An existence theorem for restrictions on the mean in the presence of a restriction on the dispersion

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This article analyzes, from the purely mathematical point of view, a general practical problem. The problem consists in the influence of the scatter of experimental data on their mean values (and, possibly, on the probability) near the borders of intervals. The second central moment, the dispersion is a common measure of a scatter. Suppose, for instance, a nonnegative random variable X takes values in a finite interval $[A,B]$. Write M for its mean. If there is a non-zero restriction on a central moment $|E(X\text{-}M)^n|\geq|r^n{}_{Disp.n}|>0$ under the condition $2\leq n<\infty$, then $A<\left(A+\frac{|r^n{}_{Disp.n}|}{(B-A)^{n-1}}\right)\leq M\leq\left(B-\frac{|r^n{}_{Disp.n}|}{(B-A)^{n-1}}\right)<B$.

That is, $\frac{|r^n{}_{Disp.n}|}{(B-A)^{n-1}}>0$ is the width of a non-zero "forbidden zone" for the mean M near a border of the interval. Here, in the case of $n=2$, this non-zero restriction is a restriction on the dispersion $E(X\text{-}M)^2\geq r^2{}_{Disp.2}=\sigma^2{}_{Min}>0$. So, if there is a non-zero restriction on the dispersion, then a non-zero "forbidden zone" exists for the mean near a border of the interval.

Keywords: Mean, dispersion, scatter, scattering, noise, probability, economics, utility theory, prospect theory, decision theories, human behavior,

Contents

1. Introduction

This article analyzes, from the purely mathematical point of view, a general practical problem. The problem is the influence of a scatter, scattering of experimental data on the mean value of a characteristic of a real object. One of the next goals of the research is to extend this consideration to the probability.

The second central moment, the dispersion is a common measure of a scattering. The scattering can be caused by noise and/or uncertainty, measurement errors, etc.

Suppose that a characteristic of a real object is described by the values of a function f whose domain consists of a set of points within a finite interval of the real numbers. Suppose further that f is non-negative, a dispersion of f is defined and is non-zero. It is proved that there are certain geometric non-zero restrictions which the mean of f must satisfy.

By a “non-zero restriction on the mean,” we will refer to the existence of a non-zero distance from a border of the interval. Within this distance, it is impossible for the mean of the function to be located. In other words, this distance is the width of a non-zero “forbidden zone” near a border of the interval. The “restriction” for one border and the “restriction” for another border constitute the “restrictions” for the borders.

Here, the non-zero dispersion of the function models the consequence of existence of real scattering. More rigorously, the non-zero dispersion signifies that the minimal dispersion of the function is limited (restricted) from below by a non-zero value. In other words, this signifies “a non-zero restriction on the dispersion.”

The size of the non-zero “forbidden zone” for the mean of the function is determined by the value of this minimal dispersion (restriction on the dispersion) of the function. The greater the scattering (due to noise and/or uncertainty, measurement errors, etc.), for a minimal dispersion, the larger the “forbidden zone.”

The article presents the first part of a whole research program. The reasons that initially motivated this research were practical problems of decision, utility and prospect theories, and human behavior.

These problems play a fundamental part in utility and prospect theories. Their analysis was started by Bernoulli in [1] in 1738. Examples include the Allais paradox [2], the Ellsberg paradox [3], etc. In 2002 Kahneman received the Prize in Economic Sciences in Memory of Alfred Nobel for research in this field. In 2006, Kahneman and Thaler [4] pointed out that preference inconsistencies in such problems have still not been adequately explained.

One possible way to explain these problems has been widely discussed, e.g., in [5], [6]. Its essence consists in a proper attention to the widespread noise, imprecision, and other reasons that may cause scattering of real data.

The essential feature of these problems is their intense manifestation near the borders of the scale of probability (see, e.g., [7]). Steingrimsson and Luce [8] and Aczél and Luce [9] emphasized a fundamental question: whether Prelec's weighting function is equal to 1 at the border $p=1$ of the probability scale). This question opens one more way which consists in paying proper attention to borders, boundaries and interfaces.

The presented research program synthesizes these two ways. That is, it considers the influence of a scattering of the data on the mean and probability near the borders of intervals, particularly near the borders of the scale of probability.

An analysis of the synthesis of these two ways has led to purely mathematical purposes. The purely mathematical purposes lead to possible explanation of not only the initial particular problems but of a general practical problem (of the influence of a scattering of experimental data on the mean value of a characteristic of a real object) as well.

The ultimate particular practical objectives are possible explanations of the problems of utility and prospect theories. Human behavior is a very complicated phenomenon. To be more thorough and unbiased, the objective of the research is to explain the problems by methods as purely mathematical as possible.

This article is an applied one. Its general methods are analytical ones.

The data used in the research are extracted from the literature on economic experiments, and are of a very general character. In essence, they evidence that there is an inalienable non-zero scattering of data (due to noise and uncertainties) and that a wide use of probability takes place in economic researches, especially in utility and prospect theories.

In this article, these data are used to appropriately choose the general properties of the function (to use it further for the probability estimation and probability) and the condition of the existence of a non-zero dispersion of the function. The methods for drawing conclusions are mainly mathematical proofs.

This article deals with the existence theorem for non-zero restrictions on the mean of a nonnegative function defined on a discrete set within a finite interval in the presence of a non-zero analog of a central moment of the function.

The immediate purely mathematical particular result of the article is this existence theorem. The theorem can and should be applied in both mathematics and practice. Applications are possible in probability theory, general science, industry and business, utility and prospect theories.

2. Preliminary notes

2.1. The function

Let us specify the properties of the main function of this article.

Definition 2.1. Let us suppose given:

a) an interval $X=[A, B]$ satisfying

$$0 < Const_{Min.AB} \le (B - A) \le Const_{Max.AB} ,$$

b) a set of points $\{x_k\} : A \le x_k \le B,\ k=1, 2, \dots K : 2 \le K \le \infty$,

c) a function f_K (a set of values $\{f_K(x_k)\}$), defined on $\{x_k\}$, satisfying

$$0 \le f_K(x_k) \quad \text{and} \quad \sum_{k=1}^{K} f_K(x_k) = W_K ,$$

where W_K (the total weight of f_K) is a constant satisfying

$$0 < W_K .$$

This function f_K will be referred to as the “original” function or the “original” set.

Note, for the first sketches of the theorem, a continuous function was preliminary considered (see, e.g., [10]). A discrete function is preferable for the purposes of the first parts of the whole research.

Without loss of generality, f_K may be and is normalized so that $W_K=1$. Under this condition, this function will be referred to as a unitary function.

2.2. Analog of the moment

Let us define an analog of a moment.

Definition 2.2. An analog of the moment of n^{th} order of the function f_K relative to a point x_0 is the expression

$$E(X-X_0)^n \equiv \frac{1}{W_K}\sum_{k=1}^{K}(x_k-x_0)^n f_K(x_k) = \sum_{k=1}^{K}(x_k-x_0)^n f_K(x_k).$$

From now on, for brevity, I refer to this analog of the moment of n^{th} order as simply the moment of n^{th} order.

Further, let us suppose the moment of first order, the mean $M \equiv E(X)$ of the function f_K exists

$$E(X) \equiv \frac{1}{W_K}\sum_{k=1}^{K}x_k f_K(x_k) = \sum_{k=1}^{K}x_k f_K(x_k) \equiv M\ .$$

Furthermore, let us suppose at least one central moment $E(X\text{-}M)^n : 2 \le n < \infty$, of the function f_K exists

$$E(X-M)^n = \frac{1}{W_K}\sum_{k=1}^{K}(x_k-M)^n f_K(x_k) = \sum_{k=1}^{K}(x_k-M)^n f_K(x_k).$$

3. Maximality

Let us search for a function which attains the maximal possible modulus of a central moment. It is intuitively evident that the maximal possible absolute value of a central moment is obtained for the function which is concentrated at the borders of the interval. Nevertheless, for the sake of mathematical rigor, this statement must be proved.

3.1. Pairs

Let us consider two possible values of f_K.

Consider three points x_A, x_B, M satisfying $A \le x_A \le M \le x_B \le B$. Consider two values $f_K(x_A)$ and $f_K(x_B)$ such that $(M - x_A) f_K(x_A) = (x_B - M) f_K(x_B)$ and $f_K(x_A) + f_K(x_B) = w = Const_w$.

Definition 3.1. Given a constant point M as above, two values $f_K(x_A)$ and $f_K(x_B)$ of f_K are called a "pair" or a "couple,"

$$f_{Pair}(x_A, x_B) \equiv \{ f_2(x_A), f_2(x_B) \} ,$$

if they satisfy:

a)

$$A \le x_A \le M \le x_B \le B ,$$

b)

$$(M - x_A) f_K(x_A) = (x_B - M) f_K(x_B) ,$$

c) and for a given constant total weight w_{Pair} or simply w, we have

$$f_K(x_A) + f_K(x_B) \equiv w_{Pair} \equiv w .$$

Further, the values of the pairs can be also referred to as "elements" and "elements of pairs."

It is evident that the pair (or couple) $\{f_K(x_A), f_K(x_B)\}$ is an example of the original function f_K, defined in Chapter 2, with $K=2$ and $w_{Pair} \equiv W_K$.

The central moment $E_{Pair}(X-M)^n$ of this pair is

$$E_{Pair}(X - M)^n \equiv (x_A - M)^n f_2(x_A) + (x_B - M)^n f_2(x_B) .$$

Its absolute value is limited by the sum of the absolute values of their elements

$$\begin{aligned} &| E_{Pair}(X - M)^n | \le | (x_A - M)^n f_2(x_A) | + | (x_B - M)^n f_2(x_B) | = \\ &= (M - x_A)^n f_2(x_A) + (x_B - M)^n f_2(x_B) \end{aligned} .$$

3.2. Limiting functions

Let us determine a limiting, bounding function for $E_{Pair}(X\text{-}M)^n$. After replacing $f_K(x_B)$ from the expressions of the weight w_{Pair} and mean point M by

$$f_K(x_B) = \frac{M - x_A}{x_B - M} f_K(x_A) = w_{Pair} - f_K(x_A)$$

and

$$f_K(x_A) + f_K(x_B) = \frac{M - x_A + x_B - M}{x_B - M} f_K(x_A) = $$
$$= \frac{x_B - x_A}{x_B - M} f_K(x_A) = w_{Pair} ,$$

one may replace $f_K(x_A)$ and $f_K(x_B)$ by functions of x_A, M, x_B and w_{Pair}

$$f_K(x_A) = \frac{x_B - M}{x_B - x_A} w_{Pair} \quad \text{and} \quad f_K(x_B) = \frac{M - x_A}{x_B - x_A} w_{Pair}$$

and obtain

$$| E_{Pair}(X - M)^n | \le (M - x_A)^n f_K(x_A) + (x_B - M)^n f_K(x_B) =$$
$$= (M - x_A)^n \frac{x_B - M}{x_B - x_A} w_{Pair} + (x_B - M)^n \frac{M - x_A}{x_B - x_A} w_{Pair} \equiv \quad .$$
$$\equiv (M - x_A)^n \frac{x_B - M}{x_B - x_A} w + (x_B - M)^n \frac{M - x_A}{x_B - x_A} w$$

Definition 3.2. One may define a limiting function $L_{Pair}(x_A, M, x_B, n, w_{Pair})$ or, abbreviated, $L(x_A, M, x_B, n, w)$ which depends only on x_A, M, x_B, n, w_{Pair}

$$L_{Pair}(x_A, M, x_B, n, w_{Pair}) \equiv L(x_A, M, x_B, n, w) \equiv$$
$$\equiv (M - x_A)^n \frac{x_B - M}{x_B - x_A} w_{Pair} + (x_B - M)^n \frac{M - x_A}{x_B - x_A} w_{Pair} \quad .$$

Note, here M, n, and w are parameters, and x_A, x_B may range over the interval.

The absolute value of a central moment, say $|E_{Pair}(X\text{-}M)^n|$, of the pair is, evidently, limited (bounded) by this limiting function $L_{Pair}(x_A, M, x_B, n, w_{Pair})$

$$| E_{Pair}(X - M)^n | \le L_{Pair}(x_A, M, x_B, n, w_{Pair}) \ .$$

3.3. Derivatives. Search for the maximum

Let us find the maximum of the limiting function $L_{Pair}(x_A, M, x_B, n, w_{Pair})$ for x_A and x_B.

3.3.1. Differentiation with respect to x_A

Let us differentiate $L_{Pair}(x_A, M, x_B, n, w_{Pair})$ with respect to x_A

$$\frac{\partial L(x_A, M, x_B, n, w)}{\partial x_A} =$$

$$= \frac{\partial\left((M - x_A)^n \frac{x_B - M}{x_B - x_A} w + (x_B - M)^n \frac{M - x_A}{x_B - x_A} w \right)}{\partial x_A} =$$

$$= \{[-n(x_B - x_A) + (M - x_A)](M - x_A)^{n-1} +$$

$$+ [-(x_B - x_A) + (M - x_A)](x_B - M)^{n-1}\} \frac{(x_B - M)}{(x_B - x_A)^2} w =$$

$$= \{[(M - x_A) - n(x_B - x_A)](M - x_A)^{n-1} - (x_B - M)^n\} \frac{(x_B - M)}{(x_B - x_A)^2} w \quad .$$

At $n \geq 1$, if $(M-x_A)<(x_B-x_A)$, that is, if $x_B>M$, then

$$(M - x_A) - n(x_B - x_A) < 0$$

and (keeping in mind $x_B-x_A>0$)

$$\frac{\partial L(x_A, M, x_B, n, w)}{\partial x_A} < 0 \, .$$

So, at $n \geq 1$, for $M<x_B \leq B$ (and, as can easily be seen, for $A \leq x_A<M$) the first derivative with respect to x_A is strictly less than zero. That is, for $A \leq x_A<M<x_B \leq B$ or for $[A, B]$ except for the specific point M, we have

$$L(A, M, x_B, n, w) > L(x_A, M, x_B, n, w) \ .$$

If $(M-x_A)=(x_B-x_A)$, that is, if $x_B=M$, then from

$$(M - x_A) f_K(x_A) = (x_B - M) f_K(x_B) \ ,$$

we obtain

$$(M - x_A) = (M - M) \frac{f_K(x_B)}{f_K(x_A)} = 0$$

or $x_A=M$.

To include the specific point M into the ranges of variation of the arguments x_A and x_B of this inequality, let us estimate the derivative $\partial L(x_A, M, x_B, n, w)/\partial x_A$ for both $x_A \to M$ and $x_B \to M$. One may impose some natural conditions of finite values of elements: $0<Const_{Min}\leq f_K(x_A)\leq Const_{Max}$ and $0<Const_{Min}\leq f_K(x_B)\leq Const_{Max}$ (and, hence, of finite value of their sum w).

Let, say, M-x_A be the basic term. Then

$$(x_B - M) = \frac{f_K(x_A)}{f_K(x_B)}(M - x_A)$$

and

$$x_B - x_A = (x_B - M) + (M - x_A) = \left(\frac{f_K(x_A)}{f_K(x_B)} + 1\right)(M - x_A) = \frac{w}{f_K(x_B)}(M - x_A)\ .$$

If $x_A \to M$ then the derivative

$$\{[(M - x_A) - n(x_B - x_A)](M - x_A)^{n-1} - (x_B - M)^n\}\frac{(x_B - M)}{(x_B - x_A)^2}w =$$

$$= \left\{\left[1 - n\frac{w}{f_K(x_B)}\right] - \left(\frac{f_K(x_A)}{f_K(x_B)}\right)^n\right\}(M - x_A)^n\,\frac{f_K(x_A)}{f_K(x_B)}\left(\frac{f_K(x_B)}{w}\right)^2 w(M - x_A)^{-1} = .$$

$$= \left\{\left[1 - n\frac{w}{f_K(x_B)}\right] - \left(\frac{f_K(x_A)}{f_K(x_B)}\right)^n\right\}\frac{f_K(x_A)\,f_K(x_B)}{w}(M - x_A)^{n-1} \xrightarrow[n>1;\ x_A \to M]{} 0$$

So (at $n>1$, if M-x_A tends to 0, then the derivative)

$$\frac{\partial L(x_A, M, x_B, n, w)}{\partial x_A} \xrightarrow[n>1;\ x_A \to M]{} 0\ .$$

Therefore, for $A\leq x_A\leq M\leq x_B\leq B$, the derivative $\partial L(x_A, M, x_B, n, w)/\partial x_A \leq 0$.

Let us include the point M into the ranges of variation of the arguments x_A and x_B of the inequality $L(A, M, x_B, n, w)>L(x_A, M, x_B, n, w)$. Let us consider an intermediate point, say $x_A=(A+M)/2$.

If, for $A\leq x_A\leq M\leq x_B\leq B$, the derivative $\partial L(x_A, M, x_B, n, w)/\partial x_A\leq 0$, then, for $A\leq x_A\leq M\leq x_B\leq B$, the function $L(x_A, M, x_B, n, w)\geq L(M, M, x_B, n, w)=L(M, M, M, n, w)$ (and $L((A+M)/2, M, x_B, n, w)\geq L(M, M, M, n, w)$).

If, for $A\leq x_A<M<x_B\leq B$, the derivative $\partial L(x_A, M, x_B, n, w)/\partial x_A<0$ then, for $A<x_A<M<x_B\leq B$, the function $L(A, M, x_B, n, w)>L(x_A, M, x_B, n, w)$ and $L(A, M, x_B, n, w)>L((A+M)/2, M, x_B, n, w)$.

Therefore,

$$L(A, M, x_B, n, w) > L\left(\frac{A+M}{2}, M, x_B, n, w\right) \geq L(M, M, M, n, w)$$

or

$$L(A, M, x_B, n, w) > L(M, M, M, n, w) \ .$$

We have included the specific point M into the ranges of variation of arguments of the inequality $L(A, M, x_B, n, w)>L(x_A, M, x_B, n, w)$ and the inequality is true for $A\leq x_A\leq M\leq x_B\leq B$.

So, at $n>1$, the limiting function $L_{Pair}(x_A, M, x_B, n, w_{Pair})$ has a maximum $L_{Pair}(A, M, x_B, n, w_{Pair})$ for x_A at $x_A=A$ for the total interval $[A, B]$.

3.3.2. Differentiation with respect to x_B

Let us differentiate $L_{Pair}(x_A, M, x_B, n, w_{Pair})$ with respect to x_B

$$\frac{\partial L(x_A, M, x_B, n, w)}{\partial x_B} =$$

$$= \frac{\partial\left((M - x_A)^n \frac{x_B - M}{x_B - x_A} w + (x_B - M)^n \frac{M - x_A}{x_B - x_A} w \right)}{\partial x_B} =$$

$$= \{[(x_B - x_A) - (x_B - M)](M - x_A)^{n-1} + \quad .$$

$$+ [n(x_B - x_A) - (x_B - M)](x_B - M)^{n-1}\} \frac{(M - x_A)}{(x_B - x_A)^2} w =$$

$$= \{(M - x_A)^n + [n(x_B - x_A) - (x_B - M)](x_B - M)^{n-1}\} \frac{(M - x_A)}{(x_B - x_A)^2} w$$

At $n \geq 1$, if $(x_B-x_A)>(x_B-M)$, that is, if $x_A<M$, then

$$n(x_B - x_A) - (x_B - M) > 0$$

and (if $x_B-x_A>0$)

$$\frac{\partial L(x_A, M, x_B, n, w)}{\partial x_B} > 0 \, .$$

If $(x_B-x_A)=(x_B-M)$, that is, if $x_A=M$, then $x_B=M$ (see above).

So, at $n \geq 1$, for $A \leq x_A<M<x_B<B$ the first derivative with respect to x_B is strictly greater than zero. That is, for $A \leq x_A<M<x_B<B$ or for $[A, B]$ except for the specific point M, we have

$$L(x_A, M, x_B, n, w) < L(x_A, M, B, n, w) \, .$$

To include the specific point M into the ranges of variation of the arguments x_A and x_B, let us estimate the derivative $\partial L(x_A, M, x_B, n, w)/\partial x_B$ for both $x_B \rightarrow M$ and $x_A \rightarrow M$ under the same natural conditions of finite values of elements as we imposed before: $0<Const_{Min} \leq f_K(x_A) \leq Const_{Max}$ and $0<Const_{Min} \leq f_K(x_B) \leq Const_{Max}$ (and, hence, of finite value of their sum w).

Let, say, x_B-M be the basic term. Then

$$(M - x_A) = \frac{f_K(x_B)}{f_K(x_A)}(x_B - M)$$

and

$$x_B - x_A = \left(1 + \frac{f_K(x_B)}{f_K(x_A)}\right)(x_B - M) = \frac{w}{f_K(x_A)}(x_B - M) \, .$$

If $x_B \rightarrow M$, then the derivative

$$\{(M-x_A)^n + [n(x_B-x_A)-(x_B-M)](x_B-M)^{n-1}\}\frac{(M-x_A)}{(x_B-x_A)^2}w =$$

$$= \left\{\left(\frac{f_K(x_B)}{f_K(x_A)}\right)^n + \left[n\frac{w}{f_K(x_A)}-1\right]\right\}\frac{f_K(x_B)}{f_K(x_A)}\left(\frac{f_K(x_A)}{w}\right)^2 w(x_B-M)^{n-1} = \quad .$$

$$= \left\{\left(\frac{f_K(x_B)}{f_K(x_A)}\right)^n + \left[n\frac{w}{f_K(x_A)}-1\right]\right\}\frac{f_K(x_B)f_K(x_A)}{w}(x_B-M)^{n-1} \xrightarrow[n>1;\ x_B\to M]{} 0$$

So (for $n>1$, if x_B (and x_A) tend to M, then)

$$\frac{\partial L(x_A, M, x_B, n, w)}{\partial x_B} \xrightarrow[n>1;\ x_B\to M]{} \ .$$

Let us include the specific point M into the ranges of variation of the arguments x_A and x_B of the inequality $L(x_A, M, B, n, w) > L(x_A, M, x_B, n, w)$. Let us consider an intermediate point, say $x_B = (M+B)/2$.

If, for $A \le x_A \le M \le x_B \le B$, the derivative $\partial L(x_A, M, x_B, n, w)/\partial x_B \ge 0$ then, for $A \le x_A \le M \le x_B \le B$, the function $L(x_A, M, M, n, w) = L(M, M, M, n, w) \le L(x_A, M, x_B, n, w)$ (and $L(M, M, M, n, w) \le L((x_A, M, (M+B)/2, n, w))$.

If, for $A \le x_A < M < x_B \le B$, the derivative $\partial L(x_A, M, x_B, n, w)/\partial x_B > 0$ then, for $A \le x_A < M < x_B < B$, the function $L(x_A, M, x_B, n, w) < L(x_A, M, B, n, w)$ and $L((A+M)/2, M, x_B, n, w) < L(x_A, M, B, n, w)$.

Therefore,

$$L(M, M, M, n, w) \le L\left(x_A, M, \frac{A+M}{2}, n, w\right) < L(x_A, M, B, n, w)$$

or

$$L(M, M, M, n, w) < L(x_A, M, B, n, w) \ .$$

We have included the specific point M into the ranges of variation of arguments of the inequality $L(x_A, M, x_B, n, w) < L(x_A, M, B, n, w)$ and the inequality is true for $A \le x_A \le M \le x_B \le B$.

So, at $n>1$, the limiting function $L_{Pair}(x_A, M, x_B, n, w_{Pair})$ has a maximum $L_{Pair}(x_A, M, B, n, w_{Pair})$ for x_B at $x_B=B$ for the total interval $[A, B]$.

So, at $n>1$, the limiting function $L_{Pair}(x_A, M, x_B, n, w_{Pair})$ has a maximum $L_{Pair}(A, M, B, n, w_{Pair})$ for x_A at $x_A=A$ and for x_B at $x_B=B$ for the total interval $[A, B]$.

3.3.3. The maximum

So, at $n>1$, for $A \leq x_A \leq M \leq x_B \leq B$, the limiting function

$$L(x_A, M, x_B, n, w) = (M - x_A)^n \frac{x_B - M}{x_B - x_A} w + (x_B - M)^n \frac{M - x_A}{x_B - x_A} w$$

attains its maximum at the borders of the interval $[A, B]$

$$Max(L_{Pair}(x_A, M, x_B, n, w_{Pair})) = L_{Pair}(A, M, B, n, w_{Pair}) =$$
$$= (M - A)^n \frac{B - M}{B - A} w_{Pair} + (B - M)^n \frac{M - A}{B - A} w_{Pair} \quad .$$

So, the absolute value $|E_{Pair}(X-M)^n|$ of a central moment of the pair of values $\{f_2(x_A), f_2(x_B)\}$ is limited by the maximal limiting function, that is concentrated at the borders $x_A=A$ and $x_B=B$ of the interval $[A, B]$

$$| E_{Pair}(X - M)^n | \leq L_{Pair}(A, M, B, n, w_{Pair}) =$$
$$= (M - A)^n \frac{B - M}{B - A} w_{Pair} + (B - M)^n \frac{M - A}{B - A} w_{Pair} \, .$$

3.4. Representation by pairs. Succession of situations

3.4.1. Preliminary considerations

Let us analyze whether the total weight and central moments of any original function f_K of Chapter 2 can be exactly represented by those of a set of pairs.

In fact, the function f_K defined in Chapter 2 is a set of values $\{f_K(x_k)\}$ defined on the set of points $\{x_k\}$. A pair $f_{Pair}(x_A, x_B) = \{f_2(x_A), f_2(x_B)\}$ defined in this chapter is the original function f_K defined in Chapter 2 and satisfying $K = 2$. This pair is also the set of two values $\{f_2(x_A), f_2(x_B)\}$. If there are P pairs then one can denote the p^{th} pair as

$$f_{p.Pair}(x_{p.A}, x_{p.B}) \equiv \{ f_2(x_{p.A}), f_2(x_{p.B})\} \ .$$

(The multiple notation, e.g. $x_{p.A}$, is used to avoid numerous three-storey indices in the text).

Let us analyze whether the total weight and central moments of any function f_K of Chapter 2 as of a set of values $\{f_K(x_k)\} : k=1, \ldots, K : K\geq 2$, can be exactly represented by the total weight and central moments of a set of some P pairs of values $\{f_{p.Pair}(x_{p.A}, x_{p.B})\}=\{f_2(x_{p.A}), f_2(x_{p.B})\} : p=1, \ldots, P : P\geq 1$, of the same function.

Let us mention the linearity of the total weight and moments.

The total weight

$$W_K = \sum_{k=1}^{K} f_K(x_k) \, ,$$

and the moments

$$E(X - X_0)^n = \sum_{k=1}^{K} (x_k - x_0)^n f_K(x_k) \, .$$

of a set $\{f_K(x_k)\}$ depend linearly on the values $f_K(x_k)$ of the members of this set. The sum is a linear function also. Therefore:

1) the total weight of the sum equals the sum of the weights and

2) a moment of the sum equals the sum of the moments.

Therefore, the total weight and moments of the set $\{f_{p.Pair}(x_{p.A}, x_{p.B})\}$ of pairs are equal to the corresponding sums of the weights and moments of the pairs of this set. The sum of the central moments of the pairs $f_{p.Pair}(x_{p.A}, x_{p.B})$ is limited by the sum of the maximal limiting functions $L_{p.Pair}(A, M, B, n, w_{p.Pair})$ (those are linear functions of $f_K(x_k)$ as well) of these pairs. One can see, indeed, that if for one pair

$$|E_{p.Pair}(X-M)^n| \le L_{p.Pair}(A, M, B, n, w_{p.Pair}) \ ,$$

then for P pairs

$$\sum_{p.Pair=1}^{P.Pair} |E_{p.Pair}(X-M)^n| \le \sum_{p.Pair=1}^{P.Pair} L_{p.Pair}(A, M, B, n, w_{p.Pair}) \ .$$

The final goal of this chapter is to exactly represent the modulus of any central moment of any original function f_K of Chapter 2 by a sum of moduli of central moments of such pairs of the same function and to estimate this sum by the limiting functions.

3.4.2. Situations

Let us divide the points x_k into three groups:

1) $x_{k.A}<M$,
2) $x_{k.M}=M$ (zero central moment(s)),
3) $x_{k.B}>M$.

Let us introduce the numbers $K.A$, $K.M$ and $K.B$, such that $k.A \le K.A$, $k.M \le K.M$, $k.B \le K.B$ and

$$K.A + K.M + K.B = K \ .$$

Owing to $x_{k.M}-M \equiv 0$, an arbitrary non-zero central moment depends only on $K.A$ and $K.B$. Let us consider in turn situations with various numbers $K.AB \equiv K.A+K.B$ from $K.AB=0$ to the general situation.

Situation $K.AB=0$

Due to the condition $K\geq 2$ of Chapter 2 and $K.M\leq 1$, the case $K.AB<1$ cannot exist.

Nevertheless, let us consider optionally more general (or fictitious) cases of $K=1$ and of mutually coincident points $\{x_{k.M}=M\} : k.M=1, ..., K.M : K.M\geq 2$.

If $K.AB=0$, then only one point M (or mutually coincident points $\{x_{k.M}=M\}$) and the corresponding value $f_K(M)$ (or the values $f_K(x_{k.M})$) can exist. Evidently, the value $f_K(M)$ (or the values $f_K(x_{k.M})$) do not contribute to the non-zero central moments.

All the mutually coincident points $\{x_{k.M}=M\}$ (or the single point) may be represented as only one aggregated point $x_{Aggr.M}=M$ and the corresponding value

$$f_{K.Aggr}(M) \equiv \sum_{k.M=1}^{K.M} f_K(x_{k.M}) \ .$$

We may formally divide the value $f_{K.Aggr}(x_{Aggr.M})\equiv f_K(M)$ into two parts $f_{K,1}(M)$ and $f_{K,2}(M)$ satisfying $f_{K,1}(M)=f_{K,2}(M)=f_K(M)/2$. The two values $f_{K,1}(M)$ and $f_{K,2}(M)$ are the required pair $f_{Pair}(M, M)$ of the previous subchapters of this chapter. The balance formally remains

$$(M-M)\,f_{K,1}(M) = (M-M)\,f_{K,2}(M)$$

or

$$(M-M)\frac{f_K(M)}{2} = (M-M)\frac{f_K(M)}{2} \ .$$

Evidently, the total weight of the formal pair $f_{Pair}(M, M)$ equals the total weight of the value $f_K(M)$ (or the values $f_K(x_{k.M})$). The central moments equal zero for both the pair $f_{Pair}(M, M)$ and the value $f_K(M)$ (or the values $f_K(x_{k.M})$). So, the total weight and central moments of the value $f_K(M)$ (or the values $f_K(x_{k.M})$) can be exactly represented by a pair of the previous subchapters.

Further, as a rule, we will not consider the point(s) $x_k=M$ and corresponding value $f_K(M)$ (values $f_K(x_{k.M})$).

Situation *K.AB=1*

Here, only two possible cases can take place: the case *K.A=1* and *K.B=0* or the case *K.A=0* and *K.B=1*.

Generally, the first central moment

$$\sum_{k=1}^{K}(x_k - M)\,f_K(x_k) \equiv 0$$

may be transformed to

$$\sum_{k=1}^{K}(x_k - M)\,f_K(x_k) = \sum_{k.A\le K.A}(x_{k.A} - M)\,f_K(x_{k.A}) + $$
$$+ \sum_{k.M\le K.M}(x_{k.M} - M)\,f_K(x_{k.M}) + \sum_{k.B\le K.B}(x_{k.B} - M)\,f_K(x_{k.B}) = 0 \;,$$

where the limits of the sums *k.A≤K.A*, *k.M≤K.M* and *k.B≤K.B* denote, that *K.M* or *K.A* or *K.B* can equal zero. That is, generally, there can be cases with no members of the sum(s) of *k.M* or *k.A* or *k.B*.

Now, since

$$x_{k.M} - M \equiv 0 \;,$$

this central moment may be transformed to the balance

$$\sum_{k.A\le K.A}(M - x_{k.A})\,f_K(x_{k.A}) = \sum_{k.B\le K.B}(x_{k.B} - M)\,f_K(x_{k.B}) \;.$$

Suppose *K.A=1* and *K.B=0*. Then

$$\sum_{k.A\le K.A}(M - x_{k.A})\,f_K(x_{k.A}) = 0 \;.$$

There are only two possible cases: $f_K(x_{k.A})>0$ and $f_K(x_{k.A})=0$. Evidently, for *K.AB=1*, the case $f_K(x_{k.A})>0$ cannot exist. If $f_K(x_{k.A})=0$ then the balance can formally hold, but this case does not contribute to the non-zero central moments $E(X\text{-}M)^n>0$.

The consideration of the case *K.A=0* and *K.B≥1* is fully analogous to the preceding one.

So, the case *K.A=0* and *K.B≥1* and the case *K.A≥1* and *K.B=0* either cannot occur or do not contribute to the non-zero central moments $E(X\text{-}M)^n>0$.

So, Situation *K.AB=1* cannot occur or does not contribute to the non-zero central moments.

Further, as a rule, we will not consider those cases that do not contribute to the non-zero central moments, namely $f_K(x_k)=0$ and $f_K(M)$.

Situation *K.AB=2*

Here, the only possible case which contributes to the non-zero central moments, is the case *K.A=1* and *K.B=1*.

If *K.A=1* and *K.B=1*, then

$$(M - x_{1.A}) f_K(x_{1.A}) = (x_{1.B} - M) f_K(x_{1.B}) \ .$$

Therefore, $f_K(x_{1.A})$ and $f_K(x_{1.B})$ are the required pair of the previous subchapters.

Here, the set of values $\{f_K(x_{1.A}), f_K(x_{1.B})\}$ of the pair $f_{1.Pair}(x_{1.A}, x_{1.B})$ and the set of values $\{f_K(x_{1.A}), f_K(x_{1.B})\}$ of the original function f_K are the same sets. Therefore, the total weight of the pair is the same as that of the function.

Moreover, the set of values $\{f_K(x_{1.A}), f_K(x_{1.B})\}$ of the pair $f_{1.Pair}(x_{1.A}, x_{1.B})$ and the set of values $\{f_K(x_{1.A}), f_K(x_{1.B})\}$ of the original function f_K are the same sets of values defined on the same sets of points $\{x_{1.A}, x_{1.B}\}$. Therefore, an arbitrary total moment of the pair is the same as that of the function.

This can be seen, indeed, in more detail for an arbitrary total moment:

$$\begin{aligned}
&E(X - M)^n = \\
&= \sum_{k.A=1}^{K.A} (x_{k.A} - M)^n f_K(x_{k.A}) + \sum_{k.B=1}^{K.B} (x_{k.B} - M)^n f_K(x_{k.B}) = \\
&= \sum_{k.A=1}^{1} (x_{k.A} - M)^n f_K(x_{k.A}) + \sum_{k.B=1}^{1} (x_{k.B} - M)^n f_K(x_{k.B}) = \\
&= (x_{1.A} - M)^n f_K(x_{1.A}) + (x_{1.B} - M)^n f_K(x_{1.B}) = \\
&= E_{1.Pair}(X - M)^n
\end{aligned} \ .$$

So, the total weight and central moments of Situation *K.AB=2* can be exactly represented by the total weight and central moments of a pair of the previous subchapters.

Remark 3.3

Let us further, for definiteness, enumerate the points $x_{k.A}$ and $x_{k.B}$, for example, from those furthest from M and with maximal weights, to those closest to M and with minimal weights.

Situation *K.AB=3*

Here, there are only two possible cases those can contribute to the non-zero central moments: the case of *K.A=2* and *K.B=1*, or the case of *K.A=1* and *K.B=2*.

If, for example, *K.A=2* and *K.B=1*, then the value $f_K(x_{1.B})$ can be exactly divided into two parts $f_{K,1}(x_{1.B})$ and $f_{K,2}(x_{1.B})$ satisfying

$$(M - x_{1.A})\, f_K(x_{1.A}) = (x_{1.B} - M)\, f_{K,1}(x_{1.B})$$

and

$$f_{K,2}(x_{1.B}) = f_K(x_{1.B}) - f_{K,1}(x_{1.B}) \ .$$

Definition 3.4. We will define a “divided” or “exactly divided” set.

Let us suppose given an original, initial set of values $\{f_K(x_k)\}$, as in Chapter 2.

A divided or exactly divided set (with respect to the initial set) is defined as a modification of the initial set such that at least one value $f_K(x_k)$ is exactly divided into, at least, two parts $f_{K,1}(x_k)$ and $f_{K,2}(x_k)$ satisfying

$$f_K(x_k) \equiv f_{K,1}(x_k) + f_{K,2}(x_k)$$

or

$$f_K(x_k) \equiv f_{K,1(k)}(x_k) + f_{K,2(k)}(x_k)$$

or, more generally,

$$f_K(x_k) \equiv \sum_{d(k)=1}^{D(k)} f_{K,d(k)}(x_k) \ ,$$

where $2 \le D(k) \le \infty$.

More generally, every member $f_K(x_k)$ (that will be either divided or not divided in the divided set) of the initial set $\{f_K(x_k)\}$ may be written via the members $f_{K,d(k)}(x_k)$ of the exactly divided set $\{f_{K,d(k)}(x_k)\}$, by definition, as

$$f_K(x_k) \equiv \sum_{d(k)=1}^{D(k)} f_{K,d(k)}(x_k) \ ,$$

where $1 \le D(k) \le \infty$.

Note, that a divided set can serve as the new initial set for a subsequent division, i.e., modification.

Definition 3.5. A “divided member” is defined as one of the members of the divided set. A “divided value” is defined as the value of one of these members.

Let us consider the total weight and moments of an exactly divided set $\{f_{K,d(k)}(x_k)\}$.

A weight is a sum of values. The sum of the divided values $f_{K,d(k)}(x_k)$ for a value $f_K(x_k)$

$$f_K(x_k) = \sum_{d(k)=1}^{D(k)} f_{K,d(k)}(x_k)$$

is a linear function with respect to the values $f_{K,d(k)}(x_k)$. Due to the linearity of the total weight for $f_{K,d(k)}(x_k)$ (see the Preliminary consideration above), the total weight of the sum of divided values $f_{K,d(k)}(x_k)$ is equal to $f_K(x_k)$.

The sum of moments of the divided values $f_{K,d(k)}(x_k)$ for a value $f_K(x_k)$ is a linear function with respect to the values $f_{K,d(k)}(x_k)$. The divided values $f_{K,d(k)}(x_k)$ for an initial value $f_K(x_k)$ are defined on the same point x_k as the initial value $f_K(x_k)$. Therefore, the moments of the whole divided set $\{f_{K,d(k)}(x_k)\}$ are equal to those of the whole initial set $\{f_K(x_k)\}$

$$\sum_{d(k)=1}^{D(k)} (x_k - x_0)^n f_{K,d(k)}(x_k) = (x_k - x_0)^n \sum_{d(k)=1}^{D(k)} f_{K,d(k)}(x_k) = (x_k - x_0)^n f_K(x_k) \ .$$

One can see, indeed, that, by definition, the total weight W_D of the exactly divided set is

$$W_D \equiv \sum_{k=1}^{K} \sum_{d(k)=1}^{D(k)} f_{K,d(k)}(x_k) \equiv \sum_{k=1}^{K} f_K(x_k) \equiv W_K$$

and the total moment $E_D(X-X_0)^n$ of the exactly divided set is

$$\begin{aligned} &E_D(X - X_0)^n \equiv \\ &\equiv \sum_{k=1}^{K} \sum_{d(k)=1}^{D(k)} (x_k - x_0)^n f_{K,d(k)}(x_k) \equiv \sum_{k=1}^{K} (x_k - x_0)^n f_K(x_k) \equiv \ . \\ &\equiv E(X - X_0)^n \end{aligned}$$

So, we have specified the properties of the divided sets: the total weight and moments of a divided set are equal to the total weight and moments of the initial set.

Let us recur to Situation *K.AB=3*.

The above considerations as to divided sets are true, in particular, when the value $f_K(x_{1.B})$ is exactly divided into two parts $f_{K,1}(x_{1.B})$ and $f_{K,2}(x_{1.B})$. Namely, the total weight and moments of the divided set are equal to the total weight and moments of the initial set.

Let us make the first step of the representation of the total weight and central moments of the set of values of the function by the total weight and central moments of the set of values of the pairs.

Since

$$(M - x_{1.A})\, f_K(x_{1.A}) = (x_{1.B} - M)\, f_{K,1}(x_{1.B}) \ ,$$

two values $f_K(x_{1.A})$ and $f_{K,1}(x_{1.B})$ of the divided set are the required pair of the previous subchapters. The portion $\{f_K(x_{1.A}), f_{K,1}(x_{1.B})\}$ of the set of values of the pairs and the portion $\{f_K(x_{1.A}), f_{K,1}(x_{1.B})\}$ of the divided set of values of the function are the same portions defined on the same points $x_{1.A}$ and $x_{1.B}$. So, the total weight and moments of these portions are the same.

This can be seen for the total weight: the weights of the same sets are the same weights. This can also be seen for the central moments. The moments of the same sets defined on the same points are the same moments. In more detail

$$\begin{aligned}
&E(X - M)^n = \\
&= \sum_{k.A=1}^{K.A} (x_{k.A} - M)^n f_K(x_{k.A}) + \sum_{k.B=1}^{K.B} (x_{k.B} - M)^n f_K(x_{k.B}) = \\
&= \sum_{k.A=1}^{2} (x_{k.A} - M)^n f_K(x_{k.A}) + \sum_{k.B=1}^{1} (x_{k.B} - M)^n f_K(x_{k.B}) = \\
&= (x_{1.A} - M)^n f_K(x_{1.A}) + (x_{1.B} - M)^n f_{K,1}(x_{1.B}) + (x_{1.B} - M)^n f_{K,2}(x_{1.B}) + \\
&+ \sum_{k.A=2}^{2} (x_{k.A} - M)^n f_K(x_{k.A}) = \\
&= (x_{1.A} - M)^n f_K(x_{1.A}) + (x_{1.B} - M)^n f_{K,1}(x_{1.B}) + \\
&+ \sum_{k.A=2}^{2} (x_{k.A} - M)^n f_K(x_{k.A}) + (x_{1.B} - M)^n f_{K,2}(x_{1.B}) = \\
&= E_{1.Pair}(X - M)^n + \\
&+ (x_{2.A} - M)^n f_K(x_{2.A}) + (x_{1.B} - M)^n f_{K,2}(x_{1.B})
\end{aligned} \ .$$

So, the first step of the representation has been done.

The balance remains

$$\begin{aligned}
&(M - x_{1.A})\, f_K(x_{1.A}) + (M - x_{2.A})\, f_K(x_{2.A}) = \\
&= (x_{1.B} - M)\, f_{K,1}(x_{1.B}) + (x_{1.B} - M)\, f_{K,2}(x_{1.B})
\end{aligned} \ ,$$

and we come to Situation *K.AB=2* for $f_K(x_{2.A})$ and $f_{K,2}(x_{1.B})$

$$(M - x_{2.A})\, f_K(x_{2.A}) = (x_{1.B} - M)\, f_{K,2}(x_{1.B}) \ .$$

As has been proved above, the total weight and central moments of Situation *K.AB=2* can be exactly represented by the total weight and central moments of a pair of the previous subchapters. So, this is the final situation.

For the central moments in the scope of Situation *K.AB=3,* this can be seen, indeed, in more detail

$$
\begin{aligned}
&E(X-M)^n = \\
&= E_{1.Pair}(X-M)^n + \\
&+ (x_{2.A}-M)^n f_K(x_{2.A}) + (x_{1.B}-M)^n f_{K,2}(x_{1.B}) = \\
&= E_{1.Pair}(X-M)^n + E_{2.Pair}(X-M)^n
\end{aligned}
.
$$

So, in Situation *K.AB=3,* at *K.A=2* and *K.B=1,*

$$
\begin{aligned}
&E(X-M)^n = E_{1.Pair}(X-M)^n + E_{2.Pair}(X-M)^n = \\
&= \sum_{p.Pair=1}^{2} E_{p.Pair}(X-M)^n
\end{aligned}
.
$$

Let us resume the consideration of Situation *K.AB=3,* at *K.A=2* and *K.B=1.*

There are three initial values: $f_K(x_{1.A})$, $f_K(x_{2.A})$ and $f_K(x_{1.B})$.

One should divide the value $f_K(x_{1.B})$ of the initial set into two values $f_{K,1}(x_{1.B})$ and $f_{K,2}(x_{1.B})$ such that $f_{K,1}(x_{1.B})$ constitutes the pair with $f_K(x_{1.A})$. Due to the properties of divided sets, the total weight and moments of the divided values $f_{K,1}(x_{1.B})$ and $f_{K,2}(x_{1.B})$ are equal to those of the initial value $f_K(x_{1.B})$.

Then one should perform the first step of the representation of the total weight and moments.

The values $f_K(x_{1.A})$ and $f_{K,1}(x_{1.B})$ of the pair $f_{1.Pair}(x_{1.A}, x_{1.B})$ and the values $f_K(x_{1.A})$ and $f_{K,1}(x_{1.B})$ of portion of the divided set $\{f_{K,d(k)}(x_k)\}$ are the same values defined on the same points. Therefore, the total weight and moments of the pair are the same as those of the portion of the divided set.

As a result of the first step, the number of unpaired values is diminished by one and we come to the resulting Situation $K.AB_{Diminished}=K.AB\text{-}1=2$. As it has been proved above, the total weight and moments of Situation *K.AB=2* can be exactly represented by the total weight and moments of a pair and, hence, this is the final situation.

So, Situation *K.AB=3,* at *K.A=2* and *K.B=1,* can be represented by the sum of the first step and the final situation. Both the total weight and moments depend linearly on the values of the members of the sets. Therefore, the total weight and moments of the sum are equal to the sum of the constituent weights and moments correspondingly.

If *K.A=1* and *K.B=2,* then the consideration is analogous to the preceding one.

So, the total weight and central moments of Situation *K.AB=3* can be exactly represented by the total weight and central moments of a set of pairs of the previous subchapters.

General Situation *K.AB*. Suppose $K.AB \geq 4$, $K.A \geq 1$ and $K.B \geq 1$ (according to the consideration of Situation $K.AB=1$, the case of $K.A=0$ and $K.B \geq 1$ and the case of $K.B=0$ and $K.A \geq 1$ cannot exist or do not contribute to the non-zero central moments).

Let us consider $f_K(x_{1.A})$ and $f_K(x_{1.B})$. There are only two possible variants:
Variant 1 (inequality)

$$(M - x_{1.A})\, f_K(x_{1.A}) \neq (x_{1.B} - M)\, f_K(x_{1.B})$$

and Variant 2 (equality)

$$(M - x_{1.A})\, f_K(x_{1.A}) = (x_{1.B} - M)\, f_K(x_{1.B}) \ .$$

Let us make a general step of the representation of the total weight and moments. Evidently, this general step may be implemented in one of the two forms depending on whether Variant 1 (inequality) or Variant 2 (equality) takes place.

Variant 1 (inequality). If

$$(M - x_{1.A})\, f_K(x_{1.A}) \neq (x_{1.B} - M)\, f_K(x_{1.B}) \ ,$$

then there are only two possible cases as well:

$$(M - x_{1.A})\, f_K(x_{1.A}) < (x_{1.B} - M)\, f_K(x_{1.B})$$

and

$$(M - x_{1.A})\, f_K(x_{1.A}) > (x_{1.B} - M)\, f_K(x_{1.B}) \ .$$

Suppose, for example, that

$$(M - x_{1.A})\, f_K(x_{1.A}) < (x_{1.B} - M)\, f_K(x_{1.B}) \ .$$

Then one should divide the value $f_K(x_{1.B})$ into two parts $f_{K,1}(x_{1.B})$ and $f_{K,2}(x_{1.B})$ satisfying

$$(M - x_{1.A})\, f_K(x_{1.A}) = (x_{1.B} - M)\, f_{K,1}(x_{1.B})$$

and

$$f_{K,2}(x_{1.B}) = f_K(x_{1.B}) - f_{K,1}(x_{1.B}) \ .$$

The value $f_K(x_{1.B})$ is exactly divided into two parts $f_{K,1}(x_{1.B})$ and $f_{K,2}(x_{1.B})$. Due to the properties of the divided sets, the total weight and moments of the divided set $\{f_{K,1}(x_{1.B}), f_{K,2}(x_{1.B})\}$ are equal to the total weight and moments of the initial set $\{f_K(x_{1.B})\}$.

Due to the above

$$(M - x_{1.A})\, f_K(x_{1.A}) = (x_{1.B} - M)\, f_{K,1}(x_{1.B}) \ ,$$

two values $f_K(x_{1.A})$ and $f_{K,1}(x_{1.B})$ of the divided set are the required pair of the previous subchapters.

Here, the set of values of the pair and the portion of the divided set of values are the same sets defined on the same points. Therefore, the total weight and an arbitrary total moment of the pair are the same as those of the portion of the divided set.

This can be seen for the total weight: the weights of the same sets are the same weights. This can be seen for the central moments. The moments of the same sets defined on the same points are the same moments. In more detail

$$E(X-M)^n =$$
$$= \sum_{k.A=1}^{K.A}(x_{k.A}-M)^n f_K(x_{k.A}) + \sum_{k.B=1}^{K.B}(x_{k.B}-M)^n f_K(x_{k.B}) =$$
$$= (x_{1.A}-M)^n f_K(x_{1.A}) + (x_{1.B)}-M)^n f_{K,1}(x_{1.B}) + (x_{1.B}-M)^n f_{K,2}(x_{1.B}) +$$
$$+ \sum_{k.A=2}^{K.A}(x_{k.A}-M)^n f_K(x_{k.A}) + \sum_{k.B=2}^{K.B}(x_{k.B}-M)^n f_K(x_{k.B}) = \quad .$$
$$= E_{1.Pair}(X-M)^n +$$
$$+ \sum_{k.A=2}^{K.A}(x_{k.A}-M)^n f_K(x_{k.A}) + (x_{1.B}-M)^n f_{K,2}(x_{1.B}) + \sum_{k.B=2}^{K.B}(x_{k.B}-M)^n f_K(x_{k.B})$$

As a result of this general step within the scope of Variant 1 (of the inequality and divided set), the number of unpaired values is diminished by one (taking into account the part of the other value) and we come to Situation $K.AB_{Diminished}=K.AB-1$. Note, that the number *K.AB* is composed of *1, …, K.A* and *1, …, K.B*. And here, the number $K.AB_{Diminished}=K.AB-1$ is composed of *2, …, K.A* and *2, …, K.B* plus one.

If

$$(M-x_{1.A})f_K(x_{1.A}) > (x_{1.B}-M)f_K(x_{1.B})$$

then the argument is analogous to the preceding one.

Variant 2 (equality). If

$$(M - x_{1.A}) f_K(x_{1.A}) = (x_{1.B} - M) f_K(x_{1.B})$$

then the two values $f_K(x_{1.A})$ and $f_K(x_{1.B})$ are the required pair (couple) of the previous subchapters.

Here, the set of values of the pair and the portion of the initial original set of values are the same sets defined on the same points. Therefore, the total weight and an arbitrary total moment of the pair are the same as those of the portion of the initial set.

This can be seen for the total weight: the weights of the same sets are the same weights. This can be seen for the central moments: the moments of the same sets defined on the same points are the same moments. In more detail,

$$\begin{aligned}
&E(X - M)^n = \\
&= \sum_{k.A=1}^{K.A} (x_{k.A} - M)^n f_K(x_{k.A}) + \sum_{k.B=1}^{K.B} (x_{k.B} - M)^n f_K(x_{k.B}) = \\
&= (x_{1.A} - M)^n f_K(x_{1.A}) + (x_{1.B} - M)^n f_K(x_{1.B}) + \\
&+ \sum_{k.A=2}^{K.A} (x_{k.A} - M)^n f_K(x_{k.A}) + \sum_{k.B=2}^{K.B} (x_{k.B} - M)^n f_K(x_{k.B}) = \\
&= E_{1.Pair}(X - M)^n + \\
&+ \sum_{k.A=2}^{K.A} (x_{k.A} - M)^n f_K(x_{k.A}) + \sum_{k.B=2}^{K.B} (x_{k.B} - M)^n f_K(x_{k.B})
\end{aligned}.$$

As a result of this general step within the scope of Variant 2 (of the equality and initial set), the number of unpaired (uncoupled) values is diminished by two and from Situation *K.AB* we come to Situation $K.AB_{Diminished}=K.AB-2$. Here, the number $K.AB_{Diminished}=K.AB-2$ is composed of *2, …, K.A* and *2, …, K.B.*

So, we have considered the general step of diminishing the number *K.AB* for general Situation $K.AB \geq 4$ within the scopes of both parallel variants. It diminishes *K.AB* by one or two. Evidently, this general step may be repeated as many times as needed to reach the final Situations $K.AB_{Diminished}=3$ or $K.AB_{Diminished}=2$.

Let us resume the consideration of general Situation $K.AB\geq 4$, at $K.A\geq 1$ and $K.B\geq 1$. There are at least four initial values: $f_K(x_{1.A})$, … and $f_K(x_{1.B})$, …

One should perform the general step of the representation of the total weight and moments. One should consider two first values $f_K(x_{1.A})$ and $f_K(x_{1.B})$ of the initial set $\{f_K(x_k)\}$ of the function f_K. There are only two possible variants:

In Variant 1, these values are not the pair. Therefore, one should divide one of the values $f_K(x_{1.A})$ or $f_K(x_{1.B})$ into two values such that one of them constitutes a pair with the remaining other value out of $f_K(x_{1.A})$ and $f_K(x_{1.B})$. Due to the properties of divided sets, the total weight and moments of the divided values are equal to those of the initial value. This pair may be a portion of the representation.

In Variant 2 these two initial values are the pair. This pair may be a portion of the representation as well.

The values of the pair (in both variants) and the divided (or initial) values are the same values defined on the same points. Therefore, the total weight and moments of the pair are the same as those of the first portion of the divided (or initial) set.

As a result of this general step of the representation, the number of unpaired values is diminished by one or two and we come to one of the resulting Situations $K.AB_{Diminished}=K.AB-1$ or $K.AB_{Diminished}=K.AB-2$. There are only two possible cases: one case is $K.AB_{Diminished}\leq 3$ ($K.AB_{Diminished}=3$ or $K.AB_{Diminished}=2$ and they are the final Situations) and another case is $K.AB_{Diminished}>3$ (they are the intermediate Situations).

If an intermediate Situation takes place, then one should repeat the general step of the representation as many times as it needs to reach the final Situations $K.AB_{Diminished}=3$ or $K.AB_{Diminished}=2$.

Both in Situation $K.AB=3$ and Situation $K.AB=2$, the total weight and central moments of the set of the initial values can be exactly represented by the set of values of the pairs, as has been shown above.

So, the total weight and central moments of the general Situation can be represented by the sum of those of all the portions and of the final Situation. Both the total weight and moments depend linearly on the values of the members of the sets. Therefore, the total weight and moments of the sum are equal to the sum of the constituent weights and moments correspondingly.

So, in the general Situation $K.AB : K.AB\geq 4$, at $K.A\geq 1$ and $K.B\geq 1$, the total weight and central moments of an arbitrary original function of Chapter 2 may be exactly represented by the total weight and central moments of the pairs of this chapter.

One can see in more detail that the total weights

$$W_{Pair.AB} \equiv \sum_{p.Pair.AB=1}^{P.Pair.AB} w_{p.Pair.AB} = \sum_{k.A=1}^{K.A} f_K(x_{k.A}) + \sum_{k.B=1}^{K.B} f_K(x_{k.B})$$

and central moments (keeping in mind that the central moments of $f_K(M)$ equal zero)

$$E(X-M)^n \equiv \sum_{k=1}^{K} (x_k - M)^n f_K(x_k) =$$

$$= \sum_{k.A=1}^{K.A} (x_{k.A} - M)^n f_K(x_{k.A}) + \sum_{k.B=1}^{K.B} (x_{k.B} - M)^n f_K(x_{k.B}) =$$

$$= \sum_{p.Pair.AB=1}^{P.Pair.AB} E_{p.Pair.AB}(X-M)^n$$

of the arbitrary original function and of the set of pairs are equal to each other.

So, the exact representation of the total weight and central moments of an arbitrary original function of Chapter 2 by the total weight and central moments of a set of pairs of the previous subchapters of this chapter has been performed.

3.5. General limitations

3.5.1. Weights

Let us consider the weights of groups of values of the function and general limitations on them.

Remembering

$$K.A + K.M + K.B = K$$

of the preceding subchapter, the total weights of these groups may be denoted as W_A, W_M and W_B

$$W_A \equiv \sum_{k.A \le K.A} f_K(x_{k.A}) \ , \ W_M \equiv \sum_{k.M \le K.M} f_K(x_{k.M}) \ , \ W_B \equiv \sum_{k.B \le K.B} f_K(x_{k.B})$$

and the sum of the weights is

$$W_A + W_M + W_B = W_K \ .$$

Let us denote the total weight of the total set of the pairs as W_{Pair}, the weight of the pair $f_{Pair.M}(M, M)$ as $W_{Pair.M}$ and the total weight of the pairs $f_{p.Pair.AB}(x_{p.A}, x_{p.B})$ as $W_{Pair.AB}$. By this definition, the weight of, e.g., a *p*-th pair $f_{p.Pair.AB}(x_{p.A}, x_{p.B})$ is denoted as $w_{p.Pair.AB}$,

$$\sum_{p.Pair=1}^{P.Pair} w_{p.Pair} \equiv W_{Pair} \ , \quad \sum_{p.Pair.M \le P.Pair.M} w_{p.Pair.M} \equiv W_{Pair.M} \ , \quad \sum_{p.Pair.AB=1}^{P.Pair.AB} w_{p.Pair.AB} \equiv W_{Pair.AB}$$

and we have

$$W_{Pair} = W_{Pair.M} + W_{Pair.AB} \ .$$

Evidently,

$$W_{Pair.M} = W_M \ ,$$

$$W_{Pair.AB} = W_A + W_B$$

and

$$W_{Pair} = W_K \ .$$

3.5.2. The general limiting function

Let us consider the central moments

$$E(X-M)^n = \sum_{p.Pair=1}^{P.Pair} E_{p.Pair}(X-M)^n =$$

$$= \sum_{p.Pair.M \le P.Pair.M} E_{p.Pair.M}(X-M)^n + \sum_{p.Pair.AB=1}^{P.Pair.AB} E_{p.Pair.AB}(X-M)^n = \ .$$

$$= \sum_{p.Pair.AB=1}^{P.Pair.AB} E_{p.Pair.AB}(X-M)^n$$

The maximal limiting functions $L_{p.Pair}(A, M, B, n, w_{p.Pair})$ satisfying

$$| E_{p.Pair.AB}(X-M)^n | \le L_{p.Pair.AB}(A, M, B, n, w_{p.Pair.AB}) \ ,$$

allow estimating the central moments of the original function f_K:

$$| E(X-M)^n | \le | \sum_{p.Pair.AB=1}^{P.Pair.AB} E_{p.Pair.AB}(X-M)^n | \le$$

$$\le \sum_{p.Pair.AB=1}^{P.Pair.AB} L_{p.Pair.AB}(A, M, B, n, w_{p.Pair.AB}) \ .$$

This estimate can be easily simplified. From

$$L_{p.Pair.AB}(A, M, B, n, w_{p.Pair.AB}) =$$

$$= (M-A)^n \frac{B-M}{B-A} w_{p.Pair.AB} + (B-M)^n \frac{M-A}{B-A} w_{p.Pair.AB} =$$

$$= \left[(M-A)^n \frac{B-M}{B-A} + (B-M)^n \frac{M-A}{B-A} \right] w_{p.Pair.AB}$$

there follows

$$\sum_{p.Pair.AB=1}^{P.Pair.AB} L_{p.Pair.AB}(A, M, B, n, w_{p.Pair.AB}) =$$

$$= \sum_{p.Pair.AB=1}^{P.Pair.AB} \left[(M-A)^n \frac{B-M}{B-A} + (B-M)^n \frac{M-A}{B-A} \right] w_{p.Pair.AB} = \ .$$

$$= \left[(M-A)^n \frac{B-M}{B-A} + (B-M)^n \frac{M-A}{B-A} \right] \sum_{p.Pair.AB=1}^{P.Pair.AB} w_{p.Pair.AB}$$

Since

$$\sum_{p.Pair.AB=1}^{P.Pair.AB} w_{p.Pair.AB} \equiv W_{Pair.AB} \ ,$$

using

$$W_{Pair.AB} = W_A + W_B \le W_K \ ,$$

it follows for a unitary function f_K (i.e., assuming $W_K=1$) that

$$W_{Pair.AB} \le 1$$

and

$$\sum_{p.Pair.AB=1}^{P.Pair.AB} L_{p.Pair.AB}(A, M, B, n, w_{p.Pair.AB}) \le$$

$$\le (M-A)^n \frac{B-M}{B-A} + (B-M)^n \frac{M-A}{B-A}$$

and

$$| E(X-M)^n | \le$$

$$\le (M-A)^n \frac{B-M}{B-A} + (B-M)^n \frac{M-A}{B-A} \ .$$

So, we have proved that the maximal possible modulus of a central moment of any function f_K of Chapter 2 is obtained for the function which is concentrated at the borders of the interval. We have also found this general limiting function: for any unitary function f_K of Chapter 2, the modulus of any central moment of f_K is not greater than

(3.1). $$| E(X-M)^n | \le (M-A)^n \frac{B-M}{B-A} + (B-M)^n \frac{M-A}{B-A}$$

4. Two notes

One may denote the unitary function

$$(M-A)^n \frac{B-M}{B-A} + (B-M)^n \frac{M-A}{B-A}$$

as a general, total limiting function *L(A, M, B, n, 1)≡L(A, M, B, n)* or *L(M, n)*.

Let us analyze the total limiting function *L(A, M, B, n)* for

$$M = \frac{B-A}{2}$$

and for

$$M \approx A \qquad \text{and} \qquad M \approx B.$$

4.1. The mean is in the center of the interval

Let us analyze the total limiting function *L(A, M, B, n)* when

$$M = \frac{B-A}{2}.$$

Let us differentiate *L(A, M, B, n)* with respect to *M*:

$$\frac{1}{B-A}\frac{\partial((M-A)^n(B-M)+(B-M)^n(M-A))}{\partial M} =$$

$$= \frac{1}{B-A}[n(M-A)^{n-1}(B-M)-(M-A)^n -$$

$$-n(B-M)^{n-1}(M-A)+(B-M)^n]$$

and, at *M=(B-A)/2*,

$$\frac{1}{B-A}[n(M-A)^{n-1}(B-M)-(M-A)^n -$$

$$-n(B-M)^{n-1}(M-A)+(B-M)^n] = \qquad .$$

$$= \frac{1}{B-A}\left(\frac{B-A}{2}\right)^n [n-1-n+1] = 0$$

So, at *M=(B-A)/2*, for any *n≥2* there is an extremum or a point of inflection.

Let us differentiate *L(A, M, B, n)* once more

$$\frac{1}{B-A}\frac{\partial^2 L(A,M,B,n)}{\partial M^2}=$$

$$=\frac{1}{B-A}[n(n-1)(M-A)^{n-2}(B-M)-2n(M-A)^{n-1}+$$

$$+n(n-1)(B-M)^{n-2}(M-A)-2n(B-M)^{n-1}]$$

and, at *M=(B-A)/2*,

$$\frac{1}{B-A}[n(n-1)(M-A)^{n-2}(B-M)-2n(M-A)^{n-1}+$$

$$+n(n-1)(B-M)^{n-2}(M-A)-2n(B-M)^{n-1}]=$$

$$=\frac{1}{B-A}\left(\frac{B-A}{2}\right)^{n-1}[n(n-1)-2n+n(n-1)-2n]= \quad .$$

$$=\left(\frac{B-A}{2}\right)^{n-2}n(n-3)$$

That is, at *M=(B-A)/2*:

For *n=2* there is a well-known maximum, the moment of inertia of two material points whose weights are equal to each other

$$L\left(A,\frac{B-A}{2},B,2\right)=\left(\frac{B-A}{2}\right)^2\frac{1}{2}+\left(\frac{B-A}{2}\right)^2\frac{1}{2}=\left(\frac{B-A}{2}\right)^2 \; .$$

For *n=3* there is a point of inflection.

For *n>3* there are minima.

4.2. The mean is near a border of the interval

Let us search for maxima which are close to the borders of the interval.

Let us differentiate the total limiting function *L(A, M, B, n)* with respect to *M* for $M \approx A$ and $n >> 1$

$$\frac{1}{B-A}\frac{\partial((M-A)^n(B-M)+(B-M)^n(M-A))}{\partial M} =$$
$$= \frac{1}{B-A}[n(M-A)^{n-1}(B-M)-(M-A)^n -$$
$$-n(B-M)^{n-1}(M-A)+(B-M)^n] \approx .$$
$$\approx \frac{1}{B-A}[(B-M)^n - n(B-M)^{n-1}(M-A)]$$

For $\partial L(A, M, B, n)/\partial M = 0$ we obtain

$$\frac{1}{B-A}[(B-M)^n - n(B-M)^{n-1}(M-A)] =$$
$$= \frac{(B-M)^{n-1}}{B-A}[(B-M)-n(M-A)] =$$
$$= \frac{(B-M)^{n-1}}{B-A}[(B-A)-(n+1)(M-A)] = 0$$

and

$$M - A \approx \frac{B-A}{n+1}.$$

The second derivative is

$$\frac{1}{B-A}\frac{\partial^2 L(A,M,B,n)}{\partial M^2} =$$

$$\frac{\partial[n(M-A)^{n-1}(B-M)-(M-A)^n -}{\partial M}$$
$$\frac{-n(B-M)^{n-1}(M-A)+(B-M)^n]}{\partial M} =$$
$$= \frac{1}{B-A}[n(n-1)(M-A)^{n-2}(B-M)-2n(M-A)^{n-1} + .$$
$$+n(n-1)(B-M)^{n-2}(M-A)-2n(B-M)^{n-1}]$$

For $M \approx A$ and $n >> 1$,

$$\frac{1}{B-A}[n(n-1)(M-A)^{n-2}(B-M) - 2n(M-A)^{n-1} +$$
$$+ n(n-1)(B-M)^{n-2}(M-A) - 2n(B-M)^{n-1}] \approx$$
$$\approx n\frac{(B-M)^{n-2}}{B-A}[(n-1)(M-A) - 2(B-M)] =$$
$$= n\frac{(B-M)^{n-2}}{B-A}[(n+1)(M-A) - 2(B-A)]$$

and, for $M \approx A+(B-A)/(n+1)$ and $n >> 1$,

$$n\frac{(B-M)^{n-2}}{B-A}[(n+1)(M-A) - 2(B-A)] \approx$$
$$\approx n\frac{(B-M)^{n-2}}{B-A}[(n+1)\frac{B-A}{n+1} - 2(B-A)] = .$$
$$= n(B-M)^{n-2}[1-2] < 0$$

So, the second derivative is negative and there are maxima at the points $M \approx A+(B-A)/(n+1)$.

The total limiting function

$$L(A, M, B, n) = (M-A)^n \frac{B-M}{B-A} + (B-M)^n \frac{M-A}{B-A}$$

and the maxima at the points $M \approx A+(B-A)/(n+1)$ and for $n >> 1$ may be taken as

$$L\left(A, A+\frac{B-A}{n+1}, B, n\right) \approx (B-M)^n \frac{M-A}{B-A} \approx \left(B-A-\frac{B-A}{n+1}\right)^n \frac{1}{n+1} = .$$
$$= \left(1-\frac{1}{n+1}\right)^n \frac{(B-A)^n}{n+1}$$

For $n >> 1$

$$\left(1-\frac{1}{n+1}\right)^n = \left(1-\frac{1}{n+1}\right)^{n+1}\left(1-\frac{1}{n+1}\right)^{-1} \approx \frac{1}{e}.$$

So, for $M \approx A+(B-A)/(n+1)$ and $n >> 1$, the maxima (those can be attained by $E(X-M)^n$ for even n) of $L(A, M, B, n)$ are, curiously, with a coefficient $\approx 1/e$

$$L\left(A, A+\frac{B-A}{n+1}, B, n\right) \approx \frac{1}{e}\frac{(B-A)^n}{n+1}.$$

Evidently, for $M \approx B-(B-A)/(n+1)$, at $n >> 1$, the maxima (those can be attained by $E(X-M)^n$ for even n) of $L(A, M, B, n)$ are analogously

$$L\left(A, B-\frac{B-A}{n+1}, B, n\right) \approx \frac{1}{e}\frac{(B-A)^n}{n+1}.$$

5. Theorem

5.1. Lemma about the tendency to zero for central moments

Lemma 5.1. If, for the nonnegative function f_K defined in Section 2, $M \equiv E(X)$ tends to A or to B, then, for $n: 2 \leq n < \infty$, $E(X-M)^n$ tends to zero.

Proof 1. For $M \rightarrow A$, the estimate (3.1) gives

$$
\begin{aligned}
&| E(X-M)^n | \leq (M-A)^n \frac{B-M}{B-A} + (B-M)^n \frac{M-A}{B-A} < \\
&< [(B-A)^{n-1} + (B-A)^{n-1}] \frac{(M-A)(B-M)}{B-A} \leq \\
&\leq 2(B-A)^{n-1}(M-A) \xrightarrow[M \to A]{} 0 .
\end{aligned}
$$

This rough estimate is already sufficient for the purpose of this article. But a more precise estimate may be obtained:

Proof 2. Let us transform

$$
\begin{aligned}
&[(M-A)^{n-1} + (B-M)^{n-1}] \frac{(M-A)(B-M)}{B-A} = \\
&= \left[\left(\frac{M-A}{B-A} \right)^{n-1} + \left(\frac{B-M}{B-A} \right)^{n-1} \right] (B-A)^{n-1} \frac{(M-A)(B-M)}{B-A} .
\end{aligned}
$$

Let us consider the terms $(M-A)/(B-A)$ and $(B-M)/(B-A)$. Keeping in mind that $A \leq M \leq B$ we obtain $0 \leq (M-A)/(B-A) \leq 1$ and $0 \leq (B-M)/(B-A) \leq 1$. For $n \geq 2$ we have

$$
\begin{aligned}
&\left(\frac{M-A}{B-A} \right)^{n-1} + \left(\frac{B-M}{B-A} \right)^{n-1} \leq \\
&\leq \frac{M-A}{B-A} + \frac{B-M}{B-A} = \frac{B-A}{B-A} \equiv 1
\end{aligned} .
$$

So,

$$
\begin{aligned}
&\left[\left(\frac{M-A}{B-A} \right)^{n-1} + \left(\frac{B-M}{B-A} \right)^{n-1} \right] (B-A)^{n-1} \frac{(M-A)(B-M)}{B-A} \leq \\
&\leq (B-A)^{n-1} \frac{(M-A)(B-M)}{B-A} \leq (B-A)^{n-1}(M-A)
\end{aligned} .
$$

So,

$$
| E(X-M)^n | \leq (B-A)^{n-1}(M-A) \xrightarrow[M \to A]{} 0 \qquad (5.1).
$$

For $M \rightarrow B$, the proof is similar and gives

$$
| E(X-M)^n | \leq (B-A)^{n-1}(B-M) \xrightarrow[M \to B]{} 0 \qquad (5.2).
$$

So, if $(B-A)$ and n are finite and $M \rightarrow A$ or $M \rightarrow B$, then $E(X-M)^n \rightarrow 0$.

The lemma has been proved.

5.2. Existence theorem for restrictions on the mean

Let us define two terms for the purposes of this article:

Definition 5.2. A "**non-zero restriction on the mean** $\boldsymbol{r_{Mean}}$" (or, simply, a "non-zero restriction") signifies the impossibility for the mean to be located closer to a border of the interval than some non-zero distance.

In other words, a non-zero restriction designates the existence of a non-zero distance from a border of the interval. Within this distance, it is impossible for the mean to be located.

This restriction may be denoted also as a "forbidden zone" for the mean near a border of the interval.

The "restriction" for one border and the "restriction" for another border constitute the "restrictions" for the borders.

The value of a non-zero restriction (or the width of a non-zero "forbidden zone") signifies the minimal possible distance between the mean and a border of the interval. For brevity, the term "the value of a restriction" may be shortened to "the restriction."

Definition 5.3. At the beginning, let us define a "non-zero restriction on the dispersion $r^2_{Dispersion.2} \equiv r^2_{Disp.2} = \sigma^2_{Min}$" to be the minimal value of the analog of the dispersion $E(X\text{-}M)^2$ satisfying $E(X\text{-}M)^n \geq r^2_{Disp.2} > 0$.

Let us define analogously a general "**non-zero restriction on the $\boldsymbol{n^{th}}$ order central moment $\boldsymbol{|r^n_{Disp.n}|}$**" to be the minimal absolute value of the analog of the n^{th} order central moment $E(X\text{-}M)^n$ satisfying $|E(X\text{-}M)^n| \geq |r^n_{Disp.n}| > 0$.

Theorem 5.2. If, for a nonnegative function f_K as in Section 2, such that its mean $M \equiv E(X)$ and its analog of the $n^{th} : 2 \leq n < \infty$, order central moment $E(X\text{-}M)^n$ exist, there exists a non-zero restriction on this analog of the n^{th} order central moment $|r^n_{Disp.n}| = Const_{Disp.n} > 0 : |E(X\text{-}M)^n| \geq |r^n_{Disp.n}|$, then a non-zero restriction on the mean $r_{Mean} = Const_{Mean} > 0$ exists and

$$A < (A + r_{Mean}) \leq M \equiv E(X) \leq (B - r_{Mean}) < B\,.$$

Proof. From the conditions of the theorem and from Lemma (5.1), for $M \rightarrow A$, we have

$$0 < | r^n{}_{Disp.n} | \le | E(X-M)^n | \le (B-A)^{n-1}(M-A)$$

and

$$0 < \frac{| r^n{}_{Disp.n} |}{(B-A)^{n-1}} \le (M-A).$$

So,

(5.3) $$(M-A) \ge r_{Mean} \equiv \frac{| r^n{}_{Disp.n} |}{(B-A)^{n-1}} > 0 \ .$$

For $M \rightarrow B$, the proof is similar and gives

(5.4) $$(B-M) \ge r_{Mean} \equiv \frac{| r^n{}_{Disp.n} |}{(B-A)^{n-1}} > 0 \ .$$

The results (5.3) and (5.4) may be rewritten as

(5.5) $$A < \left(A + \frac{| r^n{}_{Disp.n} |}{(B-A)^{n-1}} \right) \le M \le \left(B - \frac{| r^n{}_{Disp.n} |}{(B-A)^{n-1}} \right) < B \ .$$

So, as long as $(B\text{-}A)$ and n are finite and $|r^n{}_{Disp.n}|=Const_{Disp.n}>0$, then $r_{Mean}=Const_{Mean}>0$ and $A<(A+r_{Mean})\le M\le(B\text{-}r_{Mean})<B$, which proves the theorem.

6. Remarks

Remark 6.1

The estimation of a probability possesses the properties assumed for the function f of Chapter 2. In fact, these properties have been chosen to be those satisfied by the estimation of a probability. This opens prospects to develop the theorem to the probability.

Sketches of the proof of the existence theorem for non-zero restrictions on the probability were made (see, e.g., [11]-[12]) and used in particular practical cases in items connected with utility and prospect theories. Some well-known old problems of utility and prospect theories were explained, at least partially, with the help of these sketches (see, e.g., [12]-[14]).

Remark 6.2

For $n=2$ the analog of the central moment is the analog of the dispersion, and r_{Mean} at A may be rewritten for the minimum σ_{Min} of the analog of the standard deviation σ, i.e., $\sigma \geq \sigma_{Min} \equiv r_{Disp.2} > 0$, as

$$(M-A) \geq r_{Mean} \equiv \frac{r^2{}_{Disp.2}}{(B-A)} \equiv \frac{\sigma^2{}_{Min}}{(B-A)} > 0 \ .$$

The value of the restriction r_{Mean} at B may be also rewritten for the minimum σ_{Min} of the analog of the standard deviation σ as

$$(B-M) \geq r_{Mean} \equiv \frac{r^2{}_{Disp.2}}{(B-A)} \equiv \frac{\sigma^2{}_{Min}}{(B-A)} > 0 \ .$$

These results may be rewritten as

$$A < \left(A + \frac{\sigma^2{}_{Min}}{B-A} \right) \leq M \leq \left(B - \frac{\sigma^2{}_{Min}}{B-A} \right) < B \ ,$$

or

$$(5.6) \quad A < \left(A + \sigma_{Min} \frac{\sigma_{Min}}{B-A} \right) \leq M \leq \left(B - \sigma_{Min} \frac{\sigma_{Min}}{B-A} \right) < B \ .$$

Remark 6.3

The estimates (5.3)—(5.6) are rather reliable ones, especially the estimate (5.6) for $\frac{\sigma_{Min}}{B-A} \to 0$. They are, in a sense, as reliable as the Chebyshev inequality. Preliminary calculations [15] which were performed for real cases such as the normal, uniform and exponential distributions with the minimal values σ^2_{Min} of the analog of the dispersion, gave much stronger restrictions r_{Mean} on the mean of the function (for $\frac{\sigma_{Min}}{B-A} \to 0$, for the unitary interval $[A, B] : (B\text{-}A)=1$) which are not worse than

$$r_{Mean} \geq \frac{\sigma_{Min}}{3} .$$

So, the inequalities $A<(A+r_{Mean})\leq M\leq(B\text{-}r_{Mean})<B$ for these cases may be rewritten as

$$(5.7)\quad A < \left(A + \frac{\sigma_{Min}}{3} \right) \leq M \leq \left(B - \frac{\sigma_{Min}}{3} \right) < B .$$

7. Conclusions

The possibility of the existence of non-zero restrictions on the mean in the presence of a non-zero restriction on the dispersion has been analyzed.

It has been proved that there are non-zero restrictions on the mean of a nonnegative function defined on a discrete set of points within an interval $[A, B]$ when there is a non-zero restriction on the analog of some central moment of the function. Suppose $|r^n_{Dispersion.n}| \equiv |r^n_{Disp.n}| = Const_{Disp.n} > 0$ is a non-zero restriction on the modulus of the analog of n^{th} central moment $|E(X-M)^n|$ of a nonnegative function, where $2 \leq n < \infty$. That is, $|E(X-M)^n| \geq |r^n_{Disp.n}| \geq Const_{Disp.n} > 0$. Then other non-zero restrictions r_{Mean} on the mean M of this function exist at the borders A and B of the interval $[A, B]$, satisfying

$$A < (A + r_{Mean}) \leq M \equiv E(X) \leq (B - r_{Mean}) < B \ ,$$

or (see (5.5))

$$A < \left(A + \frac{|\, r^n{}_{Disp.n} \,|}{(B-A)^{n-1}} \right) \leq M \leq \left(B - \frac{|\, r^n{}_{Disp.n} \,|}{(B-A)^{n-1}} \right) < B \ .$$

For $n=2$ the analog of the central moment is the analog of the dispersion. So, restrictions on the mean M may be rewritten in terms of the minimum σ_{Min} of the analog of the standard deviation σ, i.e., $\sigma \geq \sigma_{Min} \equiv r_{Disp.2} > 0$, as

$$A < \left(A + \frac{\sigma^2{}_{Min}}{B-A} \right) \leq M \leq \left(B - \frac{\sigma^2{}_{Min}}{B-A} \right) < B \ .$$

The above estimates are, in a sense, as reliable as the Chebyshev inequality. For real cases such as the normal distribution, for $\dfrac{\sigma_{Min}}{B-A} \to 0$, the preliminary calculations of [15] gave much stronger restrictions r_{Mean} on the mean (see (5.7)).

Sketches of the proof of the existence theorem for non-zero restrictions on the probability were made from 2010 to 2014 and were used to explain, at least partially, some problems of utility and prospect theories (see, e.g., [11]-[16]).

Additionally, properties of the accessory limiting function

$$L(A, M, B, n) = (M - A)^n \frac{B - M}{B - A} + (B - M)^n \frac{M - A}{B - A}$$

were analyzed near the center of the interval *[A, B]* at *M=(B-A)/2* and near its borders at *A* and *B*.

Near the center of the interval *[A, B]* at *M=(B-A)/2* one can obtain:

For *n=2* there is a well-known maximum, the moment of inertia of two material points whose weights are equal to each other

$$L\left(A, \frac{B - A}{2}, B, 2\right) = \left(\frac{B - A}{2}\right)^2 .$$

For *n=3* there is a point of inflection. For *n>3* there are minima.

Near the borders of the interval *[A, B]* at *A* and *B*, one can obtain:

For *M≈A+(B-A)/(n+1)* and *n>>1*, there are the maxima of *L(A, M, B, n)* (those can be attained by $E(X\text{-}M)^n$ for even *n*). Curiously, they have a coefficient *≈1/e*

$$L\left(A, A + \frac{B - A}{n + 1}, B, n\right) \approx \frac{1}{e} \frac{(B - A)^n}{n + 1} .$$

For *M≈B-(B-A)/(n+1)*, at *n>>1*, the maxima (those can be attained by $E(X\text{-}M)^n$ for even *n*) of *D(A, M, B, n)* are analogously

$$L\left(A, B - \frac{B - A}{n + 1}, B, n\right) \approx \frac{1}{e} \frac{(B - A)^n}{n + 1} .$$

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