International diversification and dependence structure of equity portfolios during market crashes: the Archimedean copula approach

Muteba Mwamba, John and Mokwena, Paula

University of Johannesburg, University of Johannesburg

9 August 2013

International Diversification and Dependence Structure of Equity Portfolios during Market Crashes: The Archimedean Copula Approach

John Muteba Mwamba, and Paula Mokwena
Tel: +27 11 559 43 71
Email: johnmu@uj.ac.za
POBox: 524 Auckland Park 2006
Department of Economics and econometrics
University of Johannesburg
South Africa

Abstract

This paper analyzes the effect of the recent market crash on the international diversification of equity portfolios from the perspective of dependence structure. We use the generalized Pareto distribution to fit the left and the right tail of each return distribution in order to evaluate the upside and the downside risk measures separately after removing both autocorrelation and heteroscedasticity in the historical returns. We thereafter build a multivariate generalized Pareto distribution and draw one million simulated returns for each time series using three Archimedean copulas – Gumbel, Clayton and Frank. Using the data from emerging and developed countries; we find that the Clayton copula exhibits strong left tail dependence structure with higher Sharpe ratio and relatively weak right tail dependence after the subprime crisis. We also find that the Clayton copula is ultimately useful in modelling the left tail dependence structure in bear markets only. In addition; our empirical results show that both the Gumbel and Frank copulas produce the same magnitude of Sharpe ratio in bull and bear markets. The Frank copula is found to be useful in modelling returns with strong positive or negative dependence; while the Gumbel copula is found to be useful in modelling the upper tail of the return distribution in bull markets only.

Keywords: Archimedean copula, Gumbel, Frank, Clayton copulas, dependence structures, international diversification
JEL: G01, G11, G15, C15, C02

1. INTRODUCTION

This paper analyzes the effect of the subprime crisis on portfolio allocation from the perspective of dependence structure. Empirical evidence has proved that the multivariate normal distribution is inadequate to model portfolio asset return distribution - firstly because the empirical marginal distributions of asset returns are skewed and fat tailed; and secondly because it does not consider the possibility of extreme joint co-movement of asset returns (Fama and French, 1993; Richardson and Smith, 1993; Géczy, 1998; Longin and Solnik, 2001; Mashal and Zeevi, 2002). This paper employs Archimedean copulas to capture both the dependence structure and the asymmetry of asset returns in the tails of the empirical distributions.

We fit both the left and right tails in order to evaluate upside and downside risk separately. Hence, each return distribution is segmented into the left tail and the right tail in order to capture the potential distributions associated with the empirical data in each segment of the distribution more accurately. To fit the left and right tails of the distribution, the Extreme Value Theory (EVT) is used. We assume that the marginals follow the Generalised Pareto Distribution (GPD) due to the ease with which it can be adapted to modelling financial returns.

The standard mean-variance framework introduced by Markowitz (1952) uses correlation as a measure of dependence between different assets. The theory is based on an assumption of multivariate normally distributed returns in order to arrive at an optimal portfolio selection. However, empirical research in finance shows that the distributions of the real world are non-normal. As Jondeau and Rockinger (2006) point out, when financial returns are non-normal, it is impossible to specify the multivariate distribution of two or more return series. Previous research has investigated how the correlation between stock market returns varies over time. Longin and Solnik (1995) examine correlations between stock markets over a long time period using the constant conditional correlation model proposed by Bollerslev (1990). They find that correlations are generally higher during more volatile periods and depend on several economic variables, such as the dividend yield and interest rate. Longin and Solnik (2001) find that international stock markets are more correlated in bear markets, using extreme value theory, and that the multivariate normality of the joint distributions can be rejected in a statistical test.

Patton (2004) finds dependence asymmetry of financial returns both in the marginal distributions and in the dependence structure. Boyer et al. (1997) reported that correlations can provide little information about the underlying dependence structure in the cases of asymmetric dependence. Therefore, these studies show that simple correlation analysis can be misleading when studying financial market dependence, as also shown by Boyer et al (1997) and Embrechts et al (1999). Costinot et al (2000) suggested that dependence among financial markets was better modelled using copulas rather than correlation analysis. In the case of extreme returns, they found that the probability of joint exceedance for the Dow Jones and the French CAC40 stock market indexes increased dramatically when copulas were used rather than the bivariate normal distribution. Embrechts et al. (2002) used copulas in risk management, showing that standard Pearson correlations can go dangerously wrong as a risk measure. They then suggested the copula function as a flexible alternative to correlation, as the copula can capture dependence throughout the entire distribution of asset returns. Rodriguez (2007) modelled dependence with switching-parameter copulas to study financial contagion. Using daily returns from five East Asian stock indices during the Asian crisis, and from four Latin American stock indices during the Mexican crisis, he found evidence of changing dependence during periods of turmoil. He found that Asian countries were characterized by increased tail dependence and asymmetry, while Latin American countries were described symmetry and tail independence.

Other studies that have also used copulas for portfolio selection include Fernandez (2008) who presented a model to select the optimal hedge ratios of a portfolio composed of an arbitrary number of commodities, using copula to account for returns co-movement. Wang et al (2010) introduced the GARCH-EVT-Copula model and applied it to study the risk of foreign exchange portfolio. Multivariate Copulas, including Gaussian, t-Student and Clayton ones, were used to describe a portfolio risk structure, and to extend the analysis from a bivariate to an n-dimensional asset allocation problem. They applied this methodology to study the returns of a portfolio of four major foreign currencies in China, including USD, EUR, JPY and HKD. Their results suggested that the optimal investment allocations are similar across different Copulas and confidence levels and that the optimal investment concentrates on the USD investment. They found that the t-Student Copula and Clayton Copula better portray the correlation structure of multiple assets than Normal Copula. Ning (2010) investigated the dependence structure between the equity market and the foreign exchange market by using different copulas. The study showed that there exists significant symmetric upper and lower tail dependence between the two financial markets, and the dependence remains significant but weaker after the launch of the Euro. Harris and Küçüközmen (2001) investigated the dynamic behaviour of daily aggregate returns on the Istanbul Stock Exchange (ISE) and found that ISE returns exhibit significant linear and nonlinear dependence. They found that the nonlinear dependence is primarily due to linear dependence in the conditional variance of the returns.

International diversification involves balancing benefits and costs, and this balance is determined by the degree of asset dependence. In light of theoretical research linking diversification and dependence, Chollete et al (2010) examined international diversification using two measures of dependence: correlations and copulas. They found that dependence has increased over time and that the regions with maximal dependence or worst diversification did not have large returns. Their results suggest international limits to diversification, which is also consistent with a possible tradeoff between international diversification and systemic risk.

2. METHODOLOGY

The theory of copula allows us to study nonlinear dependences between selected assets and to build a unified distribution function based on the distribution functions of each asset.

2.1. Markowitz's Approach to Portfolio Selection

Markowitz (1952) postulates that an investor should maximize expected portfolio return $E(R_p) = \mu_p$ while minimizing portfolio variance of return σ_p^2. Hence the portfolio selection problem is summarized as:

$$\mathrm{Max}\,\mu_p = \sum_{i=1}^{n} \omega_i \mu_i \tag{1}$$

Subject to:

$$\sigma_p^2 = \sum_{i=1}^{n} \omega_i^2 \sigma_i^2 + \sum_{i=1}^{n}\sum_{j=1}^{m} \omega_i \omega_j \rho_{ij} \sigma_i \sigma_j \le \lambda \qquad i \ne j \tag{2}$$

where μ_i is asset i's expected return and ω_i is the weight of asset i in the portfolio, σ_i^2 is the variances of asset *I*, ρ_{ij} is the pairwise correlation of the returns of assets i and j *and* λ is the minimum targeted portfolio risk respectively. Equation (2) shows that the variance which is an important input in this optimization problem depends solely on the linear correlation structure which is unable to model dependence of asset during financial crisis (Muteba Mwamba, 2012).

Dependence Analysis

2.2. Copulas

2.2.1. Definition of Copula

Copulas are functions that join univariate distribution functions to form multivariate distribution functions. They were first introduced in by Sklar (1959). A copula function is defined as a multivariate distribution function F of random variables $X_1, \ldots, X_n$ with standard uniform marginal distribution functions $F_1, \ldots, F_n$ (margins). The joint distribution function C of $\{F_1(X_1), \ldots, F_n(X_n)\}$ is then called the copula of the random vector $(X_1, \ldots, X_n)$ or the multivariate distribution F. It follows that:

$$\begin{aligned} F(x_1, \ldots, x_n) &= P[F_1(X_1) \le F_1(x_1), \ldots, F_n(X_n) \le F_n(x_n)] \\ &= C(F_1(x_1), \ldots, F_n(x_n)) \end{aligned} \tag{3}$$

Alternatively, a copula is defined as any function $C : [0,1]^n \to [0,1]$ that satisfies the following properties:

- $C(u_1, \ldots, u_n)$ is increasing in each component $u_i = F_i(x_i)$.

- $C(u_1,\ldots,u_{m-1},0,u_{m+1},\ldots,u_n)=0$ *for all* $u_i \in [0,1]; i \neq m, m=1,\ldots,n$
- $C(1,\ldots,1,u_i,1,\ldots,1)=u_i$ *for all* $u_i \in [0,1], i=1,\ldots,n$. This property follows from the fact that the marginal distributions are uniform.
- For all $(a_1,\ldots,a_n),(b_1,\ldots,b_n)\in[0,1]^n$ with $a_i \le b_i$ we have
  $\sum_{i_1=1}^{2}\cdots\sum_{i_n=1}^{2}(-1)^{i_1+\ldots+i_n}C(u_{1i_1},\ldots,u_{ni_n})\ge 0$
  where $u_{j1}=a_j$ and $u_{j2}=b_j$ for all $j\in 1,\ldots,n$

For any continuous multivariate distribution Equation (3) holds for a unique copula C. If $F_1,\ldots,F_n$ are not all continuous it can still be shown that the joint distribution function can always be expressed as in Equation (3), although in this case the copula C is no longer unique and it is referred to it as a possible copula of F (Schweizer & Sklar, 1983: Chapter 6).

2.2.2. Types of Copulas: Elliptical and Archimedean Copula Families

Elliptical Copulas

Elliptical copulas are simply the copulas of elliptical distributions. The class of elliptical distributions provides useful examples of multivariate distributions because they share many of the tractable properties of the multivariate normal distribution. Simulation from elliptical distributions is easy to perform. Let F_i be the distribution function of the i^{th} margin and F_i^{-1} be its inverse function (quantile function), $i=1,\ldots,n$. The elliptical copula determined by F is:

$$C(u_1,\ldots,u_n)=F\left(F_1^{-1}(u_1),\ldots,F_n^{-1}(u_n)\right) \tag{4}$$

Differentiating Equation (4) gives the density of an elliptical copula:

$$c(u_1,\ldots,u_n)=\frac{f\left(F_1^{-1}(u_1),\ldots,F_n^{-1}(u_n)\right)}{\prod_{i-1}^{n} f_i\left(F_i^{-1}(u_i)\right)} \tag{5}$$

where f is the joint probability distribution function of the elliptical distribution and $f_1,\ldots,f_n$ are marginal density functions.

Examples of elliptical copulas are the Gaussian (normal) copula and the t-Student copula, specified by the multivariate normal and multivariate t-Student distributions respectively. Both copulas have a correlation matrix inherited from the elliptical distributions and t-copula has one more parameter, the degrees of freedom (df). Since copulas are invariant to monotonic transformation of the margins, the standardized correlation matrix determines the dependence structure. It is important to note that the Gaussian and t-copulas are copulas of elliptical distributions, but they are not elliptical distributions themselves.

Archimedean Copulas

Embrechts et. al. (2001) show that there many pitfalls to the normality assumption. Empirical evidence suggests that the use of the multi-normal distribution is inadequate (Fama and French, 1993; Longin and Solnik, 2001). There is clear evidence that financial returns have unconditional fat tails. Therefore, extreme events are more probable than anticipated by normal distribution, not only in marginals but also in higher dimensions. Elliptical distributions capture only linear dependencies and are therefore inadequate in many multivariate analyses of data with probability density concentrated on tails (extreme values). A class of copulas called Archimedean copulas are used to model nonlinear dependencies.

A copula C is termed Archimedean if there exists a generator function ψ such that C has the form:

$$C(u_1,\ldots,u_n)=\psi^{-1}(\psi(u_1)+\ldots+\psi(u_n)) \tag{6}$$

for all $0\le u_1,\ldots,u_n\le 1$, where $\psi:[0,1]\to[0,\infty]$ is continuous and strictly decreasing such that $\psi(1)=0$ and $\psi(0)=\infty$ and ψ^{-1} is the inverse function of the generator. The generator satisfies the following conditions:

- $\psi(1)=0$
- For all $t\in[0,1]$, $\psi'(t)<0$, i.e. ψ is decreasing.
- For all $t\in[0,1]$, $\psi''(t)\ge 0$, i.e. ψ is convex.

By applying ψ, both to the joint distribution and the margins, the distributions “become" independent. In order for Equation (6) to be a copula, the generator needs to be a complete monotonic function (Nelsen, 1999). A generator uniquely determines an Archimedean copula. For an Archimedean copula, the distribution and density both depend on the generator function and its inverse function. These functions are defined for each Archimedean copula. Archimedean generators associated with a particular Archimedean copula are not necessarily unique, but they are up to a constant. If ψ is a generator of an Archimedean copula, then $a\psi$, for some positive constant a, also generates the same Archimedean copula. Archimedean copulas are permutational symmetric, i.e. $C(u_1,u_2)=C(u_2,u_1)$ and associative, i.e. $C(C(u_1,u_2),u_3)=C(u_1,C(u_2,u_3))$. The density of Equation (6) can be obtained by differentiation.

The three Archimedean copulas that are going to be used in this study are the Frank, Gumbel and Clayton copulas. The Clayton and Gumbel copulas model only positive dependence, while Frank covers the whole range. The Frank copula is generated by:

$$\psi(u)=-\log\left(\frac{e^{\theta u}-1}{e^{\theta}-1}\right) \tag{7}$$

for $\theta\neq 1$ but has the independence copula as a limiting case when $\theta\to 1$. The Gumbel copula has generator:

$$\psi(u)=(-\log u)^{1/\theta} \tag{8}$$

for some $\theta > 1$ and is therefore useful for describing positive dependencies. The Clayton copula is constructed based on the generator:

$$\psi(u) = \frac{u^{-\theta} - 1}{\theta} \tag{9}$$

for $\theta > 0$.

For an Archimedean copula, Kendall's tau can be evaluated directly from the generator of the copula as follows:

$$\tau = 1 + 4\int_0^1 \frac{\psi(t)}{\psi'(t)} dt \tag{10}$$

Parameter Estimation

The method of maximum likelihood method is used to estimate the parameter of these copulas. Let f be the density of the joint distribution F:

$$f(x_1,\ldots,x_n) = c(F_1(x_1),\ldots,F_n(x_n)) \times \prod_{i=1}^{n} f_i(x_i) \tag{11}$$

where f_i is the univariate density of the marginal distribution F_i and c is the density of the copula given by the following expression:

$$c(u_1,\ldots,u_n) = \frac{\partial C(u_1,\ldots,u_n)}{\partial u_1,\ldots,\partial u_n} \tag{12}$$

We suppose to have a set of T empirical data of n financial asset log-returns, $\chi = \{(x_1^t,\ldots,x_n^t)\}_{t=1}^{T}$. Let $\vartheta = (\vartheta_1,\ldots,\vartheta_n,\boldsymbol{\alpha})$ be the parameter vector to estimate, where $\vartheta_i, i = 1,\ldots,n$ is the vector of parameters of the marginal distribution F_i and $\boldsymbol{\alpha}$ is the vector of the copula parameters. The log-likelihood function is the following:

$$l(\vartheta) = \sum_{t=1}^{T} \ln c\left(F_1(x_1^t;\vartheta_1),\ldots,F_n(x_n^t;\vartheta_n);\boldsymbol{\alpha}\right) + \sum_{t=1}^{T}\sum_{i=1}^{n} \ln f_i(x_i^t;\vartheta_i) \tag{13}$$

The ML estimator $\hat{\vartheta}$ of the parameter vector ϑ is the one which maximize the above equation, i.e.:

$$\hat{\vartheta} = \arg\max l(\vartheta) \tag{14}$$

Constructing Multivariate Distributions Using Copulas and Marginals: The Sklar's Theorem

The existence of the copula function C is established by Sklar's theorem. The first step in constructing multivariate distributions using copulas is to transform the data into uniformly distributed ones. Let T be a transformation that maps the univariate marginals X_i onto uniformly distributed random variables on $[0,1]^n$. For the univariate marginals $X_i, i = 1,\ldots,n$, let F_{X_i} denote the univariate distribution function of the i^{th} margin. Define the transformation $T : R^n \to [0,1]^n$ with $T(x_1,\ldots,x_n) = (z_1,\ldots,z_n)$ via

$$
\begin{aligned}
z_1 &= P[X_1 \le x_1] = F_{X_1}(x_1) \\
z_2 &= P[X_2 \le x_2 | X_1 = x_1] = F_{X_2|X_1}(x_2|x_1) \\
&\vdots \\
z_n &= P[X_n \le x_n | X_1 = x_1, \ldots, X_{n-1} = x_{n-1}] = F_{X_n|X_1,\ldots,X_{n-1}}(x_n|x_1,\ldots,x_{n-1})
\end{aligned} \tag{15}
$$

Then the random variables, $Z_i = T(X_i), i = 1,\ldots,n$, are independent and the random vector $\mathbf{Z} = (Z_1,\ldots,Z_n)$ is uniformly distributed on $[0,1]^n$. Things can now be formulated in terms of copulas.

Using probability integral transform, each continuous marginal $u_i = F_i(x_i)$ has a uniform distribution on $I \in [0,1]$ where $F_i(x_i)$is the cumulative integral of $f_i(x_i)$ for the random variable X_i, where X_i assume values on the extended real line $[-\infty,\infty]$. The n-dimensional probability distribution function F has a unique copula representation:

$$
\begin{aligned}
F(x_1,\ldots,x_n) &= P[F_1(X_1) \le F_1(x_1),\ldots,F_n(X_n) \le F_n(x_n)] \\
&= C(F_1(x_1),\ldots,F_n(x_n)) \\
&= C(u_1,\ldots,u_n)
\end{aligned} \tag{16}
$$

The corresponding density function is:

$$
f(x) = c(F_1(x_1), F_2(x_2),\ldots,F_n(x_n)) \times \prod_{i=1}^{n} f_i(x_i) \tag{17}
$$

where $f_i(x_i)$is the density of the marginal $F_i(x_i)$ and $c(u)$ is the density of the copula $C(u)$

$$
c(u_1,\ldots,u_n) = \frac{\partial C(u_1,\ldots,u_n)}{\partial u_1,\ldots,\partial u_n} \tag{18}
$$

This result is useful for estimating the parameters of both the copula function and the marginal distribution using parameter estimation methods such as the maximum likelihood technique.

Equation (15) above shows that the joint distribution function F, can be described by the margins $F_1,\ldots,F_n$ and the copula C, which captures the dependency structure among $X_1,\ldots,X_n$.

2.3. Simulating from Multivariate Distributions

The considered copulas - Gumbel, Clayton and Frank copulas - fall into the class of so-called Laplace transform Archimedean copulas (or LT-Archimedean copulas). For this class, the inverse of the generator ψ has a nice representation as a Laplace transform of some function G. The simulation algorithm uses that such pseudo random variables may be generated easily. To consider this approach in more detail, consider a cumulative distribution function G and denote its Laplace transform by:

$$\hat{G}(t) := \int_0^\infty e^{-tx} dG(x), t \ge 0 \tag{19}$$

We set $\hat{G}(\infty) := 0$ and realize that $\hat{G}$ is a continuous and strictly decreasing function, thus may serve well as a candidate for ψ^{-1}. Indeed, generate a pseudo random variable V with cumulative distribution function G and i.i.d. standard uniform pseudo random variables $X_1, \ldots, X_n$ (also independent of V).

Set

$$U_i := \hat{G}\left(-\frac{\ln X_i}{V}\right) \tag{20}$$

then the vector $\mathbf{U}$ has the desired Archimedean copula dependence structure with generator $\psi = \hat{G}^{-1}$. A proof is given in McNeil et al. (2005).

Portfolio Optimisation Using Copula Distributions

Before building a joint distribution using copulas, we need to find a proper specification for marginal distributions of individual series, as misspecified marginal distributions will lead to a misspecified joint distribution. First, we test the marginal distributions for normality using the Jarque Bera test. If the i.i.d. hypopaper is rejected, we fit a time series model to each margin and work on the residuals. When dealing with financial log-returns, time series are usually filtered with ARCH/GARCH process to remove long-term serial dependence in the variance. We then use the Generalised Pareto Distribution (GDP) to model the marginal distributions and we use various copulas (Frank, Gumbel and Clayton) to build the multivariate distribution using Sklar (1952)'s theorem.

We assume an AR(1) process for conditional mean and a GARCH(1,1) setup for conditional variance as in Muteba Mwamba (2012). This is a standard model for financial returns introduced by Bollerslev (1987), and which is widely used in the literature (Patton, 2002, 2006); Jondeau & Rockinger (2006) and Hu (2006)). Let $X_{i,t}$ be the return of index *i* at time *t*, and the model of marginal distributions is given by the following:

$$X_{i,t} = \mu + \alpha X_{i,t-1} + \sigma_{i,t}\varepsilon_{i,t} \qquad for\ t \ge 2;\ \varepsilon_{i,t} \sim i.i.d.t(v_i) \tag{21}$$

where $\mu \in R$ and $|\alpha| < 1$. The conditional heteroscedasticity is specified by:

$$\sigma_{i,t}^2 = \beta_{i,0} + \beta_{i,1}\varepsilon^2{}_{i,t-1} + \beta_{i,2}\sigma_{i,t-1}^2 \qquad for\ t \ge 2 \tag{22}$$

Where $\beta_{i,0} > 0$ and $\beta_{i,1}, \beta_{i,2} \ge 0$ and $\sigma_{i,t}$ is conditional standard deviation and random error $\varepsilon_{i,t}$ is i.i.d. Equation (20) is a necessary condition for stationarity.

The Extreme Value Theory (EVT) provides a framework for modelling only the tails of the return distribution. There are two approaches to EVT-based modelling. The block maxima method, leading to a generalized extreme value distribution (GEV), divides the data into consecutive blocks and focuses on the series of the maxima (minima) in these blocks. The peaks-over-threshold (POT), leading to a generalized Pareto distribution (GPD), models those events in the data that exceed a high threshold, which in a financial context, means losses larger than a given high level. The distributional model for exceedances over thresholds is GPD given by

$$F(x)=\begin{cases}1-\left(1+\xi x/\beta\right)^{-1/\xi} & \xi\neq 0\\ 1-\exp\left(-x/\beta\right), & \xi=0\end{cases} \tag{23}$$

where $\beta>0$ and $x\geq 0$ when $\xi\geq 0$ and $0\leq x\leq -\beta/\xi$ when $\xi<0$. The parameters ξ and β are the shape and the scale parameter respectively. When $\xi>0$, then F is the distribution of a Pareto distribution which has a power tail decay.

By using parameters estimated from the empirical return series, we simulate a return series for each index. Using the simulated marginal distributions for each index return, we simulate a multivariate Gumbel Copula, Frank Copula and Clayton copula for the dependence structure of the index returns.

Once the estimate procedures are done, the optimal portfolio of assets can be found by maximising the return and minimising the variance in the following optimization problem:

$$\text{Maximise } \mathrm{R_p}=\omega^{t}\mu-\tfrac{1}{2}\delta\omega\Sigma\omega^{t} \tag{24}$$

$$\textit{Subject to}\begin{cases}\sum_{i=1}^{n}\omega_i=1\\ \omega_i\geq 0\end{cases}$$

Where δ is the risk aversion coefficient.

We find the optimal portfolio weights by maximizing the expected utility of investors using simulated returns.

3. EMPIRICAL RESULTS AND DISCUSSION

Data

The study is based on eight sets of data consisting of four emerging market stock indices – South Africa's JSE/FTSE All Share Index (ALSI), Brazil's Bovespa Index (Bovespa), Mexico's Indice de Precios y Cotizaciones (IPC) index and China's Shanghai Composite Index (SCI) – and four developed market indices – the United States' S&P500, the United Kingdom's FTSE100, Germany's DAX and France's CAC40. These emerging markets were chosen for their diversity and economic growth prospects. China and Brazil are amongst the largest emerging economies in the world. South Africa is the largest economy on the African continent. Mexico was included because of its recent solid economic growth.

All the data were collected and sampled at a daily frequency from 1 January 2005 to 31 December 2010. This time period was chosen in order to capture the dependence structure of the indices 3 years before the subprime crisis (from 1 January 2005 to 31 December 2007) and 3 years after the subprime crisis (from 1 January 2008 to 31 December 2010). To eliminate spurious correlation generated by holidays, those observations when a holiday occurred at least for one country were eliminated from the database. Note that such an observation would not affect the dependency between stock markets during extreme events. All the modelling is done using the R software package.

The descriptive statistics for all the time series is shown in Table 1 below.

Table 1: Descriptive Statistics

Descriptive Statistics								
	S&P500	**FTSE100**	**DAX**	**CAC40**	**ALSI**	**BOVESPA**	**SCI**	**IPC**
Mean	0.0000	0.0001	0.0003	0.0001	0.0008	0.0007	0.0004	0.0008
Standard Error	0.0004	0.0004	0.0004	0.0004	0.0004	0.0005	0.0005	0.0004
Standard Deviation	0.0147	0.0136	0.0145	0.0153	0.0145	0.0198	0.0193	0.0149
Sample Variance	0.0002	0.0002	0.0002	0.0002	0.0002	0.0004	0.0004	0.0002
Kurtosis	10.0516	8.3458	8.3945	8.0510	3.2483	5.7477	2.8749	5.3450
Skewness	-0.2514	-0.0623	0.2413	0.2149	-0.1471	0.0835	-0.3925	0.1789

This table shows that The S&P500, FTSE100, ALSI and SCI have negative skewness, and the DAX, CAC40, Bovespa and IPC have positive skewness. Moreover, all the time series exhibit excess kurtosis except for the ALSI and SCI. This indicates that most of the series display fatter tails than the Gaussian distribution; this finding is similar to the one obtained by Muteba Mwamba (2011) when modelling stock price behaviour. The Jarque-Bera's test (Cromwell *et al.* 1994), a joint statistic using skewness and kurtosis coefficients, is also used to reject the null hypothesis of normality for these time series.

But relying on numerical summaries alone for checking the distribution of the sampled means can be misleading. Therefore, we also rely on a powerful graphical technique known as a quantile-quantile plot or QQ-plot which helped us assess whether a data set is consistent with a known distribution. The quantile function Q is the generalized inverse function of the cumulative distribution function F :

$$Q(p) = F^{\leftarrow}(p) \qquad \textit{for } p \in (0,1)$$

where the generalized inverse function F^{-1} is defined as

$$F^{-1}(p) = \inf\{x \in \mathbf{R} : F(x) \geq p\}, \quad 0 < p < 1$$

The quantity $x_p = F^{-1}$ defines the pth quantile of the distribution function F. Suppose that our data set consists of the points $x_1, x_2, \ldots, x_n$. Let $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$ denote our data sorted in increasing order. We also use the convention that $x_{(i)}$ is the $p_i = {i-0.5}/{n}$ quantile. To check if the distribution of our empirical data is consistent with the distribution function F we plot the points $(Q(p_i), x_{(i)})$; that is, the quantiles of F against the quantiles of our data set. If the empirical distribution is a good approximation of the theoretical distribution, then all the points would lie very close to the line $y = x$; departures from this line give us information on how the empirical distribution differs from the theoretical distribution. Figure 1 below shows the QQ-plot for the ALSI.

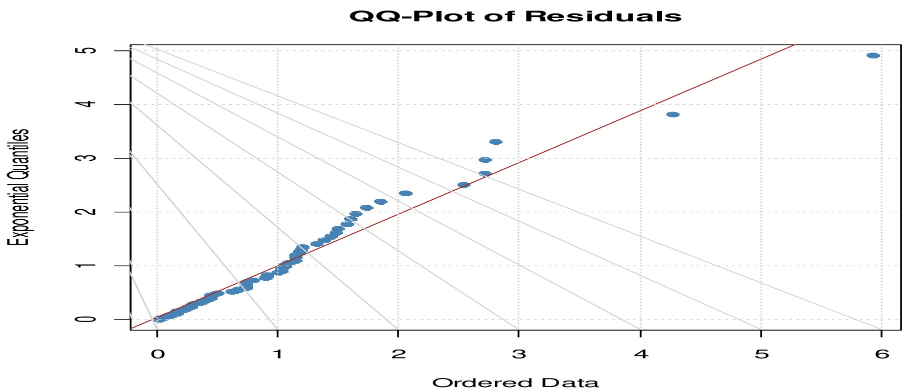

Figure 1: QQ-Plot for ALSI

Figure 1 shows clearly that the distribution of the maximum does not follow a normal distribution. If it did the data would fall approximately on a straight line. Rather the points form a concave line. At the upper right-hand corner the data are below the straight line. This implies that the distribution of the maximum is thicker tailed than the normal distribution. The same graphical method was used for all the data sets, and showed that none of the data sets converged to a normal distribution.

For each time series, the data was split into two periods – from 1 January 2005 to 31 December 2007 (before the subprime crisis) and from 1 January 2008 to 31 December 2010 (after the subprime crisis). This was done in order to determine the impact of the subprime crisis on the tails of return distributions for stock indices.

Empirical Results and Discussion

However, most financial return series exhibit some degree of autocorrelation and, more importantly, heteroskedasticity. To produce a series of independent and identically distributed (i.i.d.) observations, we fit a first order autoregressive model to the conditional mean of the returns of each equity index and a GARCH model to the conditional variance. The first order autoregressive model compensates for autocorrelation, while the GARCH model compensates for heteroskedasticity. When dealing with financial log-returns, GARCH models are a frequent choice for attempting to remove serial dependence in the component time series, as discussed in Muteba Mwamba (2012) and Giacomini et al. (2009). Following Muteba Mwamba (2012), we fit a GARCH model to each of the marginal daily log-returns series and work on the residuals.

We fit both the left and right tails in order to evaluate upside and downside risk separately. Hence, each return distribution is segmented into the left tail and the right tail in order to capture the potential distributions associated with the empirical data in each segment of the distribution more accurately. To fit the left and right tails of the distribution, the Extreme Value Theory (EVT) is used. We assume that the marginals follow the Generalised Pareto Distribution (GPD) due to the ease with which it can be adapted to modelling financial returns. We isolate the tails on both sides by specifying lower and upper thresholds.

The Pickands, Balkema & de Haan theorem (Embrechts et al., 2005: 277) show that if we pick a high enough threshold, our data should behave like data that comes from the generalized Pareto distribution. A graphical test to establish the behaviour of the tail can be performed based on the form of the distribution of mean excess (Davison & Smith, 1990). Fat-tailed distributions yield a mean excess function that tends towards infinity for high-thresholds, i.e. linear shape with positive slope (refer to Appendix A). It is possible to choose the threshold where an approximation by the GPD is reasonable by detecting an area with a linear shape on the graph. Figure 2 shows the empirical mean excess plot for Mexico's IPC. Mean excess plots for the other indices are in Appendix A.

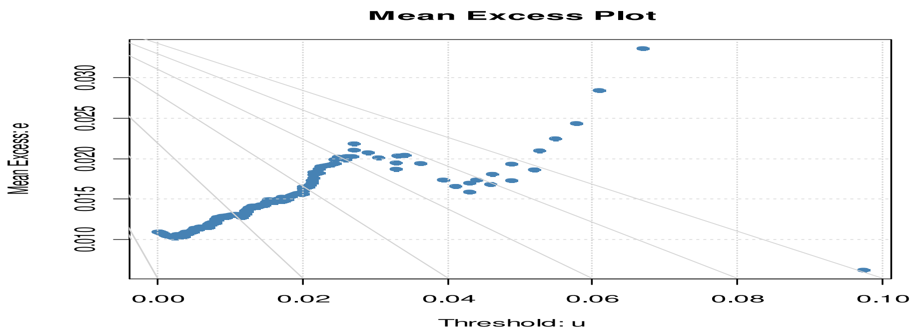

Figure 2: Mean Excess Plot for IPC

Since the mean excess function for the IPC is a straight line with positive slope, we are looking for the threshold point from which the mean excess (ME) plot follows a straight line. The choice of threshold is guided by the ME plot such that the plot is roughly linear above this threshold. According to the estimate we get from the POT model, u=0.0169, we can see that this is a good estimate because the ME plot becomes linear after u=0.0169. Linearity of the ME plot indicates there is no evidence against the hypothesis that the GPD model is a good fit for the threshol data.

The threshold is then used to estimate the parameters for the GPD. The slope of the plot is given by $\xi / 1-\xi$ and the y-intercept is given by $\beta / 1-\xi$. The GPD parameters for each index is given in the tables below.

Table 2: GPD Parameters before subprime

Before Subprime (Lower Tail)			
	ξ	μ	σ
ALSI	-0.1970	0.0100	0.0087
BOVESPA	-0.4894	0.0143	0.0109
SCI	0.1865	0.0093	0.0131
IPC	-0.2347	0.0102	0.0091
S&P 500	-0.1154	0.0062	0.0058
FTSE 100	-0.0760	0.0068	0.0063
DAX	-0.2994	0.0072	0.0066
CAC 40	-0.2902	0.0087	0.0066

Before Subprime (Upper Tail)			
	ξ	μ	σ
ALSI	0.2094	0.0056	0.0067
BOVESPA	-0.0646	0.0079	0.0093
SCI	0.0543	0.0096	0.0107
IPC	0.0735	0.0068	0.0078
S&P 500	0.0198	0.0048	0.0051
FTSE 100	-0.0779	0.0066	0.0054
DAX	-0.2373	0.0052	0.0055
CAC 40	-0.1694	0.0054	0.0054

Table 3: GPD Parameters after subprime

After Subprime (Lower Tail)			
	ξ	μ	σ
ALSI	-0.0844	0.0116	0.0115
BOVESPA	-0.0310	0.0157	0.0166
SCI	-0.2679	0.0197	0.0160
IPC	-0.0835	0.0130	0.0121
S&P 500	0.0056	0.0151	0.0150
FTSE 100	0.0150	0.0123	0.0126
DAX	0.1024	0.0116	0.0133
CAC 40	0.0074	0.0126	0.0136

After Subprime (Upper Tail)			
	ξ	μ	σ
ALSI	-0.0701	0.0138	0.0120
BOVESPA	0.2916	0.0138	0.0175
SCI	0.0479	0.0141	0.0140
IPC	0.2641	0.0111	0.0131
S&P 500	0.1376	0.0133	0.0140
FTSE 100	0.3854	0.0084	0.0126
DAX	0.3266	0.0090	0.0135
CAC 40	0.4933	0.0083	0.0146

After getting the GPD parameters for indices, we can build a multivariate generalized Pareto distribution and draw simulated returns for each index. Simulations play an important role in finance. They are used to replicate the efficient frontiers, to price options, and so on. However, the resulting risk measures computed and the conclusions drawn from the simulations depend upon the assumed model and on the quality of the data-generating algorithms.

Using the R package codes[1], 2000 simulations were performed for a multivariate generalized Pareto distribution with eight marginals representing each one of eight stock market indices. Using the concept of copulas, it is relatively easy to construct and simulate from multivariate distributions of any dimension based on almost any choice of marginals and any type of dependence structure. Three Archimedean copulas – Gumbel, Clayton and Frank – were used to fit the simulated marginals to the different copulas for each sub-period and for both tails of

[1] Available on www.analyticsresearch.net

the distributions. Figure 3, 4 and 5 below, show the plots for the Gumbel, Clayton and Frank copulas with negative returns (downside risk) before the subprime crisis.

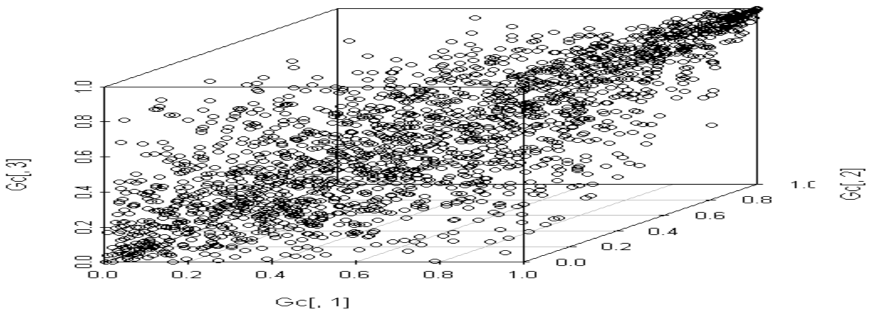

Figure 3: Gumbel Copula Before Subprime – Lower Tail

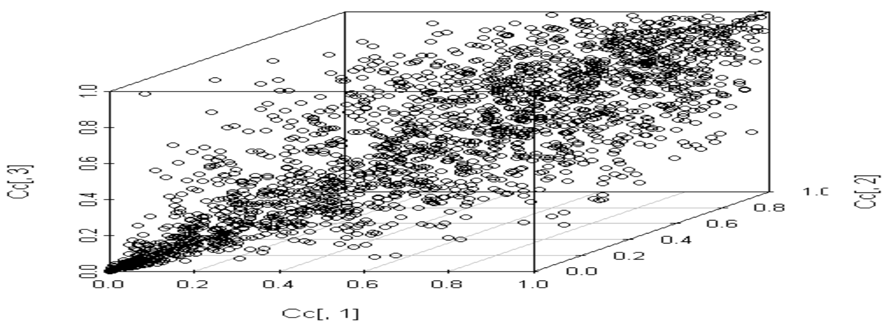

Figure 4: Clayton Copula Before Subprime – Lower Tail

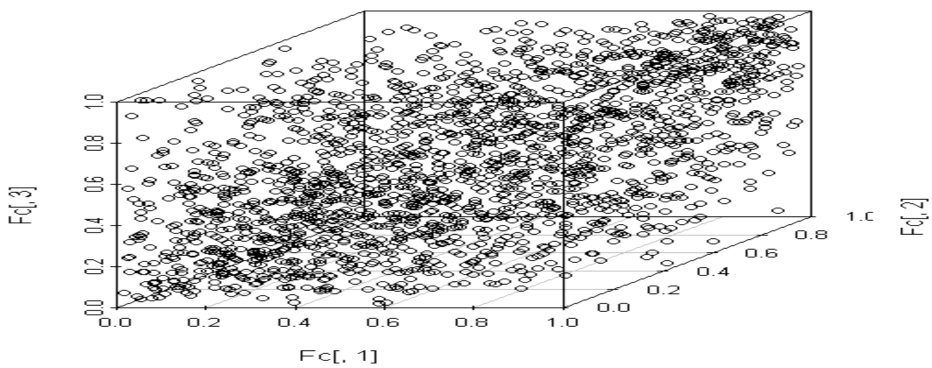

Figure 5: Frank Copula Before Subprime – Lower Tail

The scatter plots of the three copulas before the subprime crisis in the upper tail are shown in Appendix B. Appendix C and D show the scatter plots of the three copulas after the subprime crisis in the lower and upper tails, respectively.

It can be seen from these figures that, in the Clayton copula, the lower tail dependence is much stronger than in the Frank copula, where there is no tail dependence. However, as simulations illustrate, dependence in the tails of the Frank copula tends to be relatively weak compared to the Gumbel and Clayton copulas, and the strongest dependence is centred in the middle of the distribution, which suggests that the Frank copula is most appropriate for data that exhibit weak tail dependence. Similar to the Clayton copula, Gumbel does not allow negative dependence, but in contrast to Clayton, Gumbel exhibits strong right tail dependence and relatively weak left tail dependence. This can be seen by the clustering of the data points in the top right corner of the plot.

The next and final step is to optimise the portfolios using the simulated data, subject to – no short-selling. The tables below show the optimal weights of the assets for the minimum-variance portfolio and the tangency portfolio for each sub-period and in each tail of the distribution. The minimum-variance portfolio is the portfolio where the weights of the different assets results in a portfolio with the minimum standard deviation and the tangency portfolio is the portfolio – comprised of only risky assets – with the highest Sharpe ratio.

Table 4: Portfolio Weights in Lower Tail Before the Subprime

Minimum-Variance Portfolio Weights in Lower Tail Before Subprime			
	Gumbel	Clayton	Frank
S&P500	7%	25%	7%
FTSE100	2%	3%	2%
DAX	1%	2%	1%
CAC40	38%	6%	38%
ALSI	18%	29%	18%
BOVESPA	16%	0%	16%
IPC	14%	8%	14%
SCI	4%	28%	4%

Tangency Portfolio Weights in Lower Tail Before Subprime			
	Gumbel	Clayton	Frank
S&P500	7%	20%	7%
FTSE100	4%	4%	4%
DAX	0%	2%	0%
CAC40	38%	2%	38%
ALSI	15%	34%	15%
BOVESPA	14%	0%	14%
IPC	13%	10%	13%
SCI	10%	28%	10%

In this case, the Gumbel and Frank give same results – the weight of emerging markets is just over 50% for both the minimum-variance portfolio and tangency portfolio. In both cases, the CAC40 has the highest overall weighting (38%) for the Gumbel and Frank copulas. The Clayton shows a portfolio weighting of 64% in emerging markets for minimum-variance portfolio and 71% for tangency portfolio, with the ALSI having the highest weighting in both cases.

Table 5: Portfolio Weights in Upper Tail Before the Subprime

Minimum-Variance Portfolio Weights in Upper Tail Before Subprime			
	Gumbel	Clayton	Frank
S&P500	18%	0%	18%
FTSE100	0%	3%	0%
DAX	18%	0%	18%
CAC40	14%	4%	14%
ALSI	21%	32%	21%
BOVESPA	8%	0%	8%
IPC	15%	33%	15%
SCI	6%	28%	6%

Tangency Portfolio Weights in Upper Tail Before Subprime			
	Gumbel	Clayton	Frank
S&P500	18%	0%	18%
FTSE100	0%	10%	0%
DAX	18%	0%	18%
CAC40	15%	17%	15%
ALSI	20%	17%	20%
BOVESPA	8%	4%	8%
IPC	15%	25%	15%
SCI	6%	28%	6%

Again, in the upper tail the Gumbel and Frank give the same results – 50% weighting in emerging markets for both the minimum-variance portfolio and tangency portfolio, with the ALSI making up highest asset weighting in both portfolios. The Clayton copula produces a minimum-variance portfolio with 93% in emerging markets and 74% in emerging markets for tangency portfolio. Interestingly, the Clayton copula gives zero weighting for the S&P500 and DAX in both portfolios.

Table 6: Portfolio Weights in Lower Tail After the Subprime

Minimum-Variance Portfolio Weights in Lower Tail After Subprime			
	Gumbel	Clayton	Frank
S&P500	10%	21%	10%
FTSE100	8%	21%	8%
DAX	7%	3%	7%
CAC40	32%	12%	32%
ALSI	17%	0%	17%
BOVESPA	7%	18%	7%
IPC	10%	16%	10%
SCI	10%	9%	10%

Tangency Portfolio Weights in Lower Tail After Subprime			
	Gumbel	Clayton	Frank
S&P500	5%	15%	5%
FTSE100	12%	19%	12%
DAX	5%	9%	5%
CAC40	31%	13%	31%
ALSI	13%	0%	13%
BOVESPA	6%	16%	6%
IPC	10%	18%	10%
SCI	19%	9%	19%

The Gumbel and Frank give same results again in this case, with a 44% weight in emerging markets for the mean-variance efficient and 47% for tangency portfolio. Just as before the subprime crisis, the Gumbel and Frank copulas allocate the highest overall weighting in both portfolios to the CAC40 (32%). Unlike before the subprime crisis where the Clayton copula gave the ALSI has the highest overall weighting in both portfolios, the ALSI has 0% weighting in the both portfolios after the subprime crisis. Also, the weight of emerging market indices has been significantly reduced, with only 43% weighting.

Table 7: Portfolio Weights in Upper Tail After the Subprime

Minimum-Variance Portfolio Weights in Upper Tail After Subprime			
	Gumbel	Clayton	Frank
S&P500	30%	30%	30%
FTSE100	0%	0%	0%
DAX	0%	0%	0%
CAC40	50%	50%	50%
ALSI	20%	20%	20%
BOVESPA	0%	0%	0%
IPC	0%	0%	0%
SCI	0%	0%	0%

Tangency Portfolio Weights in Upper Tail After Subprime			
	Gumbel	Clayton	Frank
S&P500	27%	27%	27%
FTSE100	3%	3%	3%
DAX	0%	0%	0%
CAC40	48%	48%	48%
ALSI	19%	19%	19%
BOVESPA	0%	0%	0%
IPC	0%	0%	0%
SCI	4%	4%	4%

All three copulas give the same results in the upper tail after the subprime crisis – 20% in emerging markets for minimum-variance portfolio and 22% for tangency portfolio. The CAC40 has the highest weighting – 50% in the minimum-variance portfolio and 48% in tangency portfolio.

A risk measure which has been widely accepted since the 1990s is value-at-risk (VaR). It was approved by regulators as a valid approach for calculation of capital reserves needed to cover market risk. Even though approved by regulators and widely used in practice, VaR has major shortcomings. One of the shortcomings is that VaR is non-informative about extreme losses. A risk measure which is more informative than VaR about extreme losses is conditional value-at-risk (CVaR). It is defined as the average VaR beyond the VaR at the corresponding confidence level. CVaR measures extreme risk and calculates the risk beyond VaR and is, therefore, better suited for risk management in a fat-tailed world. CVaR is a convex function of portfolio weights, and is therefore attractive to optimize portfolios (Rockafellar and Uryasev, 2002). Table 8 and Table 9 below show the expected excess returns and risk in each tail in each sub-period using both the tangency and minimum variance methods.

Table 8: Return and Risk Trade-off using the Tangency Method

Tangency Method					
		Before Subprime Lower tail	Before Subprime Upper tail	After Subprime Lower tail	After Subprime Upper tail
Gumbel	Mean	1.62%	1.35%	2.75%	3.00%
	VaR	-0.83%	-0.64%	-1.47%	-1.21%
	CVaR	-0.77%	-0.60%	-1.39%	-1.10%
Clayton	Mean	1.63%	1.37%	2.73%	3.00%
	VaR	-0.82%	-0.63%	-1.49%	-1.21%
	CVaR	-0.76%	-0.60%	-1.40%	-1.10%
Frank	Mean	1.62%	1.35%	2.75%	3.00%
	VaR	-0.83%	-0.64%	-1.47%	-1.21%
	CVaR	-0.77%	-0.60%	-1.39%	-1.10%

Table 9: Return and Risk Trade-off using the Minimum Variance Method

Minimum Variance Method					
		Before Subprime Lower tail	Before Subprime Upper tail	After Subprime Lower tail	After Subprime Upper tail
Gumbel	Mean	1.62%	1.35%	2.75%	2.99%
	VaR	-0.82%	-0.64%	-1.48%	-1.20%
	CVaR	-0.77%	-0.60%	-1.40%	-1.09%
Clayton	Mean	1.63%	1.37%	2.73%	2.99%
	VaR	-0.82%	-0.63%	-1.48%	-1.20%
	CVaR	-0.76%	-0.60%	-1.40%	-1.09%
Frank	Mean	1.62%	1.35%	2.75%	2.99%
	VaR	-0.82%	-0.64%	-1.48%	-1.20%
	CVaR	-0.77%	-0.60%	-1.40%	-1.09%

From the two tables above, we can see that returns are higher after the subprime crisis in both tails. However, downside risk, as measured by VaR and CVaR at the 95% confidence level, has also significantly increased after the subprime crisis. The CVaR measure is always higher than the VaR measure because it is a more conservative measure of risk. The downside risk in the lower tail of the distribution is higher than in the upper tail in both sub-periods.

The return and risk measures obtained using the tangency and minimum variance methods are almost exactly the same. Furthermore, all the copulas give approximately the same return and risk measures for both tails in each sub-period. However, when we look at the returns on a risk-adjusted basis, we get a different picture. The Sharpe ratio is a measure that calculates excess return relative to the total risk of the portfolio, as measured by the standard deviation. The higher the Sharpe ratio, the higher the return on a risk-adjusted basis. Therefore, the portfolio with the highest Sharpe ratio would be preferred.

Table 10: Sharpe Ratio – Tangency Method

Sharpe Ratio - Tangency Method				
	Before Subprime Lower tail	Before Subprime Upper tail	After Subprime Lower tail	After Subprime Upper tail
Gumbel	2.25	1.67	2.12	1.09
Clayton	2.20	1.51	2.20	1.09
Frank	2.25	1.67	2.12	1.09

Table 11: Sharpe Ratio – Minimum Variance Method

Sharpe Ratio - Minimum Variance Method				
	Before Subprime Lower tail	Before Subprime Upper tail	After Subprime Lower tail	After Subprime Upper tail
Gumbel	2.25	1.67	2.12	1.09
Clayton	2.20	1.51	2.20	1.09
Frank	2.25	1.67	2.12	1.09

The two tables above show that the tangency portfolio and the minimum variance portfolio give the same risk-adjusted return for each copula in each tail and each sub-period. Also, the Gumbel and Frank copulas give the same Sharpe ratio all the time. However, the Clayton copula gives lower Sharpe ratios than the Gumbel and Frank copulas in each tail before the subprime crisis. After the subprime crisis, the Clayton copula gives a higher Sharpe ratio than both the Gumbel and Frank copulas in the lower tail and all three copulas give the same Sharpe ratio in the upper tail after the subprime crisis.

These results show that the Clayton copula is good at modelling left tail dependence in bear markets and the Gumbel copula is good at modelling right tail dependence in bull markets.

Figure 6 below shows the efficient frontier generated by the Clayton copula in the lower tail before the subprime crisis (see Appendix E and F for the rest of the efficient frontiers before and after the subprime crisis).

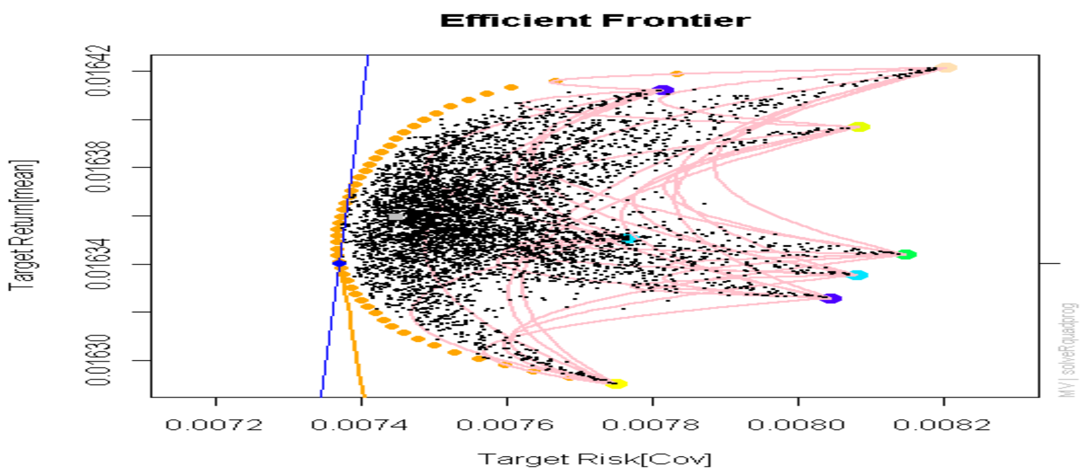

Figure 6: Efficient frontier before subprime: Clayton Copula

The tangency portfolio is the portfolio where the blue line intersects the efficient frontier. This portfolio is also the market portfolio. The weights of the assets in this portfolio are shown in the figure below (see Appendix G, H, I and J for tangent portfolio and efficient frontier weights for Clayton, Frank and Gumbel copulas before and after the subprime crisis).

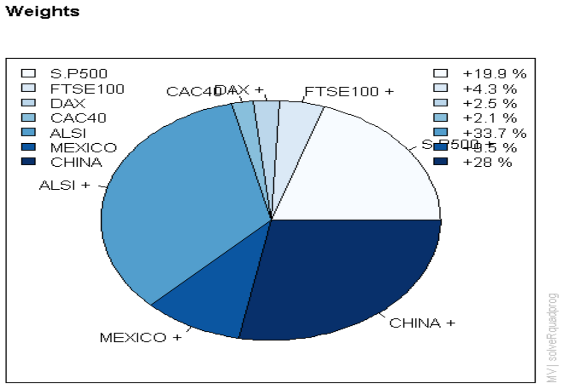

Figure 7: Weights of tangent portfolio

According to the pie graph above, the ALSI must have the greatest weighting in the market portfolio, followed by China's SCI and then the S&P500 and others. The weights of the efficient frontier are shown in the figure below.

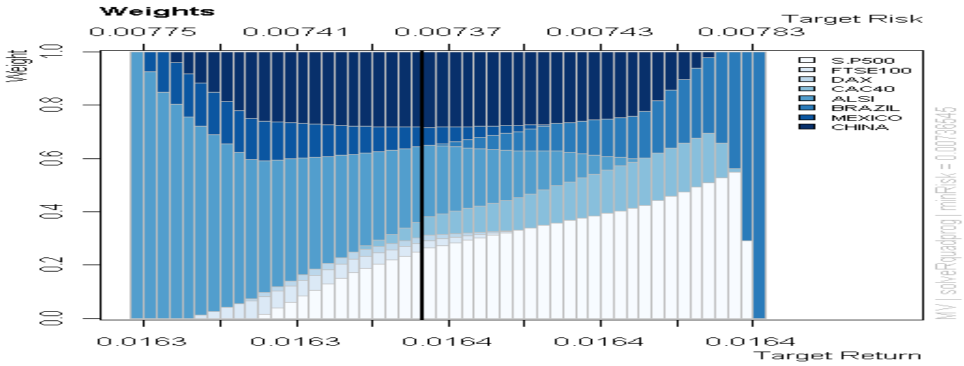

Figure 8: Efficient frontier weights

This figure also shows the asset allocation for the efficient frontier in Figure 6. As we move to the right, the portfolios become more risky, and the weights of the assets change.

4. CONCLUSION

The Gumbel, Clayton and Frank copulas have been used to study tail dependence between four emerging market and four developed market stock indices. The Clayton copula exhibits strong left tail dependence and relatively weak right tail dependence. This was highlighted in our study by the higher Sharpe ratio given by the Clayton copula in the left tail after the subprime crisis. However, it may be that the Clayton copula is good at modelling left tail dependence in bear markets and not in bull markets.

The Gumbel and Frank copulas give the same Sharpe ratio all the time. Our study is in agreement with theory because the Frank copula has shown symmetric dependence in both tails. Therefore, it can be used to model outcomes with strong positive or negative dependence. According to theory, the Gumbel copula is good at modelling the upper tail of the distribution. This was evidenced by the fact that the study shows that the Gumbel copula gives a higher risk-adjusted return in the upper tail before the subprime crisis. However, perhaps the Gumbel copula is only good at modelling the upper tail of the distribution in bull markets and not in bear markets.

Our results have also shown that the weight of emerging market stocks should be lower after the subprime crisis. This could be consistent with the flight-to-quality tendency of investors during difficult times. It also shows a major rebalancing of equity portfolios that is typical when there is a change of sentiment in financial markets. Furthermore, returns have increased after the subprime crisis, as investors demand higher returns to compensate them for the higher risk.

5. REFERENCES

1. Bekaert, G. & Harvey, C.R. (1997). Emerging Equity Market Volatility. *Journal of Financial Economics*, 43:29–77.
2. Bollerslev, T. (1987). A conditional heteroskedastic time series model for speculative prices and rates of return. *Review of Economics and Statistics,* 69:542-547.
3. Bollerslev, T. (1990). Modelling the coherence in short-run nominal exchange rates: A multivariate generalized arch model. *Review of Economics and Statistics*, 72(3):498-505.
4. Boyer, B. H., Gibson, M. S. & Loretan, M. (1999). *Pitfalls in tests for changes in correlations*. International Finance Papers No. 597, Board of Governors of the Federal Reserve System.
5. Cherubini, U., Luciano, E. & Vecchiato, W. (2004). *Copula Methods in Finance*. Chichester: Wiley.
6. Chollete, L., de la Peña, V. & Lu, C. (2010). International diversification: A copula approach. *Journal of Banking & Finance*, Forthcoming.
7. Costinot, A., Roncalli, T. & Tëiletche, J. (2000). Revisiting the Dependence Between Financial Markets with Copulas. Working paper, Crèdit Lyonnais.
8. Crichton N.J. (1999). Information point: Spearman's rank correlation. *Journal of Clinical Nursing,* 8:763.
9. Davison, A. & Smith, R. (1990). Models for exceedances over high thresholds. *Journal of the Royal Statistical Society, Series B,* 52(3):393–442.
10. Embrechts, P., McNeil, A. & Frey, R. (2005). *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton Series in Finance. New Jersey: Princeton University Press.

11. Embrechts, P., McNeil, A.J. & Straumann, D. (2002). Correlation and dependence in risk management: Properties and pitfalls. In *Risk Management: Value at Risk and Beyond*. Edited by Dempster, M.A.H. Cambridge: Cambridge University Press.
12. Embrechts, P., McNeil, A.J., and Straumann, D. (1999). Correlation and Dependence in Risk Management: Properties and Pitfalls.
13. Embrechts, P., McNeil, A.J., Straumann, D., 2001. *Correlation and dependency in risk management: properties and pitfalls*. In Risk Management. Edited by Dempster, M. & Moffatt, H. New York: Cambridge University Press.
14. Fama, E. & French, K. (1993). Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics,* 33:3-56.
15. Fernandez, V. (2008). Copula-based measures of dependence structure in assets returns. *Physica A*, 387:3615–3628.
16. Genest, C., Rémillard, B. & Beaudoin, D. (2009). Goodness-of-fit tests for copulas: A review and a power study. *Insurance Mathematics Economics,* 44:199-213.
17. Giacomini, E., Härdle, W. & Spokoiny, V. (2009). Inhomogenous dependence modeling with time-varying copulae. *Journal of Business and Economic Statistics,* 27:224–234.
18. Hu, L. (2006). Dependence patterns across financial markets: a mixed copula approach. *Applied Financial Economics*, 16:717-729.
19. Huang, J.J., Lee, K.J., Liang, H. & Lin, W.F. (2009). Estimating value at risk of portfolio by conditional copula-GARCH method. *Insurance: Mathematics and Economics*, 45:315-324.
20. Jondeau, E. & Rockinger, M. (2003). Conditional volatility, skewness and kurtosis: existence, persistence, and comovements. *Journal of Economic Dynamics & Control*, 27:1669–1737.
21. Longin, F. & Solnik, B. (1995). Is the correlation in international equity returns constant: 1960-1990. *Journal of lnternational Money and Finance*, 14:3-23.
22. Longin, F. & Solnik, B. (2001). Extreme correlation of international equity markets. *Journal of Finance*, 56(2):649-676.
23. Mandelbrot, B. (1963). The variation of certain speculative prices. *Journal of Business*, 36:394–419.
24. Markowitz, H. (1952). Portfolio selection. *Journal of Finance,* 7(1):77–91.
25. Mashal, R. & Zeevi, A. (2002). *Beyond Correlation: Extreme Co-movements between Financial Assets*. Working Paper. Columbia University.
26. McNeil, A., Frey, R. & Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton: Princeton University Press.
27. Muteba Mwamba, J. (2012). Implementing a Robust Risk Model for South African Equity Markets: A peak-over-threshold approach; *South African Journal of economics, Vol.80:4, December 2012*
28. Muteba Mwamba, J. (2011). Predictability of Stock Price behaviour in South Africa: a non-parametric approach, *African Finance Journal; Vol 13, part 1, 2011*
29. Nelsen, R.B. (1999). An Introduction to Copulas. In *Lecture Notes in Statistics, Vol. 139*. New York: Springer-Verlag.

30. Ning, C. (2010). Dependence structure between the equity market and the foreign exchange market – a copula approach. *Journal of International Money and Finance*, forthcoming.
31. Panchenko, V. (2005). Goodness-of-fit test for copulas. *Physica A*, 355:176-182.
32. Patton, A. (2004). On the out-of-sample importance of skewness and asymmetric dependence for asset allocation. *Journal of Financial Econometrics*, 2(1):130-168.
33. Patton, A. J. (2002). *Applications of copula theory in financial econometrics*. Unpublished doctoralpaper. San Diego: University of California.
34. Patton, A.J. (2006). Modelling asymmetric exchange rate dependence. *International Economics Review*, 47(2):527-556.
35. Prakash, A.J., Chang, C.H. & Pactwa, T.E. (2003). Selecting a Portfolio with Skewness: Recent Evidence from US, European, and Latin American Equity Markets. *Journal of Banking and Finance*, 27:1375–1390.
36. Richardson, M.P. & Smith, T. (1993). A test of multivariate normality of stock returns. *Journal of Business*, 66:295-321.
37. Rodriguez, J. (2007). Measuring financial contagion: A copula approach. *Journal of Empirical Finance*, 14:401-423.
38. Rousseew, P. & Molenberg, G. (1993). Transformation of non-positive semi-definite correlation matrices. *Comm. Statist.Theory Methods*, 22:965-984.
39. Schweizer, B. & Sklar, A. (1983). *Probabilistic Metric Spaces*. New York: North-Holland/Elsevier.
40. Sklar, A. (1959). *Fonctions de répartition á n dimensions et leurs marges*. Publications de l'Institut de Statistique de l'Universit´e de Paris, 8:229–231.

APPENDIX A: MEAN EXCESS FUNCTIONS

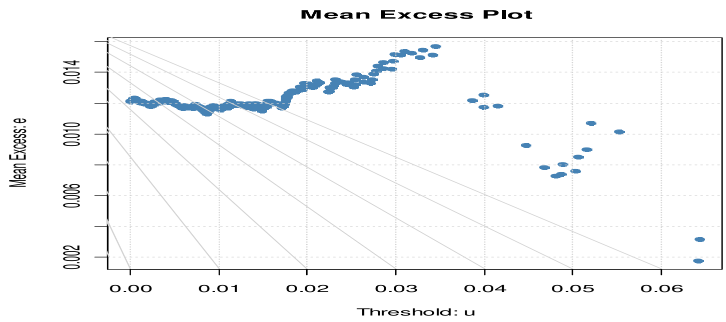

Figure A1: Mean Excess Plot for ALSI

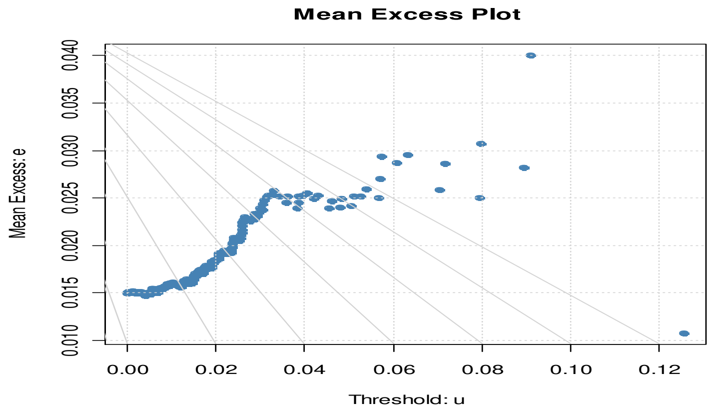

Figure A2: Mean Excess Plot for BOVESPA

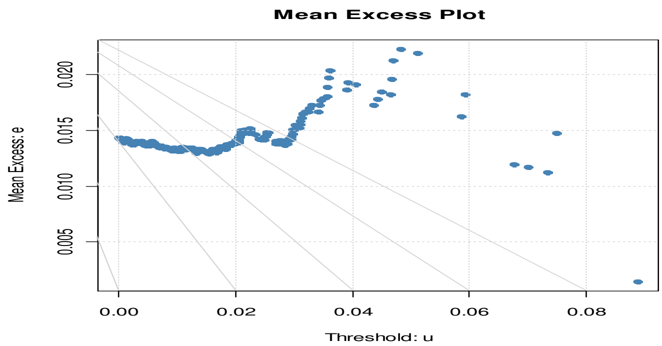

Figure A3: Mean Excess Plot for SCI

APPENDIX B: SCATTER PLOTS IN UPPER TAIL BEFORE THE SUBPRIME CRISIS

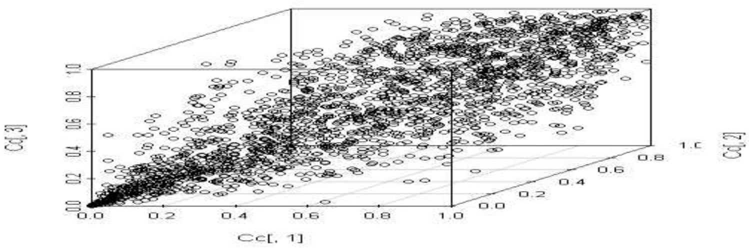

Figure B1: Clayton Copula Before Subprime – Upper Tail

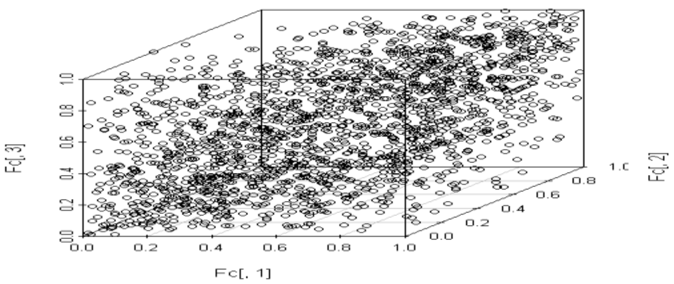

Figure B2: Frank Copula Before Subprime – Upper Tail

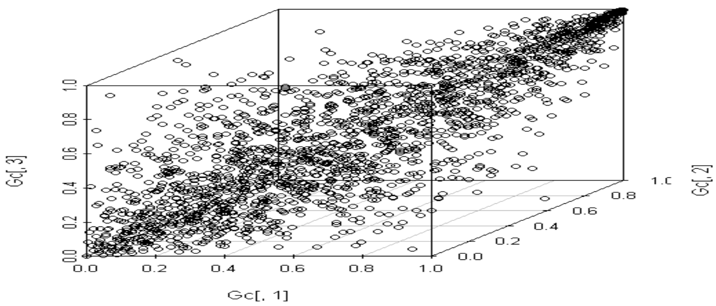

Figure B3: Gumbel Copula Before Subprime – Upper Tail

APPENDIX C: SCATTER PLOTS IN LOWER TAIL AFTER THE SUBPRIME CRISIS

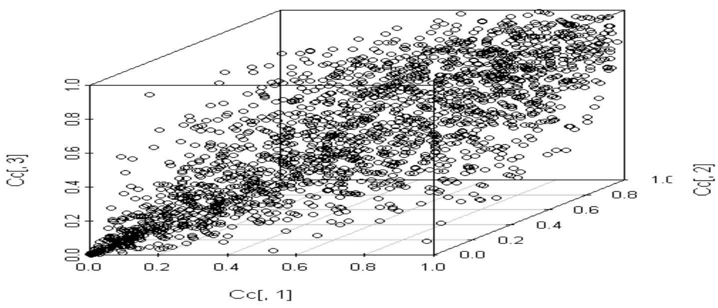

Figure C1: Clayton Copula After Subprime – Lower Tail

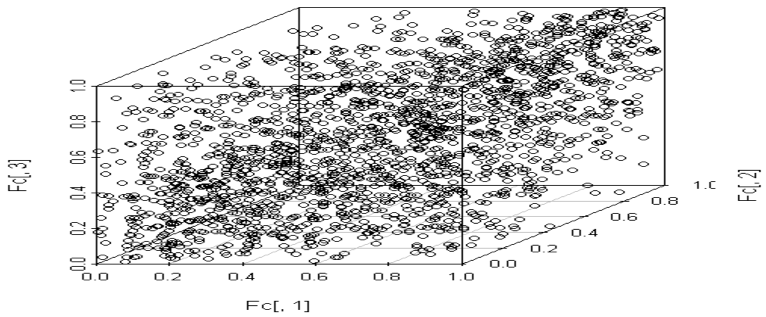

Figure C2: Frank Copula After Subprime – Lower Tail

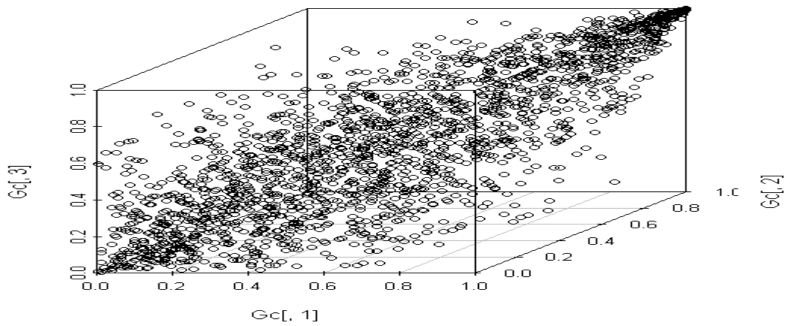

Figure C3: Gumbel Copula After Subprime – Lower Tail

APPENDIX D: SCATTER PLOTS IN UPPER TAIL AFTER THE SUBPRIME CRISIS

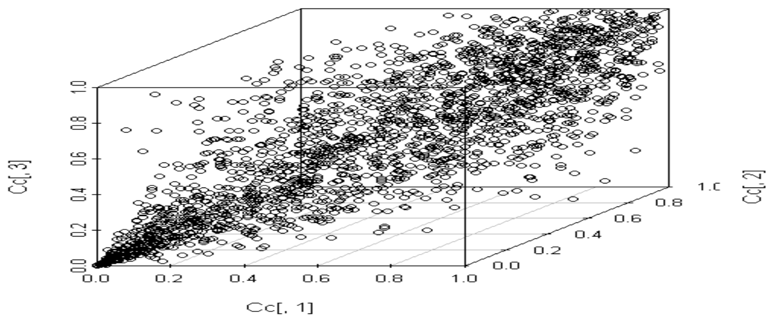

Figure D1: Clayton Copula After Subprime – Upper Tail

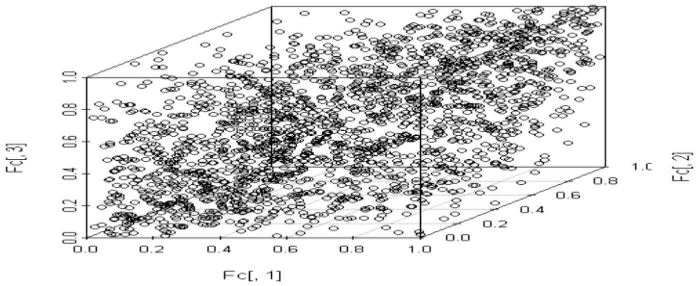

Figure D2: Frank Copula After Subprime – Upper Tail

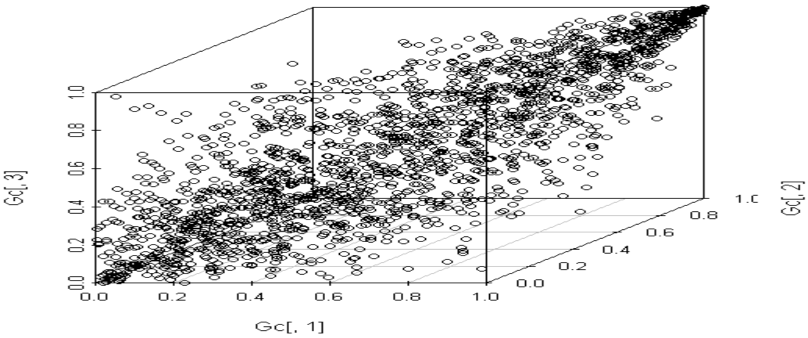

Figure D3: Gumbel Copula After Subprime – Upper Tail

APPENDIX E: EFFICIENT FRONTIERS BEFORE THE SUBPRIME CRISIS

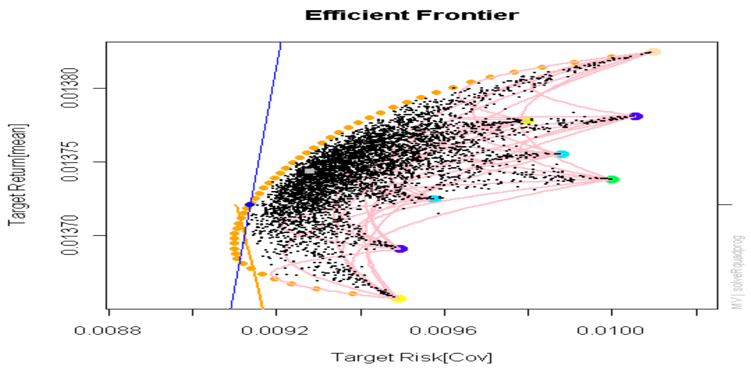

Figure E1: Clayton Copula Efficient Frontier Before Subprime – Upper Tail

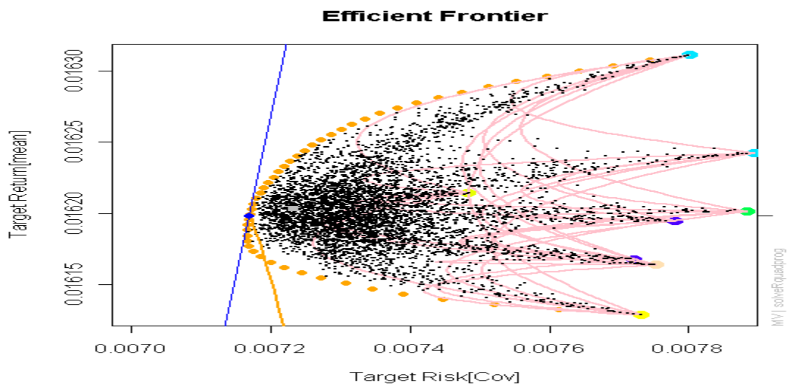

Figure E2: Frank Copula Efficient Frontier Before Subprime – Lower Tail

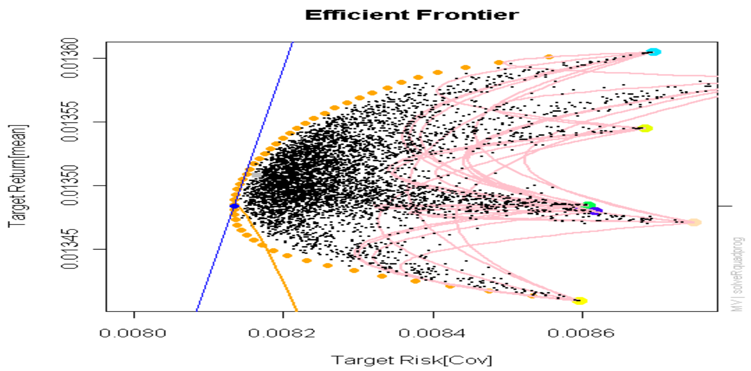

Figure E3: Frank Copula Efficient Frontier Before Subprime – Upper Tail

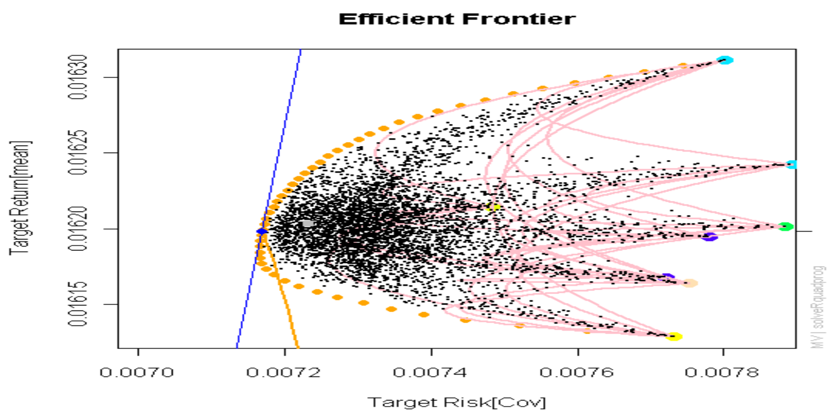

Figure E4: Gumbel Copula Efficient Frontier Before Subprime – Lower Tail

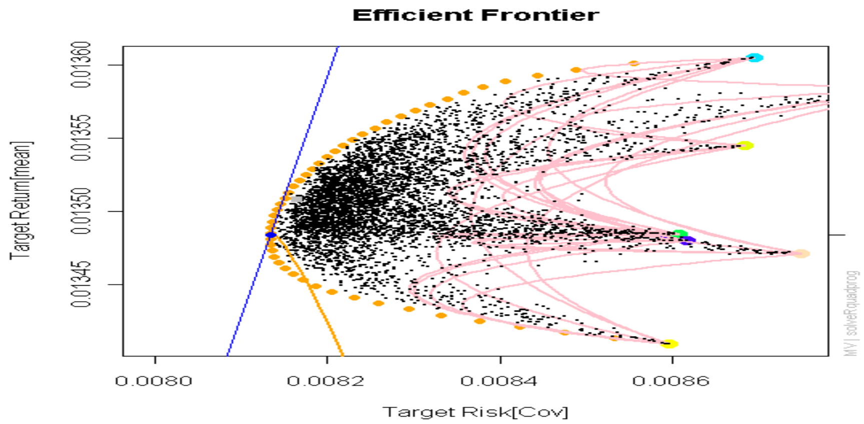

Figure E5: Gumbel Copula Efficient Frontier Before Subprime – Upper Tail

APPENDIX F: EFFICIENT FRONTIERS AFTER THE SUBPRIME CRISIS

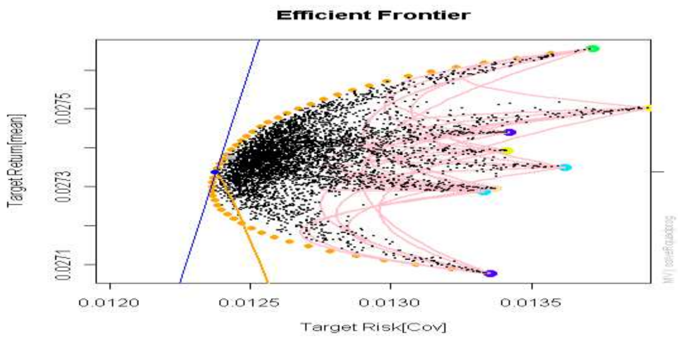

Figure F1: Clayton Copula Efficient Frontier After Subprime – Lower Tail

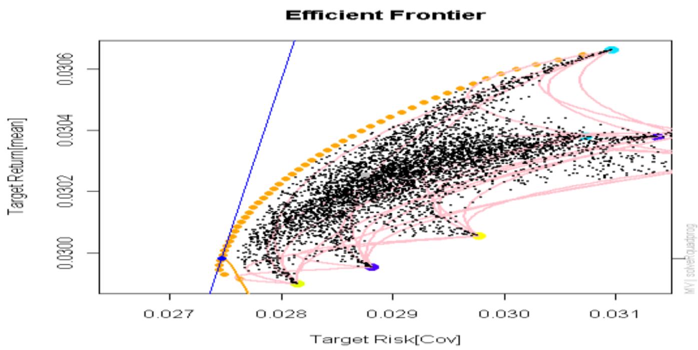

Figure F2: Clayton Copula Efficient Frontier After Subprime – Upper Tail

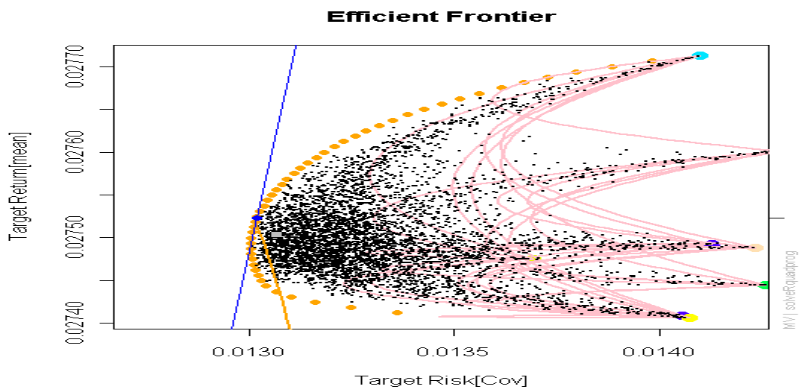

Figure F3: Frank Copula Efficient Frontier After Subprime – Lower Tail

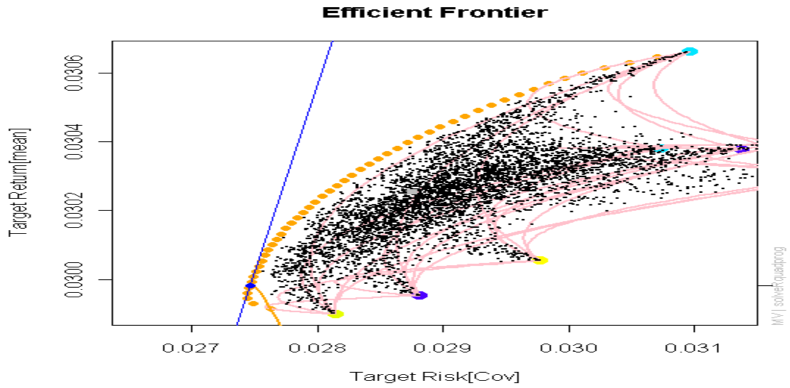

Figure F4: Frank Copula Efficient Frontier After Subprime – Upper Tail

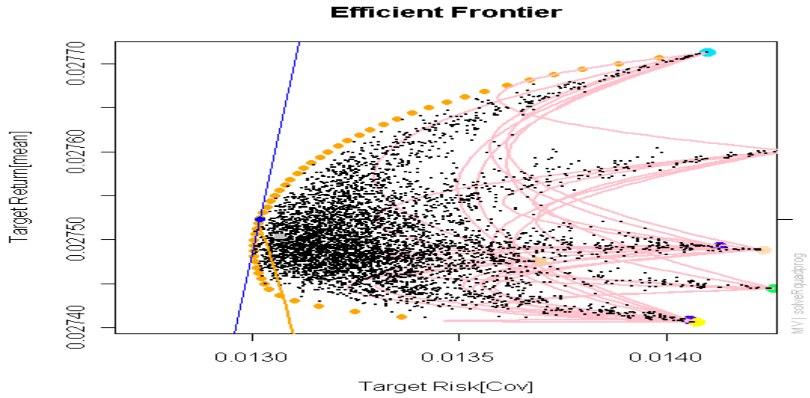

Figure F5: Gumbel Copula Efficient Frontier After Subprime – Lower Tail

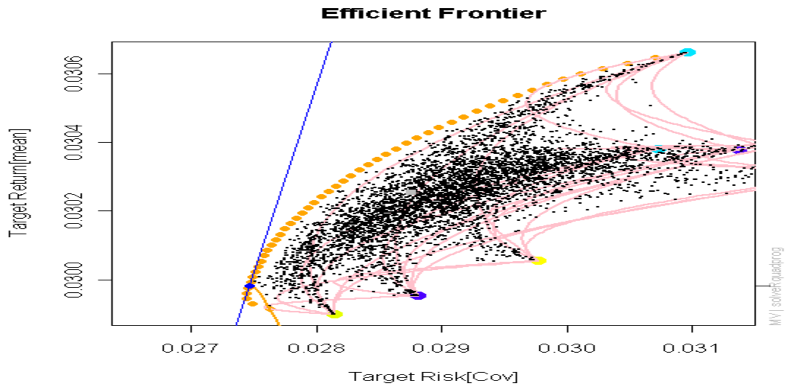

Figure F6: Gumbel Copula Efficient Frontier After Subprime – Upper Tail

APPENDIX G: TANGENT PORTFOLIO WEIGHTS BEFORE THE SUBPRIME CRISIS

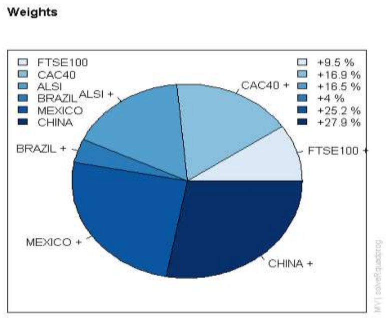

Figure G1: Clayton Copula Efficient Frontier Before Subprime – Upper Tail

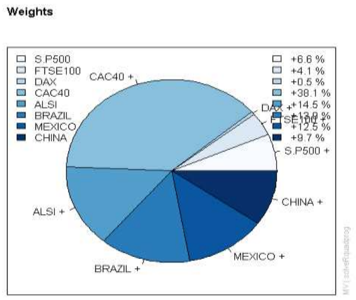

Figure G2: Frank Copula Efficient Frontier Before Subprime – Lower Tail

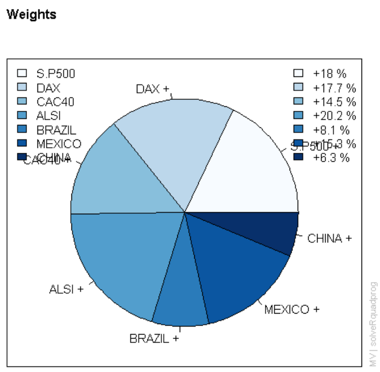

Figure G3: Frank Copula Efficient Frontier Before Subprime – Upper Tail

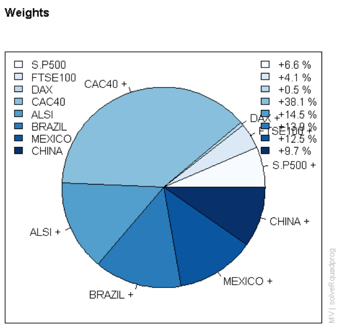

Figure G4: Gumbel Copula Efficient Frontier Before Subprime – Lower Tail

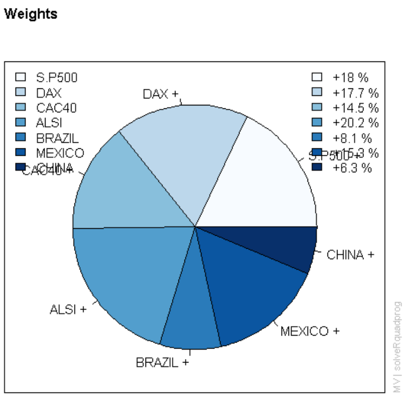

Figure G5: Gumbel Copula Efficient Frontier Before Subprime – Upper Tail

APPENDIX H: TANGENT PORTFOLIO WEIGHTS AFTER THE SUBPRIME CRISIS

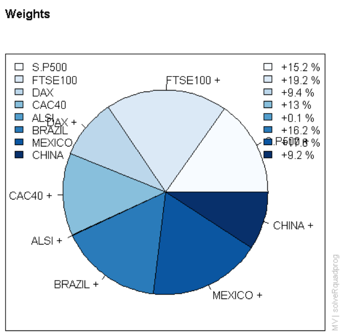

Figure H1: Clayton Copula Efficient Frontier After Subprime – Lower Tail

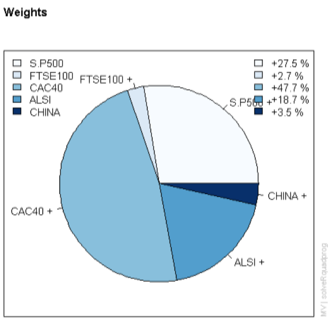

Figure H2: Clayton Copula Efficient Frontier After Subprime – Upper Tail

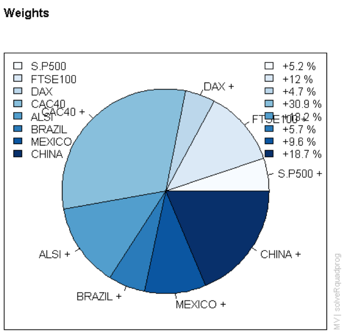

Figure H3: Frank Copula Efficient Frontier After Subprime – Lower Tail

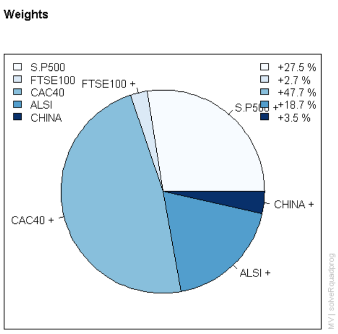

Figure H4: Frank Copula Efficient Frontier After Subprime – Upper Tail

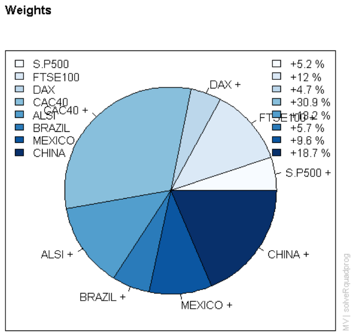

Figure H5: Gumbel Copula Efficient Frontier After Subprime – Lower Tail

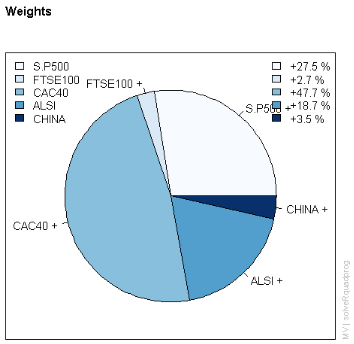

Figure H6: Gumbel Copula Efficient Frontier After Subprime – Upper Tail

APPENDIX I: EFFICIENT FRONTIER WEIGHTS BEFORE THE SUBPRIME CRISIS

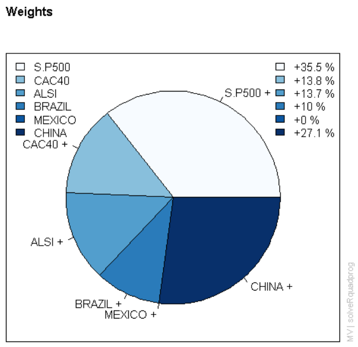

Figure I1: Clayton Copula Efficient Frontier Before Subprime – Lower Tail

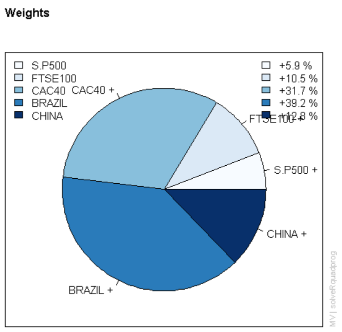

Figure I2: Clayton Copula Efficient Frontier Before Subprime – Upper Tail

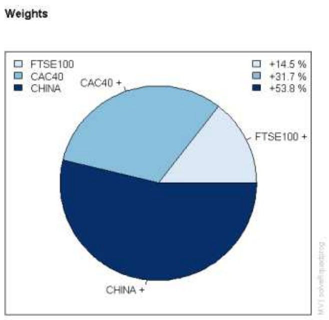

Figure I3: Frank Copula Efficient Frontier Before Subprime – Lower Tail

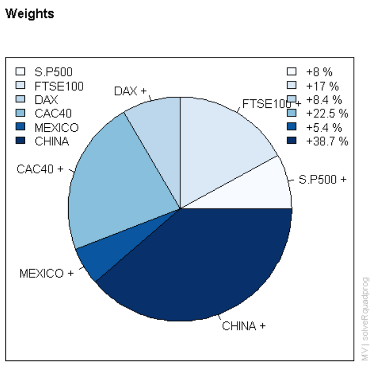

Figure I4: Frank Copula Efficient Frontier Before Subprime – Upper Tail

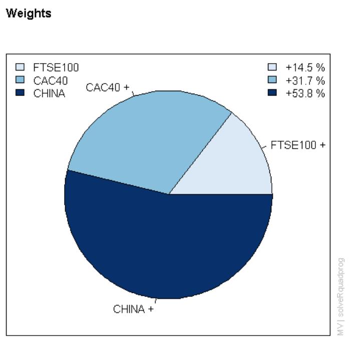

Figure I5: Gumbel Copula Efficient Frontier Before Subprime – Lower Tail

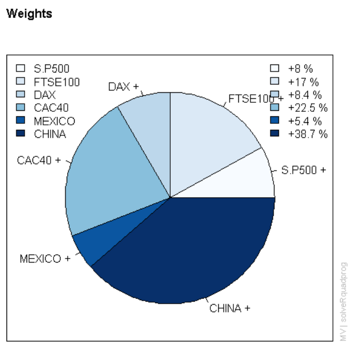

Figure I6: Gumbel Copula Efficient Frontier Before Subprime – Upper Tail

APPENDIX J: EFFICIENT FRONTIER WEIGHTS AFTER THE SUBPRIME CRISIS

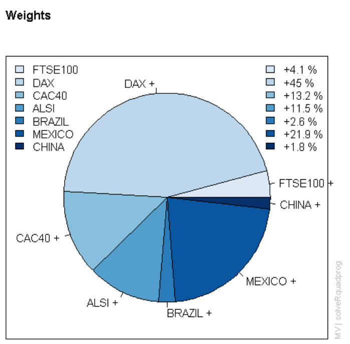

Figure J1: Clayton Copula Efficient Frontier After Subprime – Lower Tail

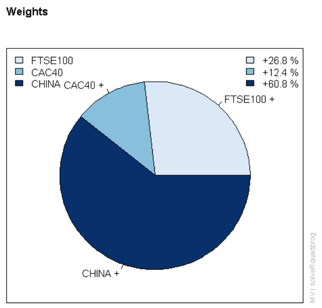

Figure J2: Clayton Copula Efficient Frontier After Subprime – Upper Tail

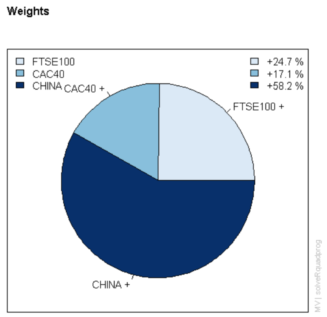

Figure J3: Frank Copula Efficient Frontier After Subprime – Lower Tail

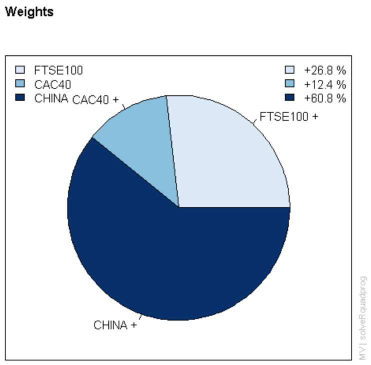

Figure J4: Frank Copula Efficient Frontier After Subprime – Upper Tail

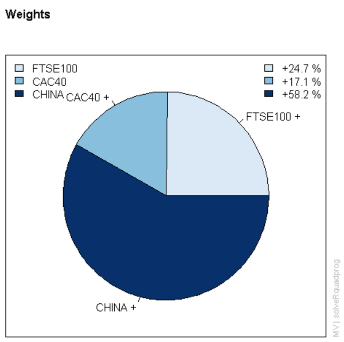

Figure J5: Gumbel Copula Efficient Frontier After Subprime – Lower Tail

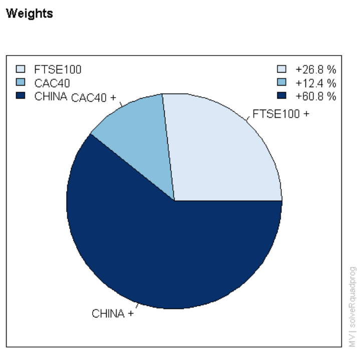

Figure J6: Gumbel Copula Efficient Frontier After Subprime – Upper Tail