MPRA
Munich Personal RePEc Archive

Designing Matching Mechanisms under General Distributional Constraints

Goto, Masahiro and Kojima, Fuhito and Kurata, Ryoji and Tamura, Akihisa and Yokoo, Makoto

Kyushu University, Stanford University, Keio University

29 April 2015

Online at https://mpra.ub.uni-muenchen.de/64000/
MPRA Paper No. 64000, posted 02 May 2015 07:03 UTC

Designing Matching Mechanisms under General Distributional Constraints

Masahiro Goto   Fuhito Kojima   Ryoji Kurata
Akihisa Tamura   Makoto Yokoo*

Abstract

In this paper, we consider two-sided, many-to-one matching problems where agents in one side of the market (schools) impose some distributional constraints (e.g., a maximum quota for a set of schools), and develop a strategyproof mechanism that can handle a very general class of distributional constraints. We assume distributional constraints are imposed on a vector, where each element is the number of contracts accepted for each school. The only requirement we impose on distributional constraints is that the family of vectors that satisfy distributional constraints must be hereditary, which means if a vector satisfies the constraints, any vector that is smaller than it also satisfies them. When distributional constraints are imposed, a stable matching may not exist. We develop a strategyproof mechanism called Adaptive Deferred Acceptance mechanism (ADA), which is nonwasteful and "more fair" than a simple nonwasteful mechanism called the Serial Dictatorship mechanism (SD) and "less wasteful" than another simple fair mechanism called the Artificial Cap Deferred Acceptance mechanism (ACDA). We show that we can apply this mechanism even

*Goto, Kurata, Yokoo: Department of Electrical Engineering and Computer Science, Kyushu University, Fukuoka 819-0395, Japan. Kojima: Department of Economics, Stanford University, Stanford, CA, 94305, United States. Tamura: Department of Mathematics, Keio University, Yokohama 223-8522, Japan. Emails: goto@agent.inf.kyushu-u.ac.jp, fkojima@stanford.edu, kurata@agent.inf.kyushu-u.ac.jp, aki-tamura@math.keio.ac.jp, yokoo@inf.kyushu-u.ac.jp. Fanqi Shi provided excellent research assistance. Tamura and Yokoo acknowledge the financial support from JSPS Kakenhi Grant Number 24220003. Kojima acknowledges the financial support from the National Research Foundation through its Global Research Network Grant (NRF-2013S1A2A2035408) as well as the Sloan Foundation.

if the distributional constraints do not satisfy the hereditary condition by applying a simple trick, assuming we can find a vector that satisfy the distributional constraints efficiently. Furthermore, we demonstrate the applicability of our model in actual application domains.
JEL Classification: C78, D61, D63
Keywords: two-sided matching, many-to-one matching, market design, matching with contracts, matching with constraints, strategyproofness, deferred acceptance.

1 Introduction

The theory of two-sided matching has been extensively developed, and it has been applied to design clearinghouse mechanisms in various markets in practice.[1] As the theory has been applied to increasingly diverse types of environments, however, researchers and practitioners have encountered various forms of distributional constraints. As these features have been precluded from consideration until recently, they pose new challenges for market designers.

The *regional maximum quotas* provide one such example. Under the regional maximum quotas, each agent on one side of the market (which we call a school) belongs to a region, and each region has an upper bound on the number of agents on the other side (who we call students) who can be matched in each region. Regional maximum quotas exist in many markets in practice. A case in point is Japan Residency Matching Program (JRMP), which organizes matching between medical residents and hospitals in Japan. Although JRMP initially employed the standard Deferred Acceptance mechanism (DA) Gale and Shapley (1962), it was criticized as placing too many doctors in urban areas and causing doctor shortage in rural areas. To address this criticism, Japanese government now imposes a regional maximum quota to each region of the country. Regulations that are mathematically isomorphic to regional maximum quotas are utilized in various contexts, such as Chinese graduate admission, Ukrainian college admission, Scottish probationary teacher matching, among others Kamada and Kojima (2015).

Furthermore, there are many matching problems in which *minimum quotas* are imposed. School districts may need at least a certain number of stu-

[1] See Roth and Sotomayor (1990) for a comprehensive survey of many results in this literature.

dents in each school in order for the school to operate, as in college admissions in Hungary Biro, Fleiner, Irving, and Manlove (2010). The cadet-branch matching program organized by United States Military Academy (USMA) imposes minimum quotas on the number of cadets who can be assigned to each branch Sönmez and Switzer (2013). Yet another type of constraints takes the form of *diversity constraints*. Public schools are often required to satisfy balance on the composition of students, typically in terms of socioeconomic status Ehlers, Hafalir, Yenmez, and Yildirim (2014). Several mechanisms have been proposed Ehlers, Hafalir, Yenmez, and Yildirim (2014); Fragiadakis, Iwasaki, Troyan, Ueda, and Yokoo (2015); Goto, Hashimoto, Iwasaki, Kawasaki, Ueda, Yasuda, and Yokoo (2014); Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014); Kamada and Kojima (2015) for each of these various constraints, but previous studies have focused on tailoring mechanisms to specific settings, rather than providing a general framework.

One notable exception is Kojima, Tamura, and Yokoo (2014), in which a general framework for handling various distributional constraints is developed, in the setting of 'matching-with-contracts' Hatfield and Milgrom (2005). Kojima, Tamura, and Yokoo (2014) assume school priorities and distributional constraints are aggregated into a preference of a representative agent, "the schools," as in Kamada and Kojima (2015).[2] They show that if the schools' aggregated preference can be represented by an M$^\natural$-concave function Murota (2003), then the generalized Deferred Acceptance mechanism (generalized DA) is strategyproof and obtains the student-optimal Hatfield-Milgrom (HM)-stable matching. Developing such a general framework and a general mechanism is important since they can contribute to the advance of practical market design (or "economic engineering") as emphasized in the recent literature (see Roth (2002) and Milgrom (2009) for instance), by providing tools for organizing matching clearinghouses in practice.

Kojima, Tamura, and Yokoo (2014) show that in order to represent schools' preferences as an M$^\natural$-concave function, the family of contracts that satisfy hard distributional constraints must form a mathematical structure called a *matroid* Murota (2003). Usually, distributional constraints are imposed on a vector, where each element is the number of contracts accepted for each school, rather than on concrete contracts. The fact that the family of contracts forms a matroid corresponds to the fact that (i) the family of vectors forms an M$^\natural$-convex set, and (ii) it is *hereditary*, which means if a

[2] Kojima, Tamura, and Yokoo (2014) use a term "hospital" instead of "school".

vector satisfies constraints, any vector that is smaller than it also satisfies constraints.

In this paper, we develop a mechanism that can handle more general distributional constraints than Kojima, Tamura, and Yokoo (2014). The only requirement we impose on distributional constraints is that the family of vectors that satisfy distributional constraints must be hereditary.

In a standard definition, a matching is stable if it is fair and nonwasteful.[3] When distributional constraints are imposed, a stable matching may not exist. If we completely ignore fairness or nonwastefulness, we can employ trivial strategyproof mechanisms in our setting.

More specifically, we can use the Serial Dictatorship mechanism (SD) to achieve nonwastefulness. In the SD, we assume a common priority ordering among students called a master list is given. Students are assigned sequentially according to the master list. A student s is allowed to be assigned to a school c if doing so would not cause any constraint violation. Then, s can choose her most preferred school within allowed schools.

Also, we can use the Artificial Cap Deferred Acceptance (ACDA) mechanism to achieve fairness, i.e., we artificially lower the maximum quota of each school such that the DA obtains a set of contracts that satisfies all distributional constraints.

However, the limitations of these mechanisms are that the SD can be extremely unfair and the ACDA can be extremely wasteful (thus it sacrifices students' welfare too much). As a result, it would be difficult to apply these mechanisms in real application domains. In this paper, we develop a strategyproof and nonwasteful mechanism called Adaptive Deferred Acceptance (ADA) mechanism, which is "more fair" than the SD, and "less wasteful" than the ACDA. This mechanism can be useful even in the case where the family of vectors forms an $\mathrm{M}^{\natural}$-convex set and we can apply the generalized DA, assuming the welfare of students is the primary concern, while the fairness among students is the secondary concern.

The rest of this paper is organized as follows. In Section 2, we show a standard model without any distributional constraints and a model with general distributional constraints. Next, in Section 3, we introduce two baseline mechanisms, i.e., the SD and the ACDA. Then, in Section 4, we introduce the ADA and show its properties. Next, in Section 5, we discuss how to han-

[3]Note that HM-stability used in Kojima, Tamura, and Yokoo (2014) is different from the standard stability when distributional constraints are imposed.

dle non-hereditary constraints. In Section 6, we investigate how to represent actual application domains in our model. Finally, Section 7 concludes this paper.

2 Model

In this section, we first show a standard model without any distributional constraints. Then, we show our model with a very general class of distributional constraints.

2.1 Standard model

A standard matching market is given by $(S, C, X, \succ_S, \succ_C, q_C)$. The meaning of each element is as follows.

- $S = \{s_1, \dots, s_n\}$ is a set of students.
- $C = \{c_1, \dots, c_m\}$ is a set of schools.
- $X \subseteq S \times C$ is a finite set of contracts. $x = (s, c) \in X$ means student s is matched to school c.
- For $X' \subseteq X$, let X'_s denote $\{(s, c) \in X' \mid c \in C\}$, and X'_c denote $\{(s, c) \in X' \mid s \in S\}$.
- $\succ_S = (\succ_{s_1}, \dots, \succ_{s_n})$ is a profile of students' preferences. Each $\succ_s$ represents the strict preference of each student s over acceptable contracts within $X_s = \{(s, c) \in X \mid c \in C\}$.
- $\succ_C = (\succ_{c_1}, \dots, \succ_{c_m})$ is a profile of schools' priorities. Each $\succ_c$ represents the strict priority of each school c over contracts within $X_c = \{(s, c) \in X \mid s \in S\}$.
- $q_C = (q_{c_1}, \dots, q_{c_m})$ is a vector of school's maximum quotas.

We assume each contract x in X_c is acceptable for c. If some contract is unacceptable for a school, we assume it is not included in X.

First, let us introduce several concepts related to feasibility.

Definition 1 (feasibility)**.** We say X' is school-feasible if for all $c \in C$, $|X'_c| \leq q_c$ holds. We say X' is student-feasible if for all $s \in S$, either (i) $X'_s = \{x\}$ and x is acceptable for s, or (ii) $X'_s = \emptyset$ holds. We say X' is feasible if it is student- and school-feasible. We call a feasible set of contracts matching.

Next, let us introduce choice functions for students and schools.

Definition 2 (choice functions)**.** For each student s, its choice function $Ch_s(X')$ specifies her most preferred contract within $X' \subseteq X$, i.e., $Ch_s(X') = \{x\}$, where x is the most preferred acceptable contract in X'_s if one exists, and $Ch_s(X') = \emptyset$ if no such contract exists. Then, the choice function of all students is defined as $Ch_S(X') := \bigcup_{s \in S} Ch_s(X')$.

For each school c, its choice function $Ch_c(X')$ chooses top q_c contracts within X'_c according to $\succ_c$. If $|X'_c| \leq q_c$, $Ch_c(X') = X'_c$. Then, the choice function of all schools is defined as $Ch_C(X') := \bigcup_{c \in C} Ch_c(X')$.

By these choice functions, the Deferred Acceptance mechanism (DA) for the 'matching with contracts' model can be defined as follows.[4]

Mechanism 1 (Deferred Acceptance mechanism (DA))**.**

1. $R \leftarrow \emptyset$.
2. $X' \leftarrow Ch_S(X \setminus R), X'' \leftarrow Ch_C(X')$.
3. If $X' = X''$ then return X', otherwise, $R \leftarrow R \cup (X' \setminus X'')$, go to (2).

2.2 Model with distributional constraints

A matching market under distributional constraints is given by: $(S, C, X, \succ_S, \succ_C, q_C, \eta)$. The only additional element to the standard model is a function $\eta : \mathbb{Z}^m \to \{-\infty, 0\}$, where m is the number of schools. We assume η represents distributional constraints.

Definition 3 (feasibility with distributional constraints)**.** We say ζ, which is a vector of m natural numbers, is admissible in η if $\eta(\zeta) = 0$. For $X' \subseteq X$, let us define $\zeta(X')$ as $(|X'_{c_1}|, |X'_{c_2}|, \ldots, |X'_{c_m}|)$. We say X' is school-feasible if $\zeta(X')$ is admissible in η.

[4] In Hatfield and Milgrom (2005), this mechanism is called *generalized Gale-Shapley* algorithm.

We assume η respects maximum quotas and the total number of students, i.e., if $\zeta_i > q_{c_i}$ for some $i \in M$, then $\eta(\zeta) = -\infty$, and if $\sum_{i \in M} \zeta_i > n$, then $\eta(\zeta) = -\infty$. In a standard two-sided matching market, X' is school-feasible if for all $c \in C$, $|X'_c| \leq q_c$ holds. By introducing distributional constraints, in order for X' to be school-feasible, $\zeta(X')$ must be admissible in η.

For two m-element vectors $\zeta, \zeta' \in \mathbb{Z}^m$, we say $\zeta \leq \zeta'$ if for all $i \in M$, $\zeta_i \leq \zeta'_i$ holds. Also, we say $\zeta < \zeta'$ if $\zeta \leq \zeta'$ and for some $i \in M$, $\zeta_i < \zeta'_i$ holds.

Definition 4 (heredity)**.** We say η is hereditary if for all $\zeta, \zeta' \in \mathbb{Z}^m$, where $\zeta > \zeta'$, if ζ is admissible in η, then ζ' is also admissible in η.

Assume η is hereditary. Then, if X' is feasible, $X'' \subset X'$ is also feasible.

Let χ_i denote an m-element unit vector, where i-th element is 1 and other elements are 0. Let χ_0 denote an m-element zero vector $(0, \ldots, 0)$ and M denote $\{1, \ldots, m\}$.

Definition 5 (M$^\natural$-convex set)**.** We say a family of m-element vectors $F \subseteq \mathbb{Z}^m$ forms an M$^\natural$-convex set, if for all $\zeta, \zeta' \in F$, for all i such that $\zeta_i > \zeta'_i$, there exists $j \in \{0\} \cup \{k \in M \mid \zeta_k < \zeta'_k\}$ such that $\zeta - \chi_i + \chi_j \in F$ and $\zeta' + \chi_i - \chi_j \in F$ hold.

Kojima, Tamura, and Yokoo (2014) shows that to apply their framework, it is required that the family of admissible vectors is an M$^\natural$-convex set, as well as hereditary.

In the rest of this paper, we assume η is hereditary. This is the only requirement we impose on η, i.e., we do not require it forms an M$^\natural$-convex set. Thus, our model of distributional constraints is quite general, and strictly more general than the model in Kojima, Tamura, and Yokoo (2014). Kamada and Kojima (2014) also examine this general case, and characterize the condition where a stable matching exists. Some distributional constraints do not satisfy heredity. For example, if a minimal quota is imposed, i.e., a certain number of students must be assigned to a school, or each student must be assigned to some school, it is clear that heredity is not satisfied. We describe a method to handle such distributional constraints in our model in Section 5.

Let us introduce a simple example.

Example 2.1. Let us consider the following situation.

- There are six students $s_1, s_2, \ldots, s_6$.

- There are four schools c_1, c_2, c_3, and c_4.
- The preferences of all students are the same: $c_1 \succ_s c_2 \succ_s c_3 \succ_s c_4$.
- The priorities of all schools are the same: $s_6 \succ_c s_5 \succ_c \ldots \succ_c s_1$.
- The maximum quota of each school is 2.
- We require the total number of students accepted for each of the following groups must be at most 3: $\{c_1, c_2\}, \{c_3, c_4\}, \{c_1, c_3\}, \{c_2, c_4\}$.

Then, $\eta(\zeta) = 0$ if $\zeta \leq (2, 1, 1, 2)$ or $\zeta \leq (1, 2, 2, 1)$. This η is hereditary, but the family of admissible vectors does not form an M$^\natural$-convex set. For example, if we choose $\zeta = (2, 1, 1, 2)$ and $\zeta' = (1, 2, 2, 1)$, for χ_1, there exits no $j \in \{0\} \cup \{k \in M \mid \zeta_k < \zeta'_k\}$ such that $\eta(\zeta - \chi_1 + \chi_j) = 0$ holds.

With a slight abuse of notation, for two sets of contracts X' and X'', we denote $X'_s \succ_s X''_s$ if either (i) $X'_s = \{x'\}$, $X''_s = \{x''\}$, and $x' \succ_s x''$ for some $x', x'' \in X_s$ that are acceptable for s, or (ii) $X'_s = \{x'\}$ for some $x' \in X_s$ that is acceptable for s and $X''_s = \emptyset$. Furthermore, we denote $X'_s \succeq_s X''_s$ if either $X'_s \succ_s X''_s$ or $X'_s = X''_s$. Also, we use notations like $x \succ_s X'_s$ or $X'_s \succ_s x$, where x is a contract and X' is a matching. Furthermore, for $X'_s \subseteq X_s$, we say X'_s is acceptable for s if either (i) $X'_s = \{x\}$ and x is acceptable for s, or (ii) $X'_s = \emptyset$ holds. Also, when describing $\succ_s$ or $\succ_c$, we sometimes write $c_1 \succ_s c_2$ instead of $(s, c_1) \succ_s (s, c_2)$, and $s_1 \succ_c s_2$ instead of $(s_1, c) \succ_c (s_2, c)$.

Let us introduce several desirable properties of a matching and a mechanism.

Definition 6 (nonwastefulness)**.** In a matching X', a student s claims an empty seat of school c, if $(s, c) \in X \setminus X'$, (s, c) is acceptable for s, $(s, c) \succ_s X'_s$, and $(X' \setminus X'_s) \cup \{(s, c)\}$ is feasible.

We say a matching X' is nonwasteful if no student claims an empty seat. We say a mechanism is nonwasteful if it produces a nonwasteful matching for every possible profile of the preferences and priorities.

Definition 7 (fairness)**.** In a matching X', a student s has justified envy towards another student s' if $(s, c) \in X \setminus X'$, (s, c) is acceptable for s, $(s, c) \succ_s X'_s$, $(s', c) \in X'$, and $(s, c) \succ_c (s', c)$ hold.

We say a matching X' is fair if no student has justified envy. We say a mechanism is fair if it produces a fair matching for every possible profile of the preferences and priorities.

Definition 8 (stability). We say a matching X' is stable if no student has justified envy and no student claims an empty seat.

When additional distributional constraints are imposed, it is common that fairness and nonwastefulness are incompatible. This is true even for simple constraints such as for a subset of schools $C' \subset C$, the total number of students accepted in these schools is bounded (i.e., regional maximum quota) Kamada and Kojima (2014).

For mechanism φ and students' preference profile $\succ_S$, let $\varphi(\succ_S)$ denote the obtained matching of φ. Also, $\varphi_s(\succ_S)$ denote $\{(s, c) \mid (s, c) \in \varphi(\succ_S), c \in C\}$ and $\varphi_c(\succ_S)$ denote $\{(s, c) \mid (s, c) \in \varphi(\succ_S), s \in S\}$. Furthermore, let $(\succ_c, \succ_{-c})$ denote the preference profile in which the preference of student s is $\succ_s$ and the profile of other agents' preferences is $\succ_{-s}$.

Definition 9 (strategyproofness). We say a mechanism φ is strategyproof if no student ever has any incentive to misreport her preference, no matter what the other students report, i.e., for all $\succ_s$, $\tilde{\succ}_s$, and $\succ_{-s}$, $\varphi((\succ_s, \succ_{-s})) \succeq_s \varphi((\tilde{\succ}_s, \succ_{-s}))$ holds.

When no additional distributional constraints are imposed, the DA (Mechanism 1) is strategyproof, fair, and nonwasteful.

Let us introduce several concepts related to the efficiency of a matching and a mechanism.

Definition 10. We say matching X' strongly dominates another matching X'' if $X'_s \succ_s X''_s$ holds for every $s \in S$. Also, we say matching X' weakly dominates another matching X'' if $X'_s \succeq_s X''_s$ holds for every $s \in S$, and there exists $s \in S$ such that $X'_s \succ_s X''_s$ holds. We say matching X' is weakly Pareto efficient for students, if there exist no other matching X'' that strongly dominates X'. Also, we say matching X' is strongly Pareto efficient for students, if there exists no matching X'' that weakly dominates X'. Furthermore, we say mechanism ψ dominates another mechanism φ if $\varphi \neq \psi$ and for each preference profile of students $\succ_S$, $\varphi(\succ_S)$ weakly dominates $\psi(\succ_S)$ or $\varphi(\succ_S) = \psi(\succ_S)$ holds, and there exists $\succ_S$ such that $\varphi(\succ_S)$ weakly dominates $\psi(\succ_S)$.

3 Baseline mechanisms

First, we introduce a baseline mechanism that is strategyproof and nonwasteful called the Serial Dictatorship mechanism (SD). Here, we assume a common priority ordering among students called a Master List (ML) is given. Without loss of generality, we assume ML is defined as follows: $(s_1, s_2, \ldots, s_n)$.

The SD is defined as follows.

Mechanism 2 (Serial Dictatorship mechanism (SD))**.**

1. Set X' to $\emptyset$, k to 1.
2. If $k > n$, return X', otherwise, choose student s_k. Then, choose the most preferred acceptable school c for s_k such that $(s_k, c) \in X$ and $\zeta(X' + (s_k, c))$ is admissible in η. Set X' to $X' + (s_k, c)$ (if no such school satisfies these conditions, X' remains the same). Set k to $k + 1$. Go to (2).

In Example 2.1, the obtained matching by the SD is:

$$\{(s_1, c_1), (s_2, c_1), (s_3, c_2), (s_4, c_3), (s_5, c_4), (s_6, c_4)\}.$$

Theorem 1. *The SD is strategyproof and nonwasteful, and obtains a feasible set of contracts.*

Proof. It is clear that the SD is strategyproof, since each student s does not have any influence on the choices of students that are higher in ML, and she can choose the best outcome in the remaining possibilities. Also, if there exists $(s, c) \in X \setminus X'$ such that $(s, c) \succ_s X'_s$, it means that $\zeta((X' \setminus X'_s) + (s, c))$ is not admissible in η. Thus, $(X' \setminus X'_s) + (s, c)$ is not feasible and s cannot claim an empty seat of c. Furthermore, in Mechanism 2, X' is always feasible. Thus, the SD obtains a feasible set of contracts. □

Actually, the matching obtained by the SD is strongly Pareto efficient for students (Definition 10), which implies nonwastefulness.

Next, we introduce another baseline mechanism, which is strategyproof and fair, called the Artificial Cap DA mechanism (ACDA), defined as follows.

Mechanism 3 (Artificial Cap Deferred Acceptance mechanism (ACDA))**.**

1. Choose an arbitrary ζ such that ζ is admissible and maximal (i.e., there exists no $\zeta' > \zeta$ such that ζ' is admissible in η).
2. Set $\hat{q}_{c_i}$ to ζ_i for each $c_i \in C$.

3. Obtain a matching X' for a standard matching market: $(S, C, X, \succ_S, \succ_C, \hat{q}_C)$ by the DA.

In Example 2.1, assume the ACDA chooses $\zeta = (1, 2, 2, 1)$. Then, the obtained matching by the ACDA is: $\{(s_1, c_4), (s_2, c_3), (s_3, c_3), (s_4, c_2), (s_5, c_2), (s_6, c_1)\}$.

Theorem 2. *The ACDA is strategyproof and fair, and obtains a feasible set of contracts.*

Proof. Since the DA is strategyproof and fair, and ζ and the maximum quotas $\hat{q}_{c_i}$ are given exogenously, it is clear that the ACDA is also strategyproof and fair. Also, the DA obtains a matching that satisfies all maximum quotas. Thus, for an obtained matching X', $|X'_{c_i}| \leq \hat{q}_{c_i} = \zeta_i$ holds for all $c_i \in C$. Thus, $\zeta(X') \leq \zeta$ holds. Since ζ is admissible in η, $\zeta(X')$ is also admissible in η since we assume η is hereditary. Thus, X' is feasible. $\square$

The problems of these mechanisms are that the SD can be extremely unfair and the ACDA can be extremely wasteful. Since the SD completely ignores the priorities of schools, if the ML and schools' priorities disagree, many students can have justified envy. In Example 2.1, the obtained matching by the SD is: $\{(s_1, c_1), (s_2, c_1), (s_3, c_2), (s_4, c_3), (s_5, c_4), (s_6, c_4)\}$. Then, students except s_1 and s_2 have justified envy towards s_1 and s_2.

Let us consider another example.

Example 3.1. The settings are identical to Example 2.1, except that students' preferences are given as follows:

$s_1, s_2 : c_1 \succ_s c_2 \succ_s c_3 \succ_s c_4$
$s_3 : c_2 \succ_s c_3 \succ_s c_4 \succ_s c_1$
$s_4 : c_3 \succ_s c_2 \succ_s c_1 \succ_s c_4$
$s_5, s_6 : c_4 \succ_s c_3 \succ_s c_2 \succ_s c_1$

If the ACDA chooses $\zeta = (1, 2, 2, 1)$, then the obtained matching X' is: $\{(s_1, c_2), (s_2, c_1), (s_3, c_2), (s_4, c_3), (s_5, c_3), (s_6, c_4)\}$. However, there exists another fair and feasible set of contracts X'': $\{(s_1, c_1), (s_2, c_1), (s_3, c_2), (s_4, c_3), (s_5, c_4), (s_6, c_4)\}$. Also, every student weakly prefers X'' over X', and students s_1 and s_5 strictly prefer X''. In this case, the choice of ζ is "wrong" considering students' preferences, but ζ must be chosen exogenously without considering students' preferences.

There exists a case where the obtained matching by the ACDA is not weakly Pareto efficient (Definition 10). Consider a following simple case. There are two schools c_1 and c_2 and one student s_1. $\eta(\zeta) = 0$ if $\zeta \leq (1, 0)$ or $\zeta \leq (0, 1)$. Assume the ACDA chooses $\zeta = (0, 1)$, while s_1 prefers c_1 over c_2. The obtained matching $\{(s_1, c_2)\}$ is not weakly Pareto efficient, since s_1 prefers another matching $\{(s_1, c_1)\}$.

4 Adaptive Deferred Acceptance Mechanism (ADA)

In this section, we develop a strategyproof and nonwasteful mechanism called Adaptive Deferred Acceptance Mechanism (ADA), which is "more fair" than the SD. Then, we show various properties of the ADA.

4.1 Mechanism description

We first introduce a concept called *forbidden school* used in the ADA.

Definition 11 (forbidden school)**.** For η and q_C, we say school c_i is forbidden if $0 < q_{c_i}$ and χ_i is not admissible in η.

We assume initially no school is forbidden.

Mechanism 4 (Adaptive Deferred Acceptance mechanism (ADA))**.**
Let $L := (s_1, \ldots, s_n), q_C^1 := q_C, \eta^1 := \eta$. Proceed to **Stage** 1.

Stage k**:** Proceed to **Round** 1.

Round t**:** Select top t students from L. Let X' be the matching that is obtained by the DA for the selected students under q_C^k. Let $q_c^{k+1} := q_c^k - |X'_c|$ for each $c \in C$, and $\eta^{k+1}(\zeta) := \eta^k(\zeta + \zeta(X'))$.

- **(i)** If all students in L are already selected, then finalize X' and terminate the mechanism.
- **(ii)** If there exists no school c_i such that c_i is forbidden for η^{k+1} and q_C^{k+1}, then proceed to **Round** $t + 1$.
- **(iii)** Otherwise, finalize X'. Remove top t students from L. For each school c that is forbidden for η^{k+1} and q_C^{k+1}, set q_c^{k+1} to 0. Proceed to **Stage** $k + 1$.

Let us describe the execution of the ADA in the setting of Example 2.1.

Round 1 of **Stage** 1: The obtained matching is: $\{(s_1, c_1)\}$. Since no school is forbidden, the mechanism proceeds to the next round.

Round 2 of **Stage** 1: The obtained matching is: $\{(s_1, c_1), (s_2, c_1)\}$. Since no school is forbidden, the mechanism proceeds to the next round.

Round 3 of **Stage** 1: The obtained matching is: $\{(s_1, c_2), (s_2, c_1), (s_3, c_1)\}$. Now, c_2 is forbidden. Thus, the current matching is finalized. The maximum quotas for c_1 and c_2 become 0.

Round 1 of **Stage** 2: The obtained matching is: $\{(s_4, c_3)\}$. Now, c_3 is forbidden. Thus, the current matching is finalized. The maximum quotas for c_3 becomes 0 and the maximum quota of c_4 is 2.

Round 1 of **Stage** 3: The obtained matching is: $\{(s_5, c_4)\}$. Since no school is forbidden, the mechanism proceeds to the next round.

Round 2 of **Stage** 3: The obtained matching is: $\{(s_5, c_4), (s_6, c_4)\}$. Now, all students are selected. Thus, the current matching is finalized. The final result of the ADA is: $\{(s_1, c_2), (s_2, c_1), (s_3, c_1), (s_4, c_3), (s_5, c_4), (s_6, c_4)\}$.

4.2 Properties of ADA

4.2.1 Basic properties

We first examine basic properties of the ADA, e.g., strategyproofness, nonwastefulness, and time-complexity.

Theorem 3. *The ADA is nonwasteful and obtains a feasible set of contracts.*

Proof. About nonwastefulness, assume student s, who is assigned at **Stage** k prefers $(s, c) \in X \setminus X'$ over X'_s. The fact s was not accepted by c means that (i) c becomes full at or before **Stage** k, or (ii) c becomes forbidden before **Stage** k. In either case, $\zeta((X' \setminus X'_s) \cup \{(s, c)\})$ is not admissible in η. Thus, s cannot claim an empty seat of c.

About feasibility, in each round, by adding one more student, the number of students of each school c is either (i) unchanged from the previous round, or (ii) incremented by one. Since each school is full or can accept at least one more student, the obtained contracts is feasible. Thus, the accumulation of these contracts is also feasible. $\square$

Next, we show that the ADA is strategyproof. Showing the strategyproofness of the ADA is non-trivial, since a student might have an incentive to force the current stage to finish earlier, so that she can avoid competing with more rivals. The following theorem shows that such a manipulation is not profitable.

Theorem 4. *The ADA is strategyproof.*

Proof. Assume student s is assigned to a better school by misreporting. Without loss of generality, we can assume $c_1 \succ_s c_2 \succ_s \ldots \succ_s c_m$. We assume s is assigned to c_i (here, we assume c_i can be the outside option, i.e., being unassigned). in **Stage** k (which is finished at **Round** t) when s reports her true preference. Also, we assume s is assigned to c_j ($j < i$) in **Stage** k (which is finished at **Round** t') when she misreports her preference. It is clear that if $t' > t$, s cannot be assigned to a more preferred school, since $t' > t$ means she needs to compete with more students by misreporting. Thus, we assume $t \geq t'$ holds.

Let us assume a matching in **Stage** k is obtained in the following way. First, all selected students at **Round** t' except s are tentatively matched to schools by the DA. Then, continue the procedure of the DA by adding s to the current tentative matching. The final matching obtained in this way is identical to the final matching obtained by applying the DA when all students enter the market simultaneously McVitie and Wilson (1971). Note that when one student s is added, either one of the following two cases are possible: (i) s is accepted and there exists exactly one school c where the number of accepted students increases by one, and the number of accepted students of all other schools are the same, or (ii) student s' (s' can be s or another student) is rejected from all schools, and the number of accepted students of all schools are the same.

In the above procedure, when s enters the market, s first applies to some school c. Then, either c accepts all students applying to c and the current round terminates, or c rejects one student s' (s' can be s or another student) and s' applies to another school.

We call such a sequence of applications and rejections a *rejection chain*. More formally, let $\mathcal{C}_s$ denote a list of schools to which student s is going to apply. We call $\mathcal{C}_s$ a *scenario*. Assume s enters the market with scenario $\mathcal{C}_s$. $R(\mathcal{C}_s)$ is the rejection chain of $\mathcal{C}_s$, which describes the sequence of applications and rejections, until s is rejected by the last school in $\mathcal{C}_s$, or the current stage terminates. When the mechanism proceeds to the next round, we assume a new student enters the market while existing students are tentatively assigned according to the matching obtained in the previous round. Figure 1 shows an example of a rejection chain. For rejection chains, the property described in Lemma 4.1 holds.

Round	Step	Action
t'	1	student s applies to school c'
	2	school c' rejects student s'
	3	student s' applies to school c''
	$\vdots$	$\vdots$
$t'+1$	1	(new) student s'' applies to school c'''
	2	school c''' rejects student s'''
	$\vdots$	$\vdots$

Figure 1: Example of rejection chain

Let $\mathcal{C}_s$ be $(c_1, \ldots, c_{i-1})$, i.e., s's true preference truncated just before c_i. Then, the last action in $R(\mathcal{C}_s)$ must be "school c_{i-1} rejects student s." On the other hand, let $\mathcal{C}'_s$ be a sequence of schools to which s applies when s misreports and the last school in $\mathcal{C}'_s$ is c_j. Since s is accepted to c_j, the last action in $R(\mathcal{C}'_s)$ must be "student s' applies to school c'" (and is accepted) for some $s' \in S$ and $c' \in C$. From the fact that the current stage k (as well as the current round) terminates, c' becomes forbidden according to q_C^{k+1} and η^{k+1}.

For $\mathcal{C}'_s$, either one of the following two cases is possible: (i) c_j is the least preferred school for s within $\mathcal{C}'_s$ (according to the true preference of s), or (ii) $\mathcal{C}'_s$ contains a school that is less preferred for s.

In case (i), each school c that appears in $\mathcal{C}'_s$ also appears in $\mathcal{C}_s$, student s applies to every school in $\mathcal{C}_s$, and all actions in $R(\mathcal{C}'_s)$ occur in the same round. Thus, from Lemma 4.1, the last action in $R(\mathcal{C}'_s)$, i.e., "student s' applies to school c'," must also be included in $R(\mathcal{C}_s)$. However, in $R(\mathcal{C}_s)$, when student s' applies to c', if c' accepts all students currently applying to c', then the current round and stage terminate. This contradicts the fact that the last action in $R(\mathcal{C}_s)$ is "school c_{i-1} rejects student s." Also, if c' rejects s', it must be due to the fact that a new student is introduced in a new round, and c' has already accepted some student s''. However, when c' accepts s'', all students currently applying to c' are accepted and the current round and stage must have terminated. This contradicts the fact that the last action in $R(\mathcal{C}_s)$ is "school c_{i-1} rejects student s."

In case (ii), let $\mathcal{C}''_s$ be the sequence that is obtained by removing all schools that are worse than c_j (according to the true preference of s) from $\mathcal{C}'_s$. The last school in $\mathcal{C}''_s$ is c_j. It is clear that all actions in $R(\mathcal{C}''_s)$ occur in the same round. This is because if the DA terminates before s applies to all schools in $\mathcal{C}''_s$, then it implies that the DA is not strategyproof (if the true preference of s is the one which corresponds to $\mathcal{C}'_s$, then s is assigned to a better school by declaring her preference as $\mathcal{C}''_s$). Thus, we can apply Lemma 4.1 for $\mathcal{C}''_s$ and $\mathcal{C}_s$. Then, the last action in $R(\mathcal{C}''_s)$ must be "school c_j rejects student s." This is because if we assume the last action in $R(\mathcal{C}''_s)$ is "student s' applies to school c'," we encounter a contradiction using a similar argument as case (i). Each school c that appears in $\mathcal{C}''_s$ also appears in $\mathcal{C}'_s$, and student s applies to every school in $\mathcal{C}'_s$. Thus, from Lemma 4.1, the last action in $R(\mathcal{C}''_s)$, i.e., "school c_j rejects student s," must also be included in $R(\mathcal{C}'_s)$. However, this contradicts the fact that s is accepted to c_j. □

The Scenario Lemma (Lemma 4.1) is inspired by the Scenario Lemma introduced in Dubins and Freedman (1981), which proves strategyproofness of the DA in a one-to-one matching. Fragiadakis and Troyan (2013) also use a similar lemma to prove strategyproofness of their mechanism for handling individual minimum quotas.

Lemma 4.1 (Scenario Lemma). *Consider two scenarios $\mathcal{C}_s$ and $\mathcal{C}'_s$ and their rejection chains $R(\mathcal{C}_s)$ and $R(\mathcal{C}'_s)$. If each school c that appears in $\mathcal{C}'_s$ also appears in $\mathcal{C}_s$ (the order of c is irrelevant), student s applies to every school $c \in \mathcal{C}_s$ in $R(\mathcal{C}_s)$, and all actions in $R(\mathcal{C}'_s)$ occur in the same round, then every action in $R(\mathcal{C}'_s)$ also appears in $R(\mathcal{C}_s)$.*

Proof. The first action in $R(\mathcal{C}'_s)$ is "student s applies to school c," where c is the first school in $\mathcal{C}'_s$. From the assumption that every school that appears in $\mathcal{C}'_s$ also appears in $\mathcal{C}_s$, and s applies to all schools in $\mathcal{C}_s$, $R(\mathcal{C}_s)$ also includes this action. For an inductive step, let us assume from the first to $i-1$-th actions in $R(\mathcal{C}'_s)$ are included in $R(\mathcal{C}_s)$. We are going to show that i-th action in $R(\mathcal{C}'_s)$ is also included in $R(\mathcal{C}_s)$. The i-th action in $R(\mathcal{C}'_s)$ is either: (i) "student s' applies to school c'," or (ii) "school c' rejects student s'." In case (i), there must be a previous action "school c'' rejects student s'" in $R(\mathcal{C}'_s)$. From the inductive assumption, this previous action must be included in $R(\mathcal{C}_s)$. Thus, the current action must also be included in $R(\mathcal{C}_s)$. In case (ii), let $S'_{c'}$ denote a set of students who have applied to c' in the current stage by the i-th action in $R(\mathcal{C}'_s)$, and let $S_{c'}$ denote a set of students who applied to c' in the current stage by the last action in $R(\mathcal{C}_s)$. It is clear that $S'_{c'} \subseteq S_{c'}$ holds. Also, in $S'_{c'}$ and $S_{c'}$, any student whose rank is worse than $q^k_{c'}$ among those in $S'_{c'}$ and $S_{c'}$, respectively, according to $\succ_{c'}$ will eventually be rejected. Thus, any student who is rejected in $R(\mathcal{C}'_s)$ must also be rejected in $R(\mathcal{C}_s)$. Thus, the action "school c' rejects student s'" must be included in $R(\mathcal{C}_s)$. $\square$

Theorem 5. *The time-complexity of the ADA is $O(m \cdot n)$, assuming η can be calculated in a constant time.*

Proof. Here, we consider a slightly modified implementation of the ADA. For each round, instead of applying the DA from scratch, we start the DA from the situation where the current tentative assignment is identical to the assignment obtained in the previous round and a new student has just arrived. As discussed in the proof of Theorem 4, this change does not affect the final matching. The time-complexity of the ADA is determined by the total time required to run the DA repeatedly. Unless some student is rejected in step (2) in the DA, the DA terminates. A student is rejected by each school at most once in the ADA (with the above modification). Thus, step (2) in the DA is executed at most $n \cdot m$ times. Then, the worst-case time complexity is $O(m \cdot n)$. $\square$

When the family of vectors forms an M$^\natural$-convex set, we can apply the generalized DA. By appropriately choosing the schools' choice function, we can make the generalized DA fair. However, it is usually impossible to make the generalized DA nonwasteful (Kojima, Tamura, and Yokoo, 2014) when distributional constraints are imposed. Thus, when the students' welfare is the primary concern, while the fairness among students is the secondary concern, we can utilize the ADA, since it can improve the students' welfare compared to the generalized DA. We confirm this fact in Section 6.2.

4.2.2 Comparison with baseline mechanisms

It is clear that the ADA is "less wasteful" than the ACDA, since the ADA is nonwasteful, while the ACDA is not. Intuitively, the ADA seems "more fair" than the SD, i.e., the number of students who have justified envy in the ADA should be smaller than that in the SD, since the SD completely ignores the schools' priorities, while the ADA utilizes them in the DA. In particular, when no additional distributional constraint is imposed, the obtained matching by the ADA is identical to the DA, since no school becomes forbidden. Then, the obtained matching by the ADA is fair. We can expect that when the distributional constraints are not too restrictive, then the obtained matching by the ADA is "almost fair."

However, we cannot guarantee the claim "the number of students who have justified envy in the ADA is smaller than that in the SD" is always true; there exists a pathological situation where the ADA has more students with justified envy than the SD.

Example 4.1. Let us consider the following situation.

- There are five students $s_1, s_2, \ldots, s_5$.
- There are three schools c_1, c_2 and c_3.
- The priority of c_1 is: $s_2 \succ_{c_1} s_1 \succ_{c_1} s_3 \succ_{c_1} s_4 \succ_{c_1} s_5$.
- The priorities of the rest of schools are the same and given as follows: $s_2 \succ_c s_3 \succ_c s_4 \succ_c s_5 \succ_c s_1$.
- The preferences of students s_1 and s_2 are given as: $c_1 \succ_s c_2 \succ_s c_3$.
- The preferences of students s_3, s_4, and s_5 are given as: $c_2 \succ_s c_3 \succ_s c_1$.
- The maximum quota of schools are given as $q_{c_1} = 1$, $q_{c_2} = 2$, and $q_{c_3} = 3$.
- The total number of students accepted for $\{c_1, c_2\}$ must be at most 2.

Then, $\eta(\zeta) = 0$ if $\zeta \leq (1, 1, 3)$ or $\zeta \leq (0, 2, 3)$.

In the situation of Example 4.1, the obtained matching by the SD is: $\{(s_1, c_1), (s_2, c_2), (s_3, c_3), (s_4, c_3), (s_5, c_3)\}$. Only s_2 has justified envy toward another student (s_1).

Now, let us examine the execution of the ADA. In the second round of **Stage** 1, s_2 is assigned to c_1 and s_1 is assigned to c_2, since s_2 has higher priority in c_1. Then, this matching is fixed since c_2 is forbidden. The rest of students are assigned to c_3. Then, s_3, s_4, s_5 have justified envy toward s_1, i.e., three students have justified envy.

However, if we compare the worst-case, where the number of students with justified envy is maximized, we can say that the worst-case number of the ADA is smaller than or equal to the number of the SD. Formally, let $JE^{SD}(S, C, X, \succ_S, \succ_C, q_C, \eta)$ denote the number of students who have justified envy under the SD at the market $(S, C, X, \succ_S, \succ_C, q_C, \eta)$. Then, let us define

$$W^{SD}(S, C, q_C, \eta) = \max_{X, \succ_S, \succ_C} JE^{SD}(S, C, X, \succ_S, \succ_C, q_C, \eta).$$

Let us define $JE^{ADA}(S, C, X, \succ_S, \succ_C, q_C, \eta)$ and $W^{ADA}(S, C, q_C, \eta)$ similarly.[5] The following theorem holds.

Theorem 6. *For any S, C, q_C, and η, $W^{ADA}(S, C, q_C, \eta) \leq W^{SD}(S, C, q_C, \eta)$ holds. Also, there exist S, C, q_C, and η such that $W^{ADA}(S, C, q_C, \eta) < W^{SD}(S, C, q_C, \eta)$ holds.*

Proof. In the ADA, the students assigned in the first stage never have justified envy. Assume $W^{ADA}(S, C, q_C, \eta) = JE^{ADA}(S, C, X, \succ_S, \succ_C, q_C, \eta) = n - k$. If $k = n$, $W^{ADA}(S, C, q_C, \eta) \leq W^{SD}(S, C, q_C, \eta)$ directly follows. So assume $k < n$. Without loss of generality, we can assume X contains all possible contracts, i.e., each student/school considers all schools/students acceptable. Furthermore, without loss of generality, we can assume in the market $(S, C, X, \succ_S, \succ_C, q_C, \eta)$, top k students are assigned in the first stage, and all $n - k$ students who are not assigned in the first stage have justified envy. If this is not the case, we can modify the preferences of $n - k$ students and priorities of related schools such that they have justified envy towards a student assigned in the first stage.

Let X' be the matching obtained in the first stage of the ADA. There must be at least one school that is forbidden after X' is fixed. Let c^* denote such a school. Consider a slightly modified students' preference profile $\tilde{\succ}_S$, which is obtained from $\succ_S$ as follows. For each student s who is assigned in the first stage of the ADA, $\succ_s$ is modified (if necessary) so that c is her most preferred school, where $X'_s = \{(s, c)\}$. For each student s who is not assigned

[5] Comparing the maximum numbers of fairness violations may appear arbitrary. However, the concept would not be useful if one considers the *minimum* numbers, because a fair allocation is certainly possible for some preference configurations, i.e., $JE^{SD}(S, C, X, \succ_S, \succ_C, q_C, \eta) = JE^{ADA}(S, C, X, \succ_S, \succ_C, q_C, \eta) = 0$. While analytic comparisons would be elusive for the average number of fairness violations unless we impose restrictive assumptions over preference distributions, Section 6 presents simulation results that suggest the average number of fairness violations in ADA is smaller than the corresponding number in SD in typical applications.

in the first stage of the ADA, $\succ_s$ is modified (if necessary) so that c^* is her most preferred school. Also, consider a slightly modified schools' priority profile $\tilde{\succ}_C$, which is obtained from $\succ_C$ as follows. For c^*, $\succ_{c^*}$ is modified (if necessary) so that the priorities of all top k students in the ML is lower than the remaining $n-k$ students. For other schools, its priority is unchanged. By running the SD for the market $(S, C, X, \tilde{\succ}_S, \tilde{\succ}_C, q_C, \eta)$, the matching for top k students is identical to X'. After first k students are assigned, the remaining $n-k$ students cannot be assigned to c^*. Then, they have justified envy towards students assigned to c^*. Thus, $JE^{SD}(S, C, X, \tilde{\succ}_S, \tilde{\succ}_C, q_C, \eta) = n-k$ holds. As a result, $W^{ADA}(S, C, q_C, \eta) \leq W^{SD}(S, C, q_C, \eta)$ holds.

Also, let us consider a very simple case, where only one school c_1 exists with the maximum quota $q_{c_1} = 1$. There is no distributional constraint. There exist n students $s_1, \ldots, s_n$. Assume the priority of c_1 is $s_n \succ_{c_1} s_{n-1} \succ_{c_1} \ldots \succ_{c_1} s_1$, then $n-1$ students have justified envy in the SD.

Since s_1 never has justified envy, $W^{SD}(S, C, X, \succ_S, \succ_C, q_C, \eta) = n-1$ must be maximal. On the other hand, in the ADA, the obtained matching is identical to the DA regardless of students' preferences since no school becomes forbidden. Thus, $W^{ADA}(S, C, q_C, \eta) = 0$. Thus, there exists a case where $W^{ADA}(S, C, q_C, \eta) < W^{SD}(S, C, q_C, \eta)$ holds. $\square$

Let us consider another criterion to compare "fairness" of different mechanisms. Let $F^{SD}(S, C, q_C, \eta)$ denote

$$\{(\succ_S, \succ_C, X) \mid \text{the obtained matching by the SD for } (S, C, X, \succ_S, \succ_C, q_C, \eta) \text{ is fair}\}.$$

Let us define $F^{ADA}(S, C, q_C, \eta)$ similarly. The following theorem holds.

Theorem 7. *For any S, C, q_C, and η, $F^{ADA}(S, C, q_C, \eta) \supseteq F^{SD}(S, C, q_C, \eta)$ holds. Also, there exist S, C, q_C, and η, such that $F^{ADA}(S, C, q_C, \eta) \supsetneq F^{SD}(S, C, q_C, \eta)$ holds.*

Proof. Assume the SD obtains a fair matching X', i.e., if student s cannot be assigned to school c, such that $(s, c) \succ_s X'_s$, then the students in X'_c are ranked higher than s according to $\succ_c$. Then, the obtained matching by the ADA must be identical to that by the SD. Thus, $F^{ADA}(S, C, q_C, \eta) \supseteq F^{SD}(S, C, q_C, \eta)$ holds.

Also, if we consider the simple case used in the proof of Theorem 6, it is clear that $F^{ADA}(S, C, q_C, \eta) \supsetneq F^{SD}(S, C, q_C, \eta)$ holds. $\square$

This theorem means when the SD obtains a fair matching, then the ADA also obtains a fair matching.

Let us examine the efficiency of the ADA, SD and ACDA using the concepts introduced in Definition 10. The SD is strongly Pareto efficient, while the ACDA is not weakly Pareto efficient. The following theorem shows that the ADA is more efficient than the ACDA, i.e., it is weakly Pareto efficient.

Theorem 8. *The matching obtained by the ADA is weakly Pareto efficient for students.*

Proof. It is a well-known fact that the matching obtained by the DA is weakly Pareto efficient for students. Assume a set of students S' is assigned in the first stage of the ADA, and the obtained matching is X'. X' is identical to the result of applying the DA for S' (ignoring distributional constraints). Thus, it is impossible to strictly improve the assignments of all students in S' from X'. As a result, the matching obtained by the ADA is weakly Pareto efficient for students. $\square$

Furthermore, the following theorem shows that there exists no strategyproof mechanism that dominates the ADA.

Theorem 9. *There exists no mechanism that is strategyproof and dominates the ADA.*

Proof. Let φ denote the ADA. For contradiction, suppose that there exists another mechanism ψ that is strategyproof and dominates the ADA. Let s and preference profile $\succ_S$ be such that $\psi_s(\succ_S) \succ_s \varphi_s(\succ_S)$. Without loss of generality, let us choose s and $\succ_S$ such that s is assigned in the earliest stage with this property. More precisely, if another pair s' $(\neq s)$ and $\succ'_S$ satisfies this property, and s is assigned in **Stage** k, while s' is assigned in **Stage** k', then $k \leq k'$ holds. Note that such s and $\succ_S$ exist by definition. We begin by establishing the following lemma.

Lemma 4.2. *For any student s and preference profile $\succ_S$ with the above property, $\varphi_s(\succ_S) \neq \emptyset$ and $\psi_s(\succ_S) \neq \emptyset$.*

Proof. The claim $\psi_s(\succ_S) \neq \emptyset$ follows because $\psi_s(\succ_S) \succ_s \varphi_s(\succ_S)$ and φ always obtains a student-feasible matching.

To show $\varphi_s(\succ_S) \neq \emptyset$, let k be the stage at which s is assigned and S_k be the set of students who are assigned at **Stage** k in the ADA with preference profile $\succ_S$. Because of the choice of s, $\varphi_{s'}(\succ_S) = \psi_{s'}(\succ_S)$ for every s' who has been assigned in any **Stage** $k' < k$. So, the restriction of the assignment under ψ to students at S_k is feasible at school capacity profile q^k (as defined in the ADA) and this assignment is also weakly preferred to the outcome of the ADA by all students in S_k, with at least one strictly preferred. Because the assignment for every student in S_k is identical to the outcome of the DA under capacity profile q^k, by a claim in the proof of Theorem 1 of Abdulkadiroğlu, Pathak, and Roth (2009), the subsets of students in S_k who are matched to some school in $\varphi(\succ_S)$ and $\psi(\succ_S)$ are identical. This fact and $\psi_s(\succ_S) \neq \emptyset$ imply $\varphi_s(\succ_S) \neq \emptyset$. $\square$

Now we are ready to prove the theorem. To do so, let s and $\succ_S$ be a student-preference profile pair as in Lemma 4.2. Now, consider another preference $\succ'_s$ of student s, which lists only $\psi_s(\succ_S)$ as the acceptable school (note that $\psi_s(\succ_S) \neq \emptyset$ by the second statement of Lemma 4.2). Let $\succ'_S := (\succ'_s, \succ_{-s})$. Then,

1. $\psi_s(\succ'_S) = \psi_s(\succ_S)$: To show this, assume for contradiction $\psi_s(\succ'_S) \neq \psi_s(\succ_S)$. Then, under mechanism ψ, student s whose preference is $\succ'_s$ is made better off by reporting $\succ_s$ and thereby obtaining her most preferred school $\psi_s(\succ_S)$, which contradicts strategyproofness of ψ.

2. $\varphi_s(\succ'_S) = \emptyset$: To show this, note that $\varphi_s(\succ'_S) \succeq_s \emptyset$ by student-feasibility of φ. So, if $\varphi_s(\succ'_S) \neq \emptyset$, then $\varphi_s(\succ'_S) = \psi_s(\succ'_S)$. But then, student s whose preference is $\succ_s$ is made better off by reporting $\succ'_s$ under φ, which contradicts strategyproofness of φ.

The above statements contradict Lemma 4.2 with respect to s and $\succ'_S$, which completes the proof of the theorem.[6] $\square$

Let us introduce another criterion for comparing fairness. For matching X', let $Ev(X') := \{(s, s') \mid s \text{ has justified envy towards } s' \text{ in } X'\}$. Then, we say X' is more fair than X'' if $Ev(X') \subsetneq Ev(X'')$ holds. Also, we say X' is equally fair to X'' if $Ev(X') = Ev(X'')$ holds.

There exists a trade-off between fairness and efficiency. We say matching X' is in the Pareto frontier (in terms of fairness and efficiency) if there exists

[6] Note that it is with respect to $\succ'_S$, and not $\succ_S$, that we derive the desired contradiction. However, the application of Lemma 4.2 is correct because we have taken s and $\succ_S$ to be arbitrary with the restriction that $\psi_s(\succ_S) \succ_s \varphi_s(\succ_S)$ and s is assigned in the earliest stage, which holds not only for $\succ_S$ but also for $\succ'_S$.

no X'' such that X'' Pareto dominates X' and X'' is more fair than or equally fair to X''.

The following theorem holds.

Theorem 10. *The matching obtained by the ADA is in the Pareto frontier.*

Proof. Let X' be the matching obtained by the ADA and X'' is a matching that Pareto dominates X'. Let k be the first stage in the ADA such that there exists a student who strictly prefers X'' to X'. Let S_k be the set of students assigned at **Stage** k. Because of the choice of k, the assignments for students at X' and X'' in any **Stage** $k' < k$ are identical. Also, the assignment at X' to students in S_k is the student-optimal stable matching at school capacity profile q^k (as defined in the ADA). This fact implies that there exists $s, s' \in S_k$ such that $(s, s') \in Ev(X'')$ holds. Since there exists no justified envy between students assigned at the same stage in the ADA, $(s, s') \notin Ev(X')$. Thus, $Ev(X'') \subseteq Ev(X')$ does not hold. Thus, X'' cannot be more fair than or equally fair to X'. $\square$

5 Handling non-hereditary cases

There are several application domains where distributional constraints do not satisfy heredity. For example, when minimum quotas are imposed, or each student is required to be matched to some school, then η does not satisfy heredity, since $\zeta(\emptyset)$ is not admissible in η. In this section, we introduce a simple trick to handle such non-hereditary constraints.

Assume η is not hereditary. To run the ADA (or the SD), let us define another function $\tilde{\eta}$ as follows: $\tilde{\eta}(\zeta)$ is 0 if there exists $\zeta' \geq \zeta$ such that $\eta(\zeta') = 0$, and otherwise, $-\infty$. It is clear that $\tilde{\eta}$ is hereditary. If we use $\tilde{\eta}$ instead of η, we can run the ADA, but the obtained set of contracts X' might be infeasible, i.e., $\tilde{\eta}(\zeta(X')) = 0$ but $\eta(\zeta(X'))$ might be $-\infty$. However, if we can guarantee that X' is maximal, i.e., there exists no $\zeta > \zeta(X')$ such that ζ is admissible in η, then, the ADA (or the SD) always obtains a feasible set of contracts.

Also, to run the ACDA, we can simply use the original η to choose an admissible and maximal ζ in η. If we can guarantee that the obtained matching X' by the ACDA is maximal, the ACDA always obtains a feasible set of contracts.

If we can assume each student/school considers all schools/students acceptable, the ADA (or the SD) always obtains a maximal matching. To be more precise, in the ADA, assume student s is not assigned to any school at **Stage** k. Then, it is clear that no more student can be assigned in the later stages. For the obtained set of contracts X', for any $i \in M$, $\zeta(X') + \chi_i$ is not admissible in η, since if $\zeta(X') + \chi_i$ is admissible in η, s must be accepted to c_i. Thus, X' is maximal.

Also, consider the ACDA where for all $c_i \in C$, $\hat{q}_{c_i} = \zeta_i$ holds. Since we assume η respects the number of students, $\sum_{i \in M} \zeta_i \leq n$ holds. In the DA, when each student/school considers all schools/students acceptable, then all schools are full when the sum of maximum quotas is less than or equal to n. Thus, for all $i \in M$, $|X'_{c_i}| = \hat{q}_{c_i} = \zeta_i$ holds. Thus, X' is maximal.

We have shown that, in principle, our mechanism can work for any constraints that are not hereditary, as long as we can guarantee that obtained matchings are maximal. However, to guarantee the ADA is computationally efficient, we require that $\tilde{\eta}(X')$ can be computed efficiently. Also, to run the ACDA, we need to find an admissible and maximal ζ. It must be noted that even when $\eta(X')$ can be computed efficiently, there exists a case where computing $\tilde{\eta}(X')$, or finding an admissible and maximal ζ, is difficult.

For example, Goto, Hashimoto, Iwasaki, Kawasaki, Ueda, Yasuda, and Yokoo (2014) show that checking whether a feasible set of contracts exists or not is NP-complete when regional minimum/maximum quotas exist and a region can be any subset of schools. Here, calculating $\eta(X')$, i.e., checking a given set of contracts X' is feasible, is easy, i.e., it can be done in a polynomial time. However, calculating $\tilde{\eta}(X')$ is difficult, since we need to check the existence of a feasible set of contracts X'' such that $X'' \supseteq X'$, which is NP-complete. Also, finding an admissible and maximal ζ is difficult.

6 Applications

In this section, we investigate how to represent actual application domains in our model, and examine the performance of our mechanism.

6.1 Controlled school choice problem with hard minimum and maximum quotas

We first examine a controlled school choice problem with hard minimum and maximum quotas presented in Ehlers, Hafalir, Yenmez, and Yildirim (2014). School choice programs are implemented to give students/parents an opportunity to choose the public school the students attend. However, a school is required to satisfy balance on the composition of students, typically in terms of socioeconomic status. Controlled school choice programs need to provide choices for students/parents while maintaining distributional constraints. The model presented in Ehlers, Hafalir, Yenmez, and Yildirim (2014) incorporates hard minimum and maximum quotas for each type of students, so that a policy maker can precisely describe her requirements/constraints. However, Ehlers, Hafalir, Yenmez, and Yildirim (2014) show several negative results in this model. As far as the authors aware, our paper is the first to develop a non-trivial strategyproof and nonwasteful mechanism that works in this model.

6.1.1 Model

A market is a tuple $(S, C, X, \succ_S, \succ_C, q_C, T, \tau, \underline{q}_C^T, \overline{q}_C^T)$. The meaning of each element is as follows.

- The definitions of $S, C, X, \succ_S, \succ_C$, and q_C are identical to the standard model. We assume each student/school considers all schools/students acceptable.
- One major difference is that we assume each student s has her type $\tau(s) \in T = \{t_1, \ldots, t_k\}$. A type of a student may represent race, income, gender, or any socioeconomic status.
- Each school has minimum and maximum quotas for each type t, i.e., $\underline{q}_C^T = (\underline{q}_c^T)_{c \in C}$ and $\overline{q}_C^T = (\overline{q}_c^T)_{c \in C}$, where $\underline{q}_c^T = (\underline{q}_c^t)_{t \in T}$ and $\overline{q}_c^T = (\overline{q}_c^t)_{t \in T}$.
- Each $\underline{q}_c^t$ and $\overline{q}_c^t$ represent minimum and maximum quotas for type t students at school c.
- We assume $\sum_{t \in T} \underline{q}_c^t \leq q_c \leq \sum_{t \in T} \overline{q}_c^t$ holds. In words, the minimum quotas for all types in c can be satisfied without violating the maximum quota of the school, and the maximum quota of the school is not superfluous.

- Let n^t denote the number of type t students and $n = \sum_{t \in T} n^t$.
- We assume $\sum_{c \in C} \underline{q}_c^t \leq n^t \leq \sum_{c \in C} \overline{q}_c^t$ holds for all $t \in T$. In words, for each school, there exist enough type t students to satisfy its minimum quotas, and its maximum quotas are large enough to accommodate all type t students.
- For $X' \subseteq X$, let $X'_{c,t}$ denote $\{(s,c) \in X' \mid s \in S, \tau(s) = t\}$, and X'_t denote $\{(s,c) \in X' \mid c \in C, s \in S, \tau(s) = t\}$.

Next, we define feasibility and fairness in this setting.

Definition 12 (feasibility)**.** We say $X' \subseteq X$ is school-feasible if $|X'_t| = n^t$ for all $t \in T$, $|X'_c| \leq q_c$ for all c, and $\underline{q}_c^t \leq |X'_{c,t}| \leq \overline{q}_c^t$ for all c and for all t.

We say $X' \subseteq X$ is student-feasible if $|X'_s| = 1$ and X'_s is acceptable for all s. Then, we say X' is feasible if it is student- and school-feasible.

Note that when assigning students to public schools, it is required that a student must be matched to some school. Thus, we require that $|X'_t| = n^t$ holds for all $t \in T$.

Definition 13 (fairness)**.** For a matching X', we say student s has justified envy towards another student s' of the same type if $\tau(s) = \tau(s')$, $(s', c) \in X', (s,c) \succ_s X'_s$, and $(s,c) \succ_c (s',c)$. We say a matching X' is fair for same types if no student has justified envy towards another student of the same type.

Definition 14 (nonwastefulness)**.** For a matching X', we say student s, whose type is t and currently assigned to c' in X', claims an empty seat of school c if $(s,c) \succ_s (s,c')$, $|X'_c| < q_c$, $|X'_{c,t}| < \overline{q}_c^t$, and $|X'_{c',t}| > \underline{q}_{c'}^t$ hold. We say a matching X' is nonwasteful if no student claims an empty seat.

Also, we say X' is constrained nonwasteful, if whenever student s claims an empty seat of school c, the matching obtained by reassigning s to c is not fair for same types.

Ehlers, Hafalir, Yenmez, and Yildirim (2014) show that a matching that is fair for same types and nonwasteful may not exist. Also, they show that there exists no strategyproof mechanism that is guaranteed to obtain a matching that is fair for same types and constrained nonwasteful.

6.1.2 Representation in our model

We can represent this problem setting in our model as follows.

- We assume school c_i is divided into separate sub-schools for each type, i.e., there are sub-schools $c_i^{t_1}, c_i^{t_2}, \ldots, c_i^{t_k}$.
- Each sub-school c_i^t has its minimum quota $\underline{q}_{c_i}^t$, and its maximum quota $\overline{q}_{c_i}^t$.
- We assume c_i^t can accept only type t students.
- We assume ζ is an $m \times k$ matrix. We also assume $\zeta(X')$ is an $m \times k$ matrix, where each $\zeta_{i,j}(X')$ represents $|X'_{c_i,t_j}|$, i.e., the number of students allocated to sub-school $c_i^{t_j}$.
- $\eta(\zeta)$ is 0 if for all $j \in K$, $\sum_{i \in M} \zeta_{i,j} = n^{t_j}$, for all $i \in M$, $\sum_{j \in K} \zeta_{i,j} \leq q_{c_i}$, and for all $i \in M$, $j \in K$, $\underline{q}_{c_i}^{t_j} \leq \zeta_{i,j} \leq \overline{q}_{c_i}^{t_j}$ hold, where K denote $\{1, \ldots, k\}$.

Note that this η is not hereditary. To run the ADA, let us define another function $\tilde{\eta}$ as follows. $\tilde{\eta}(\zeta)$ is 0 if there exists $\zeta' \geq \zeta$ such that $\eta(\zeta') = 0$, and otherwise, $-\infty$. Assuming each type t student/school considers all type t schools/students acceptable, we can guarantee that the obtained matching X' by the ADA is maximal/feasible. The ADA is strategyproof and nonwasteful.

We can apply the ACDA if we can find an admissible and maximal ζ. We can guarantee that the ACDA, where the maximum quota for each sub-school $c_i^{t_j}$ is given by $\zeta_{i,j}$, obtains a maximal matching, since $\sum_{j \in K} \zeta_{i,j} = n^{t_j}$ holds, and each type t student/school considers all type t schools/students acceptable. Note that the ACDA guarantees the fairness within each sub-school, which corresponds to the fairness for same types.

The remaining issues are, whether we can calculate $\tilde{\eta}(X')$ efficiently, and whether we can find an admissible ζ efficiently. Note that in the current setting, any admissible ζ is also maximal, since we require for all $j \in K$, $\sum_{i \in M} \zeta_{i,j} = n^{t_j}$ holds.

Let us consider the problem of checking the existence of an admissible ζ. First, we eliminate minimum quotas as follows. We replace n^t by $n^t - \sum_{c \in C} \underline{q}_c^t$, n by $n - \sum_{c \in C, t \in T} \underline{q}_c^t$, $\overline{q}_c^t$ by $\overline{q}_c^t - \underline{q}_c^t$, and q_c by $q_c - \sum_{t \in T} \underline{q}_c^t$.

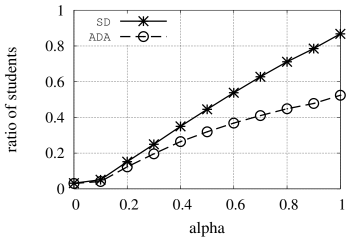

Figure 2: Ratio of students with justified envy of the same type

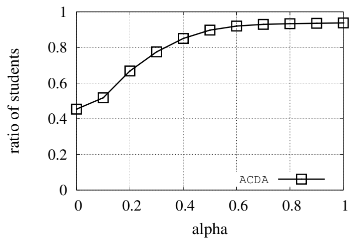

Figure 3: Ratio of students claiming an empty seat

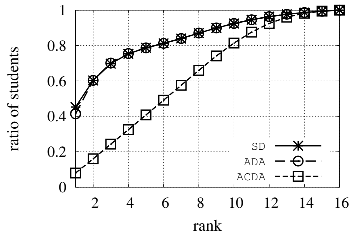

Figure 4: CDFs of students' welfare

For this modified problem with maximum quotas only, we can check the existence of an admissible ζ as follows. First, we construct a maximum flow problem as follows.

- There exists a unique source vertex v_s and a unique terminal vertex v_e.
- For each type t, there exists a type vertex v_t. From v_s to v_t, there exists a directed edge with capacity n^t.
- For each sub-school c^t, there exists a sub-school vertex v_{c^t}. From v_t to v_{c^t}, there exists a directed edge with capacity n^t.
- For each school c, there exists a school vertex v_c. From v_{c^t} to v_c, there exists a directed edge with capacity $\overline{q}_c^t$. Also, from v_c to v_e, there exists a directed edge with capacity q_c.

An admissible ζ exists if and only if the maximum flow of the above problem is equal to n. The flow from v_t to v_{c^t} represents the number of type t students assigned to school c. There exists several well-known algorithms for efficiently solving a maximum flow problem Cormen, Leiserson, Rivest, and Stein (2009).

To calculate $\tilde{\eta}(X')$, we first fix the flow corresponding to X'. Then, solve the maximum flow problem for the residual capacities.

6.1.3 Evaluation

We evaluate the performance of mechanisms via computer simulation. We consider a market with $n = 512$ students and $m = 16$ schools. The individual maximum quota for each school q_c is 48. The number of types is 4 and there are 128 students of each type. For all type t and school c, we set $\underline{q}_c^t = 4, \overline{q}_c^t = 16$.

We generate students' preferences as follows. We draw one common vector $u_{t,c}$ of the cardinal utilities for each type t from set $[0,1]^m$ uniformly at random. We then randomly draw private vector u_s of the cardinal utilities from the same set, again uniformly at random. Next, we construct cardinal utilities over all m schools for student s with type t as $\alpha u_{t,c} + (1-\alpha)u_s$, for some $\alpha \in [0,1]$. We then convert these cardinal utilities into an ordinal preference relation for each student. The higher the value of α, the more correlated the students' preferences are. In this experiment, we vary α from 0.0 to 1.0 with an increment of 0.1. School priorities $\succ_c$ are drawn uniformly at random, and ML is set to $s_1, \ldots, s_{512}$. Students $s_1, s_5, s_9, \ldots, s_{509}$ are type t_1 students, students $s_2, s_6, s_{10}, \ldots, s_{510}$ are type t_2 students, and so on. We create 100 problem instances for each parameter setting.

Figure 2 shows the ratio of students with justified envy of the same type, and Figure 3 shows the ratio of student who claim an empty seat. The x-axis denotes α and the y-axis denotes the average ratio of students. The ratio of students with justified envy of the ACDA is 0 since it is fair. Also, ratios of students claiming an empty seat of the SD and the ADA are 0 since these mechanisms are nonwasteful.

In Figure 2, we can see the ADA performs better than the SD regardless of α. The difference between the ADA and the SD becomes larger as α increases, i.e., the competition among students becomes more severe. In Figure 3, We can see that the ACDA is extremely wasteful; when $\alpha \geq 0.5$, more than 90% students claim an empty seat.

Figure 4 shows the students' welfare by plotting the Cumulative Distribution Functions (CDFs) of the average number of students matched with their k-th or higher ranked school under each mechanism when $\alpha = 0.6$. Hence, a steep upper trend line is desirable. We can see that in terms of the students' welfare, the SD and the ADA are almost identical. The performance of the ACDA is much worse than the nonwasteful mechanisms.

6.2 Regional minimum quotas

Next, we consider a case where schools are hierarchically organized into regions and for each region, a regional minimum quota is imposed Goto, Hashimoto, Iwasaki, Kawasaki, Ueda, Yasuda, and Yokoo (2014). A hierarchical structure is ubiquitous in any organization (company, university, or military). When an organization allocates human resources, it is natural to assume that the obtained matching must satisfy feasibility constraints of the various levels in the organization's hierarchy, e.g., each division, department, or section has its own minimum quota. Such a feasibility constraint can naturally be represented by regional minimum quotas.

In this setting, the family of vectors that satisfy distributional constraints forms an $\mathrm{M}^\natural$-convex set. Thus, we can apply the generalized DA of Goto, Hashimoto, Iwasaki, Kawasaki, Ueda, Yasuda, and Yokoo (2014) and Kojima, Tamura, and Yokoo (2014). We show that even in such a setting, the ADA can be useful since it can improve students' welfare.

6.2.1 Model

A market is a tuple $(S, C, R, p_R, q_C, \succ_S, \succ_C)$. The meaning of each element is as follows.

- The definitions of $S, C, X, \succ_S, \succ_C$, and q_C are identical to the standard model.
- A region $r \in R$ is a subset of schools, i.e., $r \in 2^C \setminus \{\emptyset\}$.
- We assume the set of regions R is hierarchical, i.e., $\forall r, r' \in R$ where $r \neq r'$, one of the following holds: (a) $r \cap r' = \emptyset$, (b) $r \subset r'$, or (c) $r' \subset r$.
- We can construct a tree T_R for R as follows: (i) root node C is the region that contains all schools, (ii) leaf node $\{c\}$ is a region that contains only one individual school $c \in C$, and (iii) for each node $r \in R$, where $r \neq C$, its parent node $r' \in R$ is a region that is the proper inclusion-minimal superset of r.
- $p_R = (p_r)_{r \in R}$ are regional minimum quota vectors. We assume $0 \leq p_r \leq \sum_{c \in r} q_c$ holds for all $r \in R$.
- We assume that all schools are acceptable to all students and vice versa.
- Let X'_r denote $\{(s, c) \in X' \mid s \in S, c \in r\}$.

- We assume that C, which is the region that contains all schools, is included in R with a non-binding minimum quota $p_C = n$, where n is the number of students.
- children(r) denotes a set of child nodes of r. For a leaf node, i.e., $r = \{c\}$, children(r) is $\emptyset$. It is clear that $r = \bigcup_{r' \in \text{children}(r)} r'$ holds for $|r| \geq 2$. We will often use the terms "node" and "region" interchangeably.

Next, we define school-feasibility in this setting.

Definition 15 (school-feasibility)**.** We say a set of contracts X' is school-feasible if $\forall c \in C, |X'_c| < q_c$ and $\forall r \in R, p_r \leq |X'_r|$ hold.

Without loss of generality, we can assume for each $r \in R$, the following condition holds: $p_r \geq \sum_{r' \in \text{children}(r)} p_{r'}$. If this is not true, the minimum quota of r is non-binding and we can set p_r to $\sum_{r' \in \text{children}(r)} p_{r'}$.

Goto, Hashimoto, Iwasaki, Kawasaki, Ueda, Yasuda, and Yokoo (2014) show that if $p_r \leq \sum_{c \in r} q_c$ holds for each region r, a feasible matching always exists. In this paper, we assume this condition holds. This condition implies $p_C = n \leq \sum_{c \in C} q_c$ holds. Since we assume that all schools are acceptable to all students and vice versa, a student will be matched to some school. Goto, Hashimoto, Iwasaki, Kawasaki, Ueda, Yasuda, and Yokoo (2014) show a fair mechanism called Round-robin Selection Order Deferred Acceptance mechanism for regional minimum quotas (RSDA-RQ) based on the generalized DA. The RSDA-RQ is wasteful.

6.2.2 Representation in our model

We assume $\eta(\zeta)$ is 0 if for all $c_i \in C$, $\zeta_i \leq q_{c_i}$, for all $r \in R$, $p_r \leq \sum_{c_i \in r} \zeta_i$, and $\sum_{c_i \in C} \zeta_i = n$ hold.

This η is not hereditary. To run the ADA, let us define another function $\tilde{\eta}$ as follows. $\tilde{\eta}(\zeta)$ is 0 if there exists $\zeta' \geq \zeta$ such that $\eta(\zeta') = 0$, and otherwise, $-\infty$. Assuming each student/school considers all schools/students acceptable, we can guarantee that the obtained matching X' by the ADA is maximal/feasible. The ADA is strategyproof and nonwasteful.

We can apply the ACDA if we can find an admissible and maximal ζ. We can guarantee that the ACDA, where the maximum quota for each school c_i is given by ζ_i, obtains a maximal matching, since $\sum_{i \in M} \zeta_i = n$ holds, and each student/school considers all schools/students acceptable. Thus, the ACDA obtains a feasible set of contracts.

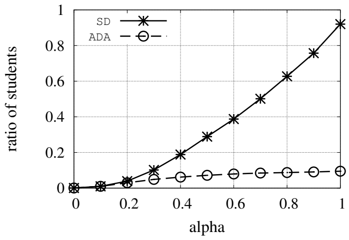

Figure 5: Ratio of students with justified envy

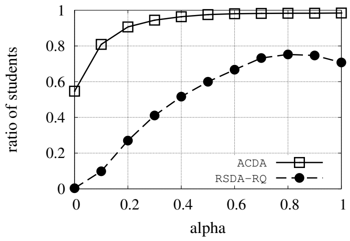

Figure 6: Ratio of students claiming an empty seat

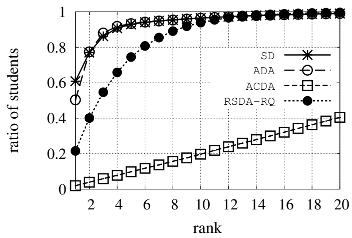

Figure 7: CDFs of students' welfare

We can calculate $\tilde{\eta}$ efficiently by using a concept called an expected minimum count.

Definition 16 (expected minimum count)**.** For each region r, $emc(r, \zeta)$, the expected minimum count of ζ for r, is defined as follows: if $|r| \geq 2$, $emc(r, \zeta) = \max(p_r, \sum_{r' \in children(r)} emc(r', \zeta))$, and if $r = \{c_i\}$, $emc(r, \zeta) = \max(p_r, \zeta_i)$.

Intuitively, the expected minimum count of ζ for r means the minimum number of students assigned to region r, when some more students are added to satisfy regional minimum quotas for all $r' \subseteq r$. Thus, if ζ already satisfies minimum quotas for all $r' \subseteq r$, i.e., $\sum_{c_i \in r'} \zeta_i \geq p_{r'}$ holds, then $emc(r', \zeta) = \sum_{c_i \in r'} \zeta_i$ holds. Also, if $\zeta' > \zeta$, then $emc(C, \zeta') \geq emc(C, \zeta)$ holds.

By using expected minimum counts, we calculate $\tilde{\eta}$ as follows: $\tilde{\eta}(\zeta)$ is 0 if for all $i \in M$, $\zeta_i \leq q_{c_i}$ and $emc(C, \zeta) = n$ hold, and otherwise, $-\infty$.

It is clear that $\zeta_i > q_{c_i}$ for some i, there exists no $\zeta' \geq \zeta$ such that ζ' is admissible in η. Also, assume $emc(C, \zeta) > n$ holds. If ζ already satisfies all minimum quotas, then $emc(C, \zeta) = \sum_{i \in M} \zeta_i > n$. Thus, there exists no $\zeta' \geq \zeta$ such that ζ' is admissible in η. If ζ does not satisfy some minimum quota, we need to add some more students to satisfy all minimum quotas. Assume $\zeta' > \zeta$ satisfies all minimum quotas. Since $emc(C, \zeta') = \sum_{i \in M} \zeta'_i \geq emc(C, \zeta) > n$, ζ' is not admissible in η. Also, as long as for all $i \in M$, $\zeta_i \leq q_{c_i}$ and $emc(C, \zeta) = n$ hold, we can find $\zeta' \geq \zeta$ such that ζ' is admissible in η.

6.2.3 Evaluation

We consider a market with $n = 512$ students and $m = 64$ schools. The individual maximum quota for each school q_c is 40. We set a tree structure to an *octary* tree. In an octary tree, each node has eight children. Thus, the height of the tree is two, i.e., schools are divided into eight regions, each of which contains eight schools. We set the individual minimum quota for each individual school p_c to 0 and regional minimum quota for each region $p_r = 32$. Students' preferences, schools' priority, and ML are generated in the same way as described in Section 6.1.3. We create 100 problem instances for each parameter setting. We follow these parameter settings according to Goto, Hashimoto, Iwasaki, Kawasaki, Ueda, Yasuda, and Yokoo (2014).

Figure 5 shows the ratio of students with justified envy, and Figure 6 shows the ratio of student who claims an empty seat. About the ratio of students with justified envy, the ADA performs better than the SD regardless of α. In the SD, the number of students with justified envy quickly increases as α increases. On the other hand, in the ADA, the ratio increases rather slowly as α increases and remains below 0.1. From Figure 6, we can see that the ACDA is extremely wasteful; when $\alpha \geq 0.2$, more than 90% students claim an empty seat. The RSDA-RQ is better than the ACDA, but it is still wasteful; when $\alpha \geq 0.5$, more than 60% students claim an empty seat.

Figure 7 shows the CDFs of students' welfare when $\alpha = 0.6$. We can see that in terms of the students' welfare, the SD and the ADA are almost identical, although the number of students who are assigned to their most preferred schools is slightly larger in the SD. Also, the ADA improves students' welfare compared to the RSDA-RQ. Thus, when the primary concern is students' welfare, and the secondary concern is the fairness, the ADA seems to be the right choice.

7 Conclusions

In this paper, we developed a strategyproof mechanism that can be applied to general distributional constraints. Our newly developed mechanism (ADA) is nonwasteful and "more fair" than the SD mechanism in several senses. Also, it is "less wasteful" than the ACDA since it is nonwasteful, while the ACDA is not. To demonstrate the applicability of our model in actual application domains, we examined a controlled school choice problem with hard minimum and maximum quotas, as well as a setting where regional minimum quotas are imposed. Simulation results showed that the number of students who have justified envy in the ADA is smaller than that in the SD, and the students' welfare is improved compared to the ACDA, as well as the generalized DA when it is applicable.

References

Abdulkadiroğlu, A., P. A. Pathak, and A. E. Roth (2009): "Strategy-proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match," *American Economic Review*, 99, 1954–1978.

Biro, P., T. Fleiner, R. Irving, and D. Manlove (2010): "The College Admissions Problem with Lower and Common Quotas," *Theoretical Computer Science*, 411(34-36), 3136–3153.

Cormen, T. H., C. E. Leiserson, R. L. Rivest, and C. Stein (2009): *Introduction to Algorithms, Third Edition*. The MIT Press.

Dubins, L. E., and D. A. Freedman (1981): "Machiavelli and the Gale-Shapley Algorithm," *The American Mathematical Monthly*, 88(7), 485–494.

Ehlers, L., I. E. Hafalir, M. B. Yenmez, and M. A. Yildirim (2014): "School Choice with Controlled Choice Constraints: Hard Bounds versus Soft Bounds," *Journal of Economic Theory*, 153, 648–683.

Fragiadakis, D., A. Iwasaki, P. Troyan, S. Ueda, and M. Yokoo (2015): "Strategyproof Matching with Minimum Quotas," *ACM Transactions on Economics and Computation*, forthcoming (an extended abstract in AAMAS, pages 1327–1328, 2012).

Fragiadakis, D., and P. Troyan (2013): "Market Design under Distributional Constraints: Diversity in School Choice and Other Applications," mimeo.

Gale, D., and L. S. Shapley (1962): "College Admissions and the Stability of Marriage," *The American Mathematical Monthly*, 69(1), 9–15.

Goto, M., N. Hashimoto, A. Iwasaki, Y. Kawasaki, S. Ueda, Y. Yasuda, and M. Yokoo (2014): "Strategy-proof matching with regional minimum quotas," in *Thirteenth International Conference on Autonomous Agents and Multiagent Systems (AAMAS-2014)*, pp. 1225–1232.

Goto, M., A. Iwasaki, Y. Kawasaki, Y. Yasuda, and M. Yokoo (2014): "Improving Fairness and Efficiency in Matching Markets with Regional Caps: Priority-list based Deferred Acceptance Mechanism," mimeo (the latest version is available at http://mpra.ub.uni-muenchen.de/53409/).

Hatfield, J. W., and P. R. Milgrom (2005): "Matching with Contracts," *American Economic Review*, 95(4), 913–935.

Kamada, Y., and F. Kojima (2014): "Stability Concepts in Matching with Distributional Constraints," mimeo.

——— (2015): "Efficient Matching under Distributional Constraints: Theory and Applications," *American Economic Review*, 105(1), 67–99.

Kojima, F., A. Tamura, and M. Yokoo (2014): "Designing Matching Mechanisms under Constraints: An Approach from Discrete Convex Analysis," in *Proceedings of the Seventh International Symposium on Algorithmic Game Theory (SAGT-2014)*. the full version is available at http://mpra.ub.uni-muenchen.de/62226).

McVitie, D. G., and L. B. Wilson (1971): "The Stable Marriage Problem," *Commun. ACM*, 14(7), 486–490.

Milgrom, P. (2009): "Assignment messages and exchanges," *American Economic Journal: Microeconomics*, 1(2), 95–113.

Murota, K. (2003): *Discrete Convex Analysis*. Society for Industrial and Applied Mathematics.

Roth, A. E. (2002): "The economist as engineer: Game theory, experimentation, and computation as tools for design economics," *Econometrica*, 70(4), 1341–1378.

Roth, A. E., and M. A. O. Sotomayor (1990): *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis (Econometric Society Monographs)*. Cambridge University Press.

Sönmez, T., and T. B. Switzer (2013): "Matching With (Branch-of-Choice) Contracts at the United States Military Academy," *Econometrica*, 81(2), 451–488.