The Mechanics of the Weitzman-Gollier Puzzles

Szekeres, Szabolcs

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by Szabolcs Szekeres[1]

Abstract: The Weitzman-Gollier Puzzle was observed in a setting of risk neutrality. This paper extends its analysis to cases of constant proportional risk aversion and finds that the phenomenon of the puzzle is not confined to the case of risk neutrality. Weitzman discounting produces declining discount rates for risk aversion values below one, but increasing ones for higher degrees of risk aversion. The finding that Weitzman's discounting is actually time reversed negative compounding is confirmed. As Weitzman certainty equivalent rates (CERs) pertain to the cost of storing resources, rather than to interest earned from investing them productively, they should not be used in the evaluation of investment projects. Discounting project net benefits with declining discount rates (DDR) is never justified.

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"Despite some puzzles along the way, the burgeoning theoretical literature on discounting distant time horizons points more or less unanimously towards the use of a declining term structure of social discount rates (DDRs) for risk free public projects," state Mark C. Freeman, Ben Groom, Ekaterini Panopoulou and Theologos Pantelidis in "Declining discount rates and the 'Fisher Effect': Inflated past, discounted future?" (2014), observing that "The ENPV approach has strongly influenced the current U.K., U.S. and Norwegian governments' guidance on long-term discounting." A recent paper supporting this conclusion is "Should Governments Use a Declining Discount Rate in Project Analysis?" by Kenneth J. Arrow, et al. (2014). "Despite some puzzles along the way" refers to the Weitzman-Gollier Puzzle, which many consider to have been solved by Christian Gollier and Marin L. Weitzman (2009) "How Should the Distant Future be Discounted When Discount Rates are Uncertain?"

A recent dissenting opinion is found in a working paper by Szabolcs Szekeres (2015a) "Governments Should Not Use Declining Discount Rates in Project Analysis" that claims that the puzzle had not been solved at the level of risk neutrality, and finds that its cause is that Weitzman's proposed method of discounting is in fact time reversed negative compounding, rather than true discounting. Szekeres (2015b) "When Should the Distant Future not be Discounted at Increasing Discount Rates?" claims that Gollier and Weizmann (2009) does not solve the puzzle nor makes a credible case for DDRs.

This paper will not review theoretical arguments that concern the Weitzman-Gollier Puzzle, but will rather use a simple numerical example to examine the characteristics of the certainty equivalent rates (CERs) computed both according to the standard method of computing CERs and the method proposed in Weitzman (1998). The latter can be used not just for the case of risk neutrality, as in the cited paper, but also for any other degree of risk aversion.

The comparison of the two CER computation methods proves to be quite instructive. Standard CERs are a negative function of the degree of risk aversion, meaning that CERs decline if the degree of risk aversion increases, while Weitzman CERs are a positive function, so that higher CERs correspond to higher degrees of risk aversion. The Weitzman-Gollier Puzzle phenomenon is not confined to the case of risk neutrality, therefore, but is present also at other levels of risk aversion. Regarding the behaviour of CERs as a function of time, the key question of the Weitzman-Gollier Puzzle, the two methods again differ. Standard CERs are an increasing function of time for degrees of risk aversion below one, and a decreasing one for degrees of risk aversion higher than one. The reverse is the case for Weitzman CERs. This leads to the following interesting finding: the

[1] http://orcid.org/0000-0003-3903-5377

phenomenon of declining CERs that proponents of DDRs consider to be supportive of Weitzman's original DDR proposal happens to be inconsistent with it, because the method that produces CERs that are a declining function of time for coefficients of risk aversion higher than one also produces a risk neutral CERs that increases with time, while the method that produces risk neutral DDRs produces CERs that are an increasing function of time for degrees of risk aversion higher than one.

The numerical example presented confirms the finding of Szekeres (2015a) that Weitzman discounting is in fact time reversed negative compounding. It will be shown that this is what explains the differences between the two methods of computing CERs.

It is organized as follows: Section 1 presents the data of the numerical example; Section 2 shows examples of standard CERs; Section 3 shows examples of Weitzman CERs; Section 4 shows that Weitzman discounting is time reversed negative compounding; Section 5 reviews the relationship between the choice of CER method and the term structure of interest rates; and Section 6 presents conclusions.

1. The numerical example

The numerical example has been structured to conform to the assumptions of the Weitzman 1998 model. It assumes a two time period, two scenario world. The interest rate is either 1% or 5%, the two scenarios being equally likely, and $1 is invested for 200 years. Notice that representing the long run with a two time period model implies assuming perfect year-to-year auto-correlation of interest rates. The conditional future monetary value (FV) of an investment of $1 is computed as follows for each scenario:

Table 1
Conditional future values of $1 invested

	Interest rate	**Calculation**	**FV**
Scenario 1	1%	EXP(0.01·200)	$ 7.39
Scenario 2	5%	EXP(0.05·200)	$ 22,026.47

CERs will be computed for both risk averse and risk neutral investors. Risk aversion is modeled using the constant-inter temporal-elasticity-of-substitution (CIES) utility function:

$$U(C)=\frac{C^{1-\sigma}-1}{1-\sigma} \quad (1)$$

where consumption C > 0, and the elasticity of marginal utility with respect to consumption σ > 0 but not equal to 1. This is also the measure of the decision maker's constant proportional risk aversion. Its inverse utility function U^{-1} is C = EXP(LN(1 + U(1 – σ))/ (1 – σ). When σ = 1 then U = LN(C) and U^{-1} is C = EXP(U).

For risk neutral investors U can simply be U = C and U^{-1} will then be C = U, but setting σ = 0 in expression (1) and its inverse function will yield the same results.

2. Calculation of standard CERs

First, the CER of a risk averse investor with σ = 2.5 will be computed. The method of computing CERs, which will probably be accepted by most as being correct, involves (1) finding the certainty equivalent of the uncertain future payoff, and (2) establishing the CER as the annual rate that will compound the initial investment to the certainty equivalent payoff.

Table 2
Step 1: compute standard certainty equivalent payoff for σ = 2.5

	Calculation	Result
Utility of the payoff of Scenario 1	$U(\$7.39) = \frac{7.39^{1-2.5} - 1}{1 - 2.5}$	0.633475288
Utility of the payoff of Scenario 2	$U(\$22026.47) = \frac{22026.47^{1-2.5} - 1}{1 - 2.5}$	0.666666463
Expected utility	[U($7.39) + U($22,026.47)] / 2	0.650070875
Certainty equivalent payoff	U^{-1}(0.650070875)= EXP([ln(1+0.650070875 (1-2.5))/(1-2.5)]	$11.73

Table 3
Step 2: compute standard CER for σ = 2.5

	Calculation	Result
CER	ln(11.73) / 200	1.23%

Step 2 in effect computes the IRR of investing $1 to gain a certain $11.73 in 200 years' time.

Next, the calculation is repeated for a risk neutral investor, *i.e.*, U(x) = x.

Table 4
Step 1: compute standard risk neutral certainty equivalent payoff

	Calculation	Result
Utility of the payoff of Scenario 1	U($7.39) =	7.39
Utility of the payoff of Scenario 2	U($22,026.47) =	22026.47
Expected utility	[U($7.39) + U($22,026.47)] / 2	11016.93
Certainty equivalent payoff	U^{-1}(11016.93)=	$11,016.93

Table 5
Step 2: compute standard risk neutral CER

	Calculation	Result
CER	ln(11016.93) / 200	4.65%

The procedure should be the same for any degree of risk aversion, and is therefore just as applicable to the risk neutrality case. The generic CE compound factor that must be used by any investor is

certainty equivalent payoff / initial investment

which in the case of risk neutrality is $\sum_i p_i e^{r_i t}$ and from which the derived CER is

$$R = \left(\frac{1}{t}\right)\ln\left(\sum_i p_i e^{r_i t}\right) \tag{2}$$

This is the same as the risk neutral CER derived in Gollier (2003).

CERs vary as a function of the degree of risk aversion assumed, as follows for selected degrees of risk aversion:

Table 6
Degrees of risk aversion and standard CERs

σ	**0**	**0.5**	**1.0**	**1.2**	**1.8**	**2.2**	**2.5**
CER	4.65%	4.33%	3.00%	2.27%	1.43%	1.29%	1.23%

Standard CERs are inversely related to risk aversion because they effectively constitute insurance. The more risk averse an investor is, the higher the yield reduction that he is willing to accept to avoid yield uncertainty. The risk neutral investor will accept none, of course (4.65% is the monetary CER of the stochastic yields).

3. Calculation of Weitzman CERs

Weitzman's method of computing CERs starts from a certain future payoff, and computes a stochastic present value. The scenario specific monetary PVs are the following:

Table 7
Conditional present values of $1 certain future payoff

	Interest rate		**Calculation**	**PV**
Scenario 1	1%		1/EXP(0.01·200)	$0.135335283
Scenario 2	5%		1/EXP(0.05·200)	$4.53999E-05

The CER of a risk averse investor with σ = 2.5 is computed as follows using the Weitzman method:

Table 8
Step 1: compute Weitzman certainty equivalent present value for σ = 2.5

	Calculation	**Result**
Utility of the investment of Scenario 1	$U(\$0.1353352\ 83) = \frac{0.13533528\ 3^{1-2.5} - 1}{1-2.5}$	–12.72369128
Utility of the investment of Scenario 2	$U(\$4.53999\text{E} - 05) = \frac{4.53999\text{E} - 05^{1-2.5} - 1}{1-2.5}$	–2179344.248
Expected utility	[U($0.135335283) + U($4.53999E-05)] / 2	–1089678.486
Certainty equivalent PV	U^{-1}(-1089678.486)= EXP([ln(1-1089678.486 (1-2.5))/(1-2.5)]	$7.20676E-05

Table 9
Step 2: compute Weitzman CER for σ = 2.5

	Calculation	**Result**
CER	ln(1/7.20676E-05) / 200	4.77%

Notice that in step 2 the logarithm of the inverse of the CE PV is taken.

The Weitzman CER for risk neutral investors is calculated as follows:

Table 10
Step 1: compute risk neutral Weitzman certainty equivalent present value

	Calculation	Result
Utility of the investment of Scenario 1	U($0.135335283) =	0.135335283
Utility of the investment of Scenario 2	U($4.53999E-05) =	4.53999E-05
Expected utility	[U($0.135335283) + U($4.53999E-05)] / 2	0.067690341
Certainty equivalent PV	U^{-1}(0.067690341)=	$0.067690341

Table 11
Step 2: compute risk neutral Weitzman CER

	Calculation	Result
CER	ln(1/0.067690341) / 200	1.35%

This corresponds, with the data of this example, to the CER proposed in Weitzman (1998)

$$R_W = -\left(\frac{1}{t}\right)\ln\left(\sum_i p_i e^{-r_i t}\right) \tag{3}$$

and which, of course, differs from the standard CER. The CERs computed using Weitzman's method for selected degrees of risk aversion are the following:

Table 12
Degrees of risk aversion and Weitzman CERs

	0	**0.5**	**1.0**	**1.2**	**1.8**	**2.2**	**2.5**
CER	1.35%	1.67%	3.00%	3.37%	4.57%	4.71%	4.77%

Notice that unlike standard CERs, Weitzman CERs are a positive function of the degree of risk aversion. Notice further, by comparing Tables 6 and 12, that the Weitzman-Gollier Puzzle phenomenon is not confined to the risk neutrality case. It is present at any degree of risk aversion, except for when $\sigma = 1$.

The reason why Weitzman CERs are a positive function of the degree of risk aversion will be explained in the next Section.

4. Weitzman discounting is time reversed negative compounding

Szekeres (2015a) showed that Weitzman discounting is time reversed negative compounding. This will be illustrated by computing the standard CER for the case in which the interest rates are –1% and –5%. The scenario specific future monetary payoffs are the following in that case:

Table 13
Conditional future values of $1 invested

	Interest rate	Calculation	FV
Scenario 1	–1%	EXP(–0.01·200)	$0.135335283
Scenario 2	–5%	EXP(–0.05·200)	$4.53999E-05

The CER for the case of σ = 2.5 is computed as follows:

Table 14
Step 1: compute standard certainty equivalent payoff for σ = 2.5

	Calculation	Result
Utility of the payoff of Scenario 1	$U(\$0.135335283) = \frac{0.135335283^{1-2.5} - 1}{1-2.5}$	–12.72369128
Utility of the payoff of Scenario 2	$U(\$4.53999E-05) = \frac{4.53999E-05^{1-2.5} - 1}{1-2.5}$	–2179344.248
Expected utility	[U($0.135335283) + U($4.53999E-05)] / 2	–1089678.486
Certainty equivalent payoff	U^{-1}(–1089678.486)= EXP([ln(1-1089678.486 (1-2.5))/(1-2.5)]	$7.20676E-05

Table 15
Step 2: compute standard CER for σ = 2.5

	Calculation	Result
CER	ln(7.20676E-05) / 200	–4.77%

Notice that the values in Table 14 are exactly the same as those of Table 8, which shows that the utilities of Weitzman PVs are exactly the same as the utilities of the negative interest rate future values. When the standard CER is computed in Table 15, –4.77% is the result. In computing Weitzman CERs, the implicit negative sign is eliminated by time reversal. By taking the future values arrived at through negative compounding to be present values, their inverses are taken to compute CERs, as in Table 9. This makes negative CERs positive, but their absolute values are unchanged: they correspond to the ones of negative compounding. Weitzman CERs differ from standard CERs not because they are based on discounting, as opposed to compounding, for that only changes the sign of the CERs obtained. The difference is due to the fact that they are computed from expected discount factors that correspond to the negatives of the interest rates explicitly assumed.

The phenomenon that Weitzman CERs are a positive function of the degree of risk aversion is a direct consequence of their being based on negative interest rates. Here too CERs are expressions of willingness to insure. Negative compounding corresponds to situations in which resources are stored for a fee. The higher the degree of risk aversion, the higher the certain fee that the "storer" is willing to pay to avoid the storing fee uncertainty. Naturally, the risk neutral one is unwilling to pay any premium over the 1.35% that is the certainty equivalent monetary annual storage fee rate.

5. Choice of CER method and the term structure of interest rates

Table 16 shows the behavior of CERs computed by the alternative methods for time periods of 100, 200 and 300 years. For σ = 1 the term structure of CERs is flat for both methods. However, whether CERs decline or grow with time varies. Standard CERs increase when σ < 1 and decline when σ > 1, whereas in the case of Weitzman CERs this relationship is reversed:

Table 16
Standard and Weitzman CERs as a function of risk aversion and time in years

	Standard CERs			**Weitzman CERs**		
	100	**200**	**300**	**100**	**200**	**300**
2.5	1.46%	1.23%	1.15%	4.54%	4.77%	4.85%
2.2	1.57%	1.29%	1.19%	4.43%	4.71%	4.81%
1.8	1.82%	1.43%	1.29%	4.18%	4.57%	4.71%
1.2	2.61%	2.27%	2.01%	3.39%	3.73%	3.99%
1.0	3.00%	3.00%	3.00%	3.00%	3.00%	3.00%
0.5	3.87%	4.33%	4.54%	2.13%	1.67%	1.46%
0.0	4.33%	4.65%	4.77%	1.67%	1.35%	1.23%

It is ironic that the phenomenon of declining CERs that proponents of DDRs consider to be supportive of Weitzman's original DDR proposal happens to be inconsistent with it, because the method that produces declining CERs for $\sigma > 1$ also produces increasing CERs when $\sigma < 1$, including $\sigma = 0$, the case of risk neutrality.

Those who consider that Weitzman discounting is correct for the risk neutral case must conclude, if they wish to be consistent, that CERs are an increasing function of time when $\sigma > 1$. This choice is only rhetorically open, however. Those who would evaluate investment projects must do so on the basis of the interest rates prevailing in the real world, and not on the basis of those of its mirror image. As Weitzman CERs pertain to storing resources, rather that investing them productively, Weitzman CERs should not be used in the evaluation of investment projects.

All paradoxes disappear when one recognizes Weitzman discounting for what it really is: time reversed negative compounding. The inequality of the CERs computed by the alternative methods, as seen in Table 16 is not a paradox, as the values shown are not alternative measures of the same concept. Rather, they are essentially the measure (except for the disguised minus sign) of two completely different situations. Standard CERs correspond to a world in which investors receive interest for their investment, whereas Weitzman CERs measure the certainty equivalents of storage fees that "storers" pay for the safekeeping of their resources. The two cannot be the same in a world of perfectly correlated interest rates of the same sign. It would be really paradoxical if they were.

6. Conclusions

The findings of the preceding sections can be summarized as follows:

- The Weitzman-Gollier Puzzle phenomenon is not confined to the risk neutrality case. It is present for any degree of risk aversion, except when $\sigma = 1$.
- The reason why Weitzman CERs differ from standard CERs is not that they are based on discounting, as opposed to compounding, for that only changes the sign of the computed CERs. The difference is due to the fact that they are computed from expected discount factors that correspond to the negatives of the interest rates explicitly assumed.
- Standard CERs are an increasing function of time when $\sigma < 1$ and a declining one when $\sigma > 1$, whereas in the case of Weitzman CERs the reverse is the case.
- The phenomenon of declining CERs that proponents of DDRs consider to be supportive of Weitzman's original DDR proposal happens to be inconsistent with it, because the method that produces declining CERs for $\sigma > 1$ also produces increasing CERs when $\sigma < 1$, including $\sigma = 0$, the case of risk neutrality.

- All paradoxes disappear when one recognizes Weitzman discounting for what it really is: time reversed negative compounding. The inequality of standard and Weitzman CERs is not a paradox because they do not measure the same concept.
- As Weitzman CERs pertain to the cost of storing resources, rather than to interest earned from investing them productively, Weitzman CERs should not be used in the evaluation of investment projects.

Before addressing the question of what the consequence of the foregoing is on the possible uses of DDRs, it is worth remembering that this entire discussion takes place within the confines of a rather unrealistic model. The assumption of perfect auto-correlation of non-negative interest rates over very long time periods is not realistic at all. The degree of auto-correlation that is needed for the effects of the Weitzman (1998) model to manifest themselves is extremely high (see Szekeres 2013, Section 3), and its presence has not been conclusively demonstrated yet. If the requisite degree of auto-correlation were absent, then the notion of DDRs would be moot, for then the term structure of interest rates would be flat.

Should such auto-correlation be detected to a sufficient degree, however, then the monetary (or risk neutral) CER of the capital markets would be an increasing function of time. If that were so, then the assertion that the opportunity cost of capital for investment projects is declining, which is what DDRs imply, could only be based on the use of incorrect CERs, or the misinterpretation of correct CERs.

- This paper has shown that the correct CER for risk neutral investors is a growing function of time. This is most important, as it is likely to pertain to most public sector projects. As Szekeres (2015a) explains, the studies that simulate DDRs on the basis of empirical data would report increasing discount rates if they were based on the correct CERs.
- There might be investors who are risk averse enough to have declining CERs. As discussed in Szekeres (2015b), however, CERs (whether low, high, growing, declining or flat) can only be used to discount Divine IOUs (a rare type of asset) or the certainty equivalent net benefits of risky investment projects, and not directly their monetary flows. The behavior of CERs is not really material. As the adjustment to monetary returns implicit in computing risk averse CERs affects project risks and market risks in the same way, if the opportunity cost of capital really is an increasing function of time, then it will be increasing for all investors, independently of their degree of risk aversion.

Discounting project net benefits with DDRs is therefore never justified.

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