Two-candidate competition with endogenous valence: a differential game approach

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Abstract

We propose a differential game approach to analyze two–candidate competition in à la Hotelling game with candidates simultaneously choosing locations and investment in valence. We find a Markov perfect equilibrium in which candidates choose divergent locations. Divergence from the median is increasing if the parameter measuring the importance of policy relative to valence is decreasing and if valence depreciates slowly. The results are generalizable to a version of the game with probabilistic voting, that is with a stochastic state equation.
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1. Introduction

Seminal works of Hotelling (1929) and Downs (1957) have predicted that in a two–candidate competition in a one–dimensional policy space, both candidates would locate at the median of the voters' distribution. Since this observation does not always correspond to empirical observations, public choice scholars have been developing theories that would reflect the reality more closely. Most recently, authors incorporate the notion of "valence" into spatial models. Valence can be understood as comprising all the non–policy characteristics of candidates, that affect the electoral outcome. Some of these characteristics remain unaffected by campaign activity (e.g., looks of candidates), whereas others might be altered during campaigns, in which parties aim to affect the views of voters with regard to, for example, their competence or experience. Intuitively, valence is a concept similar to goodwill in accounting and marketing science – it comprises the characteristics of a candidate, other than her policy, which account for the electoral success (Köppl-Turyna, 2014). Schofield (2006) describes valence as "the electorally perceived quality of the political leaders". Under this definition, valence can be decomposed into a time invariant component (e.g. looks of a candidate or party specific effects unaffected by personal investments) and the endogenous, time–varying component which can be affected by actions or expenditures of candidates. Analysis in this paper focuses on the latter kind, but inclusion of candidate time–fixed traits does not change any of the conclusions.

A number of works have incorporated valence into spatial models (e.g. Schofield and Zakharov (2010), Ansolabehere and Snyder (2000)). In most cases, however, valence is either exogenously determined, or is seen as an endogenous variable, but only in a static set–up. We believe that this approach does not capture the time-varying component of valence of a candidate: it is a stock variable, level of which can vary over time, and most importantly can be affected by the actions of the candidates. Some examples of valence affecting behavior are presented by Serra (2010): if the source of valence is competence, candidates can obtain additional education to increase their human capital; if the source of valence is experience, candidates can take jobs in the bureaucracy or the private sector before running for office; if the source of valence is campaign funding, candidates can exert effort to attract donors. These examples show the merit of understanding valence as a stock variable: one does not need to assume that certain valence affecting actions are taken only before the election takes place or before the policy choices are made. Experience of candidates obviously changes over time; politicians spend time on attracting campaign funding during their terms in office to hope for reelection, and so on. In this work we make the first approach to modelling the decision of parties to invest in their stock of valence in a spatial context. We do so by analyzing a two–candidate competition as a differential game, in which both candidates can decide to invest in their stocks of valence capital.

A word of explanation is necessary to understand the modeling technique applied in this work. Despite the fact that the game is played in continuous time, this does not mean that the elections constantly take place. In fact, the continuous time structure allows us to model in a natural way the capital formation process on the part of the candidates. Eventually, we are interested in the steady–state values of the platforms and valence levels after the initial adjustments. It is safe to assume that the election can take place after the steady–state has actually been reached.

Our main conclusions show that the candidates adopt more divergent positions, if the parameter measuring the importance of policy relative to valence is decreasing. Moreover, we show the locations of candidates are more divergent, if valence depreciates slowly, and when the candidates are more "patient". These results shed a new light on the problem of electoral campaigning, and as they can only be captured in a dynamic model, however not directly replicating any of the known results, the effect of the policy importance parameter has been identified in a static set–up (e.g. by Schofield (2007)). Moreover, our results show that the polarization of platforms is present even if there is no *difference* in valences of the candidates (indeed, in a symmetric equilibrium of our model candidates will obtain the same level of valence). The intuition for the polarization is therefore different from the results of Ashworth and Bueno de Mesquita (2009). In our set–up, the polarization is associated with the fact that investment in valence and policy choice are strategic substitutes.

A few recent theoretical works deal with the idea of investment in valence and its relation to the positions of candidates. Ashworth and Bueno de Mesquita (2009) examine a two–stage game. In the first step, candidates choose platforms and later invest in costly valences. The marginal return to valence depends on platform polarization: the closer the platforms are, the more the election outcome is affected by valence. Consequently, candidates without policy preferences choose divergent platforms in order to avoid the competition in valence. Although in both models the candidates have no policy preferences (they maximize votes), differences in valence cause them to adopt divergent positions in equilibrium. Another two–party model with similar predictions is presented in Serra (2010), who reports a positive relationship between platform polarization and differences in valence. Interpreting valence as campaign funds, one can explain an empirical fact in politics: campaign spending is positively correlated with political polarization. This relation has been observed e.g. in the United States during the recent decades (McCarty et al., 2006).

An interpretation of valence closest to this work can be found in Schofield and Zakharov (2010), who interpret valence as campaign advertising. On the other hand, the results remain in opposition to the findings of Herrera et al. (2008), who report that binding campaign spending limits would increase polarization by eliminating one of the incentives for moderation at the margin. In our set–up where a platform and campaigning are strategic substitutes, a spending limit would, by restricting the possibility of valence creation, push the parties in the direction of the median voter.

More generally, we propose inclusion of the differential game approach to modelling electoral competition models: not only in the context of valence, but also when discussing party effects, campaign expenditure and electoral outcomes, impact of the interest groups on policies and so on. These, and similar topics often have a structure similar to the model described in this work: a time–changing structure, capital investments of some kind and variables that can be modeled using state equations such as e.g. parties' and interest groups' expenditures.

This paper is structured as follows: in the next section we present a basic version of the model in which the candidates decide only on their level of valence, but remain policy–oriented. Section 3 presents the main model with candidates investing in valence and their policy choices simultaneously. Section 4 presents the case when the cost of electoral support is strictly convex and the open–loop and closed–loop solutions do not coincide. Section 5 summarizes the implications of the results and presents hypotheses about the relationship of the theory developed here to factual observations about the behavior of candidates. In Section 5 we additionally further explore the novelty and usefulness of the differential game approach in formal modeling of political games.

2. The model with investment in valence

2.1. Description of the model

The basic set–up is a standard Hotelling model, with two candidates and linear transportation cost. Consider a one–dimensional policy space $S[0, 1]$. In the first, simplified version of the model, we will assume that the two candidates with ideal points are located at the ends of the political spectrum. A reader may consider this assumption as reflecting behavior of policy–oriented candidates. Voters are uniformly distributed on the line segment, and each of them is allowed to cast one vote for either of the two candidates. We abstract from the participation effects, by assuming that voters do not abstain: there is full participation. The utility of a voter located at x from voting for a candidate located at z_i is given by

$$U(x, z_i) = \lambda_i - \beta|x - z_i|, \tag{1}$$

where λ_i is the valence of candidate i and β is the "ideological cost" parameter, which defines how strongly do the voters react to the policy relative to the valence of the party. High β means that the platform of the candidate is relatively more important than her valence. At this point it is important to note, that the presented basic model assumes deterministic voting. We will relax this unrealistic assumption further on, and show that the results can be generalized. The position of the voter indifferent between the two candidates can be found by equating the utilities, and simple algebraic manipulation yields a well known formula:

$$x_i(t) = \frac{1}{2} + \frac{\lambda_i(t) - \lambda_j(t)}{2\beta} \tag{2}$$

The position of the indifferent voter, at the same time defines the fraction of voters who vote for the candidate located at position 0. Each candidate can invest in her stock of valence, according to

$$\dot{\lambda}_i(t) = I_i(t) - \delta\lambda_i(t), \tag{3}$$

where $\delta > 0$ denotes the depreciation of the stock of valence. The positive sign of the depreciation guarantees the stability of the system, without any investments. We assume that the candidates maximize their discounted stream of votes, net of the cost of the investment in valence and other operations (x_i in the cost function) that form general costs of maintaining the electoral support. The instantaneous objectives of the candidates are therefore

$$\pi_i(t) = x_i(t) - C(I_i(t), x_i(t)) \tag{4}$$

$$\pi_j(t) = 1 - x_i(t) - C(I_j(t), 1 - x_i(t)) \tag{5}$$

where the cost function $C(\cdot)$ is $C^{(2)}$. In what follows, we drop the time index for simplicity of notation. As for the signs of the derivatives, we assume that the cost of investment is increasing and strictly convex in I_i, thus $C_{I_i}, C_{I_i,I_i} > 0$. The investment cost is also weakly increasing and weakly convex in x_i, the electoral support, thus $C_{x_i} \geq 0$ and $C_{x_i,x_i} \geq 0$. In sections 2 and 3 we analyze the case of linear cost and return to the case of increasing cost in Section 4. This cost comprises the expenses related to electoral campaigning other than the investment in valence e.g. administrative costs, costs of traveling to remote locations, etc.[1]. Additionally, let us assume that the cost function is separable in I_i and x_i, thus $C_{I_i,x_i} = 0$. A point must be raised here. The simple construction of the model assumes that there is no additional cost of the investment in valence other than the cost of activity itself and that the budget is unconstrained. In particular, in the simplified version of the model, we abstract from the fact, that party might need to be forced to move away from their ideal point, in order to raise budget for the campaigning activity, which will in turn restrict the amount of resources available.

[1] An alternative interpretation of the marginal cost of x_i is the loss utility of the party, due to a need to communicate to the voters, whose ideology is far away from the ideal point of the party.

Parties maximize their discounted stream of payoffs according to

$$\begin{aligned}
&\max_{I_i} \int_0^T \pi_i e^{\rho t} dt \\
&\max_{I_j} \int_0^T \pi_j e^{\rho t} dt \\
&\text{s.t.} \\
&\dot{\lambda}_i(t) = I_i(t) - \delta\lambda_i(t) \\
&\dot{\lambda}_j(t) = I_j(t) - \delta\lambda_j(t) \\
&\lambda_{i,j}(0) = \lambda_{0,i,j} \geq 0
\end{aligned}$$

In this work, we abstract of the consideration of interdependence of valence investments arising from e.g. negative campaigning on the part of the opponent, however such as generalized set–up is an important next step for future research. In what follows, T can be assumed finite or infinite, depending on the interpretation of the candidate's behavior. If a candidate is gathering resources to promote a certain bill, that needs to be implemented within a finite time, the finite T is a reasonable assumption. If a candidate predicts, that their operations will continue over an indefinite time–span, one should work with a model for which $T = \infty$.

2.2. Equilibrium

The current–value Hamiltonian function for candidate i is given by

$$J_i = x_i - C(I_i, x_i) + \mu_i(I_i - \delta\lambda_i) + \mu_j(I_j - \delta\lambda_j) \tag{6}$$

and the first order conditions are

$$\mu_i = C_{I_i} \tag{7a}$$

$$\dot{\mu}_i = \mu_i(\delta + \rho) - \frac{1}{2\beta}(1 - C_{x_i}) \tag{7b}$$

$$\dot{\lambda}_i = I_i - \delta\lambda_i \tag{7c}$$

$$\dot{\mu}_j = \frac{1}{2\beta}(1 - C_{x_i}) + \mu_j(\delta + \rho) \tag{7d}$$

considered together with the transversality condition. By the Mangasarian sufficiency condition, first order conditions are sufficient for maximization, since the Hamiltonian and the state equation are both jointly concave in λ_i and I_i. The Hessian matrix of the Hamiltonian is

$$\begin{pmatrix} -C_{I_i,I_i} & 0 \\ 0 & -\frac{1}{(2\beta)^2} C_{x_i,x_i} \end{pmatrix}$$

which under the assumptions is negative semi–definite[2]. Notice that since there is no interaction between the state dynamics for the two candidates, equation (7d) is redundant. Differentiating (7a) with respect to time t and substituting into (7b) together with (7c) yield the following behavior of the locus of investment in valence. Using symmetry, it is easy to derive the functions of behavior for candidate j, and the system is

$$\dot{I}_i = \frac{2\beta C_{I_i}(\delta+\rho) - (1 - C_{x_i})}{2\beta C_{I_i,I_i}} \tag{8}$$

$$\dot{I}_j = \frac{2\beta C_{I_j}(\delta+\rho) - (1 - C_{1-x_i})}{2\beta C_{I_j,I_j}} \tag{9}$$

$$\dot{\lambda}_i = I_i - \delta\lambda_i \tag{10}$$

$$\dot{\lambda}_j = I_j - \delta\lambda_j \tag{11}$$

$$\tag{12}$$

Inspection of the null–isoclines reveals that the null–isoclines $\dot{I} = 0$, under given assumption are always downward–sloping or constant. Given this and the fact that $\dot{\lambda}_{i,j} = 0$ isoclines are upward sloping, the fixed–point of the system exists, and we can linearize it in order to determine the stability. As a matter of fact, by the results of Fershtman and Muller (1984) assumptions of this and the next section imply that the saddle-path solution of this game always exists and is unique. Nevertheless, for exposition purposes we will work through the solution. The Jacobian matrix of the system is given by

$$J = \begin{pmatrix} \delta+\rho & 0 & \frac{C_{x_i,x_i}}{(2\beta)^2(C_{I_i,I_i})} & -\frac{C_{x_i,x_i}}{(2\beta)^2(C_{I_i,I_i})} \\ 0 & \delta+\rho & -\frac{C_{x_i,x_i}}{(2\beta)^2(C_{I_i,I_i})} & \frac{C_{x_i,x_i}}{(2\beta)^2(C_{I_i,I_i})} \\ 1 & 0 & -\delta & 0 \\ 0 & 1 & 0 & -\delta \end{pmatrix}_{|\dot{I}_{i,j}=0, \dot{\lambda}_{i,j}=0}$$

Under the assumptions, the eigenvalues are

$$e_1 = -\delta$$
$$e_2 = \delta + \rho$$
$$e_{3,4} = \frac{1}{2}\left(\rho \pm \frac{\sqrt{C_{I_i,I_i}C_{I_j,I_j}\beta^2\left(C_{I_j,I_j}C_{x_i,x_i} + C_{I_i,I_i}\left(C_{x_i,x_i} + C_{I_j,I_j}\beta^2(2\delta+\rho)^2\right)\right)}}{C_{I_i,I_i}C_{I_j,I_j}\beta^2}\right),$$

so the system is saddle–path stable. In what follows we consider the case of linear cost of electoral support. We will turn back to analyzing the case of $C_{x_i,x_i} > 0$ in Section 4. Under the assumptions, since both the system dynamics and the payoff functions of the

[2]The determinant is zero, if there is no cost of the electoral support or the cost is linear.

parties are linear in the states $\lambda_{i,j}$ and there is no multiplicative interaction between the state and control variables of the parties, the open–loop solution is Markov perfect.

Proposition 1. *The set of FOCs specified in (7a)-(7d), together with a transversality condition $\mu_{i,j}(T) = 0$ if T is finite and $\liminf_{t\to\infty} e^{\rho T}\mu_{i,j}[\tilde{\lambda}_{i,j}(T) - \lambda_{i,j}(t)]$ (catching–up optimality) if $T = \infty$ determines the Markov perfect solution to the game specified. The steady–state level of investment and valence is given by a solution to $\dot{I}_i = 0$ and $\dot{\lambda}_i = 0$, and it is stable in a saddle sense. The transversality condition is satisfied since the instantaneous profit is bounded and at the equilibrium I and μ are bounded.*

2.3. Comparative Statics

The steady–state level of investment for candidate i is given by a solution to $\dot{I}_i = 0$[3]:

$$C^*_{I_i} = \frac{1}{2(\delta+\rho)\beta}(1 - C_{x_i}). \tag{13}$$

Since the marginal cost of investment is assumed to be strictly increasing, higher values of C_{I_i} correspond to higher levels of optimal investment in the steady–state. *Ceteris paribus*, the steady–state level of investment is decreasing in the discount factor, the depreciation rate and the rate at which voters react to the valence of the candidate as compared to her policy. This result is hardly surprising: if the voters react strongly to a policy, the candidates will not spend resources on the non–policy campaign information. To assure a non–negative level of steady–state investment, it must be true that $C_{x_i} < 1$, but this assumption is not anyhow restricting, because it is always possible to rescale the cost function so that this condition holds.

2.4. The model with probabilistic voting

So far we have assumed that the fraction of the voters voting for candidate i at location 0 is given by (2). For well–known reasons, the assumption of deterministic voting is not realistic. Therefore, we shall instead analyze a model, in which the fraction of the voters voting for candidate i includes a random component:

$$x^S_i = \frac{1}{2} + \frac{\lambda_i - \lambda_j}{2\beta} + \varepsilon_i, \tag{14}$$

and $\varepsilon_i \sim N(0, \sigma^2)$. The kind of uncertainty imposed here can be understood similarly to what Banks and Duggan (2005) refer to as "stochastic partisanship", or a random valence shock (unlike Ashworth and Bueno de Mesquita (2009) who impose uncertainty on the median voter's location). Therefore, voters have policy preferences that are known to the candidates, but the voters also have preferences over the candidates unrelated to their policy positions. That is, in our differential game, the uncertainty enters through the state

[3]For candidate j the parenthesis changes to $1 + C_{x_i}$

variable in a form of a Wiener process. We shall therefore assume that the evolution of the state variable λ_i is given by

$$d\lambda_i(t) = (I_i(t) - \delta\lambda_i(t))dt + \sigma(x(t), \lambda(t), t)dw(t) \tag{15}$$

and the initial condition $\lambda_{i,j}(0) = \lambda_0 \geq 0$. The Markov perfect solution to this problem is given by a solution to the set of HJB equations of the form:

$$\begin{aligned}\rho V(\lambda_i, t) - \partial_t V(\lambda_i, t) = \max\{&x_i - C(x_i, I_i) + \partial_{x_i} V(\lambda_i, t)(I_i - \delta\lambda_i) \\ &+ \frac{1}{2}\mathrm{tr}[\partial_{\lambda_i}\partial_{\lambda_i} V(\lambda_i, t)\sigma(x_i, \lambda_i, t)\sigma(x_i, \lambda_i, t)']|I_i \geq 0]\}\end{aligned}$$

It can be proven (compare, e.g. Dockner et al. (2000)) that whenever a game in question is a linear state game, not only the open–loop equilibrium is a solution to the HJB equations as in the deterministic case, but it is also a solution to the stochastic version of the HJB equations. This is true, since if the game is linear in the state, the value function is also linear in the state, and the term $\frac{1}{2}\mathrm{tr}[\partial_{\lambda_i}\partial_{\lambda_i} V(\lambda_i, t)\sigma(x_i, \lambda_i, t)\sigma(x_i, \lambda_i, t)']$ vanishes, which reduces the HJB equation to the deterministic case.

Proposition 2. *The open–loop solution found in (7a)-(7d) constitutes a Markov perfect solution to the game with probabilistic voting, with state dynamics specified in (15).*

3. The model with platform choice and investment in valence

3.1. Description of the model and equilibrium

In a more realistic setting, the candidate will simultaneously choose the platform and invest in valence as to maximize her payoff. The fraction of the population voting for candidate i is now given by

$$\tilde{x}_i = \frac{u_i + u_j}{2} + \frac{\lambda_i - \lambda_j}{2\beta}, \tag{16}$$

where $u_{i,j} \in (0, 1)$ denote the platform choices of the two candidates. Again, since the game is symmetric, and the location and valence choices are independent, we need to analyze only the behavior of one candidate. As noted in the beginning, it is reasonable to assume that the candidate faces a trade–off between locating itself closer to the median of the voters' distribution and raising resources to invest in valence. We justify this assumption by the findings of Serra (2010) and McCarty et al. (2006), who link the increasing amount of resources spent on campaigning to political polarization, and hypothesize that candidates face a trade-off between raising resources from donors when they polarize and moving towards the median voter: by moving away from the ideal point the candidate loses resources necessary for campaigning, but gains additional electoral support. As we do not model directly the budget constraints of the candidates, we must be careful about this interpretation. An alternative interpretation that avoids potential complications associated with modeling the budget constraint, is to understand the cost of platform as a disutility the candidate suffers by moving away from his ideals. Additionally, the cost of platform change may comprise the actual menu costs of platform changes. In the model the cost increases along with moving away from $z_i = 0$ for candidate i and $z_j = 1$ for candidate j. The choice of u_i is, therefore costly with $C_{u_i} > 0$ and $C_{u_i,u_i} > 0$. We do not make any assumption about the sign of C_{I_i,u_i}, as whether platforms and valence are cost substitutes or complements cannot by unambiguously determined. Given the resource constraint interpretation and the disutility interpretation of the cost of platform of Serra (2010), investments in valence and platforms are substitutes (in fact Serra (2010) imposes a additively separable cost functions of parties). Yet, if a cost of a platform choice comprises the menu costs of platform changes, high valence candidates might have an advantage in this aspect (e.g. an experienced candidate might have a comparative advantage on running his office over a newcomer), in which case valence investment and platform change costs are complements. This relation clearly depends on the interpretation of the cost as well as a type of valence investment and could be established empirically.

Notice that if the locations can be chosen at no cost, an equilibrium cannot exist in our set–up. If this was the case, parties would immediately choose positions at the median, and the investment in valence would be infinite. As we will see later on, in fact, an equilibrium of the model with costly locations cannot be "too" convergent by the same reasoning.

The current–value Hamiltonians for this problem are given by

$$\tilde{J}_i = \tilde{x}_i - C(I_i, x_i, u_i) + \mu_i(I_i - \delta\lambda_i) + \mu_j(I_j - \delta\lambda_j) \tag{17}$$
$$\tilde{J}_j = 1 - \tilde{x}_i - C(I_j, (1-x_i), (1-u_j)) + \mu_i(I_i - \delta\lambda_i) + \mu_j(I_j - \delta\lambda_j) \tag{18}$$

The first order conditions for candidate i yield now

$$\mu_i = C_{I_i} \tag{19a}$$
$$1/2 = C_{u_i} \tag{19b}$$
$$\dot{\mu}_i = \mu_i(\delta + \rho) - \frac{1}{2\beta}(1 - C_{x_i}) \tag{19c}$$
$$\dot{\lambda}_i = I_i - \delta\lambda_i \tag{19d}$$
$$\dot{\mu}_j = \frac{1}{2\beta}(1 - C_{x_i}) + \mu_j(\delta + \rho) \tag{19e}$$

and are sufficient if the determinant of

$$\begin{pmatrix} -C_{I_i,I_i} & 0 & -C_{I_i,u_i} \\ 0 & -\frac{1}{(2\beta)^2}C_{x_i,x_i} & 0 \\ -C_{I_i,u_i} & 0 & -C_{u_i,u_i} \end{pmatrix}$$

is nonpositive, which is true if $(C_{I_i,u_i})^2 - C_{u_i,u_i}C_{I_i,I_i} < 0$, which is a standard assumption present in capital accumulation games of the kind analyzed here (compare with Caputo (2005)). Again, it is clear that the sign of the cost interaction partial derivative does not matter for the further analysis, but only the relative magnitude of the cost effects.

Totally differentiating (19a) and (19b) with respect to time, and substituting into (19c), yields after rearrangement:

$$\dot{I}_i = \frac{C_{u_i,u_i}(1 - C_{x_i} - 2C_{I_i}\beta(\delta+\rho))}{2[(C_{I_i,u_i})^2 - C_{u_i,u_i}C_{I_i,I_i}]\beta} \tag{20a}$$

$$\dot{\lambda}_i = I_i - \delta\lambda_i \tag{20b}$$

Given (20a), the only possibility for $\dot{I}$ to be zero, is under the assumption of strict convexity of the cost function with respect to u_i, that the second parenthesis is zero. Together with the first order condition on u_i, the steady–state investment level and positions can be found by solving

$$0 = 1 - C_{x_i} - 2C_{I_i}\beta(\delta+\rho) \tag{21}$$

$$1/2 = C_{u_i} \tag{22}$$

From the assumptions on strict monotonicity of the cost function with respect to I and u, we see that the solution is unique. By the implicit function theorem

$$\frac{du^*}{dI} < 0, \tag{23}$$

that is the platforms converge towards the median along with increasing equilibrium investments in valence. This happens also if the choice of platform and investment in valence are complements in terms of costs, as the relative magnitude of this effect is small (recall $(C_{I_i,u_i})^2 - C_{u_i,u_i}C_{I_i,I_i} < 0$) and since from the perspective of the voters' utility valence and platforms are strategic substitutes.

Again using symmetry for the dynamics of I_j and λ_j, the Jacobian matrix of the system evaluated at the steady–state is

$$J = \begin{pmatrix} \frac{d\dot{I}_i}{dI_i} & 0 & \frac{d\dot{I}_i}{d\lambda_i} & \frac{d\dot{I}_i}{d\lambda_j} \\ 0 & \frac{d\dot{I}_j}{dI_j} & \frac{d\dot{I}_j}{d\lambda_i} & \frac{d\dot{I}_j}{d\lambda_j} \\ 1 & 0 & -\delta & 0 \\ 0 & 1 & 0 & -\delta \end{pmatrix}_{|\dot{I}_i=0,\dot{\lambda}_i=0,C_{u_i}=1/2} , \tag{24}$$

where

$$\frac{d\dot{I}_i}{dI_i} = 2C_{I_i,I_i}C_{u_i,u_i}\beta(\delta+\rho)$$

$$\frac{d\dot{I}_j}{dI_j} = 2C_{I_j,I_j}C_{1-u_j,1-u_j}\beta(\delta+\rho)$$

$$\frac{d\dot{I}_i}{d\lambda_i} = \frac{-C_{u_i,u_i}C_{x_i,x_i}}{(2\beta^2)[(C_{I_i,u_i})^2 - C_{u_i,u_i}C_{I_i,I_i}]}$$
$$\frac{d\dot{I}_i}{d\lambda_j} = \frac{C_{u_i,u_i}C_{x_i,x_i}}{(2\beta^2)[(C_{I_i,u_i})^2 - C_{u_i,u_i}C_{I_i,I_i}]}$$
$$\frac{d\dot{I}_j}{d\lambda_j} = \frac{-C_{1-u_j,1-u_j}C_{x_i,x_i}}{(2\beta^2)[(C_{I_j,u_j})^2 - C_{1-u_j,1-u_j}C_{I_j,I_j}]}$$
$$\frac{d\dot{I}_j}{d\lambda_i} = \frac{C_{1-u_j,1-u_j}C_{x_i,x_i}}{(2\beta^2)[(C_{I_j,1-u_j})^2 - C_{1-u_j,1-u_j}C_{I_j,I_j}]}$$

Saddle–path stable solution exists under the assumptions about the cost functions, it is easy to see that one of the eigenvalues is equal $-\delta$, whereas the other three have complicated forms[4].

Since, the game irrespective of the complicated form of the cost structure, is still linear in the state and there are no multiplicative interactions between the state and the controls, the open–loop solution is Markov perfect and by the argument stated in Section 2.4, it is also a solution to a model with probabilistic voting entering as a white noise.

Proposition 3. *The set of solutions to (19a)–(19e) together with an appropriate transversality condition constitute a Markov perfect Nash equilibrium of the game in question. Moreover, the solution is also a Markov perfect equilibrium of a game for which the state equation is given in (15).*

3.2. Comparative statics and dynamics, and testable predictions

We can apply the Cramer's rule to find the sensitivity of the steady state location choices and investment levels to the parameters of the model. The signs of the derivatives depend on whether valence investments and platform choices are complements or substitutes. The following holds if valence investment and platform choices are substitutes: the location choice u_i is increasing in β:

$$\frac{du_i^*}{d\beta} = -\frac{C_{I_i}C_{I_i,u_i}}{\beta[(C_{I_i,u_i})^2 - C_{u_i,u_i}C_{I_i,I_i}]} > 0, \tag{25}$$

that is the parties tend toward the median for higher values of the parameter reflecting the sensitivity of the voters to the policy message[5]. Similarly,

$$\frac{dI_i^*}{d\beta} < 0. \tag{26}$$

[4]Exact derivations of the values of the eigenvalues is available from the author on request.

[5]Notice that signs of derivatives of u^* for the second player will be the opposite, as direction towards the median corresponds to *decreasing* u

As the marginal return to $\lambda_{i,j}$ is equal to $\frac{1}{2\beta}$, the opportunity cost of investment in valence is higher if voters react strongly to the policy message.
The magnitude of β depends on many factors. The are clear differences between constituencies across countries and elections as shown e.g. by Schofield and Jeon (2010), Schofield et al. (2010b), Schofield et al. (2010a) and Turyna (2010). β is also expected to differ across issues within the same election and constituency: for salient issues voters react more strongly to the platform of a candidate compared to other characteristics. The location choice and investment level also depend on the speed of depreciation of the political capital. We have

$$\frac{du_i^*}{d\delta} > 0 \tag{27}$$

and

$$\frac{dI_i^*}{d\delta} < 0. \tag{28}$$

The candidates tend towards the median instead on investing in valence, whenever valence depreciates quickly, that is for high values of δ: investment in valence in this case does not lead to a long–term increase in the electoral support. Speed of depreciation of valence is again dependent on the type of investment in mind. As noticed in the introduction, persuasive advertisement is known to depreciate slowly compared to other sorts of electoral messages. Yet, speed of depreciation of an investment of a candidate in his education or experience could depend on many factors, which can be explored in further research on this issue.
If valence investment and platform choice cost are complementary the signs of all derivatives change. A further empirical study could explore the question of the actual cost relation. Intuitively, one should find a negative relation between β and persuasive campaign activity, but a positive one if we interpret valence as experience in running the office, as exemplified in the introduction.
To understand the results even more clearly, we apply the dynamic envelope theorem of Caputo, to analyze the sensitivity of the *accumulated* variables to the parameters of the model. Denoting $V(t; \beta, \delta)$ the value of the maximized Hamiltonian – evaluated at the optimal path, the following holds

$$V_\beta(\beta, \delta) = -\frac{1}{2} \int_0^T \frac{\lambda(t)}{\beta^2} dt \leq 0 \tag{29}$$

and

$$V_\delta(\beta, \delta) = -\int_0^T \mu(t)\lambda(t) dt \leq 0, \tag{30}$$

since $\mu(t) = \text{const.} > 0$ and $\lambda(t) \geq 0$. These results mean that the *accumulated* value of the investment is decreasing both in β and δ. This dependence of the value function on the β parameter sheds new light on the question of the impact of issue salience on

platform choices. Since the value of running for office is lower when β is high, it might be a reasonable strategy of a candidate to try to influence the importance of policy to the voters relative to them being subject to campaign persuasion. This question clearly calls for further investigation.
To evaluate the second derivatives with respect ot the parameters, we can for the case of β use Corollary 11.2 of Caputo (2005) and conclude that the since the objective function is convex in β and β does not appear in the state transition equation, the value function V is convex in β and it holds that

$$V_{\beta,\beta}(\beta,\delta) = -\frac{1}{2}\left[\int_0^T \frac{\partial\lambda(t)}{\partial\beta}\beta^{-2}dt - 2\int_0^T \lambda(t)\beta^{-3}dt\right] \geq 0. \tag{31}$$

This expression does not give an unambiguous answer as for the sign of the second derivative, nevertheless it represents the comparative dynamics of the value function with respect to the β parameter. As for the impact of δ on the value function, we cannot use the above result, since δ appears in the transition equation. We apply directly Theorem 11.2 of Caputo (2005) and it follows that

$$L_{\delta,\delta} = -\int_0^T \left[H_{\delta,\lambda}\frac{\partial\lambda(t)}{\partial\delta} + H_{\delta,\mu}\frac{\partial\mu(t)}{\partial\delta}\right]dt = \int_0^T \mu(t)\frac{\partial\lambda(t)}{\partial\delta} + \lambda(t)\frac{\partial\mu(t)}{\partial\delta}dt \leq 0 \tag{32}$$

Since the second component of the sum is zero, it follows that

$$\frac{\partial\lambda(t)}{\partial\delta} \leq 0,$$

that is the accumulated valence is decreasing in the depreciation rate.

4. Strictly increasing cost of electoral support

So far we have analyzed the case of zero or linear cost of electoral support. If the cost of electoral support is strictly increasing some of the conclusions from the previous sections do not hold. In particular, the open-loop solutions found in (13) and (21) and the comparative statics are still valid, but the solution is not necessarily Markov perfect, as for the case of strictly increasing cost of x_i the state and the controls of the two players interact multiplicatively. For the case of general cost functions it is difficult to find a closed–loop solution. Additionally, the comparative dynamics of the system have to be revised, since it is no longer true that the candidates choose a constant level of investment in valence: the multiplicative interaction implies dependence of I_i on λ_i. Determinant of (24) is in any case negative, therefore a saddle path solution exists. In order to find a closed–form solution of the feedback equilibrium, we assume a quadratic cost function of the form:

$$\pi_i = \frac{u_i + u_j}{2} + \frac{\lambda_i - \lambda_j}{2\beta} - \frac{1}{2}x_i^2 - \frac{1}{2}I_i^2. - \frac{\gamma}{2}u_i^2 \tag{33}$$

We conjecture a value function of the form

$$V^i = a_0 + a_1\lambda_i + a_2\lambda_j + \frac{a_3}{2}\lambda_j^2 + \frac{a_4}{2}\lambda_j^2 + a_5\lambda_i\lambda_j + a_6u_i + a_7(1-uj) + \frac{a_8}{2}u_i^2 + \frac{a9}{2}(1-u_j)^2 + \\ a_{10}u_i(1-u_j) + a_{11}u_i\lambda_i + a_{12}(1-u_j)\lambda_j + a_{13}(1-u_j)\lambda_i + a_{14}u_i\lambda_j \tag{34}$$

Additionally we make a simplifying assumption that $\beta = 2$ and $\rho = 0.5$, so that we can concentrate on the effect of the depreciation parameter on the locations of parties. The value function has to satisfy the HJB equation[6]:

$$\rho V^i(\lambda_i, \lambda_j, u_i, u_j) = \max_{u_i, I_i}\{\pi_i + V^i_{\lambda_i}(\lambda_i, \lambda_j)[I_i - \delta\lambda_i] + V^i_{\lambda_j}(\lambda_i, \lambda_j)[\phi_j(\lambda_i, \lambda_j) - \delta\lambda_j]\} \tag{35}$$

The steady–state level of valence λ and positions of parties in a symmetric equilibrium are

$$\lambda^{CL}_{i,j} = \frac{\gamma\left(1 + \sqrt{7 + 8\delta(1+2\delta)}\right)}{2\delta\left(1 + 4\gamma + 4(1+\gamma)\delta + 8(1+\gamma)\delta^2 + (1+\gamma)\sqrt{7 + 8\delta(1+2\delta)}\right)} \tag{36}$$

$$u^{CL}_i = \frac{1 + 4\delta + 8\delta^2 + \sqrt{7 + 8\delta(1+2\delta)}}{2\left(1 + 4\gamma + 4(1+\gamma)\delta + 8(1+\gamma)\delta^2 + (1+\gamma)\sqrt{7 + 8\delta(1+2\delta)}\right)} \tag{37}$$

$$u^{CL}_j = 1 - \frac{1 + 4\delta + 8\delta^2 + \sqrt{7 + 8\delta(1+2\delta)}}{2\left(1 + 4\gamma + 4(1+\gamma)\delta + 8(1+\gamma)\delta^2 + (1+\gamma)\sqrt{7 + 8\delta(1+2\delta)}\right)}. \tag{38}$$

Similarly to the open–loop case, it holds that

$$\frac{du^{CL}_i}{d\delta} > 0$$

and

$$\frac{du^{CL}_i}{d\gamma} < 0,$$

that is the equilibrium is more convergent for higher values of the depreciation rate and lower values of the location choice cost. The opposite holds for the equilibrium value of investment in valence. Using results of Section 3, we can compare this result to the open–loop equilibrium with increasing cost of electoral support. Substitution of the quadratic

[6]Derivation of the solution is provided in the appendix

cost function into (21) yields

$$\lambda_{i,j}^{OL} = \frac{1 + 8\gamma\delta(1+2\delta)}{2\delta(1+2\delta)\left(1 + 4\gamma + 8(1+\gamma)\delta + 16(1+\gamma)\delta^2\right)} \tag{39}$$

$$u_i^{OL} = \frac{-1 + 8(-1+\gamma)\delta(1+2\delta)}{2\gamma\left(1 + 4\gamma + 8(1+\gamma)\delta + 16(1+\gamma)\delta^2\right)} \tag{40}$$

$$u_j^{OL} = 1 - \frac{(1+4\delta)^2 + 8\gamma\left(1 + 3\delta + 6\delta^2\right)}{2\gamma\left(1 + 4\gamma + 8(1+\gamma)\delta + 16(1+\gamma)\delta^2\right)}. \tag{41}$$

For all possible values of δ the closed–loop solution is more convergent than the open–loop. The effect on the investment in equilibrium is ambiguous, as it depends on the relation between the cost of platform choice γ and the depreciation rate δ. We are able to establish numerically that the closed–loop level of investment is higher than the open–loop for $\delta > 1/2$ and $\gamma > 1/2$, that is the relationship holds for the empirically relevant values of the depreciation parameter.

5. Conclusions

In this work, we have presented a dynamic two–party spatial competition model in which parties simultaneously choose locations and invest in valence. We found that the positions of parties diverge from the median if the parameter measuring the importance of policy relative to valence β is decreasing and if the political capital depreciates slowly. Although the current literature offers results as for the relationship between the β, and valences of parties, it concentrates on the differences in valences between the candidates. In our setting, in the steady–state, parties choose the same level of investment, thus they have the same valence, yet they adopt divergent positions. This results from the fact that the benefits of investment in valence and of investment in a location are interrelated and determined endogenously in the model. An innovative result of this model, which cannot be captured in a static set–up is the effect of valence depreciation and discount factor on the positions of parties. If valence depreciates quickly, parties will tend to locate themselves closer to the median. This result reveals an important difference between a two–party system and a multiparty system. In a system such as the U.S. Congress, political capital of individual candidates depreciates quickly, since those who lose the election disappear completely from the political scene at least for a while. This is generally not true for the case of elections in Europe, where individual candidates of parties change, but the composition of the parties remains fairly constant over time and the political image of a party as a whole is much more important than the individual traits of the candidates. European parties can adopt more divergent positions, and exhaust a gathered political capital for a longer period of time.

Another implication of the effect of the depreciation parameter on polarization is associated with the observation about polarization increasing in the U.S. Congress in the last decades. A reason for that, as predicted by our model, could be parties becoming more polarized as a result of the political capital depreciating more slowly these days as compared to the past decades. Nowadays, the persuasive electoral campaigning has become an enormous professionalized industry having access to various tools of modern technology and research on human cognition and marketing. Professional politicians can tailor their campaigning messages very well to the taste of the society, and use tools like Twitter of Facebook to reach the voters directly. The scope of the marketing tools available has been constantly raising over the last decades. It seems that a costless message on Twitter can have a more significant effect on the electoral support than a costly travel around the country organizing campaign trails. With the same resources available, parties nowadays can influence the electorate more strongly than 40 years ago. The political capital depreciates more slowly: with access to tailored marketing and almost costless tools of instant marketing on the Internet, the brand name "does not have the time to depreciate".

In this paper, we introduced a new tool for analyzing electoral outcomes and behavior of office–motivated candidates. Most public choice literature concentrates on stage games, static signaling games as well as bargaining problems. We believe, that a great number of topics in electoral competition can be mostly naturally modelled as a differential game. Whenever parties or contributors involve in a dynamic process of capital accumulation (political capital, campaign contributions or alike) a structure of a capital accumulation game, similar to presented in this work seems a natural tool to explore. Additionally, other problems of collective choice could well be addressed with this method: similarly to what has been for many years explored in environmental and resource economics and modelled as differential common pool resources games. Formal models of electoral competition, campaigning, collective action and electoral behavior could explore so far unrevealed dynamic features by referring to the class of differential games as a tool of choice.

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Appendix

Derivation of the closed–loop solution for the case of increasing cost of electoral support.

The value function has to satisfy the HJB equation:

$$\rho V^i(\lambda_i,\lambda_j,u_i,u_j) = \max_{u_i,I_i}\{\pi_i + V^i_{\lambda_i}(\lambda_i,\lambda_j)[I_i - \delta\lambda_i] + V^i_{\lambda_j}(\lambda_i,\lambda_j)[\phi_j(\lambda_i,\lambda_j) - \delta\lambda_j]\} \tag{42}$$

Maximization of the RHS yields

$$I_i = \phi_i(\lambda_i,\lambda_j,u_i,u_j) = V^i_{\lambda_i}(\lambda_i,\lambda_j,u_i,u_j) = a_1 + a_3\lambda_i + a_5\lambda_j + a_{11}u_i + (1 - a_{13}u_j) \tag{43a}$$

$$I_j = \phi_j(\lambda_i,\lambda_j,u_i,u_j) = V^j_{\lambda_j}(\lambda_i,\lambda_j,u_i,u_j) = a_1 + a_3\lambda_j + a_5\lambda_i + a_{11}(1 - u_j) + a_{13}u_i \tag{43b}$$

$$V^i_{\lambda_j} = a_2 + a_4\lambda_j + a_5\lambda_i + a_{12}(1 - u_j) + a_{14}u_i \tag{43c}$$

$$I_i = -\frac{1}{a_{11}}\{1/2 + a_{12}a_{13} + a_1a_{14} + a_{11}a_{14} + a_{13}a_2 + (-1/2 - a_{12}a_{13} - a_{11}a_{14})u_j + (-1/2 + 2a_{13}a_{14} - \gamma)u_i + (-1/4 + a_{13}a_5 + a_{14}a_5 - a_{11}\delta)\lambda_i + (1/4 + a_{14}a_3 + a_{13}a4 - a_{14}\delta)\lambda_j\} \tag{43d}$$

Substitution of (43a)-(43c) into the HJB equation, yields a set of equations

$$0 = a_0/2 - a_1^2/2 - a_1 a_{12} - a_{11} a_{12} - a_1 a_{13} - a_{13}^2/2 - a_1 a_2 - a_{11} a_2 + a_7/2 + a_9/4 \tag{44a}$$
$$0 = -1/2 + a_1 a_{12} + 2a_{11} a_{12} + a_1 a_{13} + a_{13}^2 + a_{11} a_2 - a_7/2 - a_9/2 \tag{44b}$$
$$0 = 1/8 - a_{11} a_{12} - a_{13}^2/2 + a_9/4 \tag{44c}$$
$$0 = -1/2 + a_{10}/2 - a_1 a_{11} - a_{11} a_{13} - a_{12} a_{13} - a_1 a_{14} - a_{11} a_{14} - a_{13} a_2 + a_6/2 \tag{44d}$$
$$0 = 1/4 - a_{10}/2 + a_{11} a_{13} + a_{12} a_{13} + a_{11} a_{14} \tag{44e}$$
$$0 = 1/8 - a_{11}^2/2 - a_{13} a_{14} + a_8/4 + \gamma/2 \tag{44f}$$
$$0 = 1/4 + a_{12}/2 + a_2/2 - a_{12} a_3 - a_2 a_3 - a_1 a_4 - a_{11} a_4 - a_1 a_5 - a_{13} a_5 + a_{12}\delta + a_2\delta \tag{44g}$$
$$0 = -1/8 - a_{12}/2 + a_{12} a_3 + a_{11} a_4 + a_{13} a_5 - a_{12}\delta \tag{44h}$$
$$0 = -1/8 + a_{14}/2 - a_{14} a_3 - a_{13} a_4 - a_{11} a_5 + a_{14}\delta \tag{44i}$$
$$0 = 1/32 + a_4/4 - a_3 a_4 - a_5^2/2 + a_4\delta \tag{44j}$$
$$0 = -1/4 + a_1/2 + a_{13}/2 - a_1 a_3 - a_{13} a_3 - a_1 a_5 - a_{11} a_5 - a_{12} a_5 - a_2 a_5 + a_1\delta + a_{13}\delta \tag{44k}$$
$$0 = 1/8 - a_{13}/2 + a_{13} a_3 + a_{11} a_5 + a_{12} a_5 - a_{13}\delta \tag{44l}$$
$$0 = 1/8 + a_{11}/2 - a_{11} a_3 - a_{13} a_5 - a_{14} a_5 + a_{11}\delta \tag{44m}$$
$$0 = -1/16 + a_5/2 - 2a_3 a_5 - a_4 a_5 + 2a_5\delta \tag{44n}$$
$$0 = 1/32 + a_3/4 - a_3^2/2 - a_5^2 + a_3\delta \tag{44o}$$

There are six solutions that satisfy equations (44a)–(44o)[7]. Out of these, stability requires that the eigenvalues of the dynamic system resulting from manipulation of (43a) to (43d) are negative, which is satisfied by one of the solutions:

[7]The exact derivation of the solutions is available from the author upon request.

$$a_0 = 1/2 \tag{45}$$
$$a_1 = 0 \tag{46}$$
$$a_2 = 0 \tag{47}$$
$$a_3 = \delta + \frac{1}{12}\left(3 - 2(1+4\delta) - \sqrt{7+8\delta(1+2\delta)}\right) \tag{48}$$
$$a_4 = \frac{1}{12}\left(1 + 4\delta - \sqrt{7+8\delta(1+2\delta)}\right) \tag{49}$$
$$a_5 = \frac{1}{12}\left(-1 - 4\delta + \sqrt{7+8\delta(1+2\delta)}\right) \tag{50}$$
$$a_6 = 0 \tag{51}$$
$$a_7 = 0 \tag{52}$$
$$a_8 = \frac{3 - 4\delta(1+2\delta)\left(3+4\delta+8\delta^2\right) - 2\gamma\left(3+4\delta+8\delta^2\right)^2 - 3\sqrt{7+8\delta(1+2\delta)}}{\left(3+4\delta+8\delta^2\right)^2} \tag{53}$$
$$a_9 = \frac{3 - 4\delta(1+2\delta)\left(3+4\delta+8\delta^2\right) - 3\sqrt{7+8\delta(1+2\delta)}}{\left(3+4\delta+8\delta^2\right)^2} \tag{54}$$
$$a_{10} = \frac{4\delta(1+2\delta)\left(3+4\delta+8\delta^2\right) + 3\left(-1+\sqrt{7+8\delta(1+2\delta)}\right)}{\left(3+4\delta+8\delta^2\right)^2} \tag{55}$$
$$a_{11} = -\frac{1}{1+\sqrt{7+8\delta(1+2\delta)}} \tag{56}$$
$$a_{12} = -\frac{1}{1+\sqrt{7+8\delta(1+2\delta)}} \tag{57}$$
$$a_{13} = \frac{1}{1+\sqrt{7+8\delta(1+2\delta)}} \tag{58}$$
$$a_{14} = \frac{1}{1+\sqrt{7+8\delta(1+2\delta)}} \tag{59}$$