Relative profit maximization in duopoly: difference or ratio

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We compare formulations of relative profit maximization in duopoly with differentiated goods, 1) (Difference case) maximization of the difference between the profit of one firm and that of the other firm, 2) (Ratio case) maximization of the ratio of the profit of one firm to the total profit. We show that in asymmetric duopoly the equilibrium output of the more efficient (lower cost) firm in the ratio case is larger than that in the difference case and the price of the good of the more efficient firm in the ratio case is lower than that in the difference case. For the less efficient firm (higher cost firm) we obtain the converse results.

Keywords: duopoly, relative profit maximization, difference, ratio

JEL Classification code: D43, L13, L21.

1. Introduction

In recent years, maximizing relative profit instead of absolute profit has aroused the interest of economists. For analyses of relative profit maximization see Schaffer (1989), Vega-Redondo (1997), Matsumura, Matsushima and Cato (2013), Gibbons and Murphy (1990), Lu (2011), Satoh and Tanaka (2013), (2014), Tanaka (2013a) and (2013b).

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In Vega-Redondo (1997) it was shown that the equilibrium in oligopoly with a homogeneous good under relative profit maximization is equivalent to the competitive equilibrium. With differentiated goods, however, the equilibrium in duopoly under relative profit maximization is not equivalent to the competitive equilibrium.

In Tanaka (2013a) it was shown that under the assumption of linear demand and cost functions when firms in duopoly with differentiated goods maximize their relative profits, the Cournot equilibrium and the Bertrand equilibrium are equivalent. Satoh and Tanaka (2014) extended this result to asymmetric duopoly in which firms have different cost functions. Satoh and Tanaka (2013) showed that in a Bertrand duopoly with a homogeneous good under relative profit maximization and quadratic cost functions there exists a range of the equilibrium price, and that range is narrower and lower than the range of the equilibrium price in duopolistic equilibria under absolute profit maximization shown by Dastidar (1995). Tanaka (2013b) showed that under relative profit maximization the choice of strategic variables, price or quantity, is irrelevant to the equilibrium of duopoly with differentiated goods.

In these papers the relative profit of a firm in duopoly is defined as the difference between its profit and the profit of the rival firm. But we can alternatively define the relative profit as the ratio of the profit of one firm to the total profit of two firms. In this paper we compare two formulations of relative profit maximization in duopoly, 1) (Difference case) maximization of the difference between the profit of one firm and that of the other firm, 2) (Ratio case) maximization of the ratio of the profit of one firm to the total profit of two firms, under linear demand and cost functions.

We think that seeking for relative profit or utility is based on the nature of human. Even if a person earns a big money, if his brother/sister or close friend earns a bigger money than him, he is not sufficiently happy and may be disappointed. On the other hand, even if he is very poor, if his neighbor is more poor, he may be consoled by that fact.

Also firms in an industry not only seek their own performances but also want to outperform the rival firms. TV audience-rating race and market-share competition by breweries, automobile manufacturers, convenience store chains and mobile-phone carriers, especially in Japan, are examples of such behavior of firms.

Market-share competition of firms in many industries indicates that the definition of relative profit based on the ratio may be more appropriate.

We show that in symmetric duopoly these definitions of relative profit are completely equivalent, but in asymmetric duopoly the equilibrium output of the more efficient (lower cost) firm in the ratio case is larger than that in the difference case, the equilibrium price of its good in the ratio case is lower than that in the difference case, the equilibrium output of the less efficient (higher cost) firm in the ratio case is smaller than that in the difference case, and the equilibrium price of its good in the ratio case is higher than that in the difference case. Also we show that the equivalence of Cournot and Bertrand equilibria holds in the ratio case as well as in the difference case, and show that the total output in the ratio case is larger than that in the difference case.

In the next section we present the model of this paper, in Section 3 we analyze the difference case, in Section 4 we consider the ratio case, and in Section 5 we present some discussions about the results. A game of relative profit maximization in duopoly in the difference case is a zero-sum game. The game in the ratio case is a constant-sum game. It is equivalent to a zero-sum game. We present an interpretation of our result, in particular, the equivalence of Cournot and Bertrand equilibria from the point of view of zero-sum game theory.

2. The model

There are two firms, A and B. They produce differentiated substitutable goods. The outputs of Firm A and Firm B are denoted by x_A and x_B. The prices of the goods of Firm A and B are denoted by p_A and p_B.

The inverse demand functions of the goods produced by the firms are

$$p_A = a - x_A - bx_B,$$

and

$$p_B = a - x_B - bx_A,$$

where $0 < b < 1$. x_A represents the demand for the good of Firm A, and x_B represents the demand for the good of Firm B. The prices of the goods are determined so that demand of consumers for each firm's good and supply of each firm are equilibrated.

The ordinary demand functions are obtained from these inverse demand functions as follows,

$$x_A = \frac{1}{1-b^2}[(1-b)a - p_A + bp_B],$$

and

$$x_B = \frac{1}{1-b^2}[(1-b)a - p_B + bp_A].$$

Demand and inverse demand functions are symmetric for the firms.

The marginal costs of Firm A and B are denoted by c_A and c_B. In symmetric duopoly the firms have the same marginal cost, that is, $c_A = c_B$. On the other hand, in asymmetric duopoly $c_A \neq c_B$. Without loss of generality we assume $c_A < c_B$ in the asymmetric duopoly, that is, Firm A is more efficient than Firm B. There is no fixed cost. c_A and c_B are positive, and $a > \max\{c_A, c_B\}$.

In the Cournot model the absolute profits of Firm A and B are written as

$$\pi_A = (a - x_A - bx_B)x_A - c_A x_A,$$

and

$$\pi_B = (a - x_B - bx_A)x_B - c_B x_B.$$

Denote the relative profits of Firm A and B, when the relative profit of each firm is defined as the difference between its profit and the profit of the rival firm, by Π_A and Π_B. Then, we have

$$\Pi_A = \pi_A - \pi_B = (a - x_A - bx_B)x_A - c_A x_A - (a - x_B - bx_A)x_B + c_B x_B,$$

and

$$\Pi_B = \pi_B - \pi_A = (a - x_B - bx_A)x_B - c_B x_B - (a - x_A - bx_B)x_A + c_A x_A.$$

Denote the relative profits of Firm A and B, when the relative profit of each firm is defined as the ratio of its profit to the total profit, by Φ_A and Φ_B. Then, we have

$$\Phi_A = \frac{\pi_A}{\pi_A + \pi_B} = \frac{(a - x_A - bx_B)x_A - c_A x_A}{(a - x_A - bx_B)x_A - c_A x_A + (a - x_B - bx_A)x_B - c_B x_B},$$

and

$$\Phi_B = \frac{\pi_B}{\pi_A + \pi_B} = \frac{(a - x_B - bx_A)x_B - c_B x_B}{(a - x_A - bx_B)x_A - c_A x_A + (a - x_B - bx_A)x_B - c_B x_B}.$$

We call the former the *difference case* and the latter the *ratio case*.

In the Bertrand model the absolute profits of Firm A and B are written as

$$\pi_A = \frac{1}{1-b^2}[(1-b)a - p_A + bp_B](p_A - c_A),$$

and

$$\pi_B = \frac{1}{1-b^2}[(1-b)a - p_B + bp_A](p_B - c_B).$$

The relative profits of the firms in the difference case are

$$\begin{aligned}\Pi_A &= \pi_A - \pi_B \\ &= \frac{1}{1-b^2}\{[(1-b)a - p_A + bp_B](p_A - c_A) - [(1-b)a - p_B + bp_A](p_B - c_B)\},\end{aligned}$$

and

$$\begin{aligned}\Pi_B &= \pi_B - \pi_A \\ &= \frac{1}{1-b^2}\{[(1-b)a - p_B + bp_A](p_B - c_B) - [(1-b)a - p_A + bp_B](p_A - c_A)\}.\end{aligned}$$

The relative profits of the firms in the ratio case are

$$\begin{aligned}\Phi_A =& \frac{\pi_A}{\pi_A + \pi_B} \\ =& \frac{[(1-b)a - p_A + bp_B](p_A - c_A)}{[(1-b)a - p_A + bp_B](p_A - c_A) + [(1-b)a - p_B + bp_A](p_B - c_B)},\end{aligned}$$

and

$$\begin{aligned}\Phi_B =& \frac{\pi_B}{\pi_A + \pi_B} \\ =& \frac{[(1-b)a - p_B + bp_A](p_B - c_B)}{[(1-b)a - p_A + bp_B](p_A - c_A) + [(1-b)a - p_B + bp_A](p_B - c_B)}.\end{aligned}$$

3. Difference case

We consider the difference case of asymmetric duopoly[1]. In the Cournot duopoly the first order conditions for maximization of relative profits of the firms are

$$\begin{aligned}\frac{\partial \Pi_A}{\partial x_A} = \frac{\partial \pi_A}{\partial x_A} - \frac{\partial \pi_B}{\partial x_A} &= a - 2x_A - bx_B - c_A + bx_B \\ &= a - 2x_A - c_A = 0, \qquad (1)\end{aligned}$$

[1]The result in this section has been proved in Satoh and Tanaka (2014). But for comparison with the ratio case we recapitulate the analysis in the difference case.

and

$$\frac{\partial \Pi_B}{\partial x_B} = \frac{\partial \pi_B}{\partial x_B} - \frac{\partial \pi_A}{\partial x_B} = a - 2x_B - bx_A - c_B + bx_A$$
$$= a - 2x_B - c_B = 0. \tag{2}$$

The second order conditions

$$\frac{\partial^2 \Pi_A}{\partial x_A^2} = -2 < 0, \text{ and } \frac{\partial^2 \Pi_B}{\partial x_B^2} = -2 < 0$$

are satisfied.

The equilibrium outputs of Firm A and B are obtained, respectively, as

$$\tilde{x}_A^{d,C} = \frac{a - c_A}{2},$$

and

$$\tilde{x}_B^{d,C} = \frac{a - c_B}{2}.$$

d denotes *difference*, and C denotes *Cournot*. The equilibrium prices of the goods of Firm A and B are obtained, respectively, as follows.

$$\tilde{p}_A^{d,C} = \frac{(1-b)a + c_A + bc_B}{2},$$

and

$$\tilde{p}_B^{d,C} = \frac{(1-b)a + c_B + bc_A}{2}.$$

In the Bertrand duopoly the first order conditions for maximization of the relative profits of the firms are

$$\frac{\partial \Pi_A}{\partial p_A} = \frac{\partial \pi_A}{\partial p_A} - \frac{\partial \pi_B}{\partial p_A} = \frac{1}{1-b^2}[(1-b)a - 2p_A + bp_B + c_A - bp_B + bc_B]$$
$$= \frac{1}{1-b^2}[(1-b)a - 2p_A + c_A + bc_B] = 0, \tag{3}$$

and

$$\frac{\partial \Pi_B}{\partial p_B} = \frac{\partial \pi_B}{\partial p_B} - \frac{\partial \pi_A}{\partial p_B} = \frac{1}{1-b^2}[(1-b)a - 2p_B + bp_A + c_B - bp_A + bc_A]$$
$$= \frac{1}{1-b^2}[(1-b)a - 2p_B + c_B + bc_A] = 0. \tag{4}$$

The second order conditions

$$\frac{\partial^2 \Pi_A}{\partial p_A^2} = -\frac{2}{1-b^2} < 0, \text{ and } \frac{\partial^2 \Pi_B}{\partial p_B^2} = -\frac{2}{1-b^2} < 0$$

are satisfied.

The equilibrium prices of the goods of Firm A and B are obtained, respectively, as follows.

$$\tilde{p}_A^{d,B} = \frac{(1-b)a + c_A + bc_B}{2},$$

and

$$\tilde{p}_B^{d,B} = \frac{(1-b)a + c_B + bc_A}{2}.$$

B denotes *Bertrand*. The equilibrium outputs of Firm A and B are

$$\tilde{x}_A^{d,B} = \frac{a - c_A}{2},$$

and

$$\tilde{x}_A^{d,B} = \frac{a - c_B}{2}.$$

We have

$$\tilde{x}_A^{d,C} = \tilde{x}_A^{d,B},\ \tilde{x}_B^{d,C} = \tilde{x}_B^{d,B},\ \tilde{p}_A^{d,C} = \tilde{p}_A^{d,B} \text{ and } \tilde{p}_B^{d,C} = \tilde{p}_B^{d,B}.$$

Thus, we have shown the following proposition.

Proposition 1. *In the difference case the Cournot equilibrium and the Bertrand equilibrium are equivalent.*

The equilibrium absolute profits of the firms are

$$\pi_A = \frac{(a-c_A)^2}{4} - \frac{b(a-c_A)(a-c_B)}{4},$$

and

$$\pi_B = \frac{(a-c_B)^2}{4} - \frac{b(a-c_A)(a-c_B)}{4}.$$

Comparing them yields

$$\pi_A - \pi_B = \frac{(2a - c_A - c_B)(c_B - c_A)}{4} > 0.$$

Denote $\tilde{x}_A^{d,C}$ and $\tilde{x}_A^{d,B}$ by $\tilde{x}_A^d$, $\tilde{x}_B^{d,C}$ and $\tilde{x}_B^{d,B}$ by $\tilde{x}_B^d$, $\tilde{p}_A^{d,C}$ and $\tilde{p}_A^{d,B}$ by $\tilde{p}_A^d$, $\tilde{p}_B^{d,C}$ and $\tilde{p}_B^{d,B}$ by $\tilde{p}_B^d$.

4. Ratio case

Next we consider the ratio case of asymmetric duopoly. The relative profits of Firm A and B in the ratio case are denoted by Φ_A and Φ_B. Generally they are written as

$$\Phi_A = \frac{\pi_A}{\pi_A + \pi_B},$$

and

$$\Phi_B = \frac{\pi_B}{\pi_A + \pi_B}.$$

In the Cournot duopoly the condition for maximization of Φ_A is as follows.

$$\frac{\frac{\partial \pi_A}{\partial x_A}(\pi_A+\pi_B)-\pi_A(\frac{\partial \pi_A}{\partial x_A}+\frac{\partial \pi_B}{\partial x_A})}{(\pi_A+\pi_B)^2}=0.$$

Simplifying this equation under the assumption that $\pi_A > 0$ and $\pi_B > 0$ we have

$$\frac{\partial \pi_A}{\partial x_A}\pi_B-\frac{\partial \pi_B}{\partial x_A}\pi_A=0.$$

Similarly the condition for maximization of Φ_B is as follows.

$$\frac{\partial \pi_B}{\partial x_B}\pi_A-\frac{\partial \pi_A}{\partial x_B}\pi_B=0.$$

They are rewritten as

$$\frac{\partial \pi_A}{\partial x_A}-\frac{\partial \pi_B}{\partial x_A}\frac{\pi_A}{\pi_B}=0, \tag{5}$$

and

$$\frac{\partial \pi_B}{\partial x_B}-\frac{\partial \pi_A}{\partial x_B}\frac{\pi_B}{\pi_A}=0. \tag{6}$$

From the first order conditions in the Cournot duopoly of the difference case, when $x_A = \tilde{x}_A^d$ and $x_B = \tilde{x}_B^d$, we have

$$\frac{\partial \pi_A}{\partial x_A}=\frac{\partial \pi_B}{\partial x_A}=-bx_B<0,$$

and

$$\frac{\partial \pi_B}{\partial x_B}=\frac{\partial \pi_A}{\partial x_B}=-bx_A<0.$$

Since $\pi_A > \pi_B$ at the equilibrium in the difference case, the left hand sides of (5) and (6) are reduced to

$$\left.\frac{\partial \pi_A}{\partial x_A}\left(1-\frac{\pi_A}{\pi_B}\right)\right|_{x_A=\tilde{x}_A^d, x_B=\tilde{x}_B^d}>0,$$

and

$$\left.\frac{\partial \pi_B}{\partial x_B}\left(1-\frac{\pi_B}{\pi_A}\right)\right|_{x_A=\tilde{x}_A^d, x_B=\tilde{x}_B^d}<0.$$

Then, we get the following result.

Proposition 2. *In asymmetric duopoly the equilibrium output at the Cournot equilibrium of the more efficient (lower cost) firm in the ratio case is larger than that in the difference case, and the equilibrium output at the Cournot equilibrium of the less efficient (higher cost) firm in the ratio case is smaller than that in the difference case.*

In the Bertrand duopoly the conditions for maximization of Φ_A and Φ_B under the assumption that $\pi_A > 0$ and $\pi_B > 0$ are written as follows.

$$\frac{\partial \pi_A}{\partial p_A}\pi_B - \frac{\partial \pi_B}{\partial p_A}\pi_A = 0,$$

and

$$\frac{\partial \pi_B}{\partial p_B}\pi_A - \frac{\partial \pi_A}{\partial p_B}\pi_B = 0.$$

They are rewritten as

$$\frac{\partial \pi_A}{\partial p_A} - \frac{\partial \pi_B}{\partial p_A}\frac{\pi_A}{\pi_B} = 0, \tag{7}$$

and

$$\frac{\partial \pi_B}{\partial p_B} - \frac{\partial \pi_A}{\partial p_B}\frac{\pi_B}{\pi_A} = 0. \tag{8}$$

From the first order conditions in the Bertrand duopoly of the difference case, when $p_A = \tilde{p}_A^d$ and $p_B = \tilde{p}_B^d$, we have

$$\frac{\partial \pi_A}{\partial p_A} = \frac{\partial \pi_B}{\partial p_A} = \frac{b}{1-b^2}(p_B - c_B) > 0,$$

and

$$\frac{\partial \pi_B}{\partial p_B} = \frac{\partial \pi_A}{\partial p_B} = \frac{b}{1-b^2}(p_A - c_A) > 0.$$

Since $\pi_A > \pi_B$ at the equilibrium in the difference case, the left hand sides of (7) and (8) are reduced to

$$\left.\frac{\partial \pi_A}{\partial p_A}\left(1 - \frac{\pi_A}{\pi_B}\right)\right|_{p_A=\tilde{p}_A^d, p_B=\tilde{p}_B^d} < 0,$$

and

$$\left.\frac{\partial \pi_B}{\partial p_B}\left(1 - \frac{\pi_B}{\pi_A}\right)\right|_{p_A=\tilde{p}_A^d, p_B=\tilde{p}_B^d} > 0.$$

Then, we get the following result.

Proposition 3. *In asymmetric duopoly the equilibrium price at the Bertrand equilibrium of the more efficient (lower cost) firm in the ratio case is lower than that in the difference case, and the equilibrium price at the Bertrand equilibrium of the less efficient (higher cost) firm in the ratio case is lower than that in the difference case.*

Also in the ratio case we can show the following result.

Proposition 4. *In the ratio case the Cournot equilibrium and the Bertrand equilibrium are equivalent.*

Proof. See Appendix A. □

We denote the equilibrium outputs of Firm A and B in the ratio case both at the Cournot equilibrium and the Bertrand equilibrium by $\tilde{x}_A^r$ and $\tilde{x}_B^r$, and denote the equilibrium prices of the goods of Firm A and B by $\tilde{p}_A^r$ and $\tilde{p}_B^r$. r denotes *ratio*.

Explicit calculations Explicitly calculating the equilibrium outputs and prices, we obtain

$$\tilde{x}_A^r = \frac{(a-c_A)(a-c_B)[(a-c_A)-b(a-c_B)]}{2(a-c_A)(a-c_B)-b[(a-c_A)^2+(a-c_B)^2]},$$

$$\tilde{x}_B^r = \frac{(a-c_A)(a-c_B)[(a-c_B)-b(a-c_A)]}{2(a-c_A)(a-c_B)-b[(a-c_A)^2+(a-c_B)^2]},$$

$$\tilde{p}_A^r = \frac{(a-c_A)\{(1+b^2)(a-c_A)(a-c_B)-b[(a-c_A)^2+(a-c_B)^2]\}}{2(a-c_A)(a-c_B)-b[(a-c_A)^2+(a-c_B)^2]},$$

and

$$\tilde{p}_B^r = \frac{(a-c_B)\{(1+b^2)(a-c_A)(a-c_B)-b[(a-c_A)^2+(a-c_B)^2]}{2(a-c_A)(a-c_B)-b[(a-c_A)^2+(a-c_B)^2]}.$$

From them

$$\tilde{x}_A^r > \tilde{x}_A^d \text{ and } \tilde{x}_B^r < \tilde{x}_B^d,$$

and

$$\tilde{p}_A^r < \tilde{p}_A^d \text{ and } \tilde{p}_B^r > \tilde{p}_B^d$$

are derived. About details, see Appendix B.

Comparing the total output in the ratio case and that in the difference case yields

$$\begin{aligned}
\tilde{x}_A^r + \tilde{x}_B^r - \tilde{x}_A^d - \tilde{x}_B^d =& \frac{b(a-c_A)[(a-c_A)+(a-c_B)](c_B-c_A)}{2\{2(a-c_A)(a-c_B)-b[(a-c_A)^2+(a-c_B)^2]\}} \\
&+ \frac{b(a-c_B)[(a-c_A)+(a-c_B)](c_A-c_B)}{2\{2(a-c_A)(a-c_B)-b[(a-c_A)^2+(a-c_B)^2]\}} \\
=& \frac{[(a-c_A)+(a-c_B)](c_B-c_A)^2}{2\{2(a-c_A)(a-c_B)-b[(a-c_A)^2+(a-c_B)^2]\}} > 0.
\end{aligned}$$

Thus, the total output in the ratio case is larger than that in the difference case.

A note on the symmetric duopoly If the duopoly is symmetric, that is, $c_A = c_B$, in the difference case and the ratio case, the equilibrium outputs of Firm A and B satisfy

$$\tilde{x}_A^d = \tilde{x}_A^r = \tilde{x}_B^d = \tilde{x}_B^r = \frac{a-c}{2},$$

where $c = c_A = c_B$.

The equilibrium prices of the goods of Firm A and B satisfy

$$\tilde{p}_A^d = \tilde{p}_A^r = \tilde{p}_B^d = \tilde{p}_B^r = \frac{(1-b)a+(1+b)c}{2}.$$

Therefore, in symmetric duopoly maximization of relative profits in the difference case and maximization of relative profits in the ratio case are completely equivalent.

5. Some discussions

5.1. Comparison of the difference case and the ratio case

Using a weight on the absolute profit of the rival firm, define the relative profit of each firm as follows.

$$\Psi_A = \pi_A - \alpha_A \pi_B, \text{ and } \Psi_B = \pi_B - \alpha_B \pi_A,$$

with $\alpha_A > 0,\ \alpha_B > 0$ and $\alpha_A \alpha_B = 1$. Then, the first order conditions for maximization of Ψ_A and Ψ_B in the Cournot duopoly are

$$\frac{\partial \pi_A}{\partial x_A} - \alpha_A \frac{\partial \pi_B}{\partial x_A} = 0,$$

and

$$\frac{\partial \pi_B}{\partial x_B} - \alpha_B \frac{\partial \pi_A}{\partial x_B} = 0.$$

Since $\frac{\partial \pi_B}{\partial x_A} < 0$ and $\frac{\partial \pi_A}{\partial x_B} < 0$, the larger the weight on the absolute profit of the rival firm, the larger the absolute value of $\frac{\partial \pi_A}{\partial x_A}$ or $\frac{\partial \pi_B}{\partial x_B}$. This means that a firm, whose weight on the absolute profit of the rival firm is larger, is more aggressive, that is, produces larger output. The difference case corresponds to a case where $\alpha_A = \alpha_B = 1$. On the other hand, the ratio case is equivalent to a case where $\alpha_A = \frac{\pi_A}{\pi_B} > 1$ and $\alpha_B = \frac{\pi_B}{\pi_A} < 1$. Therefore, the more efficient firm (Firm A) produces larger output, and the less efficient firm (Firm B) produces smaller output in the ratio case than the difference case.

In the Bertrand duopoly we can show that the more efficient firm chooses the lower price, and the less efficient firm chooses the higher price in the ratio case than the difference case because $\frac{\partial \pi_B}{\partial p_A} > 0$ and $\frac{\partial \pi_A}{\partial p_B} > 0$. This means that the more efficient firm is more aggressive in the ratio case also in the Bertrand duopoly.

5.2. Zero-sum game interpretation of the equivalence between Cournot and Bertrand equilibria

The game of the difference case is a zero-sum game because

$$\Pi_A + \Pi_B = \pi_A - \pi_B + (\pi_B - \pi_A) = 0.$$

In the game of the ratio case

$$\Phi_A + \Phi_B = \frac{\pi_A}{\pi_A + \pi_B} + \frac{\pi_B}{\pi_A + \pi_B} = 1.$$

Thus, it is a *constant-sum game*. Of course, a constant-sum game is equivalent to a zero-sum game.

Consider a two-person zero-sum game with two strategic variables as follows. There are two players, A and B. They have two sets of strategic variables, (s_A, s_B) and (t_A, t_B). The relations of them are represented by

$$s_A = f_A(t_A, t_B), \text{and } s_B = f_B(t_A, t_B).$$

f_A and f_B are differentiable. The payoff function of Player A is $u_A(s_A, s_B)$ and the payoff function of Player B is $u_B(s_A, s_B) = -u_A(s_A, s_B)$. They are differentiable. The condition for maximization of u_A with respect to s_A and the condition for maximization of u_B with respect to s_B are

$$\frac{\partial u_A}{\partial s_A} = 0, \tag{9}$$

and

$$\frac{\partial u_B}{\partial s_B} = 0. \tag{10}$$

We assume the existence of the maximums of u_A and u_B. Substituting f_A and f_B into u_A and u_B yields

$$u_A = u_A(f_A(t_A, t_B), f_B(t_A, t_B)),\ u_B = u_B(f_A(t_A, t_B), f_B(t_A, t_B)).$$

The condition for maximization of u_A with respect to t_A and the condition for maximization of u_B with respect to t_B are

$$\frac{\partial u_A}{\partial s_A}\frac{\partial f_A}{\partial t_A} + \frac{\partial u_A}{\partial s_B}\frac{\partial f_B}{\partial t_A} = 0, \tag{11}$$

and

$$\frac{\partial u_B}{\partial s_A}\frac{\partial f_A}{\partial t_B} + \frac{\partial u_B}{\partial s_B}\frac{\partial f_B}{\partial t_B} = 0. \tag{12}$$

Under the assumption that $\frac{\partial f_A}{\partial t_A}\frac{\partial f_B}{\partial t_B} - \frac{\partial f_A}{\partial t_B}\frac{\partial f_B}{\partial t_A} \neq 0$, (11) and (12) are equivalent to (9) and (10). Therefore, competition by (s_A, s_B) and competition by (t_A, t_B) are equivalent. If we regard f_A and f_B as demand functions, s_A and s_B as outputs of firms, t_A and t_B as prices, we obtain the equivalence of Cournot equilibrium and Bertrand equilibrium.

For example, consider the ratio case of relative profit maximization in duopoly. We regard s_A and s_B as the outputs of the firms and denote them by x_A and x_B, also regard t_A and t_B as the prices of the goods and denote them by p_A and p_B. We have

$$\begin{aligned} u_A =& \frac{(p_A - c_A)x_A}{(p_A - c_A)x_A + (p_B - c_B)x_B} \\ =& \frac{(a - x_A - bx_B)x_A - c_A x_A}{(a - x_A - bx_B)x_A - c_A x_A + (a - x_B - bx_A)x_B - c_B x_B}, \end{aligned}$$

$$\begin{aligned} u_B =& \frac{(p_B - c_B)x_B}{(p_A - c_A)x_A + (p_B - c_B)x_B} \\ =& \frac{(a - x_B - bx_A)x_B - c_B x_B}{(a - x_A - bx_B)x_A - c_A x_A + (a - x_B - bx_A)x_B - c_B x_B}, \end{aligned}$$

$$f_A(p_A, p_B) = x_A = \frac{1}{1 - b^2}[(1 - b)a - p_A + bp_B],$$

$$f_B(p_A, p_B) = x_B = \frac{1}{1-b^2}[(1-b)a - p_B + bp_A],$$

$$\frac{\partial f_A}{\partial p_A} = -\frac{1}{1-b^2}, \ \frac{\partial f_B}{\partial p_A} = \frac{b}{1-b^2}, \frac{\partial f_B}{\partial p_B} = -\frac{1}{1-b^2}, \text{ and } \frac{\partial f_A}{\partial p_B} = \frac{b}{1-b^2}. \tag{13}$$

$\frac{\partial f_A}{\partial p_A}\frac{\partial f_B}{\partial p_B} - \frac{\partial f_A}{\partial p_B}\frac{\partial f_B}{\partial p_A} \neq 0$ is satisfied. (9) is reduced to

$$\frac{\partial u_A}{\partial x_A} = \frac{(p_A - c_A - x_A)(p_B - c_B)x_B + bx_Ax_B(p_A - c_A)}{(\pi_A + \pi_B)^2} = 0.$$

This is equivalent to (14) in Appendix A. Since

$$\frac{\partial u_A}{\partial x_B} = -\frac{b(p_B - c_B)x_Ax_B + (p_A - c_A)(p_B - x_B - c_B)x_A}{(\pi_A + \pi_B)^2},$$

using (13), we find that (11) means

$$\begin{aligned}&-[(p_A - c_A - x_A)(p_B - c_B)x_B + bx_Ax_B(p_A - c_A)]\\&-b[b(p_B - c_B)x_Ax_B + (p_A - c_A)(p_B - x_B - c_B)x_A] = 0.\end{aligned}$$

Arranging the terms we get

$$[(1-b^2)x_A - (p_A - c_A)]x_B - bx_A(p_A - c_A) = 0.$$

This is the same as (16) in Appendix A, which is the condition for relative profit maximization in the Bertrand duopoly of the ratio case.

Similarly we can show that (10) and (12) mean (15) and (17) in Appendix A.

The results of this paper, in particular, the relation between the difference case and the ratio case seem to be extended to a case of general demand functions. It is a theme of future research.

Appendices

A. Proof of Proposition 4

The conditions for maximization of Φ_A and Φ_B in the Cournot duopoly under the assumption that $\pi_A > 0$ and $\pi_B > 0$ are

$$(p_A - c_A - x_A)(p_B - c_B) + bx_A(p_A - c_A) = 0, \tag{14}$$

and

$$(p_B - c_B - x_B)(p_A - c_A) + bx_B(p_B - c_B) = 0. \tag{15}$$

And the conditions for maximization of Φ_A and Φ_B in the Bertrand duopoly under the assumption that $\pi_A > 0$ and $\pi_B > 0$ are

$$[(1-b^2)x_A - (p_A - c_A)]x_B - bx_A(p_A - c_A) = 0 \tag{16}$$

and

$$[(1-b^2)x_B - (p_B - c_B)]x_A - bx_B(p_B - c_B) = 0. \tag{17}$$

From (14), (15) and the inverse demand functions we obtain

$$\frac{x_B + bx_A}{x_A + bx_B} = \frac{p_B - c_B}{p_A - c_A} = \frac{a - c_B}{a - c_A}, \tag{18}$$

or

$$\frac{p_A - c_A}{x_A + bx_B} = \frac{p_B - c_B}{x_B + bx_A}, \quad \frac{a - c_A}{x_A + bx_B} = \frac{a - c_B}{x_B + bx_A}.$$

From (18) we get

$$\frac{x_B}{x_A} = \frac{p_B - c_B - b(p_A - c_A)}{p_A - c_A - b(p_B - c_B)} = \frac{a - c_B - b(a - c_A)}{a - c_A - b(a - c_B)}. \tag{19}$$

Let

$$\sigma = \frac{p_A - c_A}{x_A + bx_B} = \frac{p_B - c_B}{x_B + bx_A}. \tag{20}$$

Substituting this into (14) yields

$$(p_A - c_A - x_A)\sigma(x_B + bx_A) + \sigma bx_A(x_A + bx_B) = 0. \tag{21}$$

Assuming $\sigma \neq 0$, that is, $p_A - c_A \neq 0$ and $p_B - c_B \neq 0$, we get

$$(1-b^2)x_A x_B - (p_A - c_A)(x_B + bx_A) = 0.$$

This is the same as (16). Similarly substituting (20) into (15) we obtain (17). Alternatively, substituting (21) into (16) and (17) we can get (14) and (15).

Therefore, even when the relative profit of a firm is defined as the ratio of the profit of that firm to the total profit, the Cournot equilibrium and the Bertrand equilibrium are equivalent.

B. Calculations of the equilibrium outputs and prices in the ratio case

(18) implies

$$x_B + bx_A = \frac{a - c_B}{a - c_A}(x_A + bx_B).$$

Substituting this and the inverse demand functions into (21) under the assumption of $\sigma \neq 0$ yields

$$(a - 2x_A - bx_B - c_A)(a - c_B) + bx_A(a - c_A) = 0. \tag{22}$$

(19) implies

$$x_B = \frac{a - c_B - b(a - c_A)}{a - c_A - b(a - c_B)}x_A.$$

Substituting this into (22), the equilibrium output of Firm A in the ratio case is obtained as follows.

$$\tilde{x}_A^r = \frac{(a-c_A)(a-c_B)[(a-c_A)-b(a-c_B)]}{2(a-c_A)(a-c_B)-b[(a-c_A)^2+(a-c_B)^2]}.$$

Similarly, the equilibrium output of Firm B in the ratio case is

$$\tilde{x}_B^r = \frac{(a-c_A)(a-c_B)[(a-c_B)-b(a-c_A)]}{2(a-c_A)(a-c_B)-b[(a-c_A)^2+(a-c_B)^2]}$$

Comparing them with the equilibrium outputs of the firms in the difference case, we have

$$\tilde{x}_A^r - \tilde{x}_A^d = \frac{b(a-c_A)[(a-c_A)+(a-c_B)](c_B-c_A)}{2\{2(a-c_A)(a-c_B)-b[(a-c_A)^2+(a-c_B)^2]\}}$$

and

$$\tilde{x}_B^r - \tilde{x}_B^d = \frac{b(a-c_B)[(a-c_A)+(a-c_B)](c_A-c_B)}{2\{2(a-c_A)(a-c_B)-b[(a-c_A)^2+(a-c_B)^2]\}}$$

If $c_A < c_B$,

$$\tilde{x}_A^r > \tilde{x}_A^d \text{ and } \tilde{x}_B^r < \tilde{x}_B^d$$

hold.

From the inverse demand functions the equilibrium prices of the goods of Firm A and B in the ratio case are, respectively, derived as follows.

$$\tilde{p}_A^r = \frac{(a-c_A)\{(1+b^2)(a-c_A)(a-c_B)-b[(a-c_A)^2+(a-c_B)^2]\}}{2(a-c_A)(a-c_B)-b[(a-c_A)^2+(a-c_B)^2]},$$

and

$$\tilde{p}_B^r = \frac{(a-c_B)\{(1+b^2)(a-c_A)(a-c_B)-b[(a-c_A)^2+(a-c_B)^2]}{2(a-c_A)(a-c_B)-b[(a-c_A)^2+(a-c_B)^2]}.$$

Comparing them with the equilibrium prices of the goods in the difference case, we have

$$\tilde{p}_A^r - \tilde{p}_A^d = \frac{b[(a-c_A)-b(a-c_B)](2a-c_A-c_B)(c_A-c_B)}{2\{2(a-c_A)(a-c_B)-b[(a-c_A)^2+(a-c_B)^2]\}}$$

and

$$\tilde{p}_B^r - \tilde{p}_B^d = \frac{b[(a-c_B)-b(a-c_A)](2a-c_A-c_B)(c_B-c_A)}{2\{2(a-c_A)(a-c_B)-b[(a-c_A)^2+(a-c_B)^2]\}}$$

If $c_A < c_B$,

$$\tilde{p}_A^r < \tilde{p}_A^d \text{ and } \tilde{p}_B^r > \tilde{p}_B^d$$

hold.

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