 and 1988. Dividing the total income responses by unique households yields an average of $7-8$ years of responses per household. These years are not necessarily consecutive as our sample selection procedure allows households to be excluded in certain years but return to the sample if they later meet the criteria once again.

Substituting (74) back into (73) to arrive at the partial differential equation (PDE):

$$0 = -\frac{J_s}{\gamma} - \delta J + \left(rs_t + \frac{1}{\gamma} \ln(J_s) \right) J_s + \frac{1}{2} \left(J_{ss} - \vartheta_t J_s^2 \right) \sigma_s^2. \tag{75}$$

Conjecture that the value function is of the form

$$J(s_t) = -\frac{1}{\alpha_1} \exp(-\alpha_0 - \alpha_1 s_t),$$

where α_0 and α_1 are constants to be determined. Using this conjecture, we obtain that $J_s = \exp(-\alpha_0 - \alpha_1 s_t) > 0$ and $J_{ss} = -\alpha_1 \exp(-\alpha_0 - \alpha_1 s_t) < 0$, and guess that

$$\vartheta(s_t) = -\frac{\vartheta}{J(s_t)} = \frac{\alpha_1 \vartheta}{\exp(-\alpha_0 - \alpha_1 s_t)} > 0.$$

(75) can thus be reduced to

$$-\delta \frac{1}{\alpha_1} = -\frac{1}{\gamma} + \left[rs_t - \left(\frac{\alpha_0}{\gamma} + \frac{\alpha_1}{\gamma} s_t \right) \right] - \frac{1}{2} \alpha_1 (1 + \vartheta) \sigma_s^2.$$

Collecting terms, the undetermined coefficients in the value function turn out to be

$$\alpha_1 = r\gamma \text{ and } \alpha_0 = \frac{\delta}{r} - 1 - \frac{1}{2} (1 + \vartheta) r\gamma^2 \sigma_s^2.$$

Substituting them back into the first-order condition (74) yields the consumption function, (12), in the main text.

Finally, we check if the consumer's transversality condition (TVC),

$$\lim_{t \to \infty} E\left[\exp(-\delta t) \left| J(s_t) \right| \right] = 0, \tag{76}$$

is satisfied. Substituting the consumption function, c_t^*, into the state transition equation for s_t yields:

$$ds_t = Adt + \sigma dB_t,$$

where $A = -\frac{\delta - r}{r\gamma} + \frac{1}{2} r \widetilde{\gamma} \sigma_s^2$ under the approximating model. This Brownina motion with drift can be rewritten as:

$$s_t = s_0 + At + \sigma (B_t - B_0), \tag{77}$$

where $B_t - B_0 \sim N(0, t)$. Substituting (77) into $E[\exp(-\delta t)|J(s_t)|]$ yields:

$$
\begin{aligned}
E[\exp(-\delta t)|J(s_t)|] &= \frac{1}{\alpha_1} E[\exp(-\delta t - \alpha_0 - \alpha_1 s_t)] \\
&= \frac{1}{\alpha_1} \exp\left(E[-\delta - \alpha_0 - \alpha_1 s_t] + \frac{1}{2} \operatorname{var}(\alpha_1 s_t) \right) \\
&= \frac{1}{\alpha_1} \exp\left(-\delta t - \alpha_0 - \alpha_1 (s_0 + At) + \frac{1}{2} \alpha_1^2 \sigma^2 t \right) \\
&= |J(s_0)| \exp\left(-\left(\delta + \alpha_1 A - \frac{1}{2} \alpha_1^2 \sigma^2 \right) t \right)
\end{aligned}
$$

where $|J(s_0)| = \frac{1}{\alpha_1} \exp(-\alpha_0 - \alpha_1 s_0)$ is a positive constant and we use the facts that $s_t - s_0 \sim N(At, \sigma^2 t)$. Therefore, the TVC, (76), is satisfied if and only if the following condition holds:

$$
\delta + \alpha_1 A - \frac{1}{2} \alpha_1^2 \sigma^2 = r + \frac{1}{2} (r\gamma)^2 \vartheta \sigma_s^2 > 0. \tag{78}
$$

Given the parameter values we consider in the text, it is obvious that the TVC is always satisfied in both the FI-RE and RB models. It is straightforward to show that the TVC still holds under the distorted model in which $A = -\frac{\delta - r}{r\gamma} + \frac{1}{2} r \widetilde{\gamma} \sigma_s^2 - r\gamma\vartheta\sigma_s^2$ for plausible values of ϑ.

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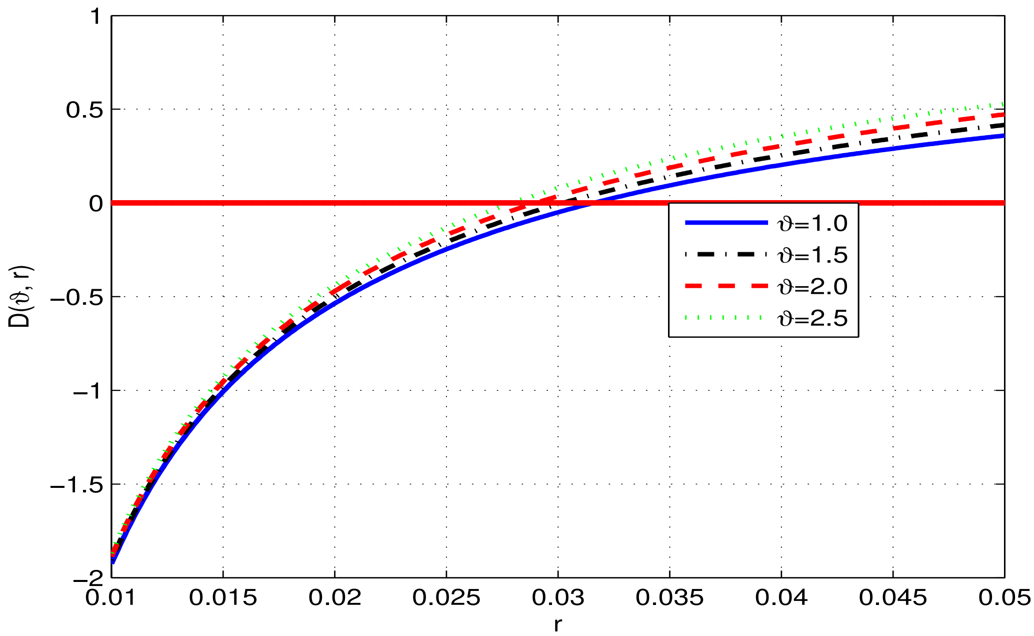


Figure 1. *Effects of RB on Aggregate Savings*

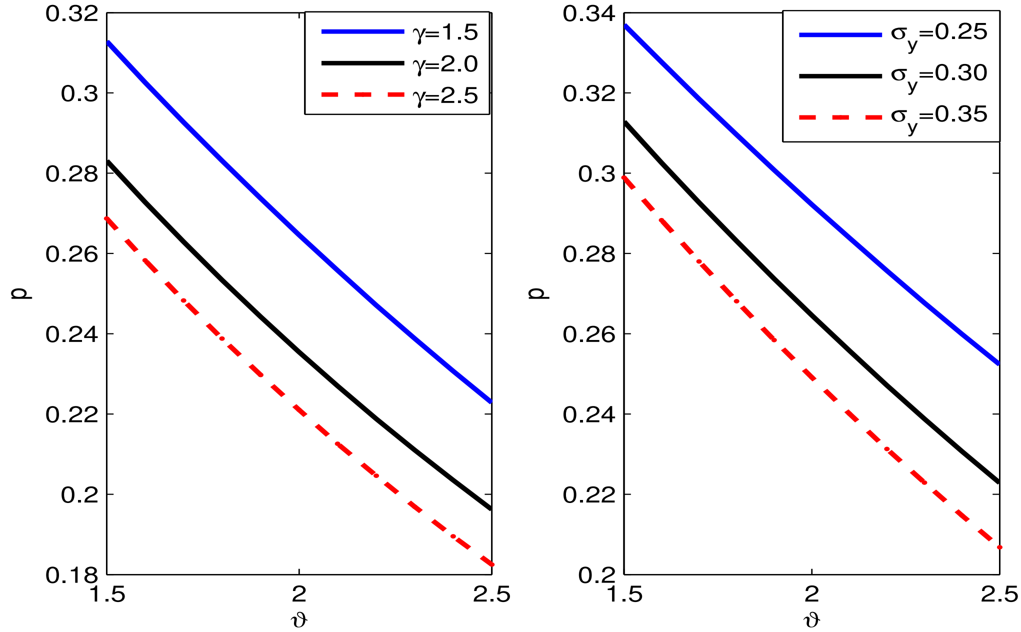


Figure 2. *Relationship between ϑ and p*

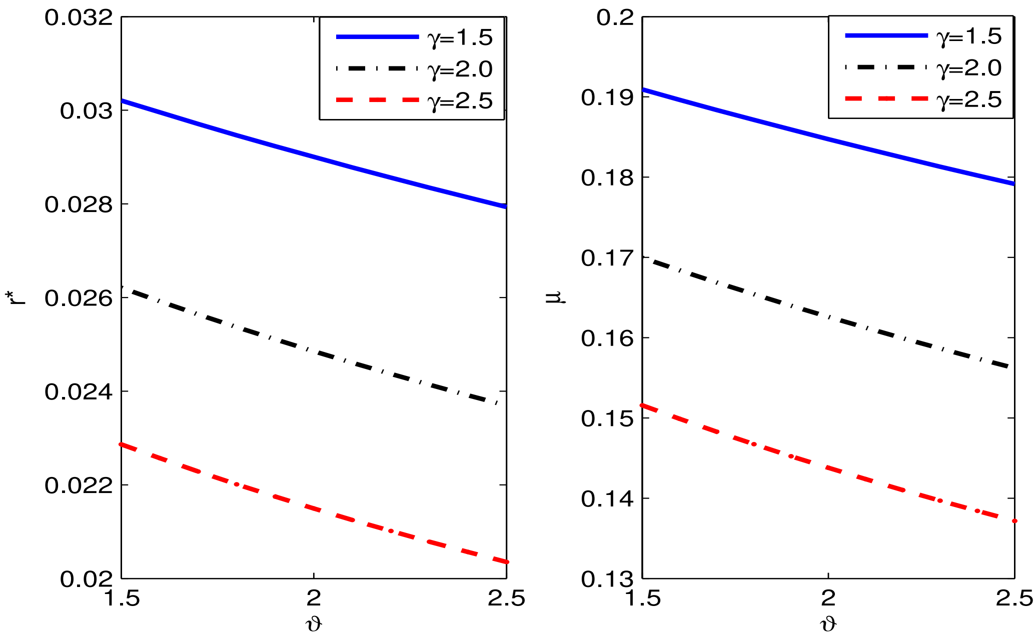


Figure 3. *Effects of RB on the Interest Rate and Consumption Volatility*

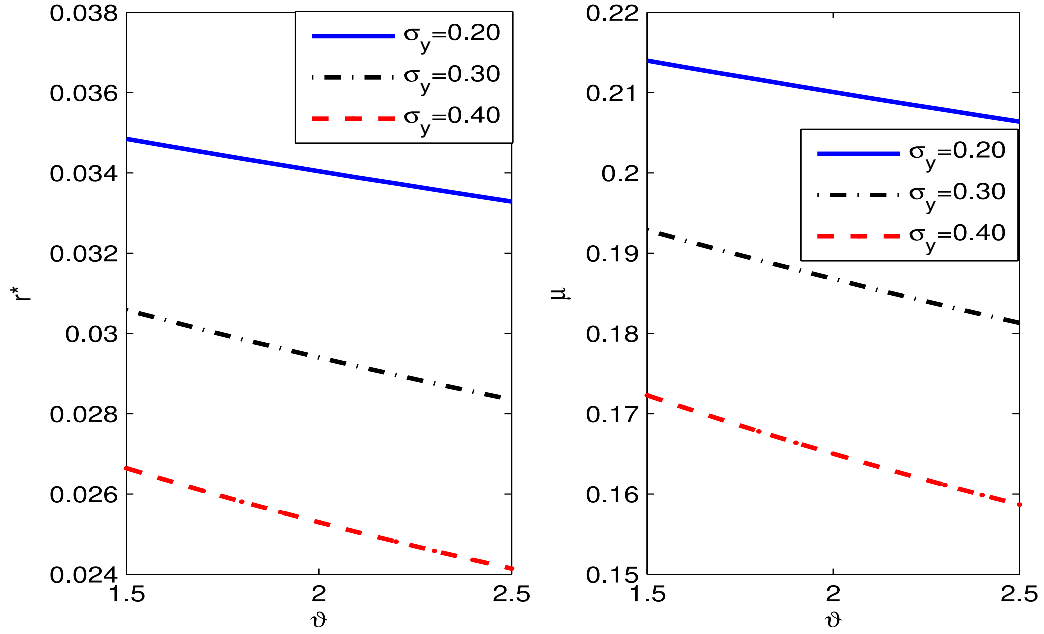


Figure 4. *Effects of RB on the Interest Rate and Consumption Volatility*

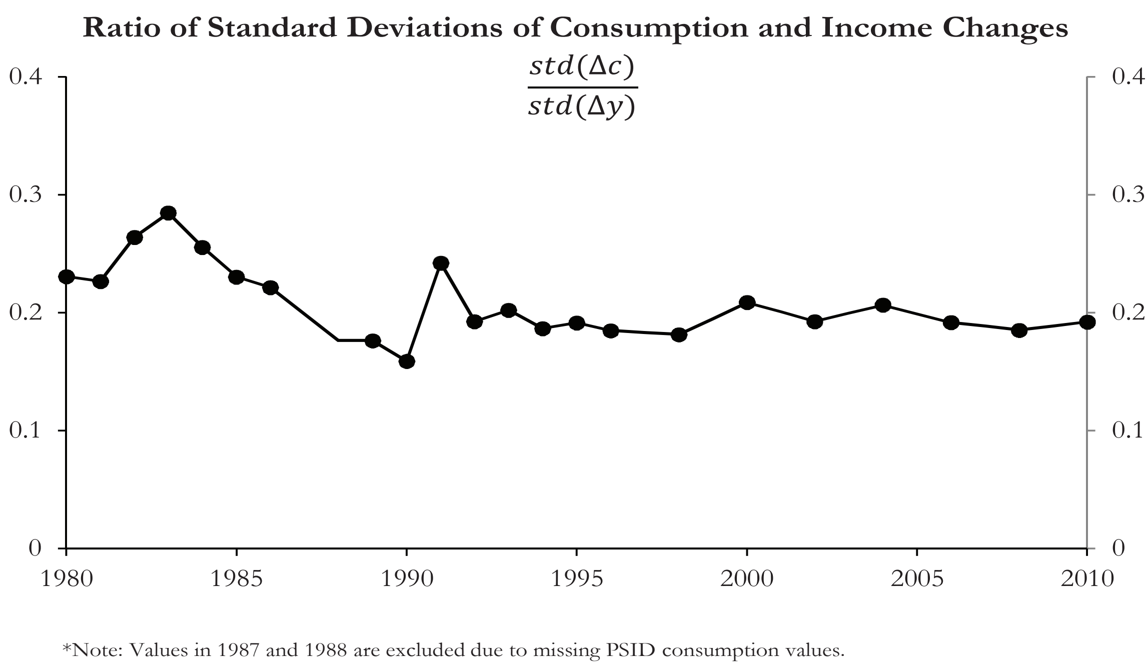


Figure 5. *Relative Consumption Dispersion*

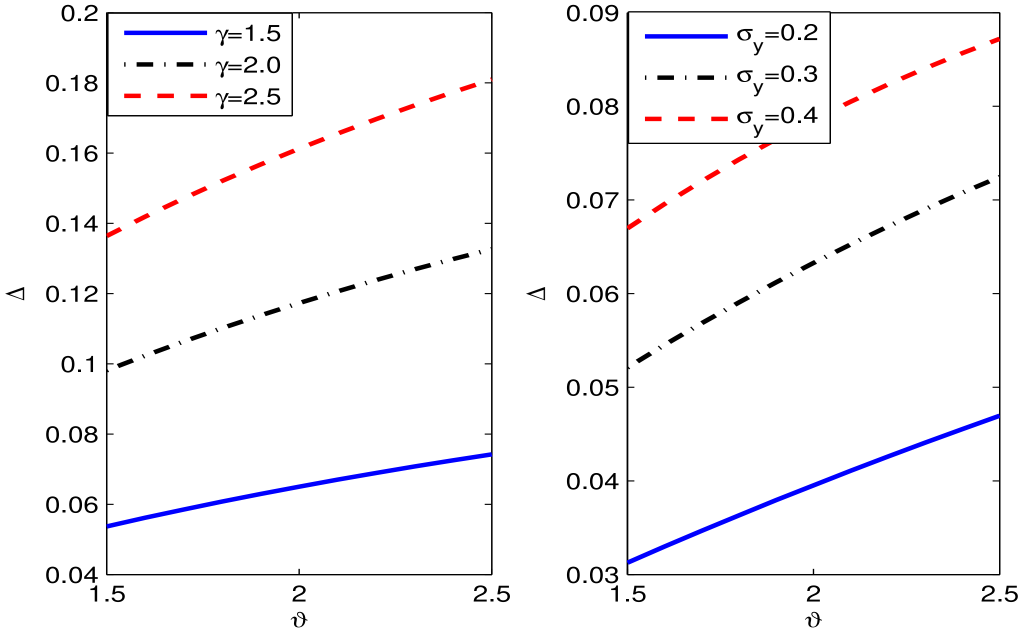


Figure 6. *Effects of RB on the Welfare Cost of Volatility*

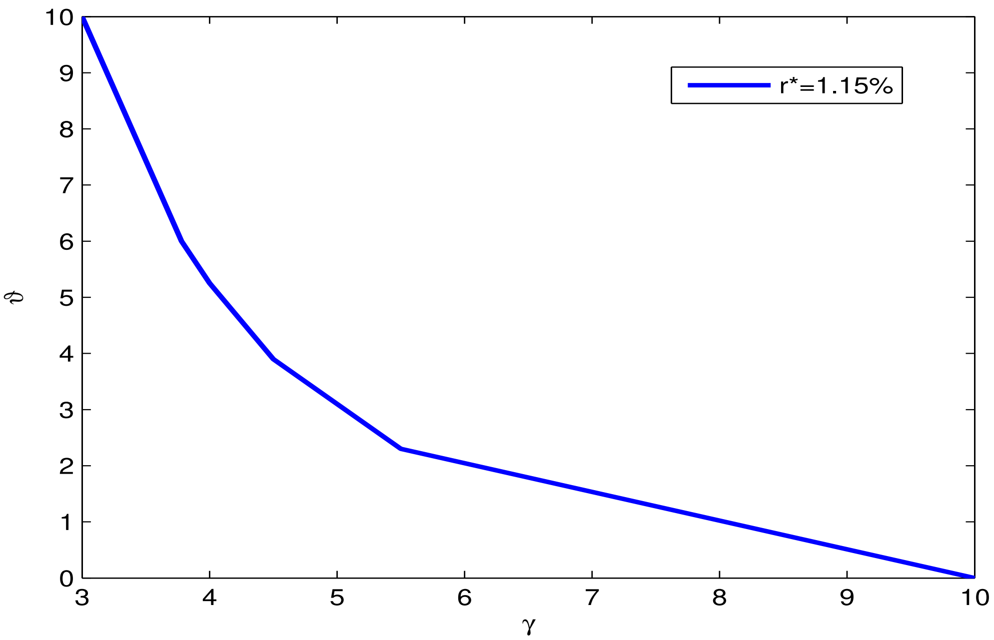
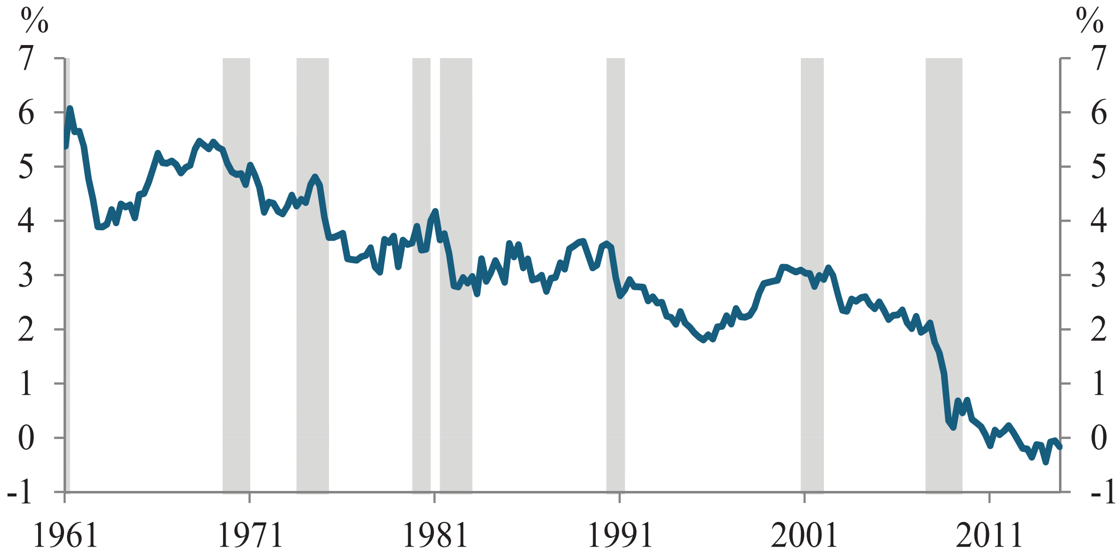


Figure 7. *Relationship between γ and ϑ.*



Note: Updated one-side estimates of the baseline model described in "Measuring the Natural rate of Interest," by Thomas Laubach and John C. Williams (2003). Estimates based on data available on February 18, 2015. Shaded bars indicate recessions. Final data point: 2014Q4.

Figure 8. *Estimates of Equilibrium Real Interest Rate in the U.S.*

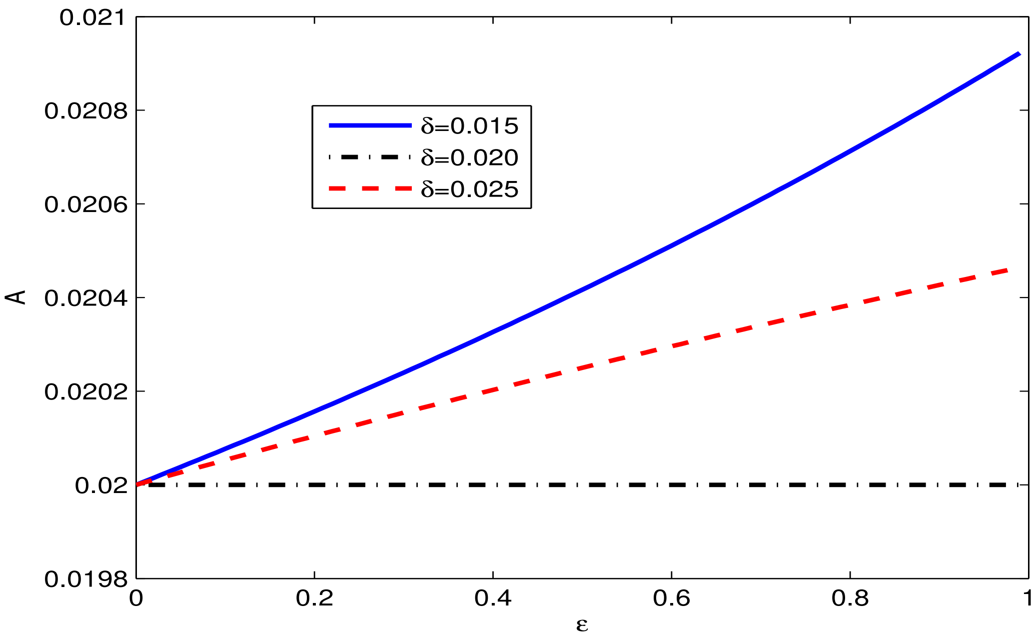


Figure 9. *Effects of ε on A*

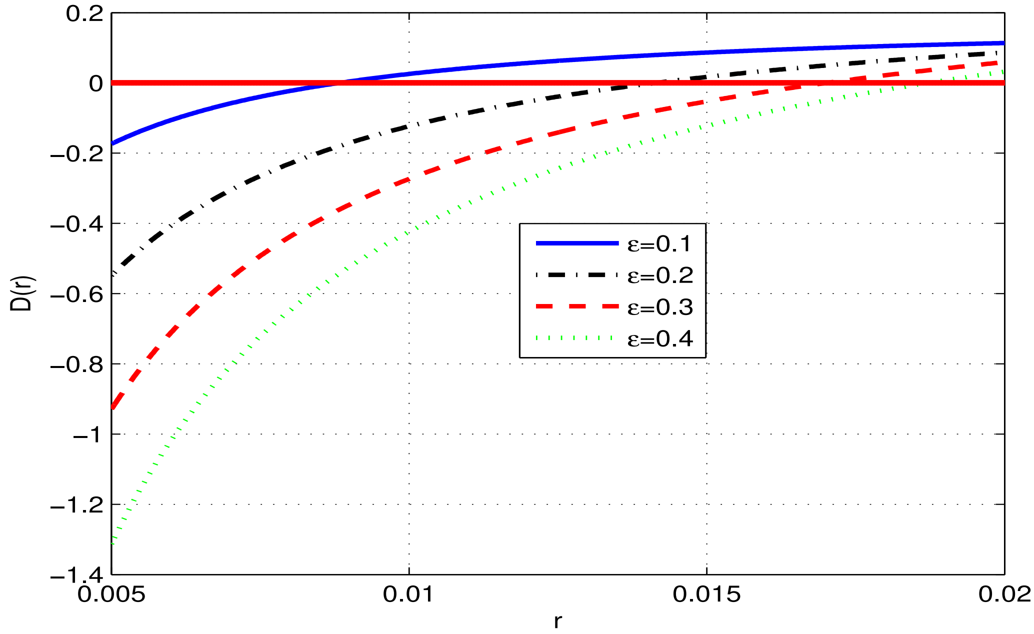


Figure 10. *Effects of EIS on Aggregate Savings*

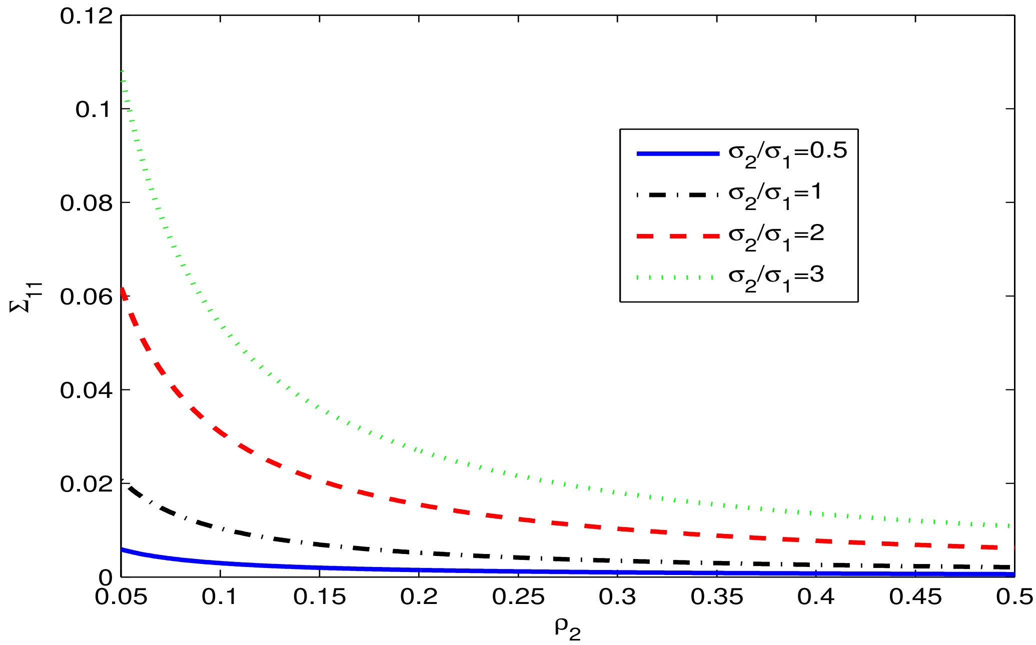


Figure 11. *Effects of Incomplete Information about Income on Estimation Risk*

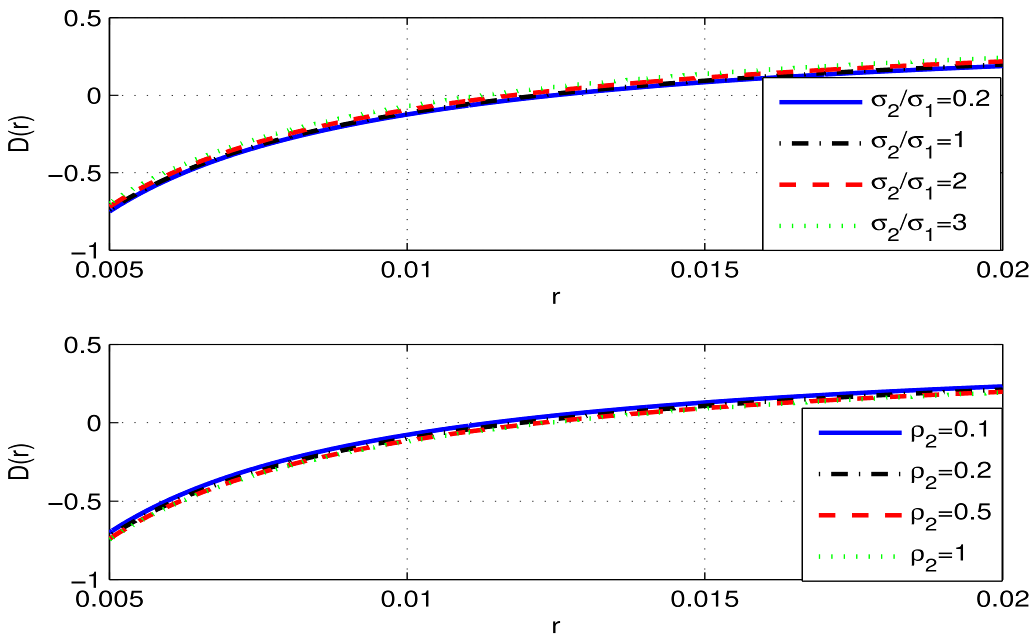


Figure 12. *Effects of Incomplete Information about Income on Aggregate Savings*

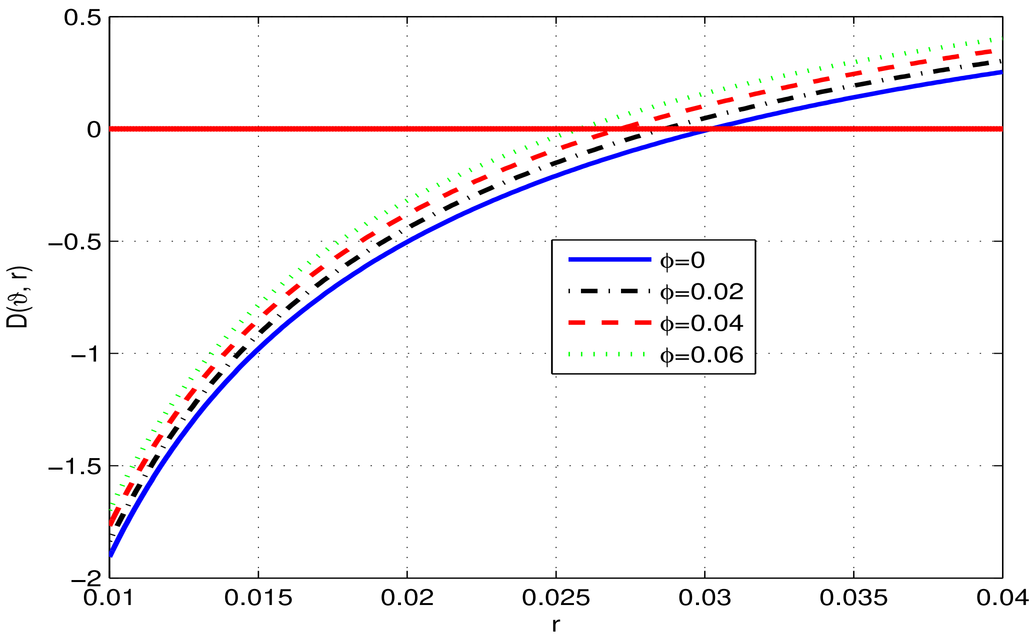


Figure 13. *Effects of RB on Aggregate Savings under Regime-Switching*

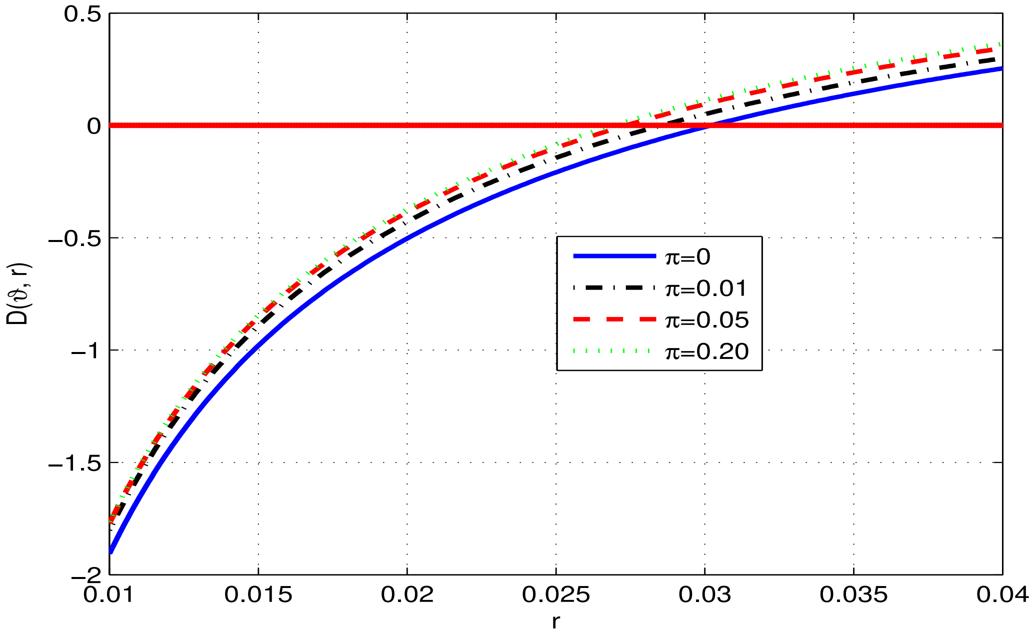


Figure 14. *Effects of RB on Aggregate Savings under Regime-Switching*

Table 1. *Estimation and Calibration Results*

	Parameter	Values
Discrete specification, eq. (23)		
constant	ϕ_0	0.0006
persistence	ϕ_1	0.88
std. of shock	σ	0.29
Continuous-time specification, eq. (1)		
persistence	ρ	0.128
std. of income changes	σ_y	0.309