Continuous or Discontinuous? Estimating Indifference Curves Inside the Marschak-Machina Triangle using Certainty Equivalents.

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Estimating Indifference Curves Inside the Marschak-Machina Triangle using Certainty Equivalents.

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Abstract

This paper presents results of a study, which shed new light on the shape of indifference curves in the Marschak-Machina triangle. The most important observation concerns (possibly discontinuous) jumps in indifference curves at the triangle legs towards the triangle origin. Such jumps, however, do not appear at the triangle hypotenuse. This points out to discontinuity in the lottery valuation when the range of the lottery outcomes changes. This observation is confirmed by an econometric analysis of six decision-making models: those models, which correctly predict jumps at the triangle legs, offer the best fit of the data collected. Focusing attention to the range of lottery outcomes appears thus one of the most important factors driving decisions under risk. The study has been made using a novel method of estimating indifference curves, which is based on linear interpolation of certainty equivalent values between adjacent points representing the lotteries under consideration.

Keywords: Marschak-Machina triangle, indifference curves, fanning-out, fanning-in, models of decision-making under risk, certainty equivalents

JEL Codes: C81, C91, D81, C14, C88

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1 Introduction

The Marschak-Machina triangle (Marschak, 1950; Machina, 1982) is a graphical tool for both theoretical and experimental considerations concerning the modeling of decision-making under risk. The triangle represents the set of all lotteries involving three fixed outcomes $x_1 < x_2 < x_3$ having probabilities p_1, p_2, and p_3 respectively. Probability p_1 is represented on the horizontal axis; probability p_3 is represented on the vertical axis; and probability p_2 is the residual from 1 (since $p_1 + p_2 + p_3 = 1$). Every point in this triangle represents a particular lottery: a point inside the triangle represents a three-outcome lottery where p_1, p_2 and p_3 are strictly positive; a point on the boundary of the triangle (but not at one of the corners) represents a two-outcome lottery since one p_i is zero; while the corners represent certainties.

A common and useful way to visualize the predictions of the various decision-making models is to inspect their indifference curves, as they connect points representing lotteries of equal utility. If the decision-maker behaves in accordance with Expected Utility Theory, then his or her preferences can be represented in the Marschak-Machina triangle by a set of indifference curves that are parallel straight lines. The triangle became widely popular after EUT paradoxes were explained using the fanning-out hypothesis (Machina, 1982, 1987). Since then, there have been many investigations that have tested hypotheses about the shape of the indifference map. Several surveys (Camerer, 1989; Harley, 1992; Harless and Camerer, 1994; Abdellaoui and Munier, 1998; Blavatskyy, 2006; Bardsley et al., 2000) have shown that Machina's fanning-out hypothesis is too simple.

This led to the development of new theories of decision-making under risk in which the straight line indifference curves fan-out (Chew and MacCrimmon, 1979), fan-in (Blavatskyy, 2006), are a mixture of both (Gul, 1991; Neilson, 1992; Jia et al., 2001; Bordalo, Gennaioli, Schleifer, 2012), or do not converge to any specific point (Dekel, 1986). The indifference curves may be concave (Kahneman and Tversky, 1979), concave or convex (Becker, Sarin, 1987), concave and convex (Tversky, Kahneman, 1992; Birnbaum, 1997). They may also be discontinuous at all boundaries (Kahneman and Tversky, 1979; Viscusi, 1989; Birnbaum, 1997; Bordalo, Gennaioli, Schleifer, 2012) or at the triangle legs only (Cohen, 1992; Kontek, Lewandowski, 2013). The Marschak-Machina triangle, with the indifference curves inside it, is thus a powerful tool to distinguish predictions of different decision-making models.

This tool becomes even more powerful when theoretical predictions are confronted with real data. There are generally two ways to identify indifference curves in experiments: ask indifference questions; ask preference questions. The former involves asking subjects to indicate those lotteries to which they are indifferent vis-à-vis a given one. This procedure allows an indifference curve to be plotted simply by connecting the points representing indifferent lotteries inside the triangle. This, however, is much more difficult to conduct and is rarely used in practice. The latter involves presenting subjects with a set of pairwise choices and asking them to indicate their preferences. Obviously, nothing can be said about the "true" shape of the individual indifference curves of the subjects. All the experimenter can do is test hypotheses regarding the shapes of indifference curves in the triangle or regions of it.

This paper contributes in at least few areas. First, it proposes a new method of estimating indifference curves, which involves indifference questions. Instead of determining lotteries to which people are indifferent vis-à-vis a given lottery, however, lottery certainty equivalents (CE) are determined. The CE values are then used to linearly interpolate the indifference curves inside the triangle. It should be pointed out that determining CEs is a known and widely used method of estimating decision-making models, e.g. it was used by Tversky and Kahneman (1992) to estimate the CPT parameters, and by Gonzales and Wu (1999) to estimate the parameters of the probability weighting function. Quite surprisingly, it has never been used (to the author's knowledge) to determine indifference curves inside the Marschak-Machina triangle.

The second contribution concerns the results of the experiment conducted using the new method. Although these results confirm many previous ones, they shed new light on the shape of indifference curves in the Marschak-Machina triangle. In particular, they show areas of: conformance to EUT; fanning-out; fanning-in; and (possibly discontinuous) jumps in the indifference curves. To the best of the author's knowledge, none of the previous experiments was able to capture all the observations in a single trial. The most important observation concerns jumps in indifference curves at the triangle legs towards the triangle origin. Such jumps, however, do not appear at the triangle hypotenuse. This points out to discontinuity in the lottery valuation when the range of the lottery outcomes changes.

To confirm this observation, an econometric analysis of six decision-making models was made. This included Expected Utility Theory (von Neumann and Morgenstern, 1944), Cumulative Prospect Theory (Tversky and Kahneman, 1992), Prospective Reference Theory (Viscusi, 1989), TAX (Birnbaum, 1997), Decision Utility Theory (Kontek and Lewandowski, 2013), and Salience Theory (Bordalo, Gennaioli, Schleifer, 2012); only the first three theories have been tested in-depth in past studies (e.g. Camerer, 1992; Hey and Orme, 1994). As shown, the best econometric fit of the stated indifference curves has been obtained by the TAX and DUT models, i.e. those that correctly predict jumps at the triangle legs.

Finally, the paper shows that excluding lotteries close to the triangle boundaries from the analysis results in a very much different shape of indifference curves, one that resembles the fanning-out pattern hypothesized by Machina. This raises a general question of how to select an optimal grid of lotteries to discriminate between the decision-making models.

The remainder of the paper is structured as follows. Section 2 presents a new method of estimating indifference curves using lottery CEs. Section 3 presents the experimental results. Section 4 discusses the results in more detail and compares them with the results so far presented in the literature. Section 5 presents estimation results of six decision-making models using the data collected in the experiment. Section 6 summarizes the research.

2 A new method of estimating indifference curves

The new method of estimating indifference curves in the Marschak-Machina triangle is based on determining lottery CEs. These values are further used to interpolate any required indifference curve(s).

The experiment involved 67 lotteries for each of two payoff schedules: $x_1 = 0$ zł, $x_2 = 150$ zł, $x_3 = 300$ zł; and $x_1 = 0$ zł, $x_2 = 450$ zł and $x_3 = 900$ zł (złoty is the Polish currency, \$1 ≈ 3.5 zł).[3] Of these 67 lotteries, 3 were located in the corners of the triangle, 24 on the boundaries, and the remaining 40 in the interior.

To verify the boundary effects and the fanning-out hypothesis, the distribution of lotteries was chosen to be more dense near the triangle boundaries and corners. The lotteries were constructed from the following list of p_1 and p_3 probabilities: {0, 0.01, 0.05, 0.2, 0.4, 0.6, 0.8, 0.95, 0.99, 1}. All combinations $\{p_1, 1 - p_1 - p_3, p_3\}$ such that $1 - p_1 - p_3 \geq 0$ resulted in the lotteries: {0, 1, 0}, {0, 0.99, 0.01}, {0, 0.95, 0.05}, etc. The following lotteries were added to verify the boundary effects close to the hypotenuse: {0.04, 0.01, 0.95}, {0.19, 0.01, 0.8}, {0.39, 0.01, 0.6}, {0.6, 0.01, 0.39}, {0.8, 0.01, 0.19}, {0.95, 0.01, 0.04} all having $p_2 = 0.01$ and {0.1, 0.05, 0.85}, {0.25, 0.05, 0.7}, {0.4 0.05, 0.55}, {0.55, 0.05, 0.4}, {0.7, 0.05, 0.25}, and {0.85, 0.05, 0.1}, all having $p_2 = 0.05$. This set of lotteries is presented as points in the Marschak-Machina triangle in Figure 2.1.

[3] Although the purchasing power for basic goods is closer to identity.

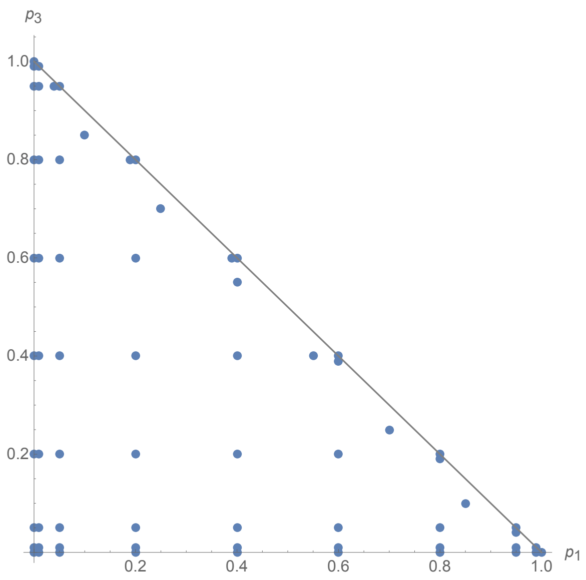

Figure 2.1 The Marschak-Machina triangle with the lotteries examined in the experiment.

The lotteries were presented in the form of urns containing black, gray and white balls (for some lotteries, the balls were only one or two colors). The problems were presented in random order and the monetary value of the balls was randomly changed for two of the payoff schedules. Moreover, for some participants, the black and white balls offered the maximum and minimum payoff respectively, while for other participants, the values of the black and white balls were reversed. The gray ball always offered an intermediate payoff.

To the right of the urn containing the balls of three colors was another urn that only contained balls with crosses. An example problem is demonstrated in Figure 2.2.

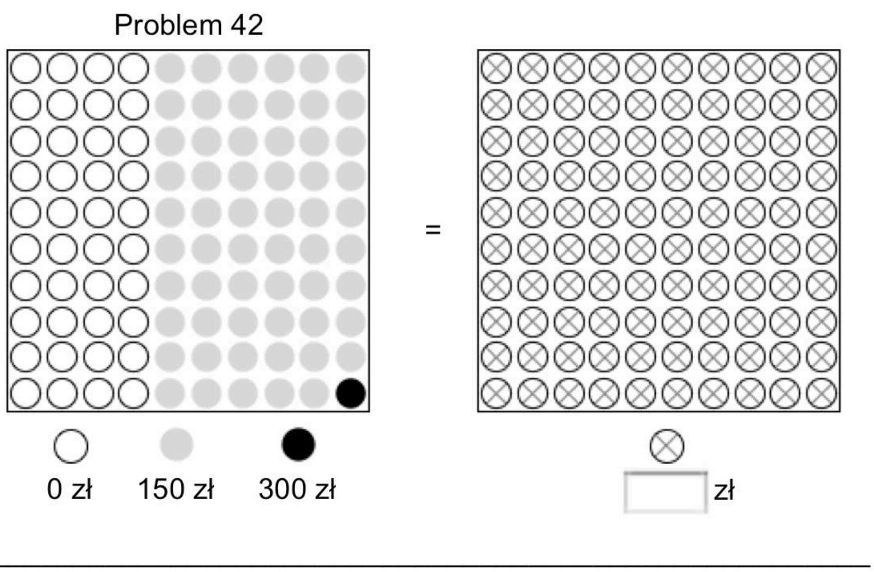

Figure 2.2 An example problem from the experiment.

In this sample problem, the value of the black ball was 300 zł, the gray ball 150 zł, and the white ball 0 zł. The participants had to state the value that a ball with a cross would need to have to make them indifferent between drawing a ball from the left or right urn. The participants thereby determined the CEs of the lotteries presented on the left side of the panel.

Fifty eight subjects took part in the experiment. Thirty four were students of the Warsaw School of Economics and the rest were students of the University of Social Sciences and Humanities in Warsaw. The age of the participants ranged from 19 to 39 years with a median of 24 years and 61% were women.

The experiment was conducted on the website: http://eksperymenty.sgh.waw.pl. Participation was voluntary, but the participants received additional marks in their exams. They were also given a 10-zł voucher that they could redeem in the campus cafeteria or bookstore.

The participants first registered and familiarized themselves with the instructions online. They were then required to solve two sample problems. The time to answer all questions was planned at 40-50 minutes, although the participants were asked to work at their own pace. The average time was about 50 minutes. This way, the value of the voucher (10 zł) exceeded the minimum hourly wage in Poland, which is about 10 zł.

3 Results.

3.1 Aggregating the data.

Median CE values are normally used to analyze data in this kind of experiments as subjects' responses are noisy, skewed and contain a large number of outliers. A trimmed mean of 20% middle CE values for each lottery, however, was also examined so as not to lose the remaining information in the sample. An average of both median and trimmed mean CE values, as a compromise between the two, was used for further analysis.

These median/trimmed mean CE values were finally combined for the two payoff schedules. The aggregated CE value was calculated as the average of the triple CE value for the 0 zł, 150 zł and 300 zł payoffs and the single CE value for the 0 zł, 450 zł and 900 zł payoffs. These aggregated CE values are presented in Table 3-1.

p1	p2	p3	CE
0.	0.	1.	900.0
0.	0.01	0.99	887.9
0.	0.05	0.95	837.3
0.	0.2	0.8	780.2
0.	0.4	0.6	706.8
0.	0.6	0.4	628.7
0.	0.8	0.2	552.7
0.	0.95	0.05	499.3
0.	0.99	0.01	463.8
0.	1.	0.	450.0
0.01	0.	0.99	887.2
0.01	0.04	0.95	828.3
0.01	0.19	0.8	749.7
0.01	0.39	0.6	646.3
0.01	0.59	0.4	598.6
0.01	0.79	0.2	523.5
0.01	0.94	0.05	475.8
0.01	0.98	0.01	450.0
0.01	0.99	0.	443.8
0.04	0.01	0.95	828.0
0.05	0.	0.95	831.8
0.05	0.15	0.8	742.4
0.05	0.35	0.6	625.6

p1	p2	p3	CE
0.05	0.55	0.4	597.8
0.05	0.75	0.2	509.7
0.05	0.9	0.05	450.6
0.05	0.94	0.01	443.8
0.05	0.95	0.	416.6
0.19	0.01	0.8	713.3
0.2	0.	0.8	710.4
0.2	0.2	0.6	616.1
0.2	0.4	0.4	531.8
0.2	0.6	0.2	463.6
0.2	0.75	0.05	431.5
0.2	0.79	0.01	400.1
0.2	0.8	0.	358.8
0.39	0.01	0.6	522.1
0.4	0.	0.6	525.6
0.4	0.2	0.4	445.9
0.4	0.4	0.2	363.9
0.4	0.55	0.05	333.7
0.4	0.59	0.01	313.0
0.4	0.6	0.	276.9
0.6	0.	0.4	362.7
0.6	0.01	0.39	353.9
0.6	0.2	0.2	269.9

p1	p2	p3	CE
0.6	0.35	0.05	244.3
0.6	0.39	0.01	235.2
0.6	0.4	0.	200.0
0.8	0.	0.2	179.3
0.8	0.01	0.19	181.7
0.8	0.15	0.05	128.9
0.8	0.19	0.01	134.6
0.8	0.2	0.	103.8
0.95	0.	0.05	63.9
0.95	0.01	0.04	67.0
0.95	0.04	0.01	58.5
0.95	0.05	0.	49.2
0.99	0.	0.01	21.7
0.99	0.01	0.	20.3
1.	0.	0.	0.0
0.1	0.05	0.85	767.9
0.25	0.05	0.7	637.7
0.4	0.05	0.55	508.7
0.55	0.05	0.4	380.8
0.7	0.05	0.25	254.5
0.85	0.05	0.1	111.9

Table 3-1. Aggregated CE values.

Although the correctness of combining data for the two payoff schedules is debatable, it was employed to further reduce the "noise" and to observe the pattern of the indifference curves, which is common to both ranges. The differences between the two payoff schedules are of less interest, at least for the results presented in this paper. It needs to be added that the use of two payoff schedules in the experiment was primarily intended to avoid restricting subjects to a single set of lottery outcomes, as this might have caused unfavorable distortion in their responses. Combining the data further reduces the danger of detecting any accidental effect.

3.2 Plotting the indifferences curves.

The experimental data was analyzed and visualized using the Wolfram Mathematica program, in particular the ListContourPlot function. This function generates a contour plot from values defined at specific points (the contours are the required indifference curves). The function smoothens the contours by linearly interpolating values between adjacent points. The function also allows arbitrary chosen contours to be plotted, either by setting the number of contours (e.g. by giving the directive "Contours -> 4" to have four contour values calculated automatically), or by setting exact contour values (e.g. by giving the directive “Contours - > {0, 100, 200, 300}” or “Contours -> Range[0, 300, 100]”). The plots presented later in this paper were generated using the directives “Contours -> Range[0, 300, 15]” or “Contours -> Range[0, 900, 45]” depending on the range of payoffs. The separate indifference curves obtained this way for each of the two payoff schedules are visualized in Figure 3.1.

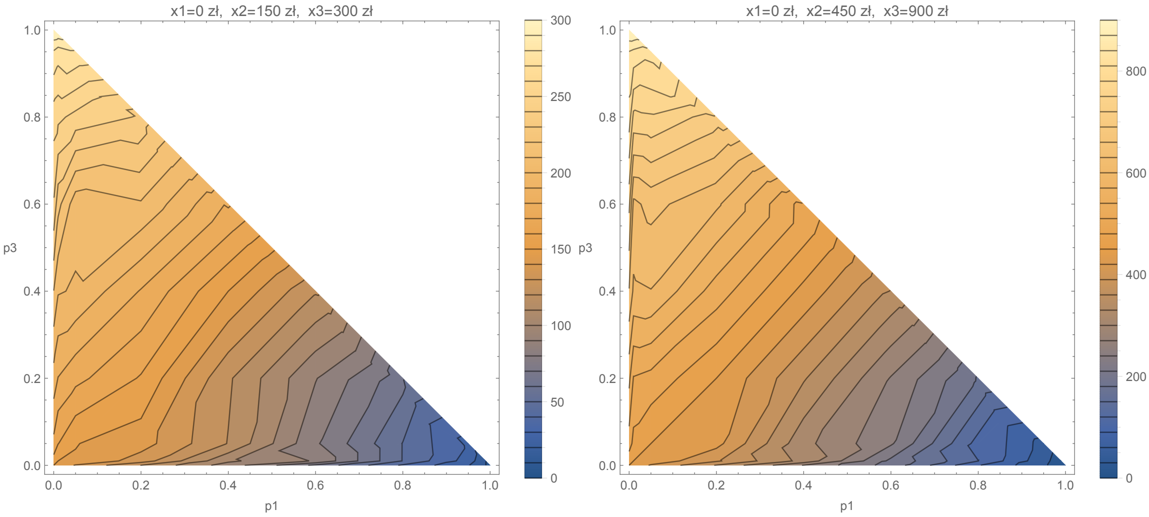

Figure 3.1 Experimental results presented on two Marschak-Machina triangles: left with payoffs x_1 = 0 zł, x_2 = 150 zł, and x_3 = 300 zł; right with x_1 = 0 zł, x_2 = 450 zł and x_3 = 900 zł.

An interesting feature of the method is that indifference curves are expressed in terms of monetary CE values, rather than hypothetical “utils” (see plot legends). The Mathematica® program draws colored contour plots, so that areas of low CE contour values are marked using “cold” colors and areas of high contour values are marked using “warm” colors (in black and white: “dark” and “light” respectively).

The indifferences curves obtained using the combined data (see Table 3-1) are presented in Figure 3.2 (left). These curves have the same general shape as those obtained for the separate payoff schedules, but the quality of the plot is much better and the curves are much smoother. This justifies the use of combined data.

Figure 3.2 (right) illustrates the operation of the Mathematica® program. The mesh lines used to derive the contours are shown. These connect adjacent points representing the lotteries under examination (see dots). As explained, the Mathematica® program linearly interpolates CE values along the mesh lines and uses these interpolated values to plot the required indifference curves.

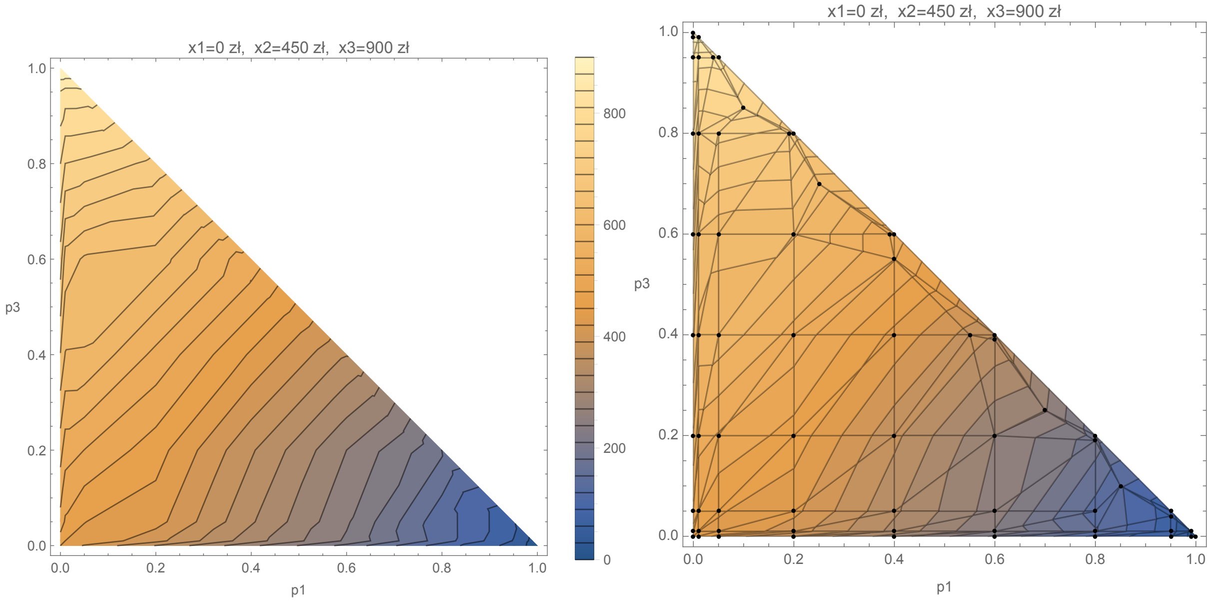

Figure 3.2 Experimental results combined for the two ranges without (left) and with (right) mesh lines.

3.3 Main observations.

Several observations need to be made.

First, the indifference curves seem to be straight parallel lines in the middle of the triangle. This is the area where the subjects' behavior conforms to EUT.

Second, the further north of the origin towards the northwest corner of the triangle, the flatter the slopes of the indifference curves, and the further east of the origin towards the southeast corner of the triangle, the steeper the slopes of the indifference curves. This results in a pattern of "fanning-in" around the two legs of the triangle, especially in the areas near the northwest and southeast corners (this effect is more pronounced in the former case). This pattern not only contradicts the predictions of EUT but also those of other theories consistent with the Machina "fanning-out" hypothesis. At the same time, however, the effect of changing the slope leads to a pattern resembling "fanning-out" in the area around the southwest corner of the triangle.

Third, the indifference curves appear to have jumps in the direction of the origin near the legs of the triangle. This is the area where boundary effects are present. Significantly, these jumps are not present near the hypotenuse. This pattern is only consistent with those few theories that predict discontinuous jumps at the legs of the triangle (but not at the hypotenuse).

4 More detailed analysis

Most of the previous experiments concerning indifference curves were based on preference questions. As such, they only tested hypotheses regarding their shapes and they did not enable any indifference curves to be plotted. While the observations presented in Section 3 confirm many previously obtained results, they also shed new light on the shape of indifference curves in the Marschak-Machina triangle. This Section discusses the results of the experiment in more detail and compares them with the results presented in the literature to date.

4.1 Conformance with EU

As noted, the indifference curves seem to be straight parallel lines in the middle part of the Marschak-Machina triangle, especially where probabilities p_1 and p_3 are both greater than or equal to 0.2. This area is presented separately in Figure 4.1 (the colors are omitted so as to give a better view of the plot details).

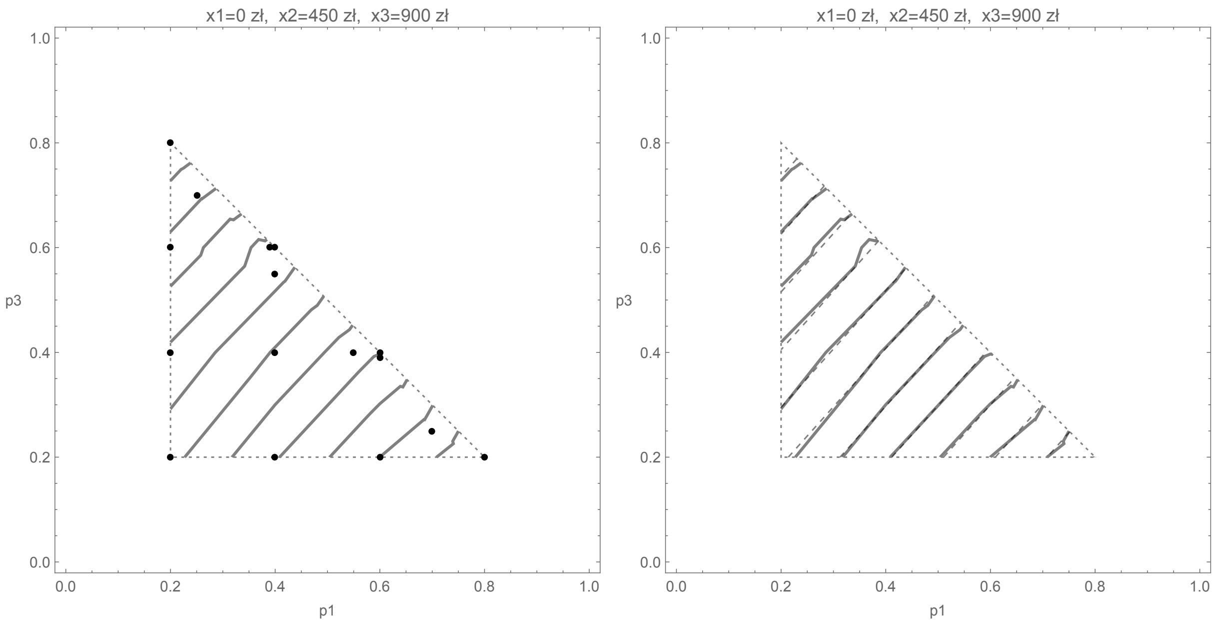

Figure 4.1 Indifference curves in the interior of the Marschak-Machina triangle, where p_1 and p_3 are both greater than or equal to 0.2. Left: with the dots representing the lotteries located in the selected area. Right: with the indifference curves (dashed) predicted by a linear model.

The solid lines represent indifference curves obtained from the experiment. The dashed lines in the right panel represent indifference curves predicted by the model $CE = 408.1p_3 - 455.1p_1 + 465.7$, which is the best-fit linear model estimated using 16 lotteries located within the presented area (see dots in the left panel). The fit is almost perfect (Adjusted $R^2 = 0.9998$).

The above linear model does not, however, provide direct information about the slope of the indifference curves. Therefore, a linear model:

$$p_3 = a + bp_1 + cCE \tag{1.1}$$

in which the required slope is given by the parameter *b* is used in this Section. The minimum least square procedure leads to the fitted model with Adjusted $R^2 = 0.999$ and the parameters presented in Table 4-1. As can be seen, the slope assumes a value of 1.10 and all the parameters are statistically significant.

	Estimate	Standard Error	t-Statistic	P-Value
a	-1.12901	0.0418632	-26.969	8.4983×10^{-13}
b	1.10405	0.0421728	26.1793	1.2425×10^{-12}
c	0.00243408	0.0000554067	43.9312	1.59779×10^{-15}

Table 4-1 ANOVA table for the linear model (1.1) estimated using 16 lotteries located in the area presented in Figure 4.1.

The observation that the EU model works fine for lotteries inside the Marschak-Machina triangle has often been reported in the literature. Hey and Orme (1994) state that the EU model appears to fit no worse than any of the other models for 39% of subjects. Similarly, Carbone and Hey (1994) find that approximately half their subjects appear to conform to the EU model. Hey and Strazzera (1989) additionally find that, for the majority of their subjects, the indifference curves were in accordance with EU theory. Abdellaoui and Munier (1998) show that indifference curves assume very different types of shapes according to their location within the triangle. The authors find that the shape of the indifference curve is compatible with the EU hypothesis along the middle part of the hypotenuse and in the "immediate" interior of that middle part. Their result is close to that obtained in the present experiment.

4.2 "Fanning-out" versus "fanning-in"

As stated in Section 5, the indifference curves seem to show a pattern of "fanning–out" near the triangle origin, and a pattern of "fanning-in" as they approach the other triangle corners. This observation is confirmed by a more detailed analysis.

The area of the Marschak-Machina triangle has been restricted to either $0.01 \le p_1 \le 0.2$ or $0.01 \le p_3 \le 0.2$ to exclude the impact of the boundary effects and central parallelism described in the former sub-section. This area, together with the dots representing lotteries within it, is presented in Figure 4.2.

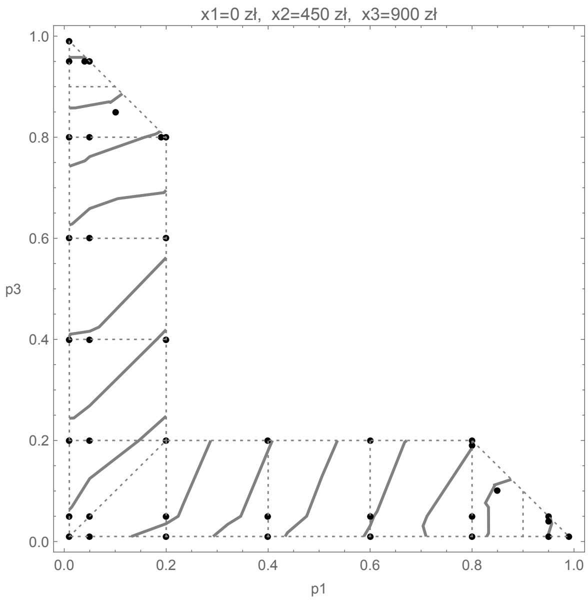

Figure 4.2 Indifference curves in the area where p_1 or p_3 are in the range [0.01, 0.2]. Dots represent lotteries located in the selected area.

The selected area has been further split into smaller regions along the horizontal and vertical legs. A linear regression procedure was performed in each of these regions to obtain a number of linear models (1.1) approximating the indifference curves locally. Estimations of the indifference curve slopes are shown in Table 4-2 and Table 4-3. It should be noted that the slopes along the horizontal leg generally assume greater, and the slopes along the vertical leg smaller, values than those in the middle of the triangle (i.e. 1.10). Moreover, these slope values increase and decrease as they approach the southeast and northwest corners respectively.

0.01≤p3≤0.2	0.01≤p1≤0.2	0.2≤p1≤0.4	0.4≤p1≤0.6	0.6≤p1≤0.8	0.8≤p1≤0.9	0.9≤p1≤1.0
No. of lotteries	6	6	6	7	5	4
Adj. R Squared	0.905	0.920	0.946	0.934	0.946	0.737
Slope	0.69	1.58	1.91	1.90	2.67	3.49
St. deviation	0.14	0.31	0.29	0.31	0.89	2.87
t-Statistics	4.79	5.09	6.49	6.05	3.01	1.21
p-value	0.017	0.015	0.007	0.004	0.095	0.439

Table 4-2. Local estimations of indifference curve slopes along the horizontal leg (where $0.01 \leq p_3 \leq 0.2$).

0.01≤p1≤0.2	0.01≤p3≤0.2	0.2≤p3≤0.4	0.4≤p3≤0.6	0.6≤p3≤0.8	0.8≤p3≤0.9	0.9≤p3≤1.0
No. of lotteries	6	6	6	7	5	4
Adj. R Squared	0.990	0.997	0.988	1.000	1.000	1.000
Slope	0.86	0.88	0.80	0.34	0.28	-0.05
St. deviation	0.09	0.11	0.35	0.06	0.01	0.05
t-Statistics	9.31	7.99	2.30	5.64	35.96	-0.92
p-value	0.003	0.004	0.105	0.005	0.001	0.525

Table 4-3. Local estimations of indifference curve slopes along the vertical leg (where $0.01 \leq p_1 \leq 0.2$).

Local, linear models of the indifference curves were finally extrapolated outside the triangle to show both “fanning-out” and “fanning-in” patterns as in Figure 4.3.

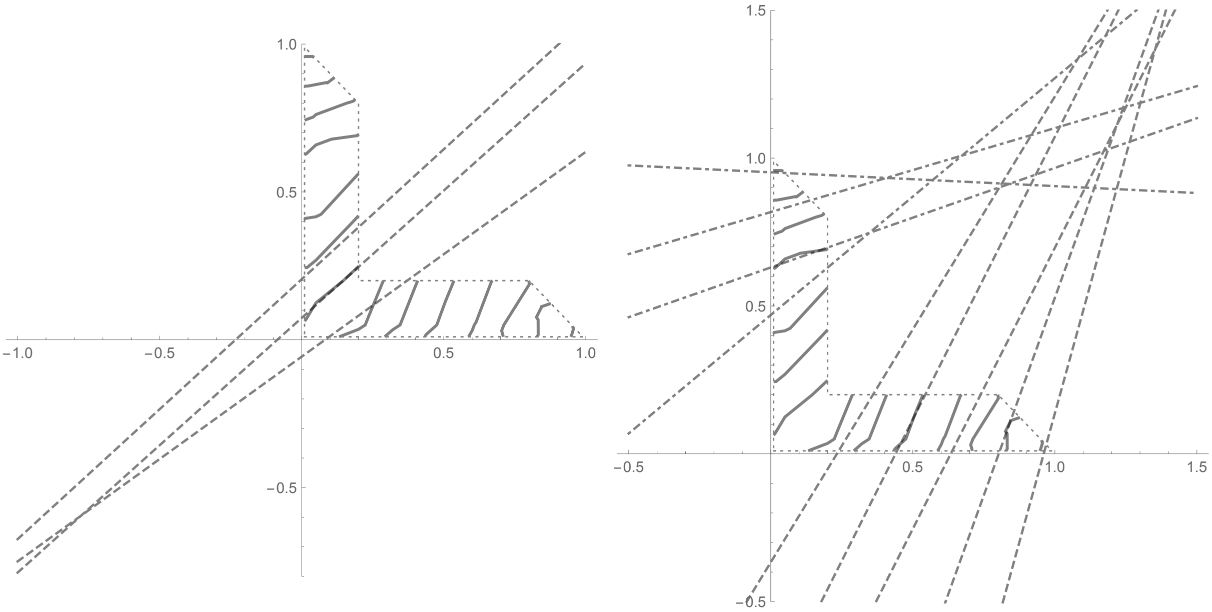

Figure 4.3 Indifference curves estimated locally in the interior of the Marschak-Machina triangle (restricted by either p_1 or p_3 in the range [0.01, 0.2]) and extrapolated outside of the triangle to show the “fanning-out” pattern near the triangle origin (left) and the “fanning-in” patterns near the other triangle corners (right).

The existence of fanning-in for different subjects has been reported in the literature by e.g. Hey and Di Cagno (1990), who observed that the fanning-in point was to the northeast of one the three triangles for 14 subjects. Moreover, the indifference curves fan in for 22 of the 56 subject/triangle pairs. Abdellaoui and Munier (1998) stated that fanning-out is present near the southwest corner and along both legs of the triangle. Although they observed concave indifference lines near the northwest corner and convex indifference curves near the southeast corner, these observations might also confirm the “fanning-in” hypothesis in those areas. Blavatskyy (2006) presents a more detailed study concerning “fanning-out” and “fanning-in”. Blavatskyy claims that a universal fanning-out hypothesis has to be rejected. There is a growing body of evidence to suggest that an individual’s indifference curves tend to fan in when probability mass is associated with the best and the worst outcomes and tend to fan out when probability mass is associated with outcomes in between. The results concerning the fanning-out and fanning-in presented in this paper are clearly close to Blavatskyy’s summary.

4.3 Boundary effects

As stated in Section 5, jumps towards the origin are observed in the indifference curves along both legs of the triangle. These boundary effects are, however, not present along the hypotenuse. So as to better visualize this observation, the area where p_1, p_2, or p_3 is less than or equal to 0.01 is presented in Figure 4.4.

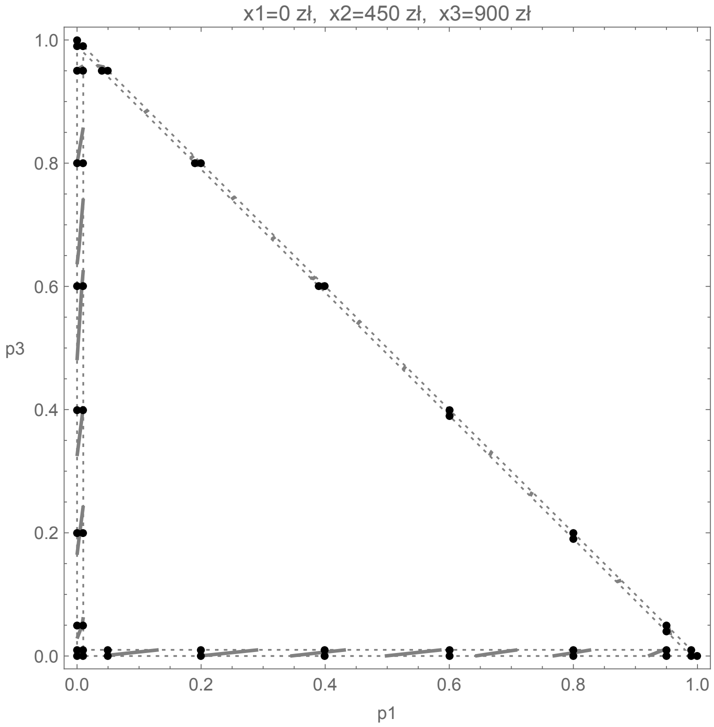

Figure 4.4 Indifference curves in the area where p_1, p_2 or p_3 is less than or equal to 0.01. Dots represent lotteries located in the selected area.

Local, linear approximations of the indifference curves for p_1 and p_3 less than or equal to 0.01 have been made to confirm this observation. The results for the slopes in the regions along the horizontal leg are presented in Table 4-4.

0.0≤p3≤0.01	0.01≤p1≤0.2	0.2≤p1≤0.4	0.4≤p1≤0.6	0.6≤p1≤0.8	0.8≤p1≤1.0
No. of lotteries	6	4	4	4	7
Adj. R Squared	0.744	0.991	1.000	0.991	0.479
Slope	0.10	0.11	0.11	0.15	0.14
St. deviation	0.04	0.01	0.00	0.01	0.08
t-Statistic	2.63	13.35	74.78	14.10	1.71
p-value	0.078	0.048	0.009	0.045	0.162

Table 4-4. Local estimations of indifference curve slopes along the horizontal leg (where $0 \le p_3 \le 0.01$).

As can be seen, the slopes of the indifference curves along the horizontal leg in the slice $0 \le p_1 \le 0.01$ assume values from 0.10 to 0.15. These values differ much from the corresponding values 0.68 to 3.49 obtained in the slice $0.01 \le p_3 \le 0.2$ (cf. Table 4-2). Importantly, except for the regions close to the triangle corners, the estimated slope values are statistically significant for both slices. Despite this, an additional estimation was made for the middle part of the horizontal leg, i.e. for $0.2 \le p_1 \le 0.8$. The results for the two slices are presented in Table 4-5.

0.2≤p1≤0.8	0.0≤p3≤0.01	0.01≤p3≤0.2
No. of lotteries	8	13
Adj. R Squared	0.941	0.939
Slope	0.11	1.65
St. deviation	0.01	0.18
t-Statistic	7.79	9.12
p-value	0.001	3.680×10^{-6}

Table 4-5. Local estimations of indifference curve slopes in the middle part of the horizontal leg ($0.2 \le p_1 \le 0.8$) for slices $0 \le p_3 \le 0.01$ and $0.01 \le p_3 \le 0.2$.

These results confirm the observation regarding a jump towards the origin in the indifference curves along the horizontal leg of the triangle. The indifference curves are steep for $0.01 \le p_3 \le 0.2$ (a slope of 1.65), but very flat for $p_3 \le 0.01$ (a slope of 0.11). These results are statistically significant.

Analogous estimation results for local slopes in regions along the vertical leg are presented in Table 4-6.

0.0≤p1≤0.01	0.01≤p3≤0.2	0.2≤p3≤0.4	0.4≤p3≤0.6	0.6≤p3≤0.8	0.8≤p3≤1.0
No. of lotteries	6	4	4	4	7
Adj. R Squared	0.961	1.000	0.992	0.998	0.999
Slope	5.27	7.84	13.59	10.00	2.32
St. deviation	2.03	0.12	5.73	3.76	2.31
t-Statistic	2.60	63.01	2.37	2.66	1.01
p-value	0.080	0.010	0.254	0.229	0.371

Table 4-6 Local estimations of indifference curve slopes along the vertical leg (where $0 \le p_1 \le 0.01$).

As can be seen, the slopes of the indifference curves along the vertical leg in the slice $0 \le p_1 \le 0.01$ assume values from 2.32 to 13.59. Again, these values differ much from the corresponding values -0.05 to 0.88 obtained for the slice $0.01 \le p_1 \le 0.2$ (cf. Table 4-3). The statistical significance of the obtained slope values is, however, mixed. An additional estimation was therefore made for the middle part of the vertical leg, i.e. for $0.2 \le p_3 \le 0.8$. The results for both slices are presented in Table 4-7.

$0.2 \le p3 \le 0.8$	$0.0 \le p1 \le 0.01$	$0.01 \le p1 \le 0.2$
No. of lotteries	8	13
Adj. R Squared	0.997	0.995
Slope	9.99	0.62
St. deviation	2.09	0.14
t-Statistic	4.79	4.52
p-value	0.005	0.001

Table 4-7. Local estimations of indifference curve slopes in the middle part of the vertical leg ($0.2 \le p_3 \le 0.8$) for slices $0 \le p_1 \le 0.01$ and $0.01 \le p_1 \le 0.2$.

These results likewise confirm the observation regarding a jump towards the origin in the indifference curves along the vertical leg of the triangle. The indifference curves are flat for $0.01 \le p_1 \le 0.2$ (a slope of 0.62), and very steep for $0 \le p_1 \le 0.01$ (a slope of 9.99). These results are also statistically significant.

The evidence concerning boundary effects is present in the literature, however it has not been clearly stated how these effects impact the shape of the indifference curves. Conlisk (1989) moved the Allais chords to the interior of the triangle, which purged the Allais example of the certainty, or double boundary effect. Conlisk concluded that EU theory violations are less frequent and cease to be systematic when boundary effects are removed. Camerer (1989), Harless (1992), and Sopher and Gigliotti (1993) obtained similar results. Harless and Camerer (1994), after analyzing a large number of experimental data sets, conclude that the EU model should be used when all the lotteries have the same number of probable outcomes (i.e. the lotteries are located in the interior of the triangle), but a different model has to be used when the lotteries have different numbers of probable outcomes (i.e. some of the lotteries are located on the boundaries or in the corners of the triangle).

Boundary effects were studied in detail by Abdellaoui and Munier (1998), who stated that indifference curves were distorted near triangle boundaries. They draw a distinction, however, between behavior near different edges of the triangle. One test, restricted to segments linking the hypotenuse to the triangle interior leads to an acceptance of the hypothesis of parallelism. By contrast, the same hypothesis concerning the segments linking the left and lower edges to the interior is strongly rejected. The present experiment not only captures this difference but additionally shows that the distortion near the triangle legs is due to jumps towards the triangle origin in the indifference curves.

4.4 Continuous or discontinuous indifference curves?

All the local estimations of the indifference curve slopes are shown in Figure 4.5.

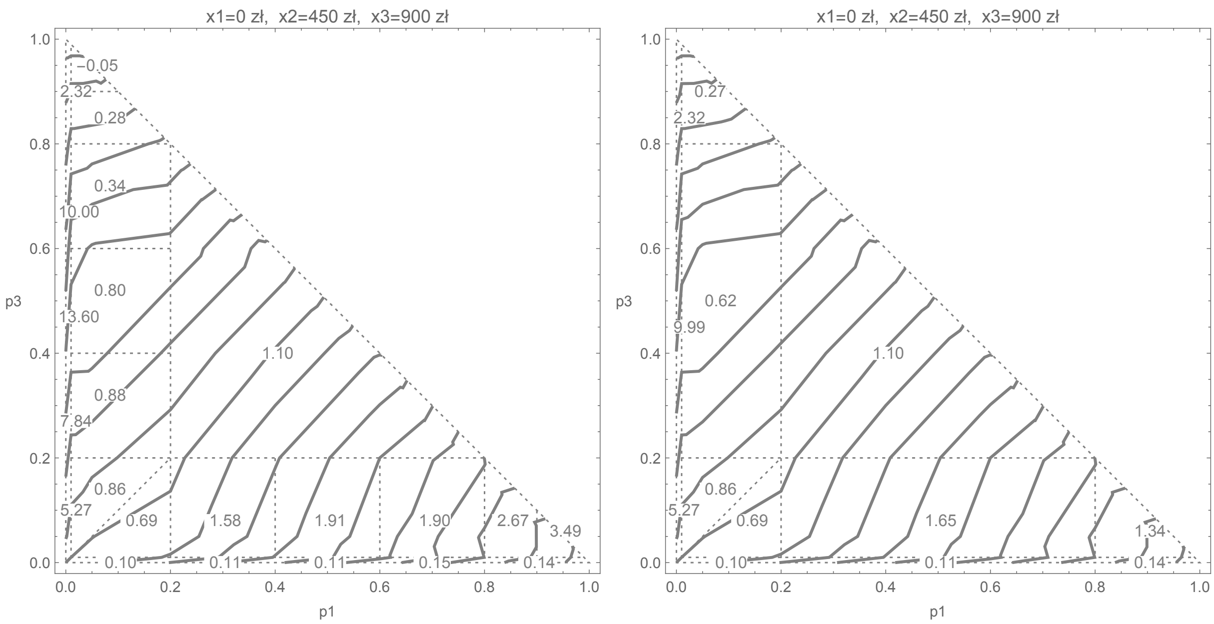

Figure 4.5. Indifference curves obtained from the experiment presented together with local estimations of their slopes.

These data raise the question as to whether these jumps at the triangle legs are continuous or discontinuous. It could be argued that 0.01, the minimum non-zero probability used in the experiment, is still far greater than 0 (at least on a logarithmic scale) and that the indifference curves might become smooth for probabilities of less than 0.01. The hypothesis regarding continuity or discontinuity of the indifference curves at the triangle legs is, however, not falsifiable. Even if the jumps observed in the experiment had occurred at a probability of say 0.001, it could still be argued that this was far greater than 0.

The path of the indifference curves near the two legs of the triangle suggests, however, that the jumps in the curves are discontinuous. It is highly unlikely that the indifference curves, which are parallel as they depart from the hypotenuse and remain so in the middle of the triangle, would first turn away from the origin, and then (somewhere between a probability of 0 and 0.01) smoothly turn back towards it. The indifference curve discontinuity hypothesis would not require such dramatic changes in the slope values: the indifference curves would remain steep near the horizontal leg for any non-zero probability p_3, and remain flat near the vertical leg for any non-zero probability p_1.

The discontinuity of indifference curves is not particularly welcomed by mathematicians and could even be regarded as a weakness in the model. This feature, however, has a solid psychological foundation. First, the tendency to overweight certain outcomes relative to merely probable ones was labeled the “certainty effect” by Kahneman and Tversky (1979). This, however, does not explain the pattern where jumps do not appear at the hypotenuse (to recapitulate: indifference curves are discontinuous at all three triangle boundaries in the original Prospect Theory). This phenomenon can, however, be explained by observing that lotteries located on the legs of the triangle do not have the same support as those located in the rest of the triangle (including the hypotenuse) (Abdellaoui and Munier, 1998). As a result, either $p_1 = 0$ or $p_3 = 0$ implies a more dramatic change in individual behavior than $p_2 = 0$. The same psychological observation underlies those models that predict that changing the security level and/or potential level (or more generally the range) of a lottery results in indifference curves that are discontinuous at the legs of the triangle, but not at the hypotenuse (Cohen, 1992; Kontek, Lewandowski, 2013).

4.5 When might fanning-in and boundary effects not be observed?

It is worth noting that the effects of fanning-in and (possibly discontinuous) jumps might not be observable in experimental set-ups where the lotteries are far from the triangle boundaries. To verify this observation, another analysis of the experimental data was performed. This time, lotteries in the interior close to the legs and the hypotenuse (i.e. having probabilities of either 0.01 or 0.05) were excluded. The resulting set of lotteries is presented in Figure 4.6 (left) and the resulting indifference curves are presented in Figure 4.6 (right).

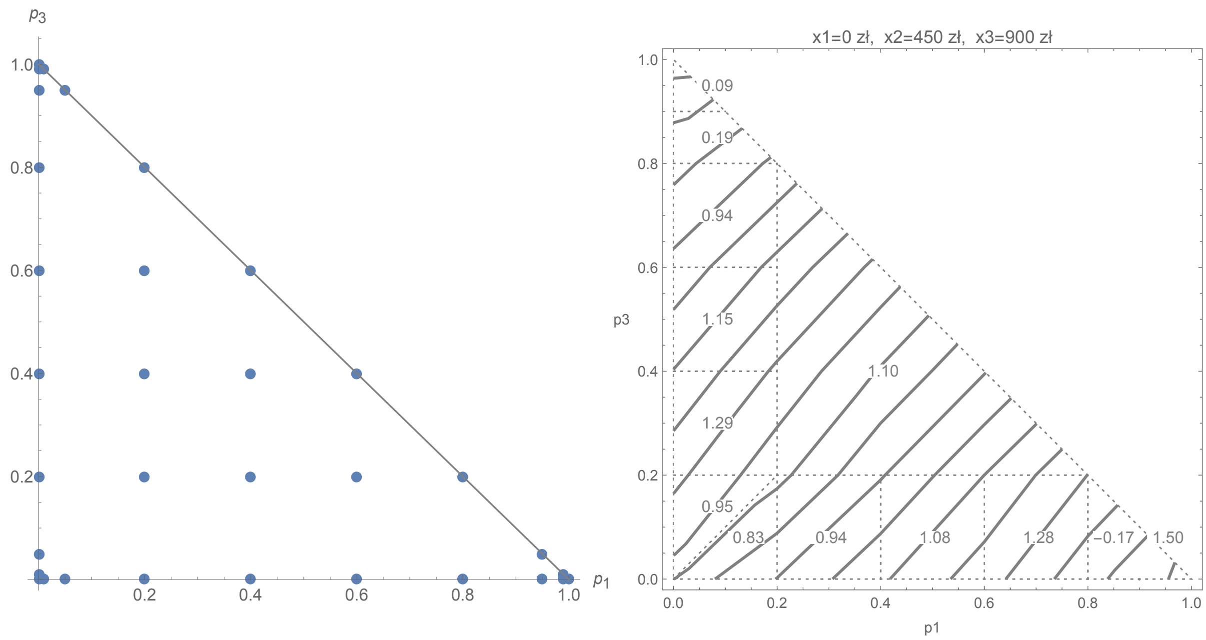

Figure 4.6 Left: lotteries on the Marschak-Machina triangle with those close to the legs and the hypotenuse excluded. Right: the resulting indifference curves presented with the local slope values.

The (discontinuous) jumps have disappeared completely. Fanning-out is observed over a much wider area near the southwest corner and along both triangle legs up to a probability of 0.6 for both p_1 and p_3. Fanning-in can only be seen near the northwest and southeast corners (and even there it is barely visible). The wider presence of the fanning-out pattern shows that the two effects are not equal in magnitude: the (discontinuous) jump towards the triangle origin is greater than the shift of the indifference curve in the opposite direction (fanning-in). The resulting fanning-in pattern is also weaker because the (discontinuous) jump cancels out a large portion of the shift of the indifference curves towards the triangle corners.

The analysis done using the given subset of lotteries would therefore support Machina's fanning-out hypothesis. At the same time, this hypothesis would, in general, be rejected using the full set of lotteries, as presented in previous subsections.

5 Econometric verification

An econometric analysis using the data collected in the experiment has been made to compare decision-making models under risk. Six models are analyzed: Expected Utility Theory (EUT), Prospective Reference Theory (PRT), Cumulative Prospect Theory (CPT), TAX, Decision Utility Theory (DUT), and Salience Theory (ST). The results are very interesting as first, similar surveys did not involve so many lotteries located in the vicinity of the triangle boundaries, second, only the first three models have been tested in-depth in the past.

As before, it is assumed that $x_1 < x_2 < x_3$. A power utility function $u(x) = x^{\alpha}$ is assumed for the first four models. The CE value in the EUT model is calculated using the formula:

$$CE_{EUT} = u^{-1}\left[\sum_{i=1}^{3} u(x_i)\, p_i\right] \tag{1.2}$$

The CE value in the PRT model is calculated using the formula:

$$CE_{PRT} = u^{-1}\left[(1-\beta)\frac{1}{n}\sum_{i=1}^{n} u(x_i) - \beta\sum_{i=1}^{n} u(x_i)\, p_i\right] \tag{1.3}$$

where n denotes the number of outcomes with a probability $p_i > 0$, and β denotes a weighting parameter. The CE value in the CPT model is calculated using the formula:

$$CE_{CPT} = u^{-1}\{u(x_3)w(p_3) + u(x_2)[w(p_3 + p_2) - w(p_3)] + u(x_1)[1 - w(p_3 + p_2)]\} \tag{1.4}$$

The probability weighting function is described using the two-parameter Prelec function:

$$w(p) = Exp\left[-\gamma(-\ln p)^{\delta}\right] \tag{1.5}$$

The formula for the CE value in the TAX model depends on the number of outcomes. For lotteries involving two outcomes, the following formula holds (subscript L denotes the lower outcome and subscript H denotes the higher outcome):

$$CE_{TAX2} = u^{-1}\left[\frac{Au(x_L) + Bu(x_H)}{A+B}\right] \tag{1.6}$$

where:

$$A = t(p_L) + \frac{\delta}{3}t(p_H)$$
$$B = \left(1 - \frac{\delta}{3}\right)t(p_H)$$

and for lotteries involving three outcomes:

$$CE_{TAX3} = u^{-1}\left[\frac{Au(x_1) + Bu(x_2) + Cu(x_3)}{A+B+C}\right] \tag{1.7}$$

where:

$$A = t(p_1) + \frac{\delta}{4}t(p_2) + \frac{\delta}{4}t(p_3)$$
$$B = \left(1 - \frac{\delta}{4}\right)t(p_2) + \frac{\delta}{4}t(p_3)$$
$$C = \left(1 - \frac{\delta}{2}\right)t(p_3)$$

The probability weighting function in the TAX model is described using a power function $t(p) = p^{\gamma}$. The CE value in the DUT model is calculated using the formula:

$$CE_{DUT} = x_1 + (x_3 - x_1)D^{-1}\left[\sum_{i=1}^{3} D\left(\frac{x_i - x_1}{x_3 - x_1}\right)p_i\right] \tag{1.8}$$

where D is the decision utility function defined in the normalized lottery range [0,1]. This function is described using the two-parameter Prelec function:

$$D(r) = Exp\left[-\gamma(-\ln r)^{\delta}\right] \tag{1.9}$$

the same as (1.5), where r denotes the relative position of x_i within the lottery range $[x_1, x_3]$. In the case of two-outcome lotteries the formula (1.8) simplifies to:

$$CE_{DUT2} = x_L + (x_H - x_L)D^{-1}(p_H) \quad (1.10)$$

where x_L and x_H are the lower and the higher outcomes, and p_H is the probability of winning the higher outcome. The preference functional in Salience Theory is context-dependent and depends on all lotteries under consideration. Therefore determining the lottery certainty equivalent is not a straightforward task. Only the formula for a binary lottery is presented below:

$$ce = \begin{cases} x_1 + (x_2 - x_1)\dfrac{p}{p + \delta(1-p)} & if \quad p < \dfrac{\delta}{\delta + A} \\ Ax_1 + \dfrac{(A-1)\theta}{2} & if \quad \dfrac{\delta}{\delta + A} < p < \dfrac{1}{1+\delta A} \\ x_1 + (x_2 - x_1)\dfrac{\delta p}{\delta p + (1-p)} & if \quad p > \dfrac{1}{1+\delta A} \end{cases} \quad (1.11)$$

where $A = \sqrt{\dfrac{2x_2 + \theta}{2x_1 + \theta}}$. Note, that the certainty equivalent assumes here a constant value within a range of probabilities. The expression for three-outcome lotteries, theoretically derived shapes of the indifference curves, and comments to the theory are given by Kontek (2015).

The estimation was performed using the Mathematica® NonlinearModelFit function, which constructs a nonlinear least-squares model and assumes that errors are independent and normally distributed. The estimation results are presented in Table 5-1.

Model	Sum of sq. err.	Adj.R^2	Adj.R^2 norm	AIC	BIC	Parameters			
						Est.value	St.err.	t-stat.	p-value
EUT	42 305.3	0.998	-0.005	626.2	630.6	α = 1.02	0.02	48.50	2.28×10^{-53}
ST	36 008.4	0.998	0.132	617.4	624.0	δ = 0.92	1.88×10^{9}	0.00	1.00
						θ = 10^{6}	0.02	43.10	1.59×10^{-49}
PRT	30 399.3	0.998	0.267	606.0	612.6	γ = 0.98	0.02	54.50	5.86×10^{-56}
						δ = 0.96	0.01	114.00	1.69×10^{-76}
CPT	26 302.4	0.998	0.356	598.3	607.1	α = 1.10	0.07	16.60	6.69×10^{-25}
						γ = 1.06	0.06	19.30	2.55×10^{-28}
						δ = 0.87	0.02	43.70	2.26×10^{-49}
TAX	24 208.9	0.999	0.407	592.8	601.6	β = 1.08	0.03	35.00	2.00×10^{-43}
						γ = 0.96	0.03	29.00	1.59×10^{-38}
						δ = 0.11	0.03	3.47	0.00×10^{-3}
DUT	17 437.9	0.999	0.579	568.8	575.4	γ = 1.03	0.01	69.40	1.10×10^{-62}
						δ = 1.17	0.02	61.60	2.20×10^{-59}

Table 5-1 Estimation results of several decision-making models under risk.

The only “Goodness-of-fit” measure to be explained is “Adj. R2 norm” (in the third column), which informs about the explained variation of the observed difference between CE and Expected Value (EV) for every lottery. This measure assumes much lower values than the standard “Adj. R2” (in the second column), which informs about the explained variation of the observed CE values.

As presented, the two-parameter DUT model offers the best fit and is able to explain 57.9% of the observed variation between the CE and EV values. TAX is the next best despite having three parameters. This model explains 40.7% of the variation. The CPT model, which also has 3 parameters, explains 35.6% of the variation. The PRT and ST models, both with two parameters, go next with respectively 26.7% and 13.2% of the explained variation. As expected, the EUT model having one parameter is the worst. A negative “Adj. R2 norm” value means that, compared to EV, adding a parameter to describe the utility function in the EUT model does not improve the fit to a degree that would justify adding this parameter. This model ranking is also confirmed by AIC and BIC measures.

Figure 5.1 presents the indifference curves predicted by the best-fit EUT, ST, PRT, CPT, DUT and TAX models together with the indifference curves observed in the experiment. The plots enable to observe how the indifference curves predicted by the models match the indifference curves obtained non-parametrically.

As seen, the DUT model predicts discontinuous jumps on both legs towards the triangle origin, while the TAX model predicts discontinuous jumps on the vertical leg only (to be correct, there are very small jumps on the other edges as well because parameter $\delta \neq 0$). This is the reason why these two models best fit the experimental data econometrically. The CPT model does not predict any jumps. The PRT model predicts jumps at all boundaries, and the jumps at the horizontal leg are directed opposite to the triangle origin. In the ST model the jumps are only seen at the hypotenuse[4]. The EUT predicts no jumps.

Note, that jumps at the hypotenuse (in any direction) and at the leg in the direction opposite to the triangle origin point out to the monotonicity violation of the model.

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[4] From the estimation point of view, the biggest problem for the ST model create the indifference curves, which seem to fan-out across the entire triangle. To be correct, the pattern is much more complex and involves also areas of fanning in, as well as areas of constant CE value (see Kontek, 2015).

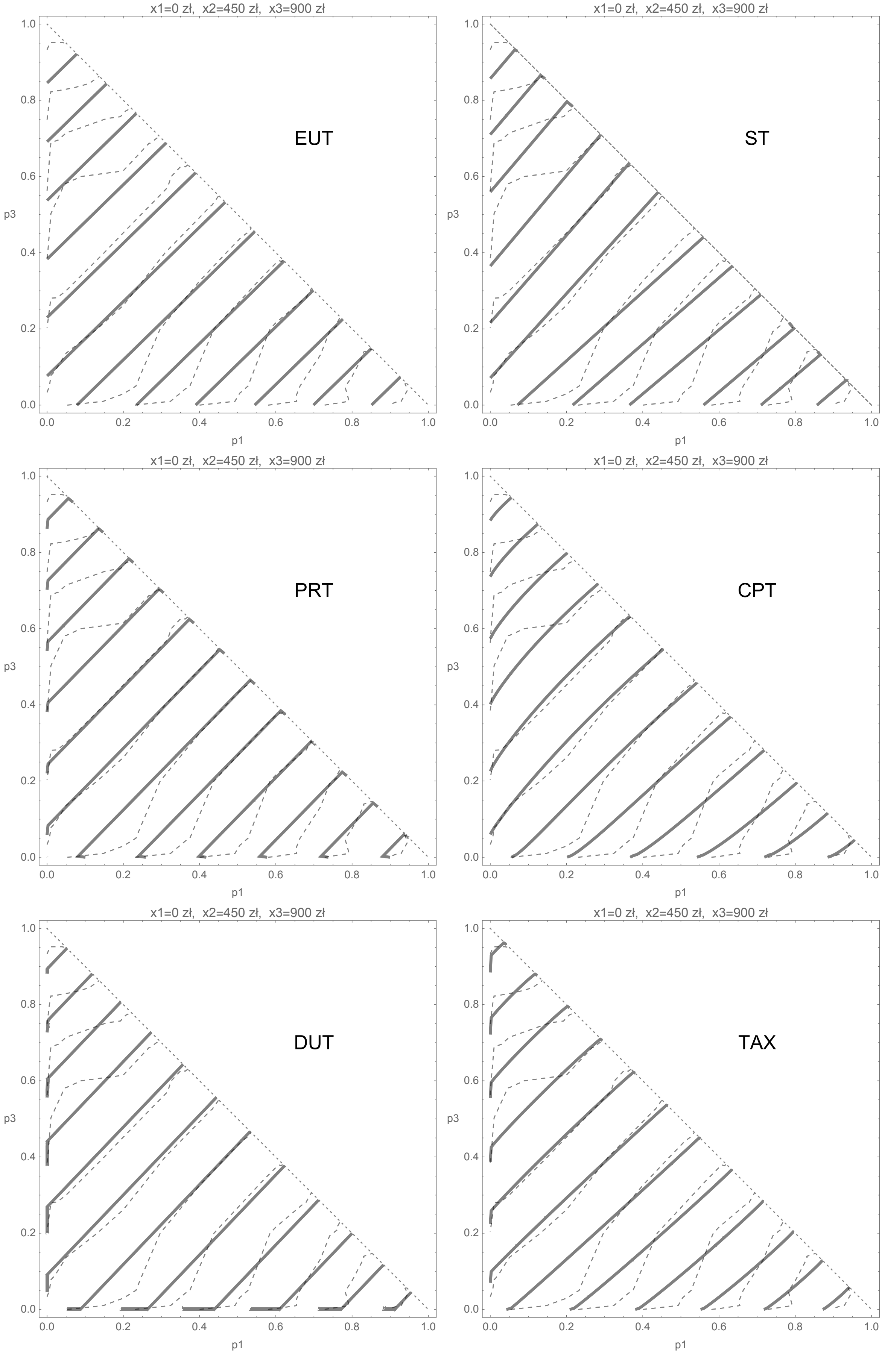

Figure 5.1 Indifference curves obtained non-parametrically (dashed) and predicted by the best-fit models.

The analysis presented above shows that the correct prediction of jumps at the triangle boundaries is the most important factor for the model accuracy, as stated in the estimation results. The results presented in this Section come, however, with a caveat that the fits and the model ranking only apply to the specific grid of lotteries in the Marschak-Machina triangle examined in the experiment. This grid involves many lotteries in the vicinity of the triangle edges, so the model ranking could be quite different for a different grid. The "smooth" indifference curves presented in Section 4.5 confirm that the result may strongly depend on the choice of lotteries. This raises a general question of how to select an optimal grid of lotteries to discriminate between the decision-making models (Cavagnaro et al., 2013), which problem is out of scope of this paper.

6 Summary

This paper proposes a new method of estimating indifference curves inside the Marschak-Machina triangle by using lottery certainty equivalents. This method has several advantages over those methods used to determine indifference curves by asking indifference and preference questions. Compared with methods based on preference questions, the new method (as do all methods based on indifference questions) yields far more information about the true shape of indifference curves. The experimenter thus obtains uniquely determined indifference curves rather than merely the results of hypothesis tests. When compared with methods based on indifference questions, it is much easier to state indifference between a multi-outcome lottery and a certain monetary value than between two multi-outcome lotteries. For this reason, many more questions can be answered during an experiment session. This results in a more precise determination of indifference curves. As it happened, the participants were able to answer 134 questions during a session that lasted less than an hour.

Significantly, interpolating CE values between adjacent points representing lotteries allows any indifference curve to be determined. A specific indifference curve can be chosen even after the experimental results have been collected. This feature stands in stark contrast to those methods where the experimenter is limited to the curves that cross the initially chosen lotteries. Another interesting feature is that indifference curves are expressed in terms of monetary CE values, rather than hypothetical "utils". Last but not least, these indifference curves can be computed using a single function of the well-known Mathematica program. There is no need to write a single line of dedicated software.

More importantly, this paper presents some interesting experimental results concerning the shape of indifference curves in the Marschak-Machina triangle. First, it was stated that indifference curves are straight parallel lines in the middle part of the Marschak-Machina triangle. This is the area of conformance to the Expected Utility model. Second, it was stated that the indifference curves “fan–out” close the triangle origin, but “fan-in” towards the other triangle corners. Third, it was stated that the indifference curves jump towards the origin along the two legs of the triangle. Moreover, as the indifference curves approach the legs of the triangle from the interior, they first turn towards the corners and then jump towards the origin. This suggests that the jumps may be discontinuous. These boundary effects are, however, not present at the hypotenuse. Finally, excluding lotteries close to the triangle boundaries from the analysis, also removes the boundary effects, reduces the fanning-in and extends the fanning-out pattern hypothesized by Machina.

The econometric analysis using the data collected in the experiment shows the importance of the boundary effects in the model comparison: those models, which correctly predict jumps at the triangle legs, offer the best fit of the indifference curves. A corollary of this is the psychological observation that focusing attention on the range of the lottery outcomes is one of the most important factors driving decisions under risk.

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