Game Form Representation for Judgement and Arrovian Aggregation

Schoch, Daniel

Nottingham University, Malaysia, School of Economics

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GAME FORM REPRESENTATION FOR JUDGEMENT AND ARROVIAN AGGREGATION

DANIEL SCHOCH

Abstract. Judgement aggregation theory provides us by a dilemma since it is plagued by impossibility results. For a certain class of logically interlinked agendas, full independence for all issues leads to Arrovian dictatorship. Since independence restricts the possibility of strategic voting, it is nevertheless a desirable property even if only partially fulfilled.

We explore a "Goldilock" zone of issue-wise sequential aggregation rules which offers just enough independence not to constrain the winning coalitions among different issues, but restrict the possibilities of strategic manipulation. Perfect Independence, as we call the associated axiom, characterises a game-form like representation of the aggregation function by a binary tree, where each non-terminal node is associated with an issue on which all voters make simultaneous decisions.

Our result is universal insofar as any aggregation rule satisfying independence for sufficiently many issues has a game-form representation. One corollary of the game form representation theorem implies that dictatorial aggregation rules have game-form representations, which can be "democratised" by simply altering the winning coalitions at every node.

1. Introduction

Sequential aggregation rules have been proposed for the field of judgement aggregation [List (2004), Dietrich (2014)] as well as for the field of Arrovian social choice [Larsson and Svensson (2006), Battaglini (2003)] as an alternative to simultaneous decisions on independent issues for agendas where alternatives are logically connected in a way that aggregation function satisfying Independence of Irrelevant Alternatives (IIA) can only be dictatorial. In judgement aggregation, they belong to the class of premise-based procedures, which have found more efficient in aggregating information than outcome-based procedures [de Clippe and Eliaz (2015)].

The goal of this paper lies in providing representations for "less than fully independent" aggregation rules in a game form of a binary tree. An axiomatic characterisation of this class of rules is provided, and a new possibility theorem is proven. By weakening Propositionwise Independence (the equivalent axiom to IIA in judgement aggregation), the classical impossibility theorems are not only avoided. The interesting point is that the only axiom involved (besides a very weak unanimity condition) does not "contagiously" constrain winning coalitions for different issues. Thus the type of voting rule - dictatorial, majority, etc. - is not determined by the tree form.

Strategy-proofness has been identified as the predominant motivation behind independence [List and Polak (2010), Ch 4.3]. It holds locally for certain outcomes, which can easily be identified without recurring to winning coalitions. This makes

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tree-form sequential rules are an alternative to dropping completeness, which leads to full strategy-proofness, but requires oligarchic agreements [Dokow und Holzman (2010), Dietrich and List (2008)].

Sequential judgement aggregation rules follow Lindenbaum's procedure in propositional logic for extending a consistent set of proposition to a maximally consistent and therefore complete set. Sequential rules aggregate judgements proposition- or issue-wise, with some given priority, depending on former decisions. At each point of the decision sequence, it is first checked if the accumulated aggregated judgements are logically consistent with the possible outcome of a decision on a proposition p (to either accept or reject p) or on an issue $\{p, \neg p\}$ (choosing between p or $\neg p$). If the outcome is already logically implied by the outcomes of past choices, then the implication is added to the outcome, otherwise a decision is being performed on the basis of the individual judgements to determine the outcome.

In our approach we adopt a harmless modification to the sequential procedure. We defer logical completion to the end of the aggregation of all proper decisions. Since logical dependence of a proposition on a set B does not change when propositions derivable from B are added to the set, or propositions $p \in B$ derivable from $B \setminus \{p\}$ are removed, postponing logical closure will not change the final outcome. Our procedures will perform all decision among logically independent alternatives and skip those whose outcome is already logically determined by the constraints of the agenda and the past decisions. This will yield an outcome, which might not be complete, but is *almost complete* in the sense it posesses a unique completion which can be generated by performing deductive closure relative to the agenda. However, for the sake of simplicity, and to guarantee almost complete outcomes we will adopt decisions on issues instead of deciding over a proposition and its negation at different time points, herewith deviating more significantly from the aforementioned literature.

In this paper we consider judgement aggregation rules which can be represented by a binary decision tree analogous to an extensive game form. We shall develop this analogy by adopting a connotation inspired by game theory. At every node of the tree, all players move simultaneously by submitting their opinion or vote on an issue $\{p, \neg p\}$. The tree then branches according to the aggregated outcome. At every node, a local form of independence on its issue relative to the past is imposed, which guarantees the existence of a set of winning coalitions for p and $\neg p$, respectively, determining a unique outcome.

Since there are no private nodes, rules for information are different compared to ordinary games. The outcome of each decision is either revealed to all players, thus common knowledge at that point, or withheld. Nodes with the same issue can appear at different places, but only nodes which assign the same winning coalitions to the same propositions can be merged in an information set. At final nodes, the deductive closure of the collected aggregated outcomes on its path is formed into the final outcome. If at all decision nodes, no player has an incentive to deviate, the outcome is strategy-proof. Strategy-proof outcomes are the analogon to Nash equilibria.

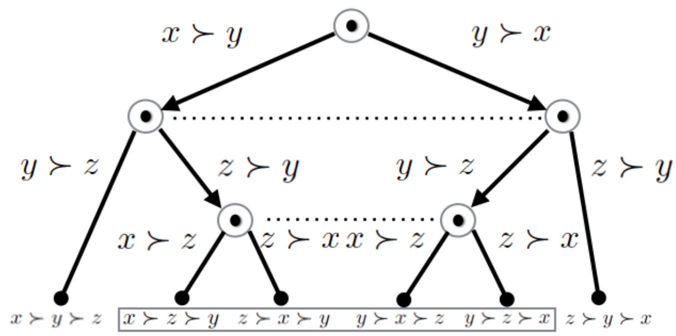
Example 1. As a simple example consider a sequential game form Arrovian aggregation rule over the profile of the Condorcet paradoxon. In terms of judgement aggregation, this corresponds to an agenda of three propositions and their negations under the strict order axioms

$$\begin{aligned} X &= \{x \succ y, \neg(x \succ y), y \succ z, \neg(y \succ z), z \succ x, \neg(z \succ x)\} \\ &= \{x \succ y, y \succ x, y \succ z, z \succ y, z \succ x, x \succ z\}, \end{aligned} \tag{1.1}$$

while the cyclic profile is given by

Player		$x \succ y$	$y \succ z$	$z \succ x$
1	$x \succ y \succ z$	√	√	×
2	$y \succ z \succ x$	×	√	√
3	$z \succ x \succ y$	√	×	√
Majority		√	√	√

Consider the sequential aggregation rule given by the following game form tree[1]On all five decision nodes (circled), simple majority voting is applied.[2]



The rule is not strategy proof. The cyclic profile of the Condorcet paradox in the above table yields the leftmost outcome $x \succ y \succ z$. Voter two has an incentive to change his vote from $y \succ z$ to $z \succ y$, which yields $z \succ x \succ y$, bringing his second best alternative to the front instead of his third. However, we can show the following weaker result.

Consider the four median outcomes inscribed within a box. Their paths are *complete* in the sense that they pass through decision nodes for all issues of the agenda. Given monotonic winning coalitions, we find that if the social outcome is reached by a complete path, then there is no incentive for any voter to deviate to reach a social outcome on a complete path. This provides us with estimated upper bounds for strategic manipulation. If the social outcome *is* reached by a complete path, any strategic manipulation could have only come from a truthful profile which yielded one of the two outer outcomes.[3]

But this is a priori unlikely: $2 \times 40 = 80$ out of $3!^3$ possible profiles, or 37%, yield one of the outer outcomes. Moreover, only a minority of these profiles are Condorcet-like and lead to actual incentives to deviate from the path. They are also less likely to occur in reality than profiles with more similar preferences leading

[1]This special case of a tree-like game form is actually equivalent to a linear sequential rule outlined in [List (2004)].

[2]Technically speaking, the winning coalitions are assigned to the child nodes of a decision node; for the details see subsection 4.1.

[3]To see that deviation from "inside the box" to "out of the box" is possible, consider the profile $x \succ z \succ y$, $y \succ z \succ x$, $z \succ x \succ y$, which yields $z \succ x \succ y$. The first voter has an incentive to change to $x \succ y \succ z$ yielding his new preference, bringing her first choice to the front. Substituting $z \to x$, $x \to y$, $y \to z$ provides an example of a deviation from "outside" to "inside".

to non-manipulated outcomes. In the majority of cases, the rule will not yield manipulated outcomes.

To keep the paper conchise, we assume the reader to be familiar with the concept of judgement aggregation as outlined in the very readable survey by List and Polak [List and Polak (2010)]. In the sprit of List and Polak we treat Arrovian social choice as a special case of judgement aggregation. We follow the logic-based approach to agendas, develop a general notion of consistent sets, and later link this to other approaches like property spaces to keep compatibility with this other major framework of generalised Arrovian choice. However, in our setting we allow agendas to be countably infinite.

2. Settings

2.1. **Judgement Spaces.** An *agenda* X is a finite or countable collection of pairs of propositions $\{p_i, \neg p_i\}_{i\in I}$, where each proposition is paired with its negation forming an *issue*. The propositions are logically constrained by the axioms of the domain, such as transitivity and antisymmetry in the case of Arrovian social choice. We consider the underlying logic to be classical and identify $\neg\neg p$ with p. Propositions are meant to be understood in the most general sense. They can stem from an underlying language, as in the model-theoretical approach [Herzberger and Eckert (2012)]. Propositions can also be seen as elementary properties, when they form a property space, which geometrical properties have been extensively studied by Nehring and Puppe [Nehring and Puppe (2007)]. A *subagenda* $Y \subseteq X$ is a subset of the agenda closed under negation ($p \in Y \Rightarrow (\neg p) \in Y$). For any $B \subseteq X$ let $\mathrm{ag}\,(B)$ denote the smallest subagenda containing B. It is important to note that subagendas form a complete set algebra.

We introduce two equivalent concepts to account for the constraints among the propositions. The first is the notion of a consistent subset of the agenda. We say that the set of all consistent subsets $\mathcal{J} \subseteq 2^X$ forms a judgement space. Our domain are profiles $\underline{J} \in \bar{\mathcal{J}}^N$ from the subset $\bar{\mathcal{J}} \subseteq \mathcal{J}$ of complete and consistent outcomes. The second approach is that of an admissible conditional, a relation $\models \subseteq 2^X \times X$ between a subset of the agenda and a proposition. This logical framework is popular within the literature, while the our own first approach is more suitable for the pupose of this paper, which relies on subspace formation on subagendas. We axiomatise both concepts and find them co-definable.

Definition 1. A *judgement space* $\langle X, \mathcal{J}\rangle$ is an agenda X together with a set of consistent subsets $\mathcal{J} \subseteq 2^X$ satisfying

- **Non-Tautology:** For all $p \in X$, $\{p\} \in \mathcal{J}$.
- **Monotonicity:** For $B \in \mathcal{J}$ and $C \subseteq B$, $C \in \mathcal{J}$.
- **Weak Consistency:** For $B \in \mathcal{J}$, $p \in B$ implies $\neg p \notin B$.
- **Dilemma:** Whenever $B \in \mathcal{J}$, either $B \cup \{p\} \in \mathcal{J}$ or $B \cup \{\neg p\} \in \mathcal{J}$.
- **Compactness:** If $B \subseteq X$ and $B \notin \mathcal{J}$, then there is a finite subset $C \subseteq B$ with $C \notin \mathcal{J}$.

Non-tautology means that no proposition or its negation should be self-contradictory. Monotonicity expresses that each subset of a consistent set is consistent. Weak consistency prohibits a consistent set to both contain a proposition and its negation. Each consistent set must either be consistent with p or with $\neg p$ or both. Inconsistencies always occur on finite subsets.

The following definition introduces the conditional compatible with the judgement space. The axioms are satisfied in classical logic.

Definition 2. An *admissible conditional* on an agenda X is a relation $\models \subseteq 2^X \times X$ satisfying

Reflexivity: $\{p\} \models p$ and $\{p\} \not\models \neg p$.
Monotonicity: $B \models p$ and $B \subseteq C$ implies $C \models p$.
Weak Consistency: $B \models p$ and $B \models \neg p$ implies $B \models q$.
Dilemma: If $B \cup \{p\} \models q$ and $B \cup \{\neg p\} \models q$ then $B \models q$.
Compactness: If $B \models p$ then there is a finite $C \subseteq B$ with $C \models p$.

Translation between these concepts is easily achieved via the following result. We can say that each judgement space has a unique associated admissible conditional, and each admissible conditional generates a judgement space.

Proposition 3. *Let $\langle X, \mathcal{J} \rangle$ be a judgement space, then by*

$$B \models_{\mathcal{J}} p \Leftrightarrow B \cup \{\neg p\} \notin \mathcal{J}$$

an admissible conditional is defined. Conversely, let $\models$ be an admissible conditional, then by

$$\mathcal{J}_{\models} = \{B \subseteq X \mid \exists q \in X : B \not\models q\}$$

a judgement space $\left\langle X, \mathcal{J}_{\models} \right\rangle$ is defined. We find that $\mathcal{J}_{\models_{\mathcal{J}}} = \mathcal{J}$ and $\models_{\mathcal{J}_{\models}}$ reproduces $\models$.

This proposition assures that each judgement space has a unique associated admissible conditional. This allows us to drop the index whenever no ambiguity can arise.

Definition 4. A set $B \subseteq X$ is *complete* if and only if for each issue $\{p, \neg p\} \subseteq X$ either $p \in X$ or $\neg p \in X$. For a given judgement space $\langle X, \mathcal{J} \rangle$ we denote by $\bar{\mathcal{J}} \subseteq \mathcal{J}$ the complete consistent sets. A consistent set $B \in \mathcal{J}$ is *almost complete* if and only if there is a unique complete consistent set $C \in \bar{\mathcal{J}}$ with $B \subseteq C$. We denote by $\tilde{\mathcal{J}}$ the almost complete sets.

One finds the Lindenbaum extension lemma as an immediate consequence of the Dilemma and Compactness axioms.

Lemma 5. *(Extension) Each consistent set $B \in \mathcal{J}$ has a completion $\bar{B} \in \bar{\mathcal{J}}$ with $B \subseteq \bar{B}$.*

2.2. **Property Spaces.** The following subsection demonstrates the compatibility of judgement spaces with the concept of property spaces as introduced by Nehring and Puppe. It can be skipped by those readers not familiar with the corresponding literature. Property spaces and Judgement spaces are equivalent in the finite case and can be made equivalent by adding a compactness axiom to the former.

Definition 6. A (compact) *property space* $\langle Y, \mathcal{H} \rangle$ with $\mathcal{H} \subseteq 2^Y$ satisfies

Non-Triviality: $\emptyset \notin \mathcal{H}$.
Negation: If $H \in \mathcal{H}$, then $Y \setminus H \in \mathcal{H}$.
Separation: For all $x, y \in Y$ with $x \neq y$ there is a $H \in \mathcal{H}$ with $x \in H$ and $y \notin H$.
Compactness: If $\mathcal{F} \subseteq \mathcal{H}$ with $\bigcap \mathcal{F} = \emptyset$ then there is a finite $\mathcal{G} \subseteq \mathcal{F}$ with $\bigcap \mathcal{G} = \emptyset$.

Proposition 7. *For a given property space $\langle Y, \mathcal{H}\rangle$ a corresponding judgement space $\langle \mathcal{H}, \mathcal{J}_{\mathcal{H}}\rangle$ can be defined with agenda $\mathcal{H}$ and*

$$\mathcal{J}_{\mathcal{H}} = \left\{ \mathcal{F} \subseteq \mathcal{H} \mid \bigcap \mathcal{F} \neq \emptyset \right\}.$$

There is a bijection which maps each $x \in Y$ to a $\mathcal{F}_x \in \bar{\mathcal{J}}_{\mathcal{H}}$ such that $\bigcap \mathcal{F}_x = \{x\}$.

2.3. **Subspaces.** Let $A \in \mathcal{J}$ be a potential outcome of a sequence of decisions. We want to develop a notion of conditioning the judgement space $\langle X, \mathcal{J}\rangle$ on A. As in probability theory, this is not possible for all sets. Even subagendas $Y \subseteq X$ might not induce a judgement space, since some elementary propositions in Y could be constrained causing the Non-Tautology axiom to fail. To keep Non-Tautology, we impose the following restrictions.

Definition 8. We say that $A \subseteq X$ is *free* if and only if for any issue $\{p, \neg p\} \subseteq X \setminus \mathrm{ag}\,(A)$ we find $A \cup \{p\}, A \cup \{\neg p\} \in \mathcal{J}$. We denote by $\hat{\mathcal{J}}$ the collection of free sets.

We can extend any consistent set to a free set, and there is a smallest among them.

Lemma 9. *For any consistent set $A \in \mathcal{J}$ there is a smallest free set $\hat{A} \in \hat{\mathcal{J}}$ with $A \subseteq \hat{A}$. It is given by the deductive closure relative to the agenda*

$$\hat{A} = \mathrm{cl}\,(A) := \{p \in X \mid A \vDash_{\mathcal{J}} p\}.$$

Together with the preceeding lemma, the following proposition guarantees the existence of a largest conditional judgement space for the smallest free set $\hat{A}$ containing $A \in \mathcal{J}$.

Proposition 10. *For a free set $A \in \hat{\mathcal{J}}$ and a subagenda $Y \subseteq X$ with $A \cap Y = \emptyset$, a conditional judgement space $\langle Y, \mathcal{J}_A\rangle$ is defined by*

$$\mathcal{J}_A = \{B \setminus A \subseteq Y \mid B \in \mathcal{J} \text{ and } A \subseteq B\}$$

The associated conditional is given by

$$B \vDash_A p \Leftrightarrow A \cup B \vDash p.$$

Example 1 continued. (from introduction). Consider the subagenda $Y = \{y \succ z, z \succ y\}$. There are four possible consistent sets A_i, $i = 1, 2, 3, 4$, which could yield Y as their conditional judgement space by proposition 10,$X_{A_i} = Y$, namely

$$\begin{aligned} A_1 &= \{x \succ y, x \succ z\}, \\ A_2 &= \{x \succ y, z \succ x\}, \\ A_3 &= \{y \succ x, x \succ z\}, \\ A_4 &= \{y \succ x, z \succ x\}. \end{aligned}$$

Not all are suitable. A_2 is $z \succ x \succ y$, which implies $z \succ y$, which settles agenda Y. Likewise, A_3 is $y \succ x \succ z$, which implies $y \succ z$, leaving no choice in Y. Both A_2 and A_3 are not free, and their closure is complete, so they can not induce a non-empty conditional subspace. However, A_1 and A_4 are compatible with both $y \succ z$ and $z \succ y$ and theirefore free, and yield Y as a conditional subspace.

3. Independence

3.1. Conditional Independence.

Definition 11. Let $\langle X, \mathcal{J} \rangle$ be a judgement space. A (judgement) *aggregation function* $F : \bar{\mathcal{J}}^N \to \bar{\mathcal{J}}$ maps profiles of complete and consistent sets to a complete and consistent set. It is *totally unanimous* if and only if for each profile $\underline{J} \in \bar{\mathcal{J}}^N$ with $J_i = J$ for all i, we have $F(\underline{J}) = J$.

Total unanimity is a very weak Pareto principle: Only if everyone agrees on everything, then the uniform common opinion is the social outcome. The analogon to the Arrovian Pareto principle, *Propositionwise Unanimity*, has been challanged alongside propositionwise independence in sequential judgement aggregation [Dietrich (2014), Ch 5]. Indeed, neither principle can hold for proposition p in the following example.

Example 2. Consider an agenda spanned by the four propositions p, q, r, s with condition $q \wedge r \wedge s \to \neg p$. The example could represent a parlamentary decision on three different and unrelated public goods q, r, s and a budget constraint p which allows the realization of only two of these goods. Consider an aggregation function defined by propositionwise majority on q, r, and s independently in an arbitrary order, followed by a conditional majority decision on p whenever consistent with the previous outcome. The profile $\{p, \neg q, r, s\}$, $\{p, q, \neg r, s\}$, $\{p, q, r, \neg s\}$ yields q, r, s by majority decisions, and $\neg p$ by the condition, although p is common opinion. Thus both unanimity and independence for p lead to a contradiction.

The central idea behind our representation theorem is the concept of conditional independence extended to subagendas containing more than one issue and relative to some previous decisions.

Definition 12. An aggregation function $F : \bar{\mathcal{J}}^N \to \bar{\mathcal{J}}$ is called *Y-independent given A* for a subagenda $Y \subseteq X$ and $A \in \mathcal{J}$ if and only if for all profiles $\underline{J}, \underline{J}' \in \bar{\mathcal{J}}^N$ with $A \subseteq F(\underline{J}) \cap F(\underline{J}')$ and $J_i \cap Y = J'_i \cap Y$ for all $i \in N$, then

$$F(\underline{J}) \cap Y = F(\underline{J}') \cap Y.$$

If $A = \emptyset$ we simply speak of *Y-independence*. The aggregation function satisfies *Propositionwise Independence* if and only if it is Y-independent for any issue $Y = \{p, \neg p\} \subseteq X$.

Conditional independence has some notable properties. Case (i) allows for the strengthenings of conditions. Case (ii) shows that issue-wise independence implies independence for all other sets. Case (iii) allows the subagenda to stretch over the agenda of the condition.

Lemma 13. *Let $F : \bar{\mathcal{J}}^N \to \bar{\mathcal{J}}$ be an aggregation function on $\langle X, \mathcal{J} \rangle$ and Y a subagenda.*

(i) If F is Y-independent given A and $A \subseteq A'$, then F is Y-independent given A'.

(ii) If F is Y_i-independent given A for all $i \in I$, then F is $\bigcup_{i \in I} Y_i$-independent given A.

(iii) If F is Y-independent given A and Y' is a subagenda of X with

$$Y \subseteq Y' \subseteq Y \cup ag(\mathrm{cl}(A)),$$

then F is Y'-independent given A.

Example 1 continued. (from subsection 2.3). All paths whose outcome contains A_1 goes through the leftmost node, which decides issue $Y = \{y \succ z, z \succ y\}$. Thus F is Y_1-independent given A_1. Similarly, all paths with oucome containing A_4 go through the rightmost node, likewise deciding issue Y, rendering F Y-independent given A_4.

Proposition 14. *Let $F : \bar{\mathcal{J}}^N \to \bar{\mathcal{J}}$ be an aggregation function on $\langle X, \mathcal{J} \rangle$. Let $A \in \hat{\mathcal{J}}$ be a free set, Y a subagenda with $A \cap Y = \emptyset$, and F being Y-independent given A. Let $\langle Y, \mathcal{J}_A \rangle$ be the conditional subspace. For all profiles $\underline{J}^A \in \bar{\mathcal{J}}_A^N$ and $\underline{J} \in \bar{\mathcal{J}}^N$ with $A \subseteq F(\underline{J})$, and $J_i^A = J_i \cap Y$ by*

$$F_A\left(\underline{J}^A\right) = F(\underline{J}) \cap Y,$$

an aggregation function on the conditional subspace is defined.

3.2. **Perfect Independence.** The idea behind the concept of perfect independence is that of a general sequential aggregation rule. Assume that several decisions have already been made, and the outcome is A. Without loss of generality, we assume that all derivable propositions of the agenda are already included in A. In other words, A is free. To make the next step in the decision, we require that there is at least one issue disjoint from A, for which F is independent given the current state of affairs A. This implies that at the beginning, we require unconditional independence only for *one* issue. All other requirements for independence are conditional on previous outcomes. This is a very weak requirement compared to full Propositionwise Independence.

Definition 15. An aggregation function $F : \bar{\mathcal{J}}^N \to \bar{\mathcal{J}}$ satisfies *Perfect Independence* if and only if for every free set A which is not complete there an issue $Y = \{p, \neg p\}$, $Y \cap \operatorname{ag}(A) = \emptyset$, such that F is Y-independent given A.

Example 1 continued. (from subsection 3.1). We demonstrate Perfect Independence for the Condorcet example. The non-complete free sets of the Condorcet example are $\emptyset$, the single propositions of the agenda, the sets A_1 and A_4 defined above, and the sets

$$\begin{aligned} A_5 &= \{x \succ y, z \succ y\}, \\ A_6 &= \{y \succ x, y \succ z\}. \end{aligned}$$

The uppermost node proves $\{x \succ y, y \succ x\}$-independence of F, which also holds given any proposition from the agenda (lemma 13 (i)). This settles the case for the empty set and for any single proposition except $x \succ y$ and $y \succ x$. The leftmost and rightmost node both decide issue $Y = \{y \succ z, z \succ y\}$, which shows that F is Y-independence given $\{x \succ y\}$ and given $\{y \succ x\}$. Sets A_1 and A_4 are extensions of these two set and disjoint from Y, thus by lemma 13, F is also Y-independent given A_1 and given A_4. The sets A_5 and A_6 both lead to the lowermost nodes, which determines the issue $Y' = \{x \succ z, z \succ x\}$ disjoint from both sets. It follows that $F$is Y'-independent given A_5 and given A_6. Since all other free sets are complete, we have shown that F is perfectly independent.

Perfect Independence is a much weaker condition than Propositionwise Independence. Indeed, lemma 13 (i) immedeately implies

Fact 16. *Propositionwise Independence implies Perfect Independence.*

4. Judgement Aggregation Game Forms

4.1. **Game Forms.** There are two possible ways to link decision nodes over an issue $\{p, \neg p\}$ to a set of winning coalitions. We could either single out one of the propositions and attaching the winning coalitions only for that proposition to the decision node, then label the child nodes as "accept" or "reject", but this would destroy the symmetry of the construction and add an unecessary bias. A different path has been layed out by Nehring and Puppe, which assign winning coalitions to a proposition and its nogation and constrain them through (4.1) to obtain a well-defined decision outcome [Nehring and Puppe (2007)]. Following this approach, we assign winning coalitions to the child nodes of a decision node. This has to be respected when defining information sets, which include parent nodes, but constrain their child nodes.

Definition 17. An *extensive game form* $\mathcal{G} = \left\langle N, T, \varphi, \{\mathcal{W}_x\}_{x\in T}, \mathcal{I}\right\rangle$ for an agenda X consists of

(i) a finite or countable set of players N,

(ii) a finite or countable binary tree T with root $r \in T$ such that every non-terminal node has exactly two child nodes,

(iii) a surjective mapping $\varphi : T \setminus \{r\} \to X$ assigning a proposition to each non-root node such if x and y are the two child nodes of a common parent node, they are mapped to complementary propositions ($\varphi(x) = \neg\varphi(y)$),

(iv) a collection of winning coalitions $\{\mathcal{W}_x\}_{x\in T}$ assigning a set $\mathcal{W}_x \subseteq 2^N$ to every node $x \in T$ (with the dummy $\mathcal{W}_r = 2^N$) such that whenever x and y are the two child nodes of a common parent node,

$$W \in \mathcal{W}_x \Leftrightarrow N \setminus W \notin \mathcal{W}_y, \tag{4.1}$$

(v) an "information set" partition $\mathcal{I}$ of T with $\{r\} \in \mathcal{I}$ such that for all $w, z \in I \in \mathcal{I}$ and all child nodes x of w there is a child node y of z with $\varphi(x) = \varphi(y)$ and $\mathcal{W}_x = \mathcal{W}_y$.

Some more notation. The game form is *monotonic* if and only if each set of winning coalitions is closed under supersets. It is called weakly neutral iff for all nodes $x, y \in T \setminus \{r\}$, $\varphi(x) = \varphi(y)$ implies $\mathcal{W}_x = \mathcal{W}_y$.

A set $P \subseteq T$ is called a *path* if with any node it contains each of its predecessors and for any two nodes $x, y \in P$ either x is a predecessor of y or y is a predecessor of x (or $x = y$). A *maximal path* is maximal with respect to set inclusion. If the path has a terminal node x, then it is unique and denoted $P(x)$. We define the *outcome* of a path by

$$\varphi[P] = \{\varphi(x) \mid x \in P\}.$$

We call a path P *complete* if and only if its outcome is complete, $\varphi[P] \in \bar{\mathcal{J}}$. Similarly, P is *almost complete* if and only if the outcome is almost complete, $\mathrm{cl}(\varphi[P]) \in \bar{\mathcal{J}}$. Clearly, an almost complete path is necessarily maximal, but the converse does not hold. We say that an extensive game form $\mathcal{G} = \left\langle N, T, \varphi, \{\mathcal{W}_x\}_{x\in T}, \mathcal{I}\right\rangle$ for an agenda X is *adapted* to the judgement space $\langle X, \mathcal{J}\rangle$ if and only if each maximal path is almost complete. The following proposition states that adaption only depends on the tree and not on the winning coalitions attached to the tree.

Proposition 18. *Assume the game form $\mathcal{G} = \left\langle N, T, \varphi, \{\mathcal{W}_x\}_{x\in T}, \mathcal{I}\right\rangle$ is adapted to the judgement space $\langle X, \mathcal{J}\rangle$, and $\{\mathcal{W}'_x\}_{x\in T}$ is another set of winning coalitions for the same tree T satisfying (4.1). Then there is a class of information sets $\mathcal{I}'$*

such that by $\mathcal{G}' = \left\langle N, T, \varphi, \{\mathcal{W}'_x\}_{x\in T}, \mathcal{I}'\right\rangle$ *another game form is defined, which is adapted to* $\langle X, \mathcal{J}\rangle$*.*

Given an extensive game form, for each profile $\underline{J} \in \bar{\mathcal{J}}^N$ a *solution path* $P(\underline{J})$ is recursively defined as follows: $r \in P(\underline{J})$, and for any $z \in P(\underline{J})$ and child node x of z we have

$$x \in P(\underline{J}) \Leftrightarrow \{i \in N \mid \varphi(x) \in J_i\} \in \mathcal{W}_x. \tag{4.2}$$

Equation (4.1) and the completeness of each J_i guarantee that for every node exactly one child node is contained in the solution path. Thus each solution path is maximal. We have just shown the first assertion of the following

Proposition 19. *If the game form* $\mathcal{G} = \left\langle N, T, \varphi, \{\mathcal{W}_x\}_{x\in T}, \mathcal{I}\right\rangle$ *is adapted to the judgement space* $\langle X, \mathcal{J}\rangle$*, then by*

$$F_{\mathcal{G}}(\underline{J}) := \mathrm{cl}\left(\varphi\left[P(\underline{J})\right]\right)$$

an aggregation function $F_{\mathcal{G}} : \bar{\mathcal{J}}^N \to \bar{\mathcal{J}}$ *is defined, which is totally unanimous whenever* $N \in \mathcal{W}_x$ *for all* $x \in T$*. For each non-terminal node* $x \in T$ *there is an issue* $Y = \{p, \neg p\}$ *such that the two child nodes have* p *and* $\neg p$ *assigned to them. and* $F_{\mathcal{G}}$ *is* Y*-independent given* $\varphi[P(x)]$ *(and thus given* $\mathrm{cl}(\varphi[P(x)])$*).*

4.2. **Representation Theorem.** The following theorem guarantees the existence of at least one game form representation. For the case of infinite agendas we introduce a weak technical condition, which is void in the finite case since aggregation functions are trivially X-independent on their agenda X (given $\emptyset$).

Definition 20. An aggregation function on a judgement space $\langle X, \mathcal{J}\rangle$ is *piecewise independent* if and only if every finite subagenda $Y \subseteq X$ is contained in a finite subagenda $X' \subseteq X$ such that the aggreagtion function is X'-independent.

Theorem 21. *A totally unanimous piecewise independent aggregation function* $F : \bar{\mathcal{J}}^N \to \bar{\mathcal{J}}$ *on a judgement space* $\langle X, \mathcal{J}\rangle$ *satisfies Perfect Independence if and only if there is an extensive game form* $\mathcal{G} = \left\langle N, T, \varphi, \{\mathcal{W}_x\}_{x\in T}, \mathcal{I}\right\rangle$ *for* X *adapted to* $\langle X, \mathcal{J}\rangle$ *such that*

$$F(\underline{J}) = F_{\mathcal{G}}(\underline{J})$$

for all profiles $\underline{J} \in \bar{\mathcal{J}}^N$*.*

As it can be easily seen, the game form representation is not unique. In example 1 we could have chosen to decide the issue $\{y \succ z, z \succ y\}$ at the root node, followed by the issue $\{x \succ y, y \succ x\}$ on both of its child nodes. The topology of the tree remains the same, and the profiles are matched to maximal paths withe the same outcomes. Therefore, independence must hold for both issues, no matter in which sequence they are. Conditions for unique representations would be harsh. For example, one could have unconditional independence exactly for one issue. Uniqueness of a game form reporesentation would be an undesired feature, since we are trying to maximise independence without falling into the trap of dictatorship. Good aggregation functions have multiple game form representations.

As a corollary from this theorem and proposition 18 we obtain the following possibility theorem. It guarantees the existence of a large class of perfectly independent aggregation functions on a given judgement space, if only one such function exists. These function can have arbitrary combinations of winning coalitions as long as condition (4.1) is satisfied. In particular, for each dictatorial aggregation function, since it is perfectly independent and totally unanimous, there are myriads of alternative aggregation functions on the same judgement spaces for each of its tree representation, among them fair sequential rules performing simple majority decisions at each non-terminal node in the tree. The interesting lesson to learn is that once decisions have been institutionalised in a consistent sequential form, dictatorship can be removed and replaced by democracy without altering the sequence of decisions.

Corollary 22. *If F is a totally unanimous aggregation function on a judgement space satifying Perfect Independence, then for each winning coalition $\{\mathcal{W}'_x\}_{x\in T}$ attached to the tree of its game form representation satisfying (4.1) there is a partition of the tree into information sets such that the game form with the new winning coalitions and information sets define another aggregation function F' on the same judgement space, which satisfies Total Unanimity and Perfect Independence.*

4.3. **Strategy-Proofness.** The formal theory on strategy-proofness for judgement aggregation rule has been developed in [Dietrich and List (2007a)]. Since we are lacking preference orders in the judgement aggregation setting, the first problem was to find a concept which could serve as a substitute for the equilibrium condition in the case of Arrovian aggregation. For a profile $\underline{J} \in \bar{\mathcal{J}}^N$ and a set $J'_i \in \bar{\mathcal{J}}$ we write $\underline{J}_{-i}J'_i$ for the profile which coincides with J_j for all $j \neq i$ and with J'_i for $j = i$. We say that an aggregation function is *non-manipulable* on an issue $Y = \{p, \neg p\}$ if and only if for each profile $\underline{J} \in \bar{\mathcal{J}}^N$, $i \in N$, and each set $J'_i \in \bar{\mathcal{J}}$,

$$J_i \cap Y = F\left(\underline{J}_{-i}J'_i\right) \cap Y \Rightarrow J_i \cap Y = F\left(\underline{J}\right) \cap Y.$$

In other words, whenever a person i agrees with the social outcome on an issue, then the social outcome should still agree when he submits his "truthful" opinions.

The relation to strategy-proofness is the following. For $J, J_1, J_2 \in \bar{\mathcal{J}}$ we say that J_1 is closer to J than J_2 if and only if J_1 and J have as least a much in common than J_2 and J. Symbolically,

$$J_1 \trianglerighteq_J J_2 \Leftrightarrow J \cap J_2 \subseteq J_1.$$

Non-manipulability on all issues is equivalent to

$$F\left(\underline{J}\right) \trianglerighteq_{J_i} F\left(\underline{J}_{-i}J'_i\right).$$

for all profiles $\underline{J} \in \bar{\mathcal{J}}^N$ and a sets $J'_i \in \bar{\mathcal{J}}$. For any preference order on $\bar{\mathcal{J}}$ with an optimal choice J which includes $\trianglerighteq_J$, the classical equilibrium condition for strategy proofness is equivalent to the latter (loc cit. Th 4).

It turns out (loc. cit. Th 1) that non-manipulability of a judgement aggregation function is equivalent to Propositionwise Independence and Monotonicity of the winning coalitions. Under the same conditions as in the case of the generalised Arrow's theorem, impossibility results can be shown (loc cit. Th 2,3). Therefore, non-manipulability or strategy-proofness is not to be expected to hold in general for sequential aggregation rules (see also [Dietrich and List (2007b), Ch 5]). However, we obtain the following weaker result.

Theorem 23. *Let $F_{\mathcal{G}}$ be the judgement aggregation form induced by a monotonic and weakly neutral game form $\mathcal{G}$ adapted to the judgement space $\langle X, \mathcal{J}\rangle$. Let $\underline{J} \in \bar{\mathcal{J}}^N$ be some profile, $i \in N$ and $J'_i \in \bar{\mathcal{J}}$. If the solution paths $P\left(\underline{J}\right)$ and $P\left(\underline{J}_{-i}J'_i\right)$ are complete, then*

$$F_{\mathcal{G}}\left(\underline{J}\right) \trianglerighteq_{J_i} F_{\mathcal{G}}\left(\underline{J}_{-i}J'_i\right).$$

The theorem demonstrates theat opportunities for strategic manipulations are mainly determined by the form of the tree. The winning coalitions can only influence the probability with which a certain solution path is taken. In the extreme case, a certain path or even all but one paths can be made unreachable by assigning an empty set of winning coalitions to one of its nodes. However, playing a complete path guarantees the absense of incentives to deviate from the results independently of the winning coalitions.

5. Conclusions

We have axiomatically characterised a class of sequential aggregation rules, thereby generalising the linear rules of [List (2004)] to tree-form rules. They are partially overlapping with the class defined by [Dietrich (2014)], however in the latter approach there is only one set of winning coalitions for each proposition, while in our settings winning coalitions depend on nodes and can differ among different paths. Tree-form rules have the practical advantage over other sequential rules that talleying the ballots is done directly at every node, without the necessity to check whether a certain issue is already decided by logically following from previous decisions. These checks are built into the tree structure. Running such a rule one just has to follow the tree.

The concluding remarks of List and Polak's survey call for two remaining challenges, one of it "characterizations of compelling non-independent aggregation rules" [List and Polak (2010), Ch 6]. We hope that this paper is a further step in this direction. We think that the tree-representable aggregation rules explored here are interesting, because they demonstrate the difference between two types of independence conditions. The first type are those which adapt to the logical structure of the agenda and lead to representations by sequential decisions. They are compatible with an almost unlimited choice of local decisions on independent issues in the form of winning coalitions. The second type does not respect logical constraints and "cuts into the flesh" by constraining the winning coalitions, in the worst case up to Arrovian dictatorship.

In sections 2.2 and 2.3 we have developed a richer framework than our main representation theorem requires. We have done this in the intention to provide an analytical framework for further research. Section 2.2 allows for an easy translation between the two major frameworks of judgement aggregation, the logical approach and the geometrical convexity approach by Nehring and Puppe. Section 2.3 might be useful to explore impossibility theorems on subspaces of the aggregation function.

Appendix: Proofs

Proof. (Proposition 3) Let $\langle X, \mathcal{J} \rangle$ be a judgement space and define

$$B \models_{\mathcal{J}} p \Leftrightarrow B \cup \{\neg p\} \notin \mathcal{J}.$$

Reflexivity: Weak Consistency assures that $\{p, \neg p\} \notin \mathcal{J}$, thus $\{p\} \models_{\mathcal{J}} p$. By Non-Tautology, $\{p\} \in \mathcal{J}$. If $\{p\} \models_{\mathcal{J}} \neg p$, then $\{p\} \cup \{\neg\neg p\} = \{p\} \notin \mathcal{J}$, a contradiction.

Monotonicity: Let $B \subseteq C$ and assume that $C \not\models_{\mathcal{J}} p$. Then $C \cup \{\neg p\} \in \mathcal{J}$, and by Monotonicity of consistent sets also $B \cup \{\neg p\} \in \mathcal{J}$, or $B \not\models_{\mathcal{J}} p$.

Weak Consistency: Assume that $B \models_{\mathcal{J}} p$ and $B \models_{\mathcal{J}} \neg p$. Then both $B \cup \{\neg p\}$ and $B \cup \{p\}$ are not in $\mathcal{J}$, and by the Dilemma axiom of consistent sets, also B is not in $\mathcal{J}$. By Monotonicity, $B \cup \{\neg q\} \notin \mathcal{J}$, thus $B \models_{\mathcal{J}} q$.

Dilemma: Assume $B \cup \{p\} \models_{\mathcal{J}} q$ and $B \cup \{\neg p\} \models_{\mathcal{J}} q$, thus $B \cup \{p\} \cup \{\neg q\}, B \cup \{\neg p\} \cup \{\neg q\} \notin \mathcal{J}$. By the Dilemma axiom for consistent sets, $B \cup \{\neg q\} \notin \mathcal{J}$. We obtain $B \models_{\mathcal{J}} q$.

Compactness: Assume $B \models_{\mathcal{J}} p$, or $B \cup \{\neg p\} \notin \mathcal{J}$. By Compactness for consistent sets there is a finite set $C \subseteq B \cup \{\neg p\}$ with $C \notin \mathcal{J}$. By Monotonicty, $C \cup \{\neg p\} \notin \mathcal{J}$, thus $C \models_{\mathcal{J}} p$.

Conversely, let $\models$ be an admissible conditional on the agenda X, and define

$$\mathcal{J}_{\models} = \{B \subseteq X \mid \exists q \in X : B \not\models q\}.$$

Non-Tautology: From Reflexivity we obtain $\{p\} \not\models \neg p$, thus $\{p\} \in \mathcal{J}_{\models}$.

Monotonicity: Assume $B \in \mathcal{J}_{\models}$, or $B \not\models q$ for some $q \in X$. For $C \subseteq B$, Monotonicity of consistent sets requires $C \not\models q$. Therefore $C \in \mathcal{J}_{\models}$.

Weak Consistency: Assume $B \in \mathcal{J}_{\models}$ and $p \in B$. Reflexivity and Monotonicity ensures $B \models p$, and Weak Consistency for conditionals entails $B \not\models \neg p$, for which the same argument as before leads to $\neg p \notin B$.

Dilemma: Assume both $B \cup \{\neg p\}$ and $B \cup \{p\}$ are not in $\mathcal{J}_{\models}$. Then for all $q \in X$, $B \cup \{\neg p\} \models q$ and $B \cup \{p\} \models q$. With the Dilemma axiom for conditionals we arrive at $B \models q$ for an arbitrary $q \in X$. Therefore, $B \notin \mathcal{J}_{\models}$.

Compactness: Assume $B \notin \mathcal{J}_{\models}$, or $B \models q$ for all $q \in X$. In particular, $B \models p$ and $B \models \neg p$ for a given p. By Compactness of the conditional there are finite sets $C_p \subseteq B$ and $C_{\neg p} \subseteq B$ with $C_p \models p$ and $C_{\neg p} \models \neg p$. Putting $C = C_p \cup C_{\neg p} \subseteq B$, Monotonicity yields $C \models p$ and $C \models \neg p$, from which by Weak Consistency we obtain $B \models q$ for all $q \in X$. We have found a finite set $C \subseteq B$ with $C \notin \mathcal{J}_{\models}$.

To show that $\mathcal{J}_{\models_{\mathcal{J}}} = \mathcal{J}$, consider that $B \in \mathcal{J}_{\models_{\mathcal{J}}}$ if and only if $B \not\models_{\mathcal{J}} q$ for some $q \in X$, or $B \cup \{\neg q\} \in \mathcal{J}$. First assume $B \in \mathcal{J}_{\models_{\mathcal{J}}}$, or $B \cup \{\neg q\} \in \mathcal{J}$. Monotonicity implies $B \in \mathcal{J}$, which gives us the "$\subseteq$" direction. For the converse direction, assume $B \in \mathcal{J}$. If $B = \emptyset$, then by reflexivity $B \cup \{\neg q\} \in \mathcal{J}$, and $B \in \mathcal{J}_{\models_{\mathcal{J}}}$. Otherwise, there is some $q \in B$, and $B \not\models_{\mathcal{J}} \neg q$ by Weak Consistency, which in turn implies $B \in \mathcal{J}_{\models_{\mathcal{J}}}$. To show that $\models_{\mathcal{J}_{\models}}$ coincides with $\models$, consider that $B \models_{\mathcal{J}_{\models}} p$ if and only if $B \cup \{\neg p\} \notin \mathcal{J}_{\models}$, which holds if and only if $B \cup \{\neg p\} \models q$ for all $q \in X$. Assume first $B \models_{\mathcal{J}_{\models}} p$, then in particular $B \cup \{\neg p\} \models p$. By Reflexivity and Monotonicity we find also $B \cup \{p\} \models p$, which by the Dilemma rule leads to $B \models p$. Now assume that $B \models p$, then by Monotonicity $B \cup \{\neg p\} \models p$. But we also find that $B \cup \{\neg p\} \models \neg p$. With Weak Consistency we arrive at $B \cup \{\neg p\} \models q$ for all $q \in X$, or $B \models_{\mathcal{J}_{\models}} p$. □

One more logical lemma for technical purposes.

Lemma 24. *If $B \cup C \models p$ and $B \models q$ for all $q \in C$, then $B \models p$.*

Proof. Assume $B \cup C \models p$ and $B \models q$ for all $q \in C$. By the Compactness axiom, C can be assumed finite without loss of generalisation. Choose $q \in C$ and set $C' = C \setminus \{q\}$. Then $B \cup C' \cup \{q\} \cup \{\neg p\} \notin \mathcal{J}$, and $B \cup C' \cup \{\neg q\} \notin \mathcal{J}$, thus by Monotonicity also $B \cup C' \cup \{\neg q\} \cup \{\neg p\} \notin \mathcal{J}$. The Dilemma axiom yields $B \cup C' \cup \{\neg p\} \notin \mathcal{J}$, or $B \cup C' \models p$. Repeating the last step for C' in place of C will after finitely many steps yield $B \models p$, what has to be shown. □

The extension lemma is standard.

Proof. (Extension lemma 5) Let $A \in \mathcal{J}$ and $\{p_i, \neg p_i\}_{i\in K}$ the collection of all issues disjoint from A. Set $A_0 = A$. define

$$A_{i+1} = \begin{cases} A_i \cup \{p_i\}, & \text{if } A_i \cup \{p_i\} \in \mathcal{J}, \\ A_i \cup \{\neg p_i,\} & \text{else.} \end{cases}$$

By the Dilemma axiom, either $A_i \cup \{p_i\} \in \mathcal{J}$ or $A_i \cup \{\neg p_i,\} \in \mathcal{J}$, therefore $A_{i+1} \in \mathcal{J}$ whenever $A_i \in \mathcal{J}$. By compactness, also $\bar{A} = \bigcup_{i\in K} A_i \in \mathcal{J}$. We conclude that $\bar{A}$ is both consistent and complete. □

We now show that property spaces can transformed into judgement spaces.

Proof. (Proposition 7) Let $\langle Y, \mathcal{H}\rangle$ be a property space. We define a corresponding judgement space $\langle \mathcal{H}, \mathcal{J}_{\mathcal{H}}\rangle$ with agenda $\mathcal{H}$ and

$$\mathcal{J}_{\mathcal{H}} = \left\{ \mathcal{F} \subseteq \mathcal{H} \mid \bigcap \mathcal{F} \neq \emptyset \right\}.$$

Non-Tautology: For $H \in \mathcal{H}$ trivially $\bigcap \{H\} = H \neq \emptyset$, thus $\{H\} \in \mathcal{J}_{\mathcal{H}}$.

Monotoniccity: Let $\mathcal{B} \in \mathcal{J}_{\mathcal{H}}$ and $\mathcal{C} \subseteq \mathcal{B}$. Since $\bigcap \mathcal{C} \supseteq \bigcap \mathcal{B} \neq \emptyset$, $\mathcal{C} \in \mathcal{J}_{\mathcal{H}}$.

Weak Consistency: Let $\mathcal{B} \in \mathcal{J}_{\mathcal{H}}$ and $H \in \mathcal{B}$, if $Y \setminus H \in \mathcal{B}$, then $\bigcap \mathcal{B} = \emptyset$, a contradiction to $\mathcal{B} \in \mathcal{J}_{\mathcal{H}}$. Therefore $Y \setminus H \notin \mathcal{B}$.

Dilemma: Let $\mathcal{B} \in \mathcal{J}_{\mathcal{H}}$ and $H \in \mathcal{H}$. Assume $\mathcal{B} \cup \{H\} \notin \mathcal{J}_{\mathcal{H}}$, then $\bigcap \mathcal{B} \cap H = \emptyset$ and further $\emptyset \neq \bigcap \mathcal{B} \subseteq Y \setminus H$, and thus $\mathcal{B} \cup \{Y \setminus H\} \in \mathcal{J}_{\mathcal{H}}$.

Compactness: Assume $\mathcal{B} \subseteq \mathcal{H}$ and $\mathcal{B} \notin \mathcal{J}_{\mathcal{H}}$, or $\bigcap \mathcal{B} = \emptyset$. There is a finite $\mathcal{C} \subseteq \mathcal{B}$ with $\bigcap \mathcal{C} = \emptyset$, or $\mathcal{C} \notin \mathcal{J}_{\mathcal{H}}$

It remains to be shown that there is a bijection which maps each $x \in Y$ to a $\mathcal{F}_x \in \bar{\mathcal{J}}_{\mathcal{H}}$ such that $\bigcap \mathcal{F}_x = \{x\}$. For $x \in Y$ denote

$$\mathcal{F}_x = \{H \in \mathcal{H} \mid x \in H\}.$$

Since $x \in \bigcap \mathcal{F}_x$, $\mathcal{F}_x \in \mathcal{J}_{\mathcal{H}}$. We have to show that $\mathcal{F}_x$ is complete. Indeed, for any issue $\{H, Y \setminus H\} \subseteq \mathcal{H}$ we have either $x \in H$ or $x \in Y \setminus H$, an therefore either $H \in \mathcal{F}_x$ or $Y \setminus H \in \mathcal{F}_x$. This demonstrates $\mathcal{F}_x \in \bar{\mathcal{J}}_{\mathcal{H}}$. Furthermore, $\bigcap \mathcal{F}_x = \{x\}$: By the Separation axiom, for any $y \in Y$, $x \neq y$ there is a $H \in \mathcal{H}$ with $x \in H$ and $y \notin H$. Thus $y \notin \mathcal{F}_x$ for all $y \neq x$, or $\bigcap \mathcal{F}_x = \{x\}$.

We have to show that the mapping $Y \to \bar{\mathcal{J}}_{\mathcal{H}}$, $x \mapsto \mathcal{F}_x$ is bijective. From $\mathcal{F}_x = \mathcal{F}_y$ we conclude $\{x\} = \bigcap \mathcal{F}_x = \bigcap \mathcal{F}_y = \{y\}$, or $x = y$. This gets us injectivity. For surjectivity, assume $\mathcal{F} \in \bar{\mathcal{J}}_{\mathcal{H}}$, or $\bigcap \mathcal{F} \neq \emptyset$. Let $x \in \bigcap \mathcal{F}$ and $y \in Y \setminus \{x\}$. Then we have to prove $\mathcal{F} = \mathcal{F}_x$. For $H \in \mathcal{F}$ clearly $x \in H$. Conversely, if $x \in H \in \mathcal{H}$, then, since $\mathcal{F}$ is complete, we must have either $H \in \mathcal{F}$ or $Y \setminus H \in \mathcal{F}$. Since $x \in \bigcap \mathcal{F}$ we can not have $Y \setminus H \in \mathcal{F}$, therefore $H \in \mathcal{F}$. This completes the proof of $\mathcal{F} = \mathcal{F}_x$. □

We now show the existence of a smallest free set containing a given consistent set.

Proof. (Lemma 9) Let $A \in \mathcal{J}$ and define

$$\hat{A} = \mathrm{cl}(A) := \{p \in X \mid A \vDash p\}.$$

We show that $\hat{A}$ is free. Let $\{p, \neg p\} \subseteq X \setminus \mathrm{ag}\left(\hat{A}\right)$ an issue disjoint from $\hat{A}$. If either $\hat{A} \cup \{p\} \notin \mathcal{J}$ or $\hat{A} \cup \{\neg p\} \notin \mathcal{J}$, then $\hat{A} \models \neg p$ or $\hat{A} \models p$, respectively. By lemma 24 this implies $A \models \neg p$ or $A \models p$, respectively, a contradiction to $\{p, \neg p\} \cap \hat{A} = \emptyset$.

It remains to show that $\hat{A}$ is the smallest free set containing A. Let $A \subseteq B \subsetneq \hat{A}$, so there is a $p \in \hat{A} \setminus B$. Since $A \models p$ and therefore, by Monotonicty, also $B \models p$, we find $B \cup \{\neg p\} \notin \mathcal{J}$. Thus by definition, B is not a free set. $\square$

The next step is to show that free sets induce well-defined conditional judgement spaces.

Proof. (Proposition 10) Assume $A \in \hat{\mathcal{J}}$ is free and Y is a subagenda with $Y \cap A = \emptyset$. We have to show that by

$$\mathcal{J}_A = \{B \setminus A \subseteq Y \mid B \in \mathcal{J} \text{ and } A \subseteq B\}$$

a conditional judgement space $\langle Y, \mathcal{J}_A \rangle$ is defined.

Non-Tautology: For $p \in Y$ we have by aasumption $A \cup \{p\} \in \mathcal{J}$ and $p \notin A$, thus $\{p\} \in \mathcal{J}_A$.

Monotonicity: Let $B \in \mathcal{J}_A$ and $C \subseteq B$. Then $B \subseteq Y$, $A \cup B \in \mathcal{J}$ and $A \cap B = \emptyset$. By Monotonicity, $A \cup C \in \mathcal{J}$. Since $C \cap A = \emptyset$ and $C \subseteq Y$, $C \in \mathcal{J}_A$.

Weak Consistency: Let $B \in \mathcal{J}_A$ and $p \in B$. Thus $p \in A \cup B \in \mathcal{J}$. By Weak Consistency, $\neg p \notin A \cup B$, thus $\neg p \notin B$.

Dilemma: Let $B \in \mathcal{J}_A$ and $\{p, \neg p\} \subseteq Y$. Then $A \cup B \in \mathcal{J}$, $B \subseteq Y$, and by the Dilemma axiom, either $A \cup B \cup \{p\} \in \mathcal{J}$ or $A \cup B \cup \{\neg p\} \in \mathcal{J}$. Since $A \cap B = \emptyset$ and $\{p, \neg p\} \subseteq Y$, either $B \cup \{p\} \in \mathcal{J}_A$ or $B \cup \{\neg p\} \in \mathcal{J}_A$.

Compactness. Let $B \subseteq Y$ and $B \notin \mathcal{J}_A$. Then $A \cup B \notin \mathcal{J}$, and, by Compactness there is a finite ser $C \subseteq A \cup B$ with $C \notin \mathcal{J}$. It follows that $C \setminus A \subseteq Y$ is a finite set contained in B with $C \setminus A \notin \mathcal{J}_A$.

For the second assertion it is sufficient to show that

$$B \models_A p \Leftrightarrow B \models_{\mathcal{J}_A} p$$

with $B \models_A p \Leftrightarrow A \cup B \models p$. Then by proposition 3, $\models_A$ is the associated conditional to the subagenda.

Assume $B \subseteq Y$ and $B \models_A p$ with $p \in Y$. Then $A \cup B \models p$, or $A \cup B \cup \{\neg p\} \notin \mathcal{J}$. Since $B \cap A = \emptyset$ and $\neg p \notin A$ we find $(A \cup B \cup \{\neg p\}) \setminus A = B \cup \{\neg p\}$. In other words, there can not be a set $J \in \mathcal{J}$ with $J \setminus A = B \cup \{\neg p\} \subseteq Y$ and $A \subseteq J$. Consequently $B \cup \{\neg p\} \notin \mathcal{J}_A$, or $B \models_{\mathcal{J}_A} p$.

Conversely, assume $B \models_{\mathcal{J}_A} p$, or $Y \supseteq B \cup \{\neg p\} \notin \mathcal{J}_A$. Then for all $J \subseteq X$ with $A \subseteq J$ and $J \setminus A = B \cup \{\neg p\} \subseteq Y$, $J \notin \mathcal{J}$. In particular, $A \cup B \cup \{\neg p\} \notin \mathcal{J}$, and thus $A \cup B \models p$, or $B \models_A p$ $\square$

The following lemma on conditional independence is neededl for the impossiblity theorem as well as the representation theorem.

Proof. (Lemma 13) (i) Let F be an aggregation function which is Y-independent given A and $A \subseteq A'$. Let $\underline{J}, \underline{J}' \in \bar{\mathcal{J}}^N$ with $A' \subseteq F(\underline{J}) \cap F(\underline{J}')$ and $J_i \cap Y = J_i' \cap Y$ for all $i \in N$. Then also $A \subseteq F(\underline{J}) \cap F(\underline{J}')$, and by assumption, $F(\underline{J}) \cap Y = F(\underline{J}') \cap Y$, what had to be shown.

(ii) Let F be Y_i-independent given A for all $i \in I$ and set $Y = \bigcup_{i \in I} Y_i$. Let $\underline{J}, \underline{J}' \in \bar{\mathcal{J}}^N$ with $A \subseteq F(\underline{J}) \cap F(\underline{J}')$ and $J_j \cap Y = J_j' \cap Y$ for all $j \in N$. Then also $J_j \cap Y_i = J_j' \cap Y_i$ for all $j \in N$ and $i \in I$, and we obtain $F(\underline{J}) \cap Y_i = F(\underline{J}') \cap Y_i$ for each $i \in I$, from which $F(\underline{J}) \cap Y = F(\underline{J}') \cap Y$ follows.

(iii) Let F be Y-independent given A and Y' a subagenda with

$$Y \subseteq Y' \subseteq Y \cup \text{ag}(\text{cl}(A)). \tag{5.1}$$

We have to show that F is Y'-independent given A. Let $\underline{J}, \underline{J}' \in \bar{\mathcal{J}}^N$ with $A \subseteq F(\underline{J}) \cap F(\underline{J}')$ and $J_i \cap Y' = J_i' \cap Y'$ for all $i \in N$. Then also $J_i \cap Y = J_i' \cap Y$, and by assumption,

$$F(\underline{J}) \cap Y = F(\underline{J}') \cap Y.$$

Moreover, since $F(\underline{J})$ and $F(\underline{J}')$ are complete, we also have $\mathrm{cl}(A) \subseteq F(\underline{J}) \cap F(\underline{J}')$ and further

$$F(\underline{J}) \cap \mathrm{ag}(\mathrm{cl}(A)) = F(\underline{J}') \cap \mathrm{ag}(\mathrm{cl}(A)).$$

Taken together, the last two equation imply

$$F(\underline{J}) \cap (Y \cup \mathrm{ag}(\mathrm{cl}(A))) = F(\underline{J}') \cap (Y \cup \mathrm{ag}(\mathrm{cl}(A))),$$

from which we infer with (5.1) that $F(\underline{J}) \cap Y' = F(\underline{J}') \cap Y'$, which had to be shown. □

The following proposition determines conditional aggregation function on conditional subspaces.

Proof. (Proposition 14) Let $A \in \hat{\mathcal{J}}$ be a free set, Y a subagenda with $A \cap Y = \emptyset$, and F being Y-independent given A. We have to show that on the conditional judgement space $\langle Y, \mathcal{J}_A \rangle$ by

$$F_A(\underline{J}^A) = F(\underline{J}) \cap Y \tag{5.2}$$

an aggregation function F_A is specified for all profiles $\underline{J}^A \in \bar{\mathcal{J}}_A^N$ and $\underline{J} \in \bar{\mathcal{J}}^N$ with $A \subseteq F(\underline{J})$, and $J_i^A = J_i \cap Y$. Indeed, let $\underline{J}, \underline{J}' \in \bar{\mathcal{J}}^N$ with $A \subseteq F(\underline{J})$, $A \subseteq F(\underline{J}')$, $J_i^A = J_i \cap Y$ and $J_i^A = J_i' \cap Y$. Then $J_i \cap Y = J_i' \cap Y$, and by Y-independence given A, we have $F(\underline{J}) \cap Y = F(\underline{J}') \cap Y$. This shows that F_A in (5.2) is well defined. □

First we introduce a short lemma on the monotonicity of path outcomes.

Lemma 25. *Let $\mathcal{G} = \langle N, T, \varphi, \{\mathcal{W}_x\}_{x \in T}, \mathcal{I} \rangle$ be a game form adapted to $\langle X, \mathcal{J} \rangle$, and P be a path and P' a maximal path in it. Then*

$$P \subseteq P' \Leftrightarrow \mathrm{cl}(\varphi[P]) \subseteq \mathrm{cl}(\varphi[P']).$$

Proof. "$\Rightarrow$": If $P \subseteq P'$, then clearly $\varphi[P] \subseteq \varphi[P']$, and the assertion follows by monotonicity of deductive closure.

"$\Leftarrow$": Assume $\mathrm{cl}(\varphi[P]) \subseteq \mathrm{cl}(\varphi[P'])$. We show inductively that $P(x) \subseteq P'$ for every $x \in P$. This is true for the root node, since every path contains the root node. Assume that we have already shown that $P(y) \subseteq P'$ for some $y \in P$. If y is the last node in P, then $P(y) = P$, and we have completed the proof. Otherwise, there is a child node x of y contained in P. Since $\varphi(x) \in \varphi[P]$, by assumption of the proof $\varphi(x) \in \mathrm{cl}(\varphi[P'])$. By induction hypothesis, $y \in P'$. Since P' is a maximal path, at least one of the child nodes of y must lie on P'. If it is not x, then we would have $\neg\varphi(x) \in \varphi[P']$, and by the assumption of the proof, $\mathrm{cl}(\varphi[P'])$ would be inconsistent. This contradicts the requirement from adaption to a judgement space that outcomes of paths are consistent. Therefore, $x \in P'$. This completes the inductive proof for $P \subseteq P'$. □

We show next that adaption to a judgement space does not depend on the winning coalitions.

Proof. (Proposition 18) Let $\mathcal{G} = \left\langle N, T, \varphi, \{\mathcal{W}_x\}_{x\in T}, \mathcal{I}\right\rangle$ be a game form is adapted to the judgement space $\langle X, \mathcal{J}\rangle$, and $\{\mathcal{W}'_x\}_{x\in T}$ be a set of winning coalitions for the same tree T satisfying (4.1). We first define the information set $\mathcal{I}'$. We say two nodes $x, y \in T$ are equivalent if x has child nodes x_1, x_2 and y has child nodes y_1, y_2 with $\varphi(x_i) = \varphi(y_i)$ and $\mathcal{W}'_{x_i} = \mathcal{W}'_{y_i}$ for $i = 1, 2$. Clearly, this is an equivalence relation, and the eqivalence classes together with $\{r\}$ form a partition $\mathcal{I}'$ of T.

Now let $\mathcal{G}' = \left\langle N, T, \varphi, \{\mathcal{W}'_x\}_{x\in T}, \mathcal{I}'\right\rangle$. Obviously, $\mathcal{G}'$ is a game form. We have to show that the outcome of every maximal path is almost complete. But this follows directly from the adaption of $\mathcal{G}$ to $\langle X, \mathcal{J}\rangle$, since the tree remains the same, and so the oucome of each path. It is therefore clear that $\mathcal{G}'$ is adapted to $\langle X, \mathcal{J}\rangle$. $\square$

We now show that every game form adapted to a judgement space defines an aggregation function, which is Y-independent for each issue associated with the childnodes of a node, given the node's history.

Proof. (Proposition 19). Let $\mathcal{G} = \left\langle N, T, \varphi, \{\mathcal{W}_x\}_{x\in T}, \mathcal{I}\right\rangle$ be a game form adapted to the judgement space $\langle X, \mathcal{J}\rangle$. We first have to show that

$$F_{\mathcal{G}}(\underline{J}) := \mathrm{cl}(\varphi[P(\underline{J})])$$

is an aggregation function. But this follows from the condition that $\mathcal{G}$ is adapted to $\langle X, \mathcal{J}\rangle$, which guarantees that all solution paths $P(\underline{J})$ are almost complete, and therefore $\mathrm{cl}(\varphi[P(\underline{J})]) \in \bar{\mathcal{J}}$.

We show independence at any non-terminal node $x \in T$. By construction there is an issue $Y = \{p, \neg p\}$ such that the two child nodes x_p and $x_{\neg p}$ have assigned p and $\neg p$ to them by φ. It is sufficient to show that $F_{\mathcal{G}}$ is Y-independent given $\varphi[P(x)]$, then independence given $\mathrm{cl}(\varphi[P(x)])$ follows from lemma 13 (i). Thus let $A = \varphi[P(x)]$ and take any profiles $\underline{J}, \underline{J}' \in \bar{\mathcal{J}}^N$ with $J_i \cap Y = J'_i \cap Y$ and $A \subseteq F_{\mathcal{G}}(\underline{J}) \cap F_{\mathcal{G}}(\underline{J}')$. We first show that $x \in P(\underline{J}) \cap P(\underline{J}')$. Since $A \subseteq \mathrm{cl}(\varphi[P(\underline{J})])$, we also have $\varphi[P(x)] \subseteq \mathrm{cl}(\varphi[P(\underline{J})])$, and lemma 25 implies $x \in P(x) \subseteq P(\underline{J})$ since $P(\underline{J})$ is a maximal path. Similarly we find $x \in P(\underline{J}')$.

For the child nodes we obtain from the construction (4.2)

$$\begin{aligned} x_p \in P(\underline{J}) \quad &\Leftrightarrow \quad \{i \in N \mid p \in J_i\} \in \mathcal{W}_{x_p}, \\ x_{\neg p} \in P(\underline{J}) \quad &\Leftrightarrow \quad \{i \in N \mid \neg p \in J_i\} \in \mathcal{W}_{x_{\neg p}}. \end{aligned}$$

Analogous equivalences hold for $\underline{J}'$. Thus we obtain from $J_i \cap Y = J'_i \cap Y$

$$\begin{aligned} x_p \in P(\underline{J}) \quad &\Leftrightarrow \quad x_p \in P(\underline{J}'), \\ x_{\neg p} \in P(\underline{J}) \quad &\Leftrightarrow \quad x_{\neg p} \in P(\underline{J}'). \end{aligned}$$

This can be expressed by

$$\begin{aligned} p \in \varphi[P(\underline{J})] \quad &\Leftrightarrow \quad p \in \varphi[P(\underline{J}')], \\ \neg p \in \varphi[P(\underline{J})] \quad &\Leftrightarrow \quad \neg p \in \varphi[P(\underline{J}')]. \end{aligned}$$

This in turn proves

$$F_{\mathcal{G}}(\underline{J}) \cap \{p, \neg p\} = F_{\mathcal{G}}(\underline{J}') \cap \{p, \neg p\},$$

which completes the proof of Y-independence given A.

Further assume that $N \in \mathcal{W}_x$ for all $x \in T$. We have to show that $F_{\mathcal{G}}$ is totally unanimous. For any complete $J \in \bar{\mathcal{J}}$ define a profile $\underline{J}$ by letting $J_i = J$ for all $i \in N$. Thus at all non-root nodes $x \in P(\underline{J})$, we have a winning majority for $\varphi(x)$

with $\varphi(x) \in J$. It follows $\varphi[P(\underline{J})] \subseteq J$, and, since $\varphi[P(\underline{J})]$ is almost complete, $F_{\mathcal{G}}(\underline{J}) = \mathrm{cl}(\varphi[P(\underline{J})]) = J$. □

We are now in a position to demonstrate the representation theorem.

Proof. (Theorem 21) We first consider the case of a finite agenda X. Assume $F : \bar{\mathcal{J}}^N \to \bar{\mathcal{J}}$ is a totally unanimous aggregation function on $\langle X, \mathcal{J}\rangle$ satisfying Perfect Independence. We define a game form $\mathcal{G} = \langle N, T, \varphi, \{\mathcal{W}_x\}_{x\in T}, \mathcal{I}\rangle$ for X adapted to $\langle X, \mathcal{J}\rangle$ by recursively constructing a binary tree starting with the root node $T = \{r\}$, $\mathcal{W}_r = 2^N$, and an empty function φ. For each terminal node $x \in T$, let $P(x)$ be its path and set $A_x = \mathrm{cl}(\varphi[P(x)])$. If A_x is complete for all terminal nodes x, the we are finished.

Otherwise there is a terminal node $x \in T$ with free set A_x which is not complete. By Perfect Independence there an issue $Y = \{p, \neg p\}$, $Y \cap \mathrm{ag}(A) = \emptyset$, such that F is Y-independent given A. We construct new nodes x_p and $x_{\neg p}$ and add them to T, and assign the propositions of the issue to them by setting $\varphi(x_p) = p$ and $\varphi(x_{\neg p}) = \neg p$. We define

$$\mathcal{W}_{x_p} = \left\{\{i \in N \mid p \in J_i\} \mid p \in F(\underline{J}), \underline{J} \in \bar{\mathcal{J}}^N\right\}, \tag{5.3}$$

$$\mathcal{W}_{x_{\neg p}} = \left\{\{i \in N \mid \neg p \in J_i\} \mid \neg p \in F(\underline{J}), \underline{J} \in \bar{\mathcal{J}}^N\right\}. \tag{5.4}$$

From $N \setminus \{i \in N \mid p \in J_i\} = \{i \in N \mid \neg p \in J_i\}$ it immediately follows that

$$W \in \mathcal{W}_{x_p} \Leftrightarrow N \setminus W \notin \mathcal{W}_{x_{\neg p}}.$$

The construction of the tree will not be completed after finitely many steps if the agenda is infinite. The tree is complete when all paths associated to terminal nodes are almost complete. Then also every maximal path is almost complete. We find that the game form is adapted to the judgement space.

We now define the information set $\mathcal{I}$. We say two nodes $x, y \in T$ are equivalent if x has child nodes x_1, x_2 and y has child nodes y_1, y_2 with $\varphi(x_i) = \varphi(y_i)$ and $\mathcal{W}_{x_i} = \mathcal{W}_{y_i}$ for $i = 1, 2$. Clearly, this is an equivalence relation, and the eqivalenc classes together with $\{r\}$ form a partition of T.

It remains to show that $F(\underline{J}) = F_{\mathcal{G}}(\underline{J})$. We show that the two sets $F(\underline{J})$ and $F_{\mathcal{G}}(\underline{J})$ agree on all issues. For issues $\{p, \neg p\}$ which are decided on a node lying on the solution path $P(\underline{J})$, this follows from (4.2), (5.3), and (5.4). The outcome of these decisions are $\varphi[P(\underline{J})] \subseteq F(\underline{J}) \cap F_{\mathcal{G}}(\underline{J})$. But since $P(\underline{J})$ is almost complete, $F(\underline{J})$ and $F_{\mathcal{G}}(\underline{J})$ must coincide.

Conversely, assume there is a game form $\mathcal{G} = \langle N, T, \varphi, \{\mathcal{W}_x\}_{x\in T}, \mathcal{I}\rangle$ for X adapted to $\langle X, \mathcal{J}\rangle$ with the associated aggregation function $F_{\mathcal{G}}$, and assume that $F_{\mathcal{G}}$ is totally unanimous. Let A be a free set which is not complete. We have to show that there is an issue $Y = \{p, \neg p\}$ disjoint from the agenda $\mathrm{ag}(A)$ such that $F_{\mathcal{G}}$ is Y-independent given A. Let $\mathcal{P}$ be the set of all paths P of T with outcome containing A, precisely $A \subseteq \mathrm{cl}(\varphi(P))$. We first have to show that $\mathcal{P} \neq \emptyset$. Indeed, choose a complete set $J \in \overline{\mathcal{J}}$ containing A (lemma 5). Define a profile $\underline{J} \in \bar{\mathcal{J}}^N$ with $J_i = J$. By total unanimity we find $F_{\mathcal{G}}(\underline{J}) = J$, thus there is a path P with $A \subseteq \mathrm{cl}(\varphi(P))$. Therefore $\mathcal{P} \neq \emptyset$.

Let $\underline{P} = \bigcap \mathcal{P}$ the largest common path. It is nonempty, since every path contains the root node. Since A is not complete, there is an issue $\{p, \neg p\}$ disjoint from the agenda $\mathrm{ag}(A)$. Since A is free, both $A \cup \{p\}$ and $A \cup \{\neg p\}$ are consistent. With the same argument as in the previous paragraph we define two profiles $\underline{J}_p, \underline{J}_{\neg p} \in \bar{\mathcal{J}}^N$

yielding $A \cup \{p\} \subseteq F_{\mathcal{G}}\left(\underline{J}_p\right)$ and $A \cup \{\neg p\} \subseteq F_{\mathcal{G}}\left(\underline{J}_{\neg p}\right)$. Thus the corresponding paths $P\left(\underline{J}_p\right)$ and $P\left(\underline{J}_{\neg p}\right)$ deviate at one node $x \in T$. Since $\underline{P} \subseteq P(x)$, $\underline{P}$ is finite. Thus there is some $x \in T$ with $\underline{P} = P(x)$. By proposition 19 there is an issue $Y = \{p, \neg p\}$ associated to its child nodes such that $F_{\mathcal{G}}$ is Y-independent given $\varphi[P(x)] = \varphi[\underline{P}] \subseteq A$. By lemma 13 (i), $F_{\mathcal{G}}$ is Y-independent given A, what has to be shown.

For a countably infinite agenda X, the condition of piecewise independence provides us a sequence $X_1 \subseteq X_2 \subseteq \cdots$ of finite subagenda with $\bigcup_i X_i = X$ such that F is X_i-independent for every i. The construction above provides us with a sequence of game forms $\mathcal{G}_1, \mathcal{G}_2, \ldots$ with a sequence of trees $T_1 \subseteq T_2 \subseteq \cdots$ such that $F_{\mathcal{G}_i}$ is the conditional function of F from proposition 14 on subagenda X_i given $\emptyset$. Setting $T = \bigcup_i T_i$, defining φ as the extension of all assignments φ_i in $\mathcal{G}_i$, with the collection $\{\mathcal{W}_x\}_{x \in T}$ we obtain at a new game form, where the information sets are given by proposition 18. As above we conclude that $F(\underline{J}) = F_{\mathcal{G}}(\underline{J})$ for all $\underline{J} \in \bar{\mathcal{J}}^N$. $\square$

The last result is on strategic manipulation.

Proof. (Theorem 23) Let $F_{\mathcal{G}}$ be the judgement aggregation form induced by a monotonic and weakly neutral game form $\mathcal{G}$ adapted to the judgement space $\langle X, \mathcal{J} \rangle$. Let $\underline{J} \in \bar{\mathcal{J}}^N$ be some profile and let $i \in N$ and $J_i' \in \bar{\mathcal{J}}$. Assume the solution paths $P(\underline{J})$ and $P\left(\underline{J}_{-i} J_i'\right)$ are complete. We have to show that $F_{\mathcal{G}}(\underline{J}) \trianglerighteq_{J_i} F_{\mathcal{G}}\left(\underline{J}_{-i} J_i'\right)$, or $J_i \cap F_{\mathcal{G}}\left(\underline{J}_{-i} J_i'\right) \subseteq F_{\mathcal{G}}(\underline{J})$. Let $p \in J_i \cap F_{\mathcal{G}}\left(\underline{J}_{-i} J_i'\right)$. Since $P\left(\underline{J}_{-i} J_i'\right)$ is complete, there is a node $y \in P\left(\underline{J}_{-i} J_i'\right)$ with $\varphi(y) = p$. Thus $W := \left\{ j \in N \mid p \in \left(\underline{J}_{-i} J_i'\right)_j \right\} \in \mathcal{W}_y$. By monotonicity, $W' := W \cup \{i\} \in \mathcal{W}_y$. Since $P(\underline{J})$ is complete, there is a node $x \in P(\underline{J})$ with child notes x_p and $x_{\neg p}$, $\varphi(x_p) = p$, and $\varphi(x_{\neg p}) = \neg p$. Weak neutrality implies that $\mathcal{W}_{x_p} = \mathcal{W}_y$, therefore $W' \in \mathcal{W}_{x_p}$. Since $W' = \{j \in N \mid p \in J_j\}$, we conclude $p \in F_{\mathcal{G}}(\underline{J})$, what had to be shown. $\square$

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