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Median and average as tools for measuring, electing and ranking: new prospects

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Abstract: *Impossibility theorems expose inconsistencies and paradoxes related to voting systems. Recently, Michel Balinski and Rida Laraki proposed a new voting theory called Majority Judgment which tries to circumvent this limitation. In Majority Judgment, voters are invited to evaluate candidates in terms taken in a well-known common language. The winner is then the one that obtains the highest median. Since the Majority Judgment proposal was made, authors have detected insufficiencies with this new voting system. This article aims at reducing these insufficiencies by proposing a voting system to decide between the median-based voting and the mean-based one. It proposes, moreover, a new tie-breaking method computing intermedian ranks mean.*

Key words: Borda Majority Count, Majority Judgment, Mean-Median Compromise Method

Mathematics Subject Classification: 91A80, 91B12, 91B14, 91-02, 91A35

1. INTRODUCTION

In our current democracies, there is, according to Nobel Laureate Kenneth J. Arrow (1950 ; 1951 ; 1963), essentially two ways by which social choices can be made: voting, typically used to make a political decision, and the market mechanism typically used to make economic decisions (Taylor, 2005). Our concern in this paper is exclusively the former, i.e. voting.

The Theory of Social Choice concerns methods for amalgamating the appreciations or evaluations of many individuals into one collective appreciation or evaluation (Balinski & Laraki, 2007). Impossibility theorems stated by Arrow (1951; 1963), Sen (1970a; 1970b), Gibbard (1973) and Satterthwaite (1975) highlight the fact that none of these aggregation methods fulfills the requirements of an ideal democracy.

Many results now known as paradoxes abound in the direction to show imperfections of the electoral systems. These results tackle usual modes of poll and those which are not practically used in any country in the world as well. Instead of discouraging researchers in Economics and Social Choice Theory, paradoxes as well as impossibility theorems have on the contrary instigated them to conceive and even try out several new methods or aggregation functions. These methods are being proposed as alternative to those which are publicly valued.

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As an example, approval voting was tested in France at the 2002 presidential elections first round by a team directed by Jean-François Laslier and Karine Van der Straeten (Laslier, 2011; Laslier & Van der Straeten, 2004). A second experiment was carried out under the same conditions in 2007 for approval voting and evaluation voting (Alcantud & Laruelle, 2014) by Antoinette Baujard and Herrade Igersheim (Baujard & Igersheim, 2007; Baujard & Igersheim, 2009; Laslier, 2011). The same day of the 2007 elections, another team led by Etienne Farvaque, Hubert Jayet and Lionel Ragot (Farvaque et al., 2009) tried out the single transferable vote. Various teams still undertook experimental votes parallel to the official one on April 22, 2012 (Baujard et al., 2013; Balinski & Laraki, 2012a; 2012b).

Among all modes of poll that are tested in France, one seems to dissociate others on philosophical level: it is about the one proposed by Michel Balinski and Rida Laraki. This new voting theory is median-based and uses several ways, more or less complicated, for ranking candidates. Its authors named it *Majority Judgment* (MJ) (Balinski & Laraki, 2007; 2010; 2012a; 2012b). MJ was tested twice in France at the presidential elections of 2007 (Balinski & Laraki, 2011; 2012a) and 2012 (Balinski & Laraki, 2012b). Philosophically, it is opposed to traditional vote which consists in operating summations of individual preferences votes in order to obtain a candidate's value in an election or – more formally – a given situation. It asks each voter to evaluate each candidate instead of simply voting for one or more candidates according to one's preferences.

The 6 Feb. 2009 “Numbers Guy” column in the Wall Street Journal (“And the Oscar Goes to...Not Its Voting System”) contained: “*Prof. Balinski... calls (mean-based) range voting a “ridiculous method”, because it can be manipulated by strategic voters*” (Bialik, 2009). However, though in its beginning, MJ already sustains praises and attacks as well from scientific community. Greeted in particular by Nobel Laureates Kenneth Arrow, Robert Aumann and Eric Maskin for giving the possibility of enriching and moderating the expression of one's political choice (Balinski & Laraki, 2011), it is also disparaged by some scientists, in particular:

- Jean-François Laslier from CNRS (Centre National de Recherche Scientifique France) who details a whole theory exposing all the weaknesses inherent to the choice based on the evaluation of the best median (Laslier, 2010 ; 2012).
- Warren D. Smith, emblematic range voting (mean-based) defender who devoted a whole Web page to MJ weaknesses.
- Manzoor Ahmed Zahid who proposes a thesis highlighting MJ pathologies and prefers to it a mean-based method which he calls Borda Majority Count (BMC). This method is similar to evaluation voting except the fact that it a particular tie-breaking mechanism (Zahid, 2012). Manzoor Ahmed Zahid has in his thesis seven paradoxical results relating to the MJ that the BMC could circumvent. However, as we will see further in this article, mean-based aggregation methods (including the BMC) are not either free from any reproach. They can, on the contrary, prove more dictatorial than usual modes of poll which are already much disparaged.

In this article, we propose a voting method reconciling both concepts: the mean-based concept and the median-based one. But before achieving this, let us recall Borda Majority Count and Majority Judgment foundations.

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2. BORDA MAJORITY COUNT

Let a be an alternative and $\{g_1, g_2, \dots, g_k\}$ a set of grades in ascending order $g_1 > g_2 > \dots > g_k$. Let p_j be the number of voters who assign grade g_j to a , where $j = 1, 2, \dots, k$. One then defines the BMC of a by the formula

$$BMC(a) = \sum_{j=1}^k p_j \cdot [k - j]$$

For example, let us suppose that we have 5 grades: Excellent (E), Very Good (V G), Good (G), Acceptable (A), Reject (R). One assigns 4 points to E, 3 points to V G, 2 points to G, 1 point to A and 0 point to R. With this coding, let us suppose that 10 voters evaluated candidate a in the following way:

E	VG	G	A	R
1	2	3	3	1

Then $BMC(a) = 1 \times 4 + 2 \times 3 + 3 \times 2 + 3 \times 1 + 1 \times 0 = 19$

3. MAJORITY JUDGMENT (MJ)

Balinski and Laraki's idea to conceive a voting theory completely based on the median is justified by the fact that – according to authors – median is not as easier to manipulate as average.

MJ goes beyond arrovian framework: voters do not decide about candidates in terms of “... is preferred to...” any more but they evaluate them individually. In practice, in skating contest and wine competitions, voters are asked to evaluate each candidate in a language understood by all in order to rank candidates. The same applies to Range Voting, though the social ranking is established in different ways. While in Range Voting simply the average of all evaluations of a candidate is taken as the social evaluation, Balinski and Laraki take the median value of all evaluations of a candidate as their starting point, in order to reduce the possibilities for successful strategic behavior. In case two or more candidates get the same median value, a tie-breaking rule then decides the final social ranking. The MJ is exactly what has been detailed.

MJ is recent and successful proposal that attracts the attention of many researchers from theoretical and experimental points of view. Some of its extensions – inclusive of cases where extension comes from the side of MJ tie-breaking rule – are given in (Falco & Garcia-Lapresta, 2011), (Falco et al., 2013) and (Falco et al., 2014).

First of all, a common language for grading is required such that all grades are well understood by all persons involved. In their experiment at the 2007 French presidential elections, Balinski and Laraki took the following common language:

{Excellent, Very Good, Good, Acceptable, Poor, Reject}

Definition 3.1 (Common Language). One calls “common language” a set $\mathcal{L} = \{g_1, g_2, \dots, g_k\}$ strictly ordered by “>” such that $g_1 > g_2 > \dots > g_k$ ($g_i \geq g_j := g_i > g_j$ or $g_i = g_j$).

Note that one can have as a common language an infinite set such as the interval $[0, 1]$ of the real numbers with its natural ordering.

We can notice the possibility for a voter to allot the same evaluation to more than two candidates. For this reason, a voter can say that he values a candidate a as *Very Good*, a candidate b as *Very Good* and candidate c as *Good*. In Arrow's framework, these evaluations will be rendered by:

“ a is at least as good as b ”, “ b is at least as good as a ”, “ a is at least as good as c ”, “ b is at least as good as c ”, “ a is preferred to c ”, “ b is preferred to c ”, “ a is indifferent to b ”.

Definition 3.2 (Method of grading). A function F is a “method of grading” if it assigns to any profile a single grade (in the same language) for every candidate. So,

$$F: \mathcal{L}^{m \times n} \rightarrow \mathcal{L}^m$$

$$(g_{11}, g_{12}, \dots, g_{1n}, g_{21}, g_{22}, \dots, g_{2m}, g_{m1}, g_{mn}) \mapsto (g_1^*, g_2^*, \dots, g_m^*)$$

Where m is the number of candidates, n the number of judges or voters and g_i^* ($1 \leq i \leq m$) the final grade obtained by the candidate i .

Properties of F . It was shown (Balinski & Laraki, 2010) that F enjoys the following properties:

- *Neutrality*: If each voter reverses his preference, the selected candidate changes too.
- *Anonymity*: If one permutes voters (or even with other voters), the elected candidate does not change.
- *Pareto-consistency*: If all the voters prefer candidate x to candidate y then x must be the election winner.
- *Monotonicity*: If one or more voters reclassify a candidate x better, then x does not have ultimately to be less better placed in this election.
- *Independence of Irrelevant Alternatives*: To classify two candidates among several others, it must be enough to know preferences of each voter for these two candidates – their choices for others do not influence the classification between these candidates as well as the addition or the withdrawal of a candidate.

Definition 3.3 (Majority grade). Let a_i a candidate or competitor with grades $g_{i1}, g_{i2}, \dots, g_{in}$ where $g_{i1} \geq g_{i2} \geq \dots \geq g_{in}$. Then the “majority grade” or “majority rank” $f^{maj}(a_i)$ is by definition:

$$f^{maj}(a_i) = \begin{cases} f^{\frac{n+1}{2}}(g_{i1}, g_{i2}, \dots, g_{in}) & \text{if } n \text{ is odd,} \\ f^{\frac{n+2}{2}}(g_{i1}, g_{i2}, \dots, g_{in}) & \text{if } n \text{ is even.} \end{cases}$$

For example, if 5 judges allot grades 4, 8, 7, 9, 5 to a_i then $f^{maj}(a_i) = f^3(9, 8, 7, 5, 4) = 7$ and if 8 judges allot grades 9, 7, 3, 6, 5, 4, 5, 8 to a_i then

$$f^{maj}(a_i) = f^5(9, 8, 7, 6, 5, 5, 4, 3) = 5$$

Tie-Breaking (Balinski & Laraki, 2007). When the majority grades of two candidates are different, the one with the higher majority grade naturally ranks ahead of the other. The majority ranking \succ_{maj} between two candidates evaluated by the same jury is determined by a repeated application of the majority grade:

- If $f^{maj}(a) > f^{maj}(b)$ then $a \succ_{maj} b$
- If $f^{maj}(a) = f^{maj}(b)$ then one grade (majority grade) is dropped from the grades of each of the competitors, and the procedure is repeated.

Balinski and Laraki (2010) give the following example to illustrate this definition:

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Suppose a and b are evaluated by a jury of 7 voters as follows:

$a :$	85	73	78	90	69	70	73
$b :$	77	70	95	81	73	73	66

Then the ordered profile is:

$a :$	90	85	78	<u>73</u>	73	70	69
$b :$	95	81	77	<u>73</u>	73	70	66

After three iterations, one obtains $f_3^{maj}(a) = 78 > f_3^{maj}(b) = 77$ and then $a \succ_{maj} b$.

It is clear that the majority ranking always ranks a candidate ahead of another unless both are assigned an identical set of grades by judges.

In the case of many judges or voters (presidential elections for example), Balinski and Laraki present another simplified tie-breaking rule.

Definition 3.4 (Majority gauge). The “majority gauge” of a candidate with $f^{maj}(a) = \alpha$ is a triplet (p_A, α^*, q_A) where p is the number or percentage of the candidate's grades above the majority grade, q is the number or percentage of the candidate's grades below the majority grade, and

$$\alpha^* = \begin{cases} \alpha^+ & \text{if } p > q, \\ \alpha^- & \text{if } p \leq q. \end{cases}$$

α^* is called the candidate's modified majority grade.

Let β another majority grade, by definition, $\alpha^* > \beta^*$ iff $\alpha > \beta$ or $(\alpha = \beta$ and $\alpha^* = \alpha^+$ and $\beta^* = \beta^-)$.

Balinski and Laraki use majority gauge to define the majority-gauge-ranking \succ_{mg} .

Definition 3.5 (Majority-gauge-ranking). Let a and b be two candidates with majority-gauges (p_a, α_a^*, q_a) and (p_b, α_b^*, q_b) respectively. Then $a \succ_{mg} b$ or $(p_a, \alpha_a^*, q_a) \succ_{mg} (p_b, \alpha_b^*, q_b)$ iff $\alpha^* > \beta^*$ or $\alpha_a^* = \alpha_b^* = \alpha^+$ and $p_a > p_b$ or $(\alpha_a^* = \alpha_b^* = \alpha^-$ et $p_a < p_b)$.

Manzoor Ahmed Zahid (2012) shows that majority-gauge-ranking may not slice between candidates in exceptional cases. A theorem stated by Balinski and Laraki (Theorem 14.1 in (Balinski & Laraki, 2010)) shows that: $a \succ_{mg} b \implies a \succ_{maj} b$.

Ahmed Zahid (2012) then takes an example which illustrates a case where $a \succ_{maj} b$, but neither $a \succ_{mg} b$ nor $b \succ_{mg} a$.

Candidate	p	Excellent	Very Good	Good	Acceptable	Poor	Reject	q	Total
a	5	2	3	<u>3</u>	1	3	3	7	15
b	6	3	3	<u>2</u>	0	2	5	7	15

The majority gauge for a is $(5, Good^-, 7)$ and that of b is $(6, Bien^-, 7)$. As $q_a = q_b = 7$, majority-gauge-ranking does not give any decision while one easily checks that $a \succ_{maj} b$.

4. MEAN-MEDIAN COMPROMISE METHOD (MMCM)

Taking into account MJ and BMC, we work out an original method able to reach as well distinctly as jointly the two theories and which would circumvent a maximum of paradoxes presented farther in this paper. This method makes it possible to decide between the median and the average.

4.1. Concepts and description of the MMCM

The method that we present in this article is based at the same time on the median and the average. One proceeds by a division of the grades distribution for a candidate in 2^k intervals with the same amplitude. This division determines 2^{k+1} grades which are the points that limit the intervals. The function returns as candidate evaluation a real number corresponding to the arithmetic mean of 2^{k+1} selected grades while extreme values are excluded (Olympic average).

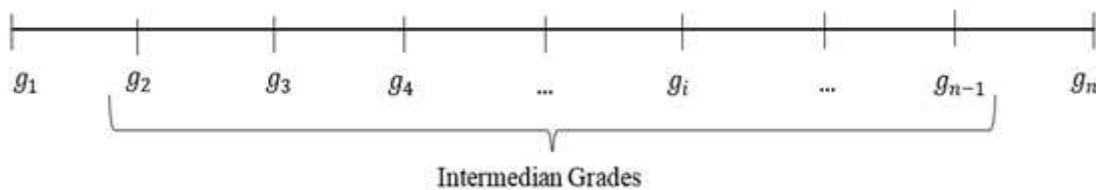


Figure 1. Division of the grades distribution in 2^k intervals.

Definition 4.1 (Olympic average). Let g_1, g_2, \dots, g_n be a succession of n data such as $g_1 \geq g_2 \geq \dots \geq g_n$. One calls “Olympic average” of these data the number:

$$\bar{g}_o = \frac{1}{n-2} \sum_{i=2}^{n-1} g_i$$

The Olympic average of n data is then the arithmetic average of these data, when the two extreme values (largest and smallest) are dropped.

Definition 4.2 (Amplitude of a division). Let $N = \{1, 2, \dots, n\}$ be the set of n judges, we call amplitude of a division the real number:

$$\varrho = \frac{n+1}{2^k} \tag{1}$$

with k a whole number called “division degree”.

Definition 4.3 (Intermedian grades). Let a_i be a candidate or competitor with grade $g_{i1}, g_{i2}, \dots, g_{in}$ such that $g_{i1} \geq g_{i2} \geq \dots \geq g_{in}$. A grade g_{ij} is called “intermedian” if and only if $\exists m \in N$ (with $1 \leq m \leq 2^k - 1$) such that $[\varrho \cdot m] = j$ where $[\varrho \cdot m]$ is the whole number that is nearest to $\varrho \cdot m$ and ϱ the amplitude of division for a fixed division degree k .

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We note \mathcal{M}_k the set of non-redundant intermedian grades obtained from a division degree k .

The so-defined \mathcal{M}_k is the set of data involved in the Olympic average calculation of points which are bounds (higher or lower) of 2^k intervals obtained after division.

Definition 4.4 (Average Majority Compromise). Let a_i be a candidate or competitor with grades $g_{i1}, g_{i2}, \dots, g_{in}$ where $g_{i1} \geq g_{i2} \geq \dots \geq g_{in}$ and $\mathcal{M}_k = \{g_{i1}^*, g_{i2}^*, \dots, g_{ij}^*\}$ the set of his or her intermedian grades obtained from division degree k . Then the ‘‘average majority compromise’’, or ‘‘average majority evaluation’’ or ‘‘average majority rank’’ $f^{mm}(a_i)$ is by definition:

$$f^{mm}(a_i) = \frac{1}{j} \sum_{m=1}^j g_{im}^* \quad (2)$$

For example, if 5 judges assign grades 4, 8, 7, 9, 5 to a_i . One fixes $k = 2$, $\varrho = \frac{5+1}{2^2} = 1.5$

The classified grades in a descending order are: 9, 8, 7, 5, 4.

$$\mathcal{M}_2 = \{f^{[1 \times 1.5]}, f^{[2 \times 1.5]}, f^{[3 \times 1.5]}\} = \{f^2, f^3, f^5\} = \{8, 7, 4\}$$

$$\text{Therefore } f^{mm}(a_i) = \frac{8+7+4}{3} = \frac{19}{3} = 6.33$$

And if 8 judges assign grades 9, 7, 3, 6, 5, 4, 5, 8 to a_i .

$$\text{For } k = 3, \varrho = \frac{8+1}{2^3} = 1.125$$

The classified grades in the descending order are: 9, 8, 7, 6, 5, 5, 4, 3

$$\begin{aligned} \mathcal{M}_3 &= \{f^{[1 \times 1.125]}, f^{[2 \times 1.125]}, f^{[3 \times 1.125]}, f^{[4 \times 1.125]}, f^{[5 \times 1.125]}, f^{[6 \times 1.125]}, f^{[7 \times 1.125]}\} \\ &= \{f^{[1.125]}, f^{[2.25]}, f^{[3.375]}, f^{[4.5]}, f^{[5.625]}, f^{[6.75]}, f^{[7.875]}\} \\ &= \{f^1, f^2, f^3, f^5, f^6, f^7, f^8\} \\ &= \{9, 8, 7, 5, 5, 4, 3\} \end{aligned}$$

$$\text{Then } f^{mm}(a_i) = \frac{9+8+7+5+5+4+3}{7} = \frac{41}{7} = 5.8$$

4.2. Tie-Breaking

When the average majority grades of two candidates are different, the one with the higher average majority grade ranks ahead of the other. The majority ranking \succ_{mm} between two candidates evaluated by the same jury is determined by a repeated application of the average majority ranking:

- One starts with $k = 2$
- If $f_k^{mm}(a) > f_k^{mm}(b)$ then $a \succ_{mm} b$
- If $f_k^{mm}(a) = f_k^{mm}(b)$ then the procedure is repeated for $k + 1$.

Let us take the following example to illustrate this definition:

Suppose that a et b are evaluated by a jury of 7 voters:

$a :$	85	73	78	90	69	70	71
$b :$	77	72	95	83	73	73	66

The ordered profile is:

$a :$	90	85	78	73	71	70	69
$b :$	95	83	77	73	73	72	66

$$q = \frac{7+1}{2^2} = \frac{8}{4} = 2$$

$$\mathcal{M}_2 = \{f^2, f^4, f^6\}$$

$$\mathcal{M}_2(a) = \{85, 73, 70\} \text{ and } f_2^{mm}(a) = \frac{85 + 73 + 70}{3} = \frac{228}{3} = 76$$

$$\mathcal{M}_2(b) = \{83, 73, 72\} \text{ and } f_2^{mm}(b) = \frac{83 + 73 + 72}{3} = \frac{228}{3} = 76$$

$f_2^{mm}(a) = f_2^{mm}(b) = 76$. By definition, one repeats the procedure for $k = 3$ and obtains:

$$q = \frac{7+1}{2^3} = \frac{8}{8} = 1$$

$$\mathcal{M}_3 = \{f^1, f^2, f^3, f^4, f^5, f^6, f^7\}$$

$$\mathcal{M}_3(a) = \{90, 85, 78, 73, 71, 70, 69\} \text{ and } f_3^{mm}(a) = \frac{90+85+78+73+71+70+69}{7} = 76.57$$

$$\mathcal{M}_3(b) = \{95, 83, 77, 73, 73, 72, 66\} \text{ and } f_3^{mm}(b) = \frac{95+83+77+73+73+72+66}{7} = 77$$

$$f_3^{mm}(b) = 77 > f_3^{mm}(a) = 76.57. \text{ Alors } b \succ_{mm} a.$$

In this example the average majority evaluation gives exactly the same result as the arithmetic average. This is due to the fact that the set of intermedian grades of each candidate is equal to the set of grades.

Definition 4.5 (Maximum division index). Let a_i be a candidate or competitor and $G_i = \{g_{i1}, g_{i2}, \dots, g_{in}\}$ set of a_i 's grades with $g_{i1} \geq g_{i2} \geq \dots \geq g_{in}$ and $\mathcal{M}_k = \{g_{i1}^*, g_{i2}^*, \dots, g_{ij}^*\}$ the set of his or her intermedian grades obtained with a degree of division k . Then, the smallest whole number k such as $G_i = \mathcal{M}_k$ is called *maximum division index* or *total division index*. It is denoted ν .

In the latter example, the maximum division index is $\nu = 3$.

Theorem 4.1. Let be a candidate or competitor and $G_i = \{g_{i1}, g_{i2}, \dots, g_{in}\}$ with $g_{i1} \geq g_{i2} \geq \dots \geq g_{in}$. If $k = 1$ then $f^{mm}(a_i) = f^{maj}(a_i)$.

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Proof :

For $\nu = 1$, $q = \frac{n+1}{2}$. The only intermedian grade is $f^{\lceil \frac{n+1}{2} \rceil}$. It is exactly the value $f^{mm}(a_i)$ will return.

If n is odd $\lceil \frac{n+1}{2} \rceil = \frac{n+1}{2}$ and if n is even $\lceil \frac{n+1}{2} \rceil = \frac{n}{2} + 1 = \frac{n+2}{2}$

It is clear that in this case $f^{mm}(a_i) = f^{maj}(a_i)$. ■

This theorem shows that if $k = 1$, the result given by the MMCM is the same as that given by the Balinski and Laraki method. The MMCM does not move away from the MJ philosophy in the sense that it is measure-based and not preference-based (Arrow's framework). Moreover on this subject, Balinski and Laraki (2007) argue that a judge may dislike a wine and yet give it a high grade because of its merit; he or she may also like a wine and yet, with great satisfaction, give it a low grade because of its demerits. A measure provides a common language, be it numerical, ordinal or verbal, to grade and classify. In this respect, Arrow's theorem means that, without a common language, there can be no consistent collective decision.

The MMCM can also give the same result as the MJ for other values of k different from 1.

Theorem 4.2. Let a_i be a candidate or competitor and $G_i = \{g_{i1}, g_{i2}, \dots, g_{in}\}$ with $g_{i1} \geq g_{i2} \geq \dots \geq g_{in}$. If $k = 1$ then $f^{mm}(a_i) = \frac{BMC(a_i)}{n}$.

Proof :

Let us consider $G_i = \{g_{i1}, g_{i2}, \dots, g_{in}\}$. For $k = \nu$, $G_i = \mathcal{M}_k$. So

$$f^{mm}(a_i) = \frac{1}{n} \sum_{j=1}^n g_{ij}$$

Yet

$$\sum_{j=1}^n g_{ij} = BMC(a_i)$$

Therefore,

$$f^{mm}(a_i) = \frac{BMC(a_i)}{n} \quad \blacksquare$$

This result shows that, when division is total, the result of the MMCM is equal to the one returned by the BMC. MMCM is in this case an intermediate method between MJ (highest median-based) and BMC (highest average-based). For values of k varying between 1 and ν , it allows to weigh and balance between the advantages and disadvantages of one or the other method.

5. MEAN-MEDIAN COMPROMISE: SIX PROBLEMS

In this section, we will apply MMCM to some examples which are considered as paradoxical results for MJ or for BMC and will see whether the MMCM is in accordance with what we hope.

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By means of scientifically founded analysis and studies, many scientists such as Jean-François Laslier, Warren D. Smith, Steven Brams or Manzoor Ahmed Zahid opposed as well MJ as any other form of aggregation based on "MaxMed".

Steven Brams (2011) is much more moderate than others because he accepts the MJ for ranking candidates when there is a small jury.

However, mean-based methods are failing as well as median-based one. The following examples stated in a form of problems illustrate weaknesses of mean-based methods and median-based one as well.

PROBLEM 1. Three friends wish to buy a common dish to share but one of them is a muslim and thus cannot eat the pork meat. All of them like the beef meat but the non-muslims slightly prefer the pork meat. Their profile is as follows:

	A1	A2	A3
Pork	9	9	0
Beef	8	8	9

The median for the pork is 9 and that for the beef is 8. However, the beef meat is the only one which meets the approval of all. In this case, the average is quite preferable to the median. The MMCM will give the same result as the BMC. Indeed, $v = 2$.

PROBLEM 2. Five judges evaluate the performance of two skaters. The table of the results is presented as follows:

	Judge 1	Judge 2	Judge 3	Judge 4	Judge 5
A	10	5	6	4	2
B	4	7	7	5	3

The average mark of skater A is 5.4 and that of the skater B is 5.2 and one can then deduce that $A > B$. Yet 4 judges out of 5 (either 80%) estimate that B is better than A. The median gives $B > A$.

Judge 1, because of his too high evaluation for A, imposed his preference to all the jury. This is not right for a community wanting to be democratic.

The MCMM will give the desired result as well. Indeed, the ordered profile is:

A	:	10	6	5	4	2
B	:	7	7	5	4	3

$$\mathcal{M}_2 = \{f^2, f^3, f^5\}$$

$$f^{mm}(A) = \frac{6+5+2}{3} = 4.33 \text{ and } f^{mm}(B) = \frac{7+5+3}{3} = 5.$$

Here, MMCM give the same result as MJ: $B \succ_{mm} A$.

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One could think that this example is only imaginary and that it cannot occur in the real world. The following example is somewhat realistic.

PROBLEM 3. In the Balinski and Laraki's framework, we get the following results for two candidates:

	5	4	3	2	1	0
Candidate A	25%	10%	18%	5%	12%	30%
Candidate B	0%	25%	30%	10%	25%	10%

$$BMC(\text{Candidate A}) = \frac{25 \times 5 + 10 \times 4 + 18 \times 3 + 5 \times 2 + 12 \times 1 + 30 \times 0}{100} = \frac{241}{100} = 2.41$$

$$BMC(\text{Candidate B}) = \frac{0 \times 5 + 25 \times 4 + 30 \times 3 + 10 \times 2 + 25 \times 1 + 10 \times 0}{100} = \frac{235}{100} = 2.35$$

The average gives candidate A victorious whereas it is the candidate B who is better accepted by the community. 30% of the population assigned mark 0 to candidate A against 10% only for the candidate B. Except for the maximum note, for all other marks higher than 0, B has a proportion of the population higher than that of A. The 25% of the population that expressed a strong preference for A carried off on 30% that rejected him. In Africa, for example, the winning of A would be likely to disturb social peace and stability.

Finally, probably Balinski was not completely wrong to qualify this method (mean-based) "ridiculous"!

According to MJ, candidate B is the winner. MCMM corroborates this result:

$$q = \frac{100 + 1}{2^2} = 25.25$$

$$\mathcal{M}_2 = \{f^{25}, f^{51}, f^{76}\}$$

$$\mathcal{M}_2(A) = \{5, 3, 0\} \text{ and } \mathcal{M}_2(B) = \{4, 3, 1\}$$

$$f_2^{mm}(A) = f_2^{mm}(B) = 2.66$$

As there is *ex aequo*, the procedure is repeated with $k = 3$.

$$q = \frac{100 + 1}{2^3} = 12.625$$

$$\mathcal{M}_3 = \{f^{13}, f^{25}, f^{38}, f^{51}, f^{63}, f^{76}, f^{88}\}$$

$$\mathcal{M}_3(A) = \{5, 5, 3, 3, 1, 0, 0\} \text{ and } f_3^{mm}(A) = \frac{17}{7} = 2.42$$

$$\mathcal{M}_3(B) = \{4, 4, 3, 3, 2, 1, 1\} \text{ and } f_3^{mm}(B) = \frac{18}{7} = 2.57$$

$$f_3^{mm}(B) > f_3^{mm}(A) \Rightarrow B \succ_{mm} A$$

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PROBLEM 4. This illustration was given by Zahid (2009).

	Excellent	Very Good	Good	Acceptable	Poor	Reject
A	9%	41%	50%	0%	0%	0%
B	4%	47%	3%	5%	12%	29%

According to the MJ, B wins. However A is judged at least “Good” by all the voters and B is more rejected than A by 29 % against 0 % for A. According to Borda Majority Count, A would be the winner. Thus, Mean-based methods are not as “ridiculous” as Balinski says it!

Here again, MMCM corroborate the BMC result:

$$\varrho = \frac{100 + 1}{2^2} = 25.25$$

$$\mathcal{M}_2 = \{f^{25}, f^{51}, f^{76}\}$$

$$\mathcal{M}_2(A) = \{Very\ Good, Good, Good\} \text{ and } \mathcal{M}_2(B) = \{Very\ Good, Very\ Good, Reject\}$$

By allotting the appropriate mark to each grade, one obtains:

$$\mathcal{M}_2(A) = \{4, 3, 3\} \text{ and } f_2^{mm}(A) = \frac{4+3+3}{3} = 3.33$$

$$\mathcal{M}_2(B) = \{4, 4, 0\} \text{ and } f_2^{mm}(B) = \frac{4+4+0}{3} = 2.66$$

$$f_3^{mm}(A) > f_3^{mm}(B) \Rightarrow A \succ_{mm} B$$

PROBLEM 5. This example is drawn from Ahmed Zahid's PhD thesis (2012).

	Excellent	Very Good	Good	Acceptable	Poor	Reject
A	1%	98%	1%	0%	0%	0%
B	50%	1%	0%	0%	0%	49%

According to the MJ, B wins whereas he is rejected by 49% of the population and A is judged at least “Good” by all the voters!

The BMC circumvent this paradox: A is the winner by the average. If we process this problem with the MMCM, we obtain exactly the same result:

$$\varrho = \frac{100 + 1}{2^2} = 25.25$$

$$\mathcal{M}_2 = \{f^{25}, f^{51}, f^{76}\}$$

$$\mathcal{M}_2(A) = \{Very\ Good, Very\ Good, Very\ Good\}$$

And

$$\mathcal{M}_2(B) = \{Excellent, Very\ Good, Reject\}$$

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By assigning the appropriate mark to each grade, one obtains:

$$\mathcal{M}_2(A) = \{4, 4, 4\} \text{ and } f_2^{mm}(A) = \frac{4+4+4}{3} = 4$$

$$\mathcal{M}_2(B) = \{5, 4, 0\} \text{ and } f_2^{mm}(B) = \frac{5+4+0}{3} = 3$$

$$f_3^{mm}(A) > f_3^{mm}(B) \Rightarrow A \succ_{mm} B$$

MMCM circumvents also this paradox of the MJ.

PROBLEM 6. Zahid (2009, 2012) presents another example similar to the precedent.

	Excellent	Very Good	Good	Acceptable	Poor	Reject
A	50%	0%	0%	0%	0%	50%
B	0%	0%	0%	0%	51%	49%

According to MJ, B wins whereas 100 % of the electorate estimate that he is at most “Poor” (with 49 % of rejection) and that A is considered to be “Excellent” by half of the electorate. The BMC is, here, preferable for MJ because the average of A is higher than that of B.

If we process this example by MMCM, we obtain:

$$q = \frac{100 + 1}{2^2} = 25.25$$

$$\mathcal{M}_2 = \{f^{25}, f^{51}, f^{76}\}$$

$$\mathcal{M}_2(A) = \{Excellent, Reject, Reject\}$$

And

$$\mathcal{M}_2(B) = \{Poor, Poor, Reject\}$$

By assigning the appropriate mark to each grade, one obtains:

$$\mathcal{M}_2(A) = \{4, 4, 4\} \text{ and } f_2^{mm}(A) = \frac{5+0+0}{3} = 1.66$$

$$\mathcal{M}_2(B) = \{5, 4, 0\} \text{ and } f_2^{mm}(B) = \frac{1+1+0}{3} = 0.66$$

$$f_2^{mm}(A) > f_2^{mm}(B) \Rightarrow A \succ_{mm} B$$

Examples 1 to 6 constitute paradoxical results detailing the individual incapacity of both mean and median approaches to provide logically plausible results in all cases. These problems are all solved by a method of compromise: the MCMM.

6. CONCLUDING REMARKS

In this article, we presented a new voting system obtained by hybridizing Borda Majority Count (2009, 2012) and Majority Judgement (Balinski & Laraki, 2007; 2010; 2012a). The extension of each one of the two above mentioned methods consists of dividing the ordered grades/marks list into m equal parts and retaining only the bounds of the internal parts. The value returned by Mean-Median Compromise Method (MMCM) is the average of selected grades/marks (intermedian grades). We propose also a tie-breaking mechanism by increasing the number of parts in the list of grades. The suggested tie-breaking rule differs as well from the BMC as the MJ.

MMCM coincides with BMC when the number of division is equal to the maximum division index and agrees with the MJ when the number of division is equal to 1. We thought fit that, as coincidence with the MJ can be also observed for values of number of division different from 1, it is necessary to begin the process with a number of division equal to 2.

The article refers to six didactic examples made from situations discussed for BMC and MJ. For each evoked paradox, if one of the two opposite methods circumvents it, MMCM circumvents it too. MMCM then appears like a better compromise between both methods. It keeps, indeed, the common properties of its parents, namely:

- it maximizes the voter-expressivity;
- it fulfills the criteria of anonymity, neutrality, unanimity, monotonicity and Arrow's Independence of the Irrelevant Alternatives;
- it is reduced to the approval voting if the allowed scores are restricted to the two values 0 and 1, i.e. each voter "approves (1)" or "disapproves (0)" each candidate;
- it proceeds in a round;
- it can cause the enthusiasm and the participation of the voters to use a new method of really democratic election where they can fully express their opinion;
- it enjoys immunity to candidate cloning ;
- it has the enjoyable property that "betraying one's favorite candidate does not pay the voter!" – voter does not take advantage of strategic vote any more.
- it easily allows having no opinion votes;
- it evades to some extent the Arrow and Gibbard-Satterthwaite impossibility theorems.

However, it also inherits the insufficiencies of its parents, namely:

- it is not Condorcet-consistent (Cf. Problem 1);
- it is not join-consistent and does not circumvent the no-show paradox (considering that MJ does not fulfill them);
- the offsetting effect of the arithmetic mean (which hurts BMC), although reduced, always exists.

The above-mentioned weaknesses are not in the same magnitude as for BMC and the MJ. Besides, Condorcet-consistency, as problem 1 shows it, is not always a desirable condition in an election. It would not be astonishing that one finds cases where the MMCM is vulnerable to join-inconsistency and the No-Show paradox which hurt MJ.

In a forthcoming publication, we will analyze MMCM and will study its vulnerability vis-à-vis paradoxes from which the BMC and the MJ suffer. We hope to show that the MMCM is

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less vulnerable to the tyranny of the majority which hurts MJ and expose the fact that BMC gives too much power to strategic voters who assign extreme grades. MCMM stands it.

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