Evolution in a Walrasian setting

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June 2015

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Abstract

This paper models the dynamic of a sector where firms imitate the technology of leading firms. While it would seem natural to expect that managers will aim at producing with the technology that produces the highest benefits, if many other managers also follow this behavior, the market structure might be modified so much that the advantage associated with a high-profit technology might be erased or even reverse. By modeling this imitation process with replicating dynamics, we find that even if the parameters of the economy are continuous through time and the economy follows a path of competitive equilibria, endogenous discrete jumps in technological choices occur.

*The authors thank Mario Ibarburu and participants of seminars at the Economics Department of the Social Sciences School, Uruguay, for useful comments.

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1 Introduction

Managers of firms are confronted continuously with the decision of whether to keep producing with their firm's existing technology or to shift to other leading alternatives. It would seem natural for a manager, even under incomplete information, to observe available technologies and to choose for his firm the one that gives rise to the highest profits to other firms that already have it. However, some markets might provide a certain advantage if not too many producers are using the same technology so that, if too many managers choose it simultaneously, the benefits of this technology might even be reversed. In other words, if a manager decides to switch to a technology that is observed to have the highest profits, it could be the case that many other managers also make the same decision simultaneously.

This paper has two main contributions. On the hand, it proposes a theoretical model to study this dynamic game of imitation under incomplete information. To do so, we propose an approach using replicating dynamics, extending this way the work of Schlag (1998), Harrington et al. (2005), Ania (2008), Bergin and Bernhardt (2009), Apesteguia et al. (2010) and Duersch et al. (2012). On the other hand, and more importantly, we characterize dynamics along its equilibrium path and find that, under robust conditions, even being always in equilibrium, endogenous jumps in equilibrium variables can occur.[1] As in (Perla and Tonetti, 2014) we consider a distribution of heterogeneous firms producing with heterogeneous technologies, but we consider the rates of profits as the real engine of the decisions of managers. Indeed, we do not focus on growth, but instead our main point of attention is on the characteristic of the equilibrium manifold and the evolution along an equilibrium path on this manifold, showing that "technological crises" can occur as a result of rational decisions.

This paper is structured as follows. Section 2 presents the model of a private-ownership economy, its parameters, and the definition of equilibrium. In particular, we will define firms, consumers and Walrasian equilibria. Section 3 defines the equilibrium set and studies its properties. To do so, we study excess demand functions, as well as regular and critical economies.

[1] It is worth mentioning that there is a vast literature on the evolution of Walrasian behaviour, although not via imitation, since the seminal work of Vega-Redondo (1997). For example, Schenk-Hopp(2000), Huang (2003), Ben-Shoham et al. (2004), Huang (2011).

The dynamical behaviour of pure system is introduced in section 4. We consider that imitation play a significant role when the decision makers have incomplete information. In section 5 we analyze the stability of the dynamical equilibrium using the Liapunov method. In section 6 we consider a numerical example and finally, some conclusions are given.

2 The economy

In this section, we introduce the model of our economy. We will consider a competitive market with two goods, two types of firms and two types of consumers. Within each type, firms will use identical technologies, while consumers will have identical preferences, and endowments of both goods and shares in companies. We will assume that markets are competitive in the sense that every good is traded in the market at a publicly known price, and consumers trade to maximize their own welfare, while firms produce to maximize profits. We will formalize the structure of consumers, firms and equilibria in this section.

Since there are two goods in the economy, the *commodity space* is $\mathbb{R}^2$ while the *consumption set* and *price set* is $\mathbb{R}^2_+ = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$. We will also use the notation $\mathbb{R}^2_{++} = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 > 0\}$.

2.1 Firms

We consider an economy with a finite but large number N of firms, which will be indexed by $\mathcal{F} = \{1, \ldots, N\}$. There will be two types $j \in \{1, 2\}$ of such firms. There are n_1 firms of type 1 and n_2 of type 2, with $n_1 + n_2 = N$. We will index firms within each type by $\mathcal{F}_j = \{0, ..., n_j\}$, with at least one firm of one of the two types. The proportion of firms of type j is given by $N_j = n_j/N$, and one should keep in mind that as $N \to \infty$, N_j takes values in the continuum $[0, 1]$.

Firms are characterized by their *technological set* Y_1 or Y_2, according to whether they are of type 1 or 2, respectively. Without loss of generality, we suppose that technology is costless to acquire. That is, in each period each manager must choose between producing according to the technology represented by Y_1 or Y_2 and this choice does not imply any additional cost or loss of profit. The set of available technologies will be denoted by $\mathcal{T} = \{Y_1, Y_2\}$. We consider that there is no principal-agent problem, that is, that the interests of managers are the same of owners, which is to maximize profits. We assume that technological sets satisfy the following standard properties, for $j = \{1, 2\}$: (i) Y_j is closed and convex; (ii) $Y_j \cap \mathbb{R}^2_+ = \{0\}$; (iii) $-\mathbb{R}^2_+ \subset Y_j$; and, (iv) Y_j is bounded from above, i.e., there exists some $a_j \in \mathbb{R}^2_+$ satisfying $y \leq a_j$ for all $y \in Y_j$.[2]

If Y_j is a production set and $p \in \mathbb{R}^2_{++}$ is a price vector, then the *profit function* at price level p is the function $\pi^j : Y_j \to \mathbb{R}$ defined by $\pi^j(y) = p \cdot y$. Thus, firm j 's goal is to solve the problem

$$\max_{y \in Y_j} p \cdot y, \tag{1}$$

The function $y^j : \mathbb{R}^2_{++} \to \mathbb{R}^2$ that solves problem (1) is the *supply of a firm* of type $j \in \{1, 2\}$. When summarising the supply functions of all firms in the economy we will use the notation $\mathbf{y} = (\mathbf{y}^1, \mathbf{y}^2) \in \mathbb{R}^{2n_1} \times \mathbb{R}^{2n_2}$, where $\mathbf{y}^j$ is the vector of supply functions of the n_j firms of type j.

2.2 Consumers

The are M consumers in the economy distributed into type 1 or type 2. We will index consumers by $I = \{1, \ldots, M\}$. There are m_i consumers of type i, $i \in \{1, 2\}$, with $m_1 + m_2 = M$. Therefore, the proportion of consumers of type i is $M_i = m_i/M$. We will index consumers in each type by $I_i = \{0, \ldots, m_i\}$ with at least one consumer of one type. All consumers of type i have identical C^∞ utilities $u_i : \mathbb{R}^2_+ \to \mathbb{R}$ and identical initial endowments of goods $w^i = (w^i_1, w^i_2) \in \mathbb{R}^2_+$. We assume utilities satisfy the following standard properties: (i) the bordered Hessian of u_i is non-zero at every x; (ii) the set $\{x' : u(x') \geq u(x)\}$ is closed in $\mathbb{R}^2$ for every $x \in \mathbb{R}^2_{++}$.[3]

The wealth of a consumers is derived from individual endowments of commodities but also from ownership claims (shares) of profits of firms, which

[2] As is standard, it is sometimes convenient to describe the production set using a function $F_j : \mathbb{R}^2 \to \mathbb{R}$ such that $Y_j \subset \mathbb{R}^2$ and $F_j(x_1, x_2) \leq 0$, $\forall (x_1, x_2) \in Y_j$, and $F(x_1, x_2) = 0$ if and only if (x_1, x_2) is an element of the boundary of Y_j. These functions are called *technological functions*. See for instance (Mas-Colell, 1989).

[3] See Mas-Colell and Nachbar (1991).

we also assume to be identical within types. Hence, each consumer $i \in I$ has a claim to a non-negative share θ_{ij} of the profit of each firm $j \in \mathcal{F}$ in a way that $\sum_{i=1}^{M} \theta_{ij} = 1$ for each j. This implies that the individual wealth W_i of a consumer of type i is given by

$$W^i(n_1, n_2, p) = p \cdot w_i + n_1\theta_{i1}\pi_1(p) + n_2\theta_{i2}\pi_2(p).$$

It is important to highlight that the wealth of an individual depends on the distribution (n_1, n_2) of firms in the economy. In the next section, when we introduce dynamics to our model, this distribution will change endogenously so that the actions of a manager looking to use the most profitable technologies will have an immediate impact over the wealth of agents.

Now, the budget set B^i of a consumer of type i is given by

$$B^i(n_1, n_2, p) = \{x^i \in \mathbb{R}^2_+ : p \cdot x^i \leq W^i(n_1, n_2, p)\}.$$

Thus, given the distribution of firms (n_1, n_2) and price level p, the optimization problem of a consumer of type i is given by

$$\max_{x \in \mathbb{R}^2_+} u_i(x) \ \ s.t. \ \ x \in B^i(n_1, n_2, p). \tag{2}$$

The function $x^i : \mathbb{R}^2_{++} \to \mathbb{R}^2$ that solves optimization problem (2) is called the *individual demand function* of a consumer of type i. When summarising the demand functions of all consumers in the economy, we will use the notation $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2) \in \mathbb{R}^{2m_1} \times \mathbb{R}^{2m_2}$, where $\mathbf{x}^i$ is the vector of demand functions of the m_i consumers of type i.

2.3 Walrasian equilibrium

As discussed above, a private-ownership economy is determined by the following factors: the number and distribution of firms (n_1, n_2) and their technologies (Y_1, Y_2); the number and distribution of consumers (m_1, m_2), their preferences (u_1, u_2), their initial endowments (w_1, w_2), and their portfolios of shares $(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})$. This gives rise to the following definition.

Definition 1 (Private-ownership economies): A *private ownership economy* $\mathcal{E}$ is a choice of parameters $(n_j, Y_j, m_i, u_i, w_i, \theta_{ij})$ with $i, j = \{1, 2\}$. We will write

$$\mathcal{E} = \mathcal{E}(n_j, Y_j, m_i, u_i, w_i, \theta_{ij}).$$

Now, the notion of a price-taking equilibrium for a competitive private ownership economy is that of a Walrasian equilibrium, which we now define.

Definition 2 (Walarsian equilibrium): Consider a private-ownership economy

$$\mathcal{E}(n_j, Y_j, m_i, u_i, w_i, \theta_{ij}).$$

We say that a vector of demand-supply functions $(\mathbf{x}^*, \mathbf{y}^*) = (\mathbf{x^{1*}}, \mathbf{x^{2*}}, \mathbf{y^{1*}}, \mathbf{y^{2*}})$ and a system of prices $p^* = (p_1^*, p_2^*)$ constitute a *Walrasian equilibrium*, or *competitive equilibrium*, if:

1. For each firm in $\mathcal{F}$, y^{j*} solves the optimisation problem (1) of a firm of type j;.
2. For each agent in I, x^{i*} solves the optimisation problem (2) of a consumer of type i;.
3. Aggregate demand equals aggregate supply. That is,

$$\sum_{i=1}^{m_1} x_i^{1*} + \sum_{i=1}^{m_2} x_i^{2*} = m_1 w^1 + m_2 w^2 + \sum_{j=1}^{n_1} y_j^{1*} + \sum_{j=1}^{n_2} y_j^{2*}.$$

Note in Definiton (2), that the market-clearing condition can be written succinctly as

$$m_1 \left(x^{1*} - w^1\right) + m_2 \left(x^{2*} - w^2\right) = n_1 y^{1*} + n_2 y^{2*}. \tag{3}$$

3 The equilibrium set

Through the rest of the paper, we will suppose $(Y_j, m_i, u_i, w_i, \theta_{ij})$ are all fixed parameters. That is, we will suppose that the only parameter of the economy is the distribution of firms (n_1, n_2) according to its type. Furthermore, we will also suppose that the total number of firms N is fixed (albeit very large). With this assumption, n_2 is determined once a choice of n_1 is made since $n_2 = N - n_1$. To highlight this choice of parameters, we will denote an economy by $\mathcal{E}_{(N,Y_j,m_i,u_i,w_i,\theta_{ij})}(n_1)$ or, if no confusion arises, simply by $\mathcal{E}(n_1)$. In other words, speaking of an economy is equivalent to choosing a parameter

n_1.

In this section we will explore the equilibrium set of an economy $\mathcal{E}(n_1)$. Recall that the Walrasian corresponde is the set-valued function that assigns to each economy $\mathcal{E}(n_1)$ its set of competitive equilibria. This idea gives rise to the notion of the *equilibrium set* as the set of all pairs *economies-equilibria*. We will show that studying this set is an important first step in our analysis. Thus, in this section we formalize the notion of excess demand functions, the equilibrium set and regular and critical economies. Intuitively, a regular economy n_1 is one where equilibrium prices will be determinate (that is, prices are locally continuous functions of n_1, while a critical economy is one where small perturbations of n_1 lead to large perturbations of its equilibirum price.

3.1 Excess demand functions

The notion of a competitive equilibrium can be rephrased in terms of zeroes of aggregate excess demand functions. To see this, consider the private-ownership economy $\mathcal{E}(n_1)$. For a fixed n_1, we say that the *individual excess demand function* $\zeta^i_{n_1} : \mathbb{R}^2_{++} \to \mathbb{R}^2$ of a consumer of type i is given by

$$\zeta^i_{n_1}(p) = x^i_{n_1}(p) - w^i.$$

Similarly, for a fixed economy n_1 we define the *aggregate excess demand function* of this economy, $\Psi_{n_1} = (\Psi^1_{n_1}, \Psi^2_{n_1}) : \mathbb{R}^2_{++} \to \mathbb{R}^2$, by

$$\Psi_{n_1}(p) = m_1\zeta^1_{n_1}(p) + m_2\zeta^2_{n_2}(p) - n_1 y^1(p) - n_2 y^2(p).$$

Notice that the aggregate excess demand function Ψ_{n_1} of economy $\mathcal{E}(n_1)$ satisfies the following properties:

- Ψ_{n_1} is homogeneous of degree zero, i.e., $\Psi_{n_1}(\lambda p) = \Psi_{n_1}(p)$ for all $\lambda > 0$;
- Ψ_{n_1} satisfies Walras law, i.e., $p \cdot \Psi_{n_1}(p) = 0$, for all p.

3.2 The equilibrium set

With the introduction of excess demand functions above, we can see that the set of equilibrium prices of economy $\mathcal{E}(n_1)$ consists of vectors p such that

$\Psi_{n_1}(p) = 0$. We denote the *equilibrium set of economy* $\mathcal{E}(n_1)$ by

$$\Gamma_{n_1} = \{p \in \mathbb{R}^2_{++} : \Psi_{n_1}(p) = 0\} \subset \mathbb{R}^2_{++}.$$

Notice that for each equilibrium price, demand and supply functions $(\mathbf{x}, \mathbf{y})$ are fully determined so equilibrium prices also determine fully Walrasian equilibria. Whenever it is convenient to highlight that the parameter n_1 is changing, we will explicitly write this dependence in the excess demand function by $\Psi_{n_1}(p) = \Psi(n_1, p)$ and its corresponding equilibrium set by

$$\Gamma = \{(n_1, p) : \Psi(n_1, p) = 0\}.$$

Since excess demand functions satisfy homogeneity of degree zero, it allows us to choose a suitable price normalization; thus, we will let prices to be in the simplex

$$\Delta = \left\{(p_1, p_2) \in \mathbb{R}^2_{+} : p_1 + p_2 = 1\right\}.$$

Note that, if p is in the simplex and p_2 is known, then p_1 is also known, since $p_1 = 1 - p_2$. Also notice that, from Walras law, $p_1 \Psi^1_{n_1}(p) + p_2 \Psi^2_{n_1}(p) = 0$ for all p. Combined, these two facts suggest we can consider a *restricted excess demand function* $\hat{\Psi}_{n_1} : [0, 1] \to \mathbb{R}$ such that $\hat{\Psi}_{n_1}(p_1) = \Psi^1_{n_1}(p)$. Hence, a price $p = (p_1, p_2)$ is an equilibrium for the economy $\mathcal{E}(n_1)$ if and only if $\hat{\Psi}_{n_1}(p_1) = 0$. Thus the set Γ_{n_1} can also be represented as

$$\Gamma_{n_1} = \left\{p \in \Delta : \hat{\Psi}_{n_1}(p_1) = 0\right\} \subset \Delta.$$

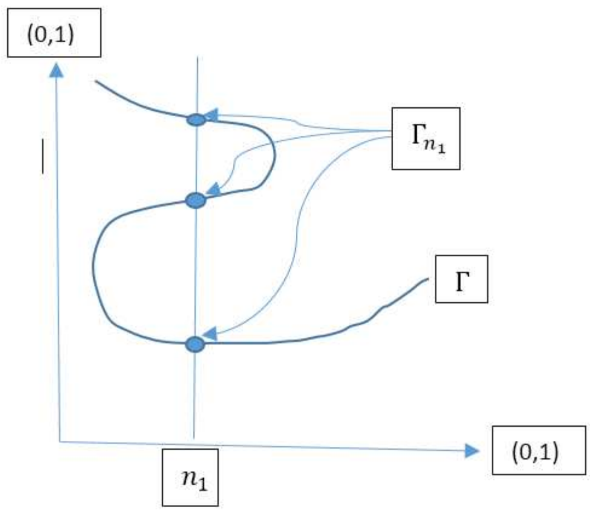
Figure 1 shows the intuition behind the equilibrium sets Γ and Γ_{n_1}. First, notice that the horizontal axis consists of the set of economies. Similarly, the vertical axis shows the possible values of prices p which is isomorphic to the set $[0, 1]$. Now, for a fixed economy n_1 as depicted, its equilibria (three in this case) form the set Γ_{n_1}. All pairs *economies-equilibria* together form the set Γ.

3.3 Regular and critical private-ownership economies

The previous figure showed the intuition behind the equilibrium set Γ.[4] If this diagram were a true representation of our model, one could see that there

[4] See also Accinelli and Puchet (2011) and Balasko (2009).

Fig. 1: The equilibrium set Γ.



are two types of economies. On the one hand, there are economies such as n_1 where each price equilibrium has the property that small (i.e. infinitesimal) perturbations to n_1 would be accompanied by small changes in this equilibrium price. A price with this property will be called a *regular price*. Notice that, under this intuition, all equilibrium prices of n_1 are regular. When this is the case, the economy (that is, n_1) will be called a *regular economy*.

On the other hand, notice that economy n_1' has a different structure. While the *lower* equilibrium is regular (since an infinitesimal perturbation of n_1' leads to a small perturbation in this equilibrium price), the *upper* equilibrium is different. In this case, a perturbation $n_1' - \epsilon$ of n_1' would be accompanied by a continuous change of the price along one of the branches, but a perturbation $n_1' + \epsilon$ of n_1' would lead to a *jump* of the equilibrium price to the lower branch. An equilibrium price with these features will be called a *critical price*. An economy that has at least one critical price will be called a *critical economy*. In other words, an economy will be called regular if and only if *all* of its prices are regular prices, or otherwise a critical economy. We formalise these ideas in the following definition.

Definition 3 (Regular and critical prices): Consider a private ownership economy $\mathcal{E}(n_1)$, with its corresponding excess demand function $\Psi_{n_1} : \Delta \to \mathbb{R}^2$, and equilibrium price set $\Gamma_{n_1} = \{p \in \Delta : \Psi_{n_1}(p) = 0\}$. Then, we say that an equilibrium price p is a *regular price* if and only if the rank of the Jacobian of the excess demand function evaluated at p is equal to 1. Otherwise, we say p is a *critical price*.

Definition 4 (Regular and critical economies): We say that an economy $\mathcal{E}(n_1)$ is regular if all of its equilibrium prices are regular. If at least one if its equilibrium prices is critical, we say the economy is critical.

As a consequence of the Sard-Smale Theorem it follows that the set of regular economies form an open and dense subset of the set of economies. This means that if a regular economy suffers a perturbation on its fundamentals the perturbed economy remains regular. This, however, stops being true if the economy is singular. Nevertheless, the set of critical economies is " meager" in the set of economies, that is, it is a subset of Lebesgue-measure zero.

4 Dynamical behaviour

Suppose that in equilibrium at time t, the distribution of firms is given by $N(t) = (N_1(t), N_2(t))$ and that the share profits of a firm of type Y_i are revealed to be higher than profits of a firm of type Y_j. Then, rational behavior would suggest a manager to produce with technology Y_i. But if, at any given time other managers also decide to change the technology under which their firms produce, certain characteristics of markets associated with each technology -including profits- can also be modified. Moreover, it is possible that if a large enough number of managers decide to migrate to the branch that is currently offering higher benefits, this advantage can be reversed in the future and the branch that currently has the greatest benefits might become the one that will offer the lowest level of profits. In this case, the best individual response to the decision of others, would be to not change the technology or branch.

The key question is then based on which factors will managers make the decision of changing technologies used, given that until now they do not know what others are doing. Regardless of the answer to this question, it seems natural to assume that the number of firms producing according to the technology that in each time appears to be the most successful increases. Equivalently, at each time, the share of firms producing according with the technology with the highest profits increases. Such evolutionary process can be modeled by *replicating dynamics*. In this way, this dynamical behaviour becomes a good tool to describe and understand the evolution of the economy. This section introduces this dynamical behaviour to a private-ownership economy, first as a static game, and then by an evolutionary process.

As in previous Sections, we suppose there is a a finite (albeit very large) number of firms N. We suppose this number remains constant (i.e., no entries or exits), although each firm will endogenously change its type through time. At each time period t, the number of firms of type $j \in \{1,2\}$, is given by n_j so that $N = n_1(t) + n_2(t)$, $\forall t$. The proportion of firms of type j at each t is thus given by $N_j(t) = n_j(t)/N$.

4.1 Static game

Consider the private-ownership economy $\mathcal{E}(n_1)$. The distribution of firms is thus given by (N_1, N_2). In this situation, the manager of each company must choose between changing its existing technology or to keep it. It is possible to model this situation as a $N-$population normal form game, where each firm has two available strategies $\mathcal{T} = \{Y_1, Y_2\}$. Recall that the supply of a firm of type $j \in \{1,2\}$ is the function $y_j : \mathbb{R}^2_{++} \to \mathbb{R}^2$, and its profit function is given by $\pi_j : Y_j \to \mathbb{R}$. The function $\pi_i^* : \Delta \to \mathbb{R}$ defined by $\pi_i^*(N_1, N_2) = \pi_i(y^{*i}(N_1, N_2))$ represents the profits in equilibrium of a firm operating according with the technology Y_j. Consequently we will let $\Pi_i^*(N_1, N_2) = \Pi_i(y^{*i}(N_1, N_2))$ to symbolize the rate of profits of a firm of type $i \in \{1,2\}$ under equilibrium conditions. The two definitions below will establish a firm's best response. Indeed, a firm's best response is naturally a choice of technology Y_j with higher rate of profits. Furthermore, a distribution of firms is a equilibrium distribution if and only if the rate of profits of all firms is the same.

Definition 5: We say that to produce according with Y_j is a best response given the distribution $N = (N_1, N_2)$, if the rate of profits producing according with technology Y_j is greater than or equal to the percentage obtained when produced according to the technology Y_i. That is,

$$\Pi_j(y^{*j}(N)) \geq \Pi_i(y^{*i}(N)).$$

Definition 6: A distribution $N^* = (N_1^*, N_2^*)$ is an equilibrium distribution (or an equilibrium in mixed strategies) for this game if and only if

$$\Pi_j(y^{*j}(N^*)) = \Pi_i(y^{*i}(N^*)) \tag{4}$$

for all $i, j \in \{1, 2\}$.

Therefore, a Nash equilibrium for the $N - population$ game

$$G = \{Y_i, N_i, i \in \{1, 2, \}\}$$

is a distribution N^* verifying equation 4.

4.2 Evolution

When introducing time to the static game defined above, dynamic competition increases difficulties for managers. Indeed, they know that from time to time they will have to make sudden decisions even if they do not feel ready to act, because otherwise one or more competitors will come up with a better solution, pushing a firm to exit. Certainly, if owners lack information or sufficient time to perform calculations, some mistakes in decisions can happen. In this framework to imitate the behavior of the agents considered most successful can be a good strategy. So, a bounded rational vision of the top competent team set the direction of the firm and define the competence of the owner or manager involved. One of the fundamental issue in evolutionary game theory concerns the relationship between predictions as consequence of myopic decisions made by simple agents and those provided by traditional rationality-based concepts.

We start this section by introducing a *replicating* dynamical system in Definition 7 below. Recall that equilibrium supply functions depend on the distribution of firms so that $y_i^* = y_i^*(N_1, N_2)$.

4.2.1 The replicator dynamical system

The degree of competition in a given economy depend on the variance on the profit rates, the larger the spread the grater the degree of dynamic competition will be. In equilibrium essentially all the firms are obtaining the same result. I the next definition we introduce a function that measure the degree of dynamical competition in an economy, from the difference in profit rates.

Definition 7: **(Augmented replicating dynamical system)** Consider an economy n_1 where prices are given by $p^*(n_1) \in \Gamma_{n_1}$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous, increasing function such that $\phi(0) = 0$. Then, the *(augmented) replicating dynamical system* is given by

$$\begin{aligned} &\tfrac{d}{dt} N_i = \left\{ \phi_i \left(\Pi_i \left(y_i^*(N) \right) - \Pi_j \left(y_j^*(N) \right) \right) \right\} N_i, && (i) \\ &\tfrac{d}{dt} N_j = - \tfrac{d}{dt} N_i, && (ii) \\ &N(t_0) = (N_1(t_0), N_2(t_0)) >> 0, && (iii) \end{aligned} \tag{5}$$

A few remarks regarding Definition 7 are in order:

- In condition (i), the function $\phi_i : \mathbb{R} \to \mathbb{R}$ represents the growth rate in the share of each technology per unit time,

  $$\frac{\dot{N}_i}{N_i} = \lim_{\Delta t \to 0} \frac{\Delta N_i}{N_i} \frac{1}{\Delta t} = \phi_i((\Pi_i(y_i^*(N))) - \Pi_j(y_j^*(N)))].$$

  Here by $\dot{N}_i(t)$ we symbolize the derivative of $N_i(t)$ we respect to time t i.e.; $\frac{d}{dt} N_i(t) = \dot{N}_i(t)$. To simplify the notation we do not write the variable t.

- To guarantee that this system of differential equations induces a well defined dynamics on the state-space given by the simplex, i.e,

  $$\Delta = \left\{ (N_1, N_2) \in \mathbb{R}^2_{++} : N_1 + N_2 = 1 \right\},$$

  $\phi_i : \mathbb{R}_+ \to \mathbb{R},\ i \in \{1, 2\}$ are Lipschitz continuous functions in $(0, \infty)$.

- Recall that $N_2(t) = 1 - N_1(t)$ at all t. Therefore, condition (ii) is automatically satisfied. This forces no creation or destruction of firms and instead allows us to focus on the redistribution of firms according to their type. Furthermore, as mentioned before, this fact also allows us to assume that y_i^* depends only on N_1.

- Condition (iii) establishes initial conditions. We assume that at time $t = t_0$, the proportion of both types of firms is positive, since if at time t_0 either n_1 or $n_2 = 0$, then the evolution given by the system (5) makes no sense. Equivalently, we could have asked for $N(t_0)$ to be in the interior of the simplex Δ.

- Finally, recall that under the assumptions of our model, the following identity is always verified,

  $$m_1(x^{1*}(N_1(t))-w^1)+m_2(x^{2*}(N_1(t))-w^2) = n_1(t)y^{1*}(N_1(t))+n_2y^{2*}(N_2(t))$$

  which implies that market clearing conditions always hold.

Note that if equilibrium prices at time t_0 are regular, then immediately after a perturbation on the distribution of firms, these prices will not change much and therefore the benefits of firms will suffer only small modifications. Conversely, if these prices correspond to a critical economy, a small perturbation on the distribution of firms can give place to large and discontinuous changes in the behavior of the economy. Thus, the assumption of the existence of Liptchitz-continuous functions ϕ_j can be considered only if prices are regular.[5]

Notice that a migration process of firms to technologies with higher expected benefits, will lead to changes in the excess demand function, and therefore also in equilibrium prices. Moreover, the whole economy $\mathcal{E}_{n_1}$ is changing along this process; that is, for each t there is a different economy. Furthermore, note that even at time $t = t_0$ the inequality $\Pi_i(y^{*i}(N_1)) > \Pi_j(y^{*j}(N_1))$ $i \neq j$ holds, but this does not mean that every manager will choose the technology Y_i, because profits at time $t > t_0$ depend on the future

[5] This was also the main reason behind the need for the lengthy discussion of continuous random selections in the previous Section.

distributions of the firms over technology types, and these future distributions are unknown at time t_0, or because not necessarily every manager will know the profits of other firms.

A distribution (N_1^e, N_2^e) defines a dynamical economic equilibrium in the sense of the replicator dynamics (5), if and only if $\Pi(y^{*1}(N_1^e)) = \Pi(y^{*2}(N_1^e))$. Under these hypothesis of our model, a Walrasian equilibrium does not verify necessarily this equality. Moreover, a Walrasian equilibrium $(p, \bar{x}, \bar{y}) \in Eq_{n_1}$ is at the same time a dynamical equilibrium, if and only if the rate of profits of the firms are the same regardless the technology used. So, if in $t = t_0$ this equality on the profits holds, then, the economy will not change in the future, unless it happens some perturbation in its fundamentals for reasons exogenous to the model. (The stability in the Lyapunov sense of this equilibrium will be considered in the next section). In other case, the economy is evolving in a transition process corresponding to a trajectory defined by the dynamical system (5), along this trajectory the economy is always in a Walrasian equilibrium. Along this trajectory the economy is always in a Walrasian equilibrium. Certainly these equilibria change according with the evolution of the distribution of the firms on the set of available technologies. This means that, in each time, prices, allocations and plans of production correspond to a Walrasian equilibria. More specifically, for each $t > t_0$ and $n_1(t)$ such that $n_1(t) = nN_1(t)$, $p^*(n_1(t)) \in Eq_{n_1}(t)$.

If the dynamical equilibrium (N_1^e, N_2^e) is an attractor and if in time $t = t_0$ the initial distribution $(N_1(t_0), N_2(t_0))$ is in the basin of attraction of this equilibrium, then $(N_1(t), N_2(t))_{t\to\infty} \to (N_1^e, N_2^e)$ and so

$$\mathcal{E}_{n_1(t),n_2(t)} \to \mathcal{E}_{n_1^e,n_2^e}.$$

This evolution will takes place along a trajectory of economies $\mathcal{E}_{n_1(t),n_2(t)}$ in equilibrium. Along this trajectory only smooth changes can occur if the economies are regular. Then the trajectory is a continuous path. Big changes necessarily take place if this trajectory crosses a singular economy.

Definition 8: Let $N(N_0, \cdot) : R_+ \to \Delta$ be the solution of the equation (5) where $N(N_0, t_0) = N_0$, being $N_0 \in \Delta^0$ the initial distribution (percentage) of the firms, then the transition path will be defined by the trajectory:

$$\mathcal{T}(t) = \{(t, N(N_0, t)) \in [t_0, \infty) \times \Delta\}.$$

This is a Walrasian equilibrium trajectory, meaning that for each t and $N(N_0, t)$ for each $n(t) = nN(N_0, t)$ the corresponding $p^*(n_1(t)) \in Eq_{n_1(t)}$ $\forall t \geq t_0$.

To simplify the notation, from now on we write $\Pi^*(N_1)$ to represent $\Pi_i(y^{*i})$ $i = 1, 2$.

It is easy to see that the trajectory remains in the simplex if the condition

$$\dot{\overline{N_1 + N_2}} = \dot{N}_1 + \dot{N}_2 =$$

$$= \phi_1 \left(\Pi_1(N_1) - \Pi_2(N_1) \right) N_1 + \left(\phi_2 \left(\Pi_2(N_1) - \Pi_1(N_1) \right) (1 - N_1) = 0. \right.$$

is verified in all time $t \geq t_0$, this means that the simplex is invariant under this dynamic, and it is true because the population remains constant.

Let $p^*(\bar{n}_1) = (p^*_1(\bar{n}_1), p^*_2(\bar{n}_1))$ be the equilibrium price of a regular economy $\mathcal{E}_{\bar{n}_1,\bar{n}_2}$. As we already shown in section (**??**) the equilibrium prices are smooth, functions of the distribution of the firms. There exist a neighborhood $V_{\bar{n}_1,\bar{n}_2} \subset R^2$ of $(\bar{n_1}, \bar{n_2})$ such that for all $(n_1, n_2) \in V_{\bar{n}_1,\bar{n}_2} \cap \Delta$ $p^*(\bar{n}_1 \in Eq_{n_1}$. Let us now introduce de definition of regular transition path:

Definition 9: Let $N(t) = N(N_0, t)$ be the solution of the dynamical system: (5). Let $\Psi : (0, 1) \times \Delta \to R^2$ be the generalized excess demand function, then a transition path, will be regular if and only if $rank J_p \Psi(N_1(t), p) = 1$ for all $(N_1(t), p) : \Psi(N_1(t), p) = 0$, where $\frac{n_1(t)}{n} = N_1(N_0, t)$ i.e. if and only for each $N_1(t)$ along a trajectory defined by a solution of the dynamical system (5) the economy $\mathcal{E}_{n_1(t),n_2(t)}$ is regular.

Note that if the economy in time t_0 is regular, then after a small perturbation on the distribution of the firms, the economy will continuous being regular, so there exist some interval $[t_0, t_1]$ such that

$$T(t_0, t_1) = \{(t, N(N_0, t)) \in [t_0, t_1] \times \Delta\}$$

is regular restricted transition path, because all economy $\mathcal{E}_{n_1(t),n_2(t)}$ is regular. We can summarize this assertion by the next proposition:

Proposition 1: The restricted transition path in a neighborhood of a regular equilibrium price is smooth.

The existence of such continuous transition paths, is a local property verifiable only in a neighborhood of a regular equilibrium prices. Recall that there is a maximal interval (α, β) containing t_0 where a solution $N(t)$ with $N(t_0) = N_0$ exists. The solutions corresponding to two intervals containing t_0 are the same in the intersection of both. In some cases $\alpha = -\infty$, $\beta = \infty$ or both, however,in general, it is not guaranteed that the solution of a differential equation can be defined for all t. For details see [11].

Contrarily, in the neighborhood of a singularity we can not ensure the existence of a solution for the dynamical system. The transition path in the neighborhood of a singularity can show discontinuous. If the economy is singular, then small perturbation in the distribution of the firms can give place to large changes in the behavior of the future economies, respect to the behavior of the actual, i.e, some equilibrium prices and consequently the corresponding profits can suffer larger changes, just as the distribution of the wealth between the consumers and so, their respective demands can can be altered. Most of these changes will be unpredictable an the consequence of a small perturbation in de distribution of the firms. Certainly, after and before that this perturbation occurs, we can describe the evolution of the economy by a dynamical system similar to the one given in (5), but it is not possible in the moment of when the trajectory attain a equilibrium price. The future evolution will start in a (previously) unforeseeable initial conditions. So, if we understand an economic crisis like an abrupt and unexpected change in the behavior of the economy, as the result of arbitrarily small changes in its fundamentals, then we can say that singular economies are thresholds of the economic crises.

4.3 Evolution by an imitative process

Now, we introduce a model of economic evolution where the engine of this process, is the imitative behavior of the owners or managers of the firms, looking for the most profitable technology. We imagine that these agents, acting strategically, stick to a given technology for a while, and occasionally, looking for technologies with higher profits, at least some of them and from time to time, review their previous decisions of production.

There are two particular elements characterizing this process. The first one is the specification of the time rate at which managers or owners review the technology under which their firms are producing. The second element is the probability that a reviewer, producing according with the technology Y_i, change effectively, to the technology Y_j. This probability is written p_{ij}. Certainly the probability that a reviewer change effectively, depends on his believes on the future behavior of the others. By p_{ii} we denote the probability that a reviewing manager using the $i-$technology does not change the technology.

In a finite but large population of firms, following [20], one may imagine that the review times of an agent (an owner or a manager) are the arrival time of a Poisson process with arrival time $r_i(\Pi^*)$, where $\Pi^* = (\Pi^*(N_1), \Pi_2^*(N_2))$. Then, if the agent become a reviewer, he selects the technology to produce in the next period, he decides to change from technology i to technology j with probability p_{ij} or he decides to maintain the previous election with probability p_{ii}. Assuming that all agents' are random variables, statistically independent, the aggregate of reviewing of each subset of firms is itself a Poisson process with arrival rate (normalized) $N_i r_i(\Pi^*)$. So, the aggregate Poisson process of switches form technology Y_i to technology Y_j is $N_i r_i(\Pi^*) p_{ij}$.

Now, imagine a large number of firms, by the law of large number we can model the aggregate stochastic process as a deterministic flows, where the outflow from the subset of firms using the technology Y_i is $N_i r_i(\Pi^*) p_{ij}(\Pi^*)$, and the inflow is $N_j r_j(\Pi^*) p_{ji}(\Pi^*)$, then we obtain:

$$\dot{N}_i = N_j r_j(\Pi^*) p_{ji} - N_i r_i(\Pi^*) p_{ij} \tag{6}$$

Each individual actor look at the world through his or her, ex-ante experience, so, it is natural to assume that less successful manager, on average, review their behavior at a higher rate than manager using more successful technologies. Then we consider that r_i is a decreasing function with respect to profits Π_i^* obtained by the firm.

$$r_i(\Pi^*) = \rho_i(\Pi_i^*). \tag{7}$$

where ρ_i is a Lipschitz continuous function, decreasing in it argument. Note that this assumption does not presume that the agent knows the expected profits associated with the technology currently in use.

So equation (6) can be rewritten as:

$$\dot{N}_i = -N_i \left[\rho_j(\Pi_j^*) p_{ji} + \rho_i(\Pi_i^*) p_{ij}\right] + \rho_j(\Pi_j^*) p_{ji}. \tag{8}$$

Moreover we can consider that the average review rate is linearly decreasing in the profit's rate

$$\rho_i(\Pi_i^*) = \alpha - \beta \Pi_i^*(N_i). \tag{9}$$

Then equation (8) become:

$$\dot{N}_i = -N_i\left[(\alpha-\beta\Pi_j^*(N_j))p_{ji} + (\alpha-\beta\Pi_i^*(N_i))(\Pi_i^*)p_{ij}\right] + \left[\alpha-\beta\Pi_j^*(N_j)\right]p_{ji}. \tag{10}$$

Suppose that each reviewing manager or owner, observe profits difference between her own profit and the profit associate with the other technology with some noise, and that the reviewer agent switches the technology if and only if $\Pi_j^*(N_1) > \Pi_i^*(N_1) + \epsilon$ where ϵ is a random variable, with a probability distribution $\phi_i : R \to [0, 1]$ continuous and differentiable. Then

$$p_{ij} = \phi_i(\Pi_j^*(N_1) - \pi_i^*(N_1)). \tag{11}$$

As a special case, assume that the error term is uniformly distributed with a support containing the range of all possible profits differences. Then ϕ_i is an affine function function over the relevant interval, $\phi_i(z) = a_i + b_i z$ for some $a_i, b_i \in R$ and $b_i > 0$ then (10) becomes:

$$\dot{N}_i = -N_i\left[(\alpha-\beta\Pi_j^*)(a_i - b_i(\Pi_j^* - \Pi_i^*) + (\alpha+\beta\pi_i^*)(a_j - b_j(\Pi_i^* - \Pi_j^*)\right] +$$
$$+\left[(\alpha-\beta\Pi_j^*)(a_j - b_j(\Pi_i^* - \Pi_j^*))\right]. \tag{12}$$

The imitative behavior: Under conditions of incomplete information, the evolution of the economy depends on the perception that owners or managers have, over the future benefits associated with each technology available. This perception depends on the skills and potential of managers to predict the future actions of others. We can assume that, under incomplete information, each owner or manager, to choose the future strategy, takes account the behavior of the nearest competitors or that followed by those he considered leaders or most successful. These types of behaviors can be called imitative, and each one gives place to a different dynamic.

Assuming that all review rates are constantly equal to one, i.e.; $r_i(\Pi^*) = 1$ we obtain the following dynamics:

$$\dot{N}_i = -N_i\left[(a_j + a_i) + (b_j + b_i)(\pi_j^* - \pi_i^*)\right] + \left[(a_j + b_j)(\pi_j^* - \pi_i^*)\right]. \tag{13}$$

The difference $\Pi_i^* - \Pi_j^*$ corresponds to the true difference, because

$$p_{ij} = P(\Pi_i^* + \epsilon < \Pi_j^*) = P(\epsilon < \Pi_j^* - \Pi_i^*) = \phi(\Pi_j^* - \Pi_i^*).$$

If in addition, we consider that along the time the difference between profits remain constant, then the solution of this dynamical system is given by:

$$N_i(t) = \frac{A}{B} + \left[N_{i0} - \frac{A}{B}\right] e^{-B(t-t_0)} \quad \forall t \geq t_0,$$

where $A = (a_j + b_j)(\Pi_j^* - \Pi_i^*),\ B = (a_j + a_i) + (b_j + b_i)(\Pi_j^* - \Pi_i^*)$ and $N_{10} = N_1(t_0)$.

Note that if the difference $\Pi_j^* - \Pi_i^*$ remains constant and positive along the time, then the solution converges to the stationary state $N_i = \frac{A}{B}$. This means that following this particular imitative process, even in the case when the rate of profits associate with the technology Y_j remain, along the time, higher than the profits associate with the technology Y_i, even in the long run, will be possible to find in the market some firms producing according with the technology Y_i.

5 Stability of the dynamical equilibrium

Both the social or natural world, the only dynamic equilibria we can see are those that are stable. In the previous section, we describe the evolution of an economy along a path of Walrasian equilibria. Once that the modification of the economy (i.e, the modification of the distribution of firms in the set of available technology) is permanent, the modification of equilibria prices will be also permanent, unless that the distribution correspond to a stable steady state of the dynamical system, only in these cases prices will remain constant. These changes will be continuous, at least until such time that a singularity appears, if such is the case, then after this moment, a discontinuity in prices or, in general in the economic behavior, can be observed.

Let us now introduce some considerations on the stability of the stationary state of the dynamical system considered in (5). A solution $N^e = (N_1^e, N_2^e) \in \Delta$ of this system (5) is a dynamical equilibrium, or steady state, if and only if

$$[\phi_i \left(\Pi_i(y_i^*(N_1^e)) - \Pi_j(y_j^*(N_1^e))\right)]N_1^e = 0 \ \ \forall\ i = 1, 2.$$

i.e if and only if the payoffs corresponding to each technology are the same:

1. $\Pi_i(y_1^*) = \Pi_j(y_2^*)$ or if
2. $N_i^e = 0$ for some $i \in \{1, 2\}$ because in this case all firms use the same technology and earn the same payoff.

3. In both cases $N(t) = N^e$, $\forall t$ and so $\dot{N} = (\dot{N}_1, \dot{N}_2) = 0$.

Each dynamical equilibrium, have associated a Walrasian equilibrium price, denoted by

$$p(N_1^e, N_2^e) = (p_1(N_1^e, N_2^e), p_2(N_1^e, N_2^e)) \in Eq_{n_1^e}.$$

The corresponding profits and the optimal plan for firms using the technology Y_i are respectively π_i^* and y_i^*. The rates of profits are the same when they are evaluate at these prices. Recall that, $N^e = \frac{1}{n}(n_1^e, n_2^e)$ then, in addition, we can say that, each dynamical equilibrium has associated an economy $\mathcal{E}_{n_1^e, n_2^e}$.

Let us consider the function $g : \Delta^2 \to R^2$

$$g_i(N) = \phi_i \left(\Pi_i(y_i^*(N_1)) - \Pi_j(y_j^*(N_1)) \right), \; i = 1, 2.$$

The differential equations system (5) can be written as:

$$\dot{N}_i = g_i(N) N_i, \;\; i = 1, 2. \tag{14}$$

It is clear that a population state $N \in \Delta^0$ is an stationary state if and only if $g_i(N) = 0, i = 1, 2$.

The following theorem provides a sufficient condition for asymptotic stability and instability in the Liapunov sense.

Theorem 1: Let N^e a fixed point of the system (14) then if there exists some neighborhood $U \subset R^2$ of N^e such that;

(a) $g(M)N^e = g_1(M)N_1^e + g_2(M)N_2^e > 0$ for all distribution $M \neq N^e$ in U then N^e is asymptotically stable in 14).

(b) $g(M)N^e = g_1(M)N_1^e + g_2(M)N_2^e < 0$ for all $M \neq N^e$, then N^e is unstable.

Proof: (Case (a)) We will show that the function

$$H_{N^e}(M) = N_1^e \log \left(\frac{N_1^e}{M_1} \right) + N_2^e \log \left(\frac{N_2^e}{M_2} \right)$$

is a Liapunov function for the system (14).

1. It is straightforward that $H_{N^e}(N^e) = \log(1) = 0$

2. If $M \in \Delta$ then the inequalities $H_{N^e}(M) \geq -N_1^e\left(1-\frac{M_1}{N_1^e}\right) - N_2^e\left(1-\frac{M_2}{N_2^e}\right) \geq 0$ hold[6]. Thus we conclude that there is a neighborhood $U \cap \Delta$ of N^e where the function $H_{N^e}(M)$ is positive.

3. Finally, since $H_{N^e}(M) \geq \dot{H}_{N^e}(M) = \left(\frac{N_1^e}{M_1}\dot{M}_1 + \frac{N_2^e}{M_2}\dot{M}_2\right)$ from condition given in item (a) the inequality

$$\dot{H}_{N^e}(M) = -\left(\frac{N_1^e}{M_1}g_1(M)M_1 + \frac{N_2^e}{M_2}g_2(M)M_2\right) = -g(M)N^e < 0$$

holds

4. The claim set out in item (b), follows similarly.

A geometrical interpretation of the asymptotic stability: First note that, since

$$\dot{N}_1 + \dot{N}_2 = g_1(N)N_1 + g_2(N)N_2 = g(N)N = 0 \qquad (15)$$

so, the vector field $g(N)$ remain orthogonal to N. Thus, the condition $g(M)N < 0$ used in the theorem (1) said that in a neighborhood of N the vector field g form an obtuse angle with N. So if N^e is asymptotically stable equilibrium for the system (14) then $g(N)$ drift locally to N^e. On the other hand, note that the identity (15) show that if at time $t = t_0$ $N(t_0) = N_0 \in \Delta$ then the trajectory $N(N_0, t) \in \Delta \; \forall \; t \geq t_0$.

Under the assumption of regularity, equilibrium prices are continuous functions of the firms see theorem (**??**).

1. The analysis of the stability in the Liapunov sense of the first type of equilibria, give place to the following 4 different cases, see figure (2). Let $N' = (N_1', N_2')$ be the distribution after a perturbation in the equilibrium distribution of the firms:

   (i) Consider the case where: $(N_1^e, N_2^e) >> 0$ if there exists a neighborhood $V_{n^e} \subset R^2 \cap \Delta_n$ of (n_1^e, n_2^e) such that $\Pi_i(y_i^*(n_1, n_2)) < \Pi_i(y^*(n_1^e, n_2^e))$ $i \in \{1, 2\}$ $\forall (n_1, n_2) \in V_{n^e}$ then the dynamical equilibrium is asymptotically stable, see theorem (1) .

   (ii) In the others three cases, where $\pi_i(y_i^*(n_1', n_2')) > \pi_i(y_i^*(n_1^e, n_2^e))$ for at least one $i \in \{1, 2\}$ the dynamical equilibrium is unstable, see theorem (1) .

[6] Recall that for $x > 0, log\ x \leq (x - 1)$.

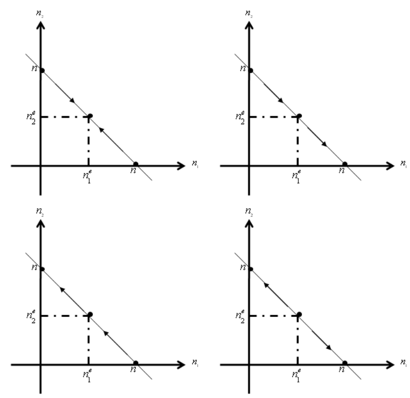


Fig. 2: Figure at the top and left, the only case of asymptotically stable equilibrium

2. For the cases on the second type, the dynamical equilibrium is asymptotically stable if, being

   (iii) $(N_1^e, N_2^e) = (1, 0)$ the inequality $\Pi_2(y_i^*(n_1', n_2')) < \Pi_1(y_i^*(n_1', n_2'))$ is verified, or if,

   (iv) being $(N_1^e, N_2^e) = (0, 1)$ the inequality $\Pi_2(y_i^*(n_1', n_2')) > \Pi_1(y_i^*(n_1', n_2'))$ is verified.

Note that for the last case considered in section (4.3) we have that $g_j(N) = a_j + b_j(\pi_j^*(N) - \pi_i^*(N),\ j \neq i \in \{1, 2\}$ and then if $\pi_j^*(N) - \pi_i^*(N) > 0$ it follows that $g(N)N^e > 0$. Then the asymptotical stability of the stationary point $N_i^e = \frac{A}{B},\ N_j^e = 1 - \frac{A}{B}$ holds.

6 An example

This example is inspired in ([3]).

Consider a neoclassical private ownership production economy with commodity space R^2 having m consumer divided in two types, m_1 of type 1 and m_2 of type 2, the set $\mathcal{F}_1 = \{1, ...m_1\}$ symbolize the consumers of type 1, and $\mathcal{F}_2 = \{(m_1+1), ..., m\}$ we symbolize the consumers of type 2. Having n firms of two types n_1 of type 1 and n_2 of type 2, with the following characteristics:

- Consumers type 1: Initial endowments $w_i = \frac{1}{m_1}(1, 3)$ and utility function $u_i(x, y) = xy, i \in \{1, ..., m_1\}$

- Consumers type 2: Initial endowments $w_i = \frac{1}{m_2}(2,3)$ and utility function $u_i(x,y) = xy^2, i \in \{(m_1+1), ..., m\}$.
- Firms type 1: Production set $Y_1 = \left\{(x,y) \in R^2 : x < 1, \text{and } y \leq \frac{x}{x-1}\right\}$.
- Firms type 2: Production set $Y_2 = \{(x,y) \in R^2 : x < 1, \text{and } y \leq g(x)\}$ where:

$$g(x) = \begin{cases} 1 - e^x & if\ x \leq 0 \\ \\ \ln(1-x) & if\ 0 < x < 1 \end{cases}$$

- Shares: Type 1 consumers have own one third of type 1 firms, and two thirds of type 2, the rest is owned by the consumers of type 2. Within each type, individuals have the same share of firms, i.e:

  $\theta_{i1} = \frac{1}{3m_1},\ \forall\, i \in \mathcal{F}_1$ and $\theta_{i2} = \frac{2}{3m_1},\ \ \forall i \in \mathcal{F}_2,\ i = 1, ...m_1;$

  $\theta_{i1} = \frac{2}{3m_2},\ \forall\, i \in \mathcal{F}_1$ and $\theta_{i2} = \frac{1}{3m_2},\ \ \in \mathcal{F}_2,\ i = m_1+1, ..., m.$

We start considering the supply function of a firms of type 1: The technological set Y_1 is strictly convex and the efficiency frontier is

$$EFF(Y_1) = \left\{\left(x, \frac{x}{x-1}\right),\ x < 1\right\}.$$

So the profit function of a firm of this type is given by:

$$\pi_1(x) = p_1 x + p_2 \frac{x}{x-1}. \tag{16}$$

The supply of a firm of type 1, is: $x_1(p) = 1 - \sqrt{\frac{p_2}{p_1}}$. Writing $t = \sqrt{\frac{p_2}{p_1}}$ we obtain that the output input vector is given by

$$y_1(t) = \left(1-t, 1-\frac{1}{t}\right).$$

Substituting $x = y_2(t)$ in (18) we obtain that the profit for a firm of type 1 is: $\pi_1(t) = p_1(1-t)^2$.

Remark 1: Note that the rate of profits is given by

$$\Pi_1(t) = \begin{cases} \frac{\pi_1(t)}{-p_2(1-\frac{1}{t})} = -1 + \frac{1}{t} & 0 < t < 1 \\ \\ \frac{\pi_1(t)}{-p_1(1-t)} = 1 - t & 1 < t \end{cases} \tag{17}$$

For a firm of type 2, we obtain

$$EFF(Y_2) = \{(x, 1-e^x),\ x \leq 0\} \cup \{(x, \ln(1-x),\ 0 < x < 1\}\,.$$

The profit function is given by:

$$\pi_2(x) = \begin{cases} p_1[x + t^2(1-e^x)] & x \leq 0 \\ \\ p_1[x + t^2 \ln(1-x)] & 0 < x < 1 \end{cases} \tag{18}$$

since $\pi_2''(x) < 0$ in both cases, then the supply of a firm of type 2 is:

$$y_2(t) = \begin{cases} (1-t^2, 2\ln t) & if\ 0 < t < 1, \\ \\ \left(-2\ln t, 1 - \frac{1}{t^2}\right) & if\ t \geq 1. \end{cases}$$

Substituting $x = y_2(t)$ in (18) we obtain that the profit for a firm of type 2 is:

$$\pi_2(t) = \begin{cases} p_1[(1-t^2) + 2t^2 \ln t] & if\ 0 < t < 1, \\ \\ p_1[-2\ln t + t^2 - 1] & if\ t > 1. \end{cases}$$

Remark 2: Note that the rate of profits of a firm of type 2 is given by:

$$\Pi_2(t) = \begin{cases} \frac{-1+t^2}{2t^2 \ln t} + 1 & if\ 0 < t < 1, \\ \\ \frac{-2\ln t + t^2 - 1}{2\ln t} & if\ t > 1 \end{cases} \tag{19}$$

The demand of each consumer of type 1, is given by:

$$x_1(t) = \frac{W_1(t)}{2p_1}\left(1, \frac{1}{t^2}\right) \tag{20}$$

and the demand of each consumer of type 2, is given by:

$$x_2(t) = \frac{W_2(t)}{3p_1}\left(1, \frac{2}{t^2}\right) \tag{21}$$

Where $W_i(t)$ denotes the income of a consumer of type i at prices $p = p_1(1, t^2)$. We shall compute:

$$W_i(t) = pw_i + n_1\theta_{i1}\pi_1(t) + n_2\theta_{i2}\pi_2(t),\ i \in \{1, 2\}$$

$$W_1(t) = \frac{p_1}{3m_1} \begin{cases} (9+n_1+2n_2)t^2 - 2n_1 t + 3 + n_1 - 2n_2 - 4n_2 \ln t & if\ t \geq 1 \\ \\ (9+n_1-2n_2)t^2 - 2n_1 t + 4n_2 t^2 \ln t + (3+n_1+2n_2) & if\ 0 < t < 1 \end{cases} \tag{22}$$

$$W_2(t) = \frac{p_1}{3m_2} \begin{cases} (9+2n_1+n_2)t^2 - 4n_1 t + 6 + 2n_1 - n_2 - 2n_2 \ln t & if\ t \geq 1 \\ \\ (9+2n_1-n_2)t^2 - 4n_1 t + 2n_2 t^2 \ln t + 6 + 2n_1 + n_2 & if\ 0 < t < 1 \end{cases} \tag{23}$$

Substituting in (20) we obtain the demand for each consumer of type 1:

$$x_1(t) = \begin{cases} \left(\frac{(9+n_1+2n_2)t^2-2n_1t+3-n_1-2n_2-4n_2\ln t,}{6m_1} \frac{(9+n_1+2n_2)t^2-2n_1t+3+n_1-2n_2-4n_2\ln t}{6m_1t^2}\right) \\ if\ t \geq 1, \\ \\ \left(\frac{(9+n_1-2n_2)t^2-2n_1t+3+n_1+2n_2+4n_2t^2\ln t}{6m_1}, \frac{(9+n_1-2n_2)t^2-2n_1t+3+n_1+2n_2+4n_2t^2\ln t}{6m_1t^2}\right) \\ if\ 0 < t < 1. \end{cases} \tag{24}$$

and substituting in (21) we obtain the demand for each consumer of type 2:

$$x_2(t) = \begin{cases} \left(\frac{(9+2n_1+n_2)t^2-4n_1t+6+2n_1-n_2-2n_2\ln t}{9m_2}, 2\frac{(9+2n_1+n_2)t^2-4n_1t+6+2n_1-n_2-2n_2\ln t}{9m_2t^2}\right) \\ if\ t \geq 1 \\ \\ \left(\frac{(9+2n_1-n_2)t^2-4n_1t+2n_2t^2\ln t+6+2n_1+n_2}{9m_2}, 2\frac{(9+2n_1-n_2)t^2-4n_1t+2n_2t^2\ln t+6+2n_1+n_2}{9m_2t^2}\right) \\ if\ 0 < t < 1 \end{cases} \tag{25}$$

The aggregate excess demand function will be:

$$\zeta(t) = m_1(x_1(t) - w_1) + m_2(x_2(t) - w_2) - n_1 y_1(t) - n_2 y_2(t).$$

$$\zeta_1(t) = \frac{1}{18}\begin{cases} (45+7n_1+8n_2)t^2+4n_1t-33-11n_1-8n_2-20n_2\ln t & if\ t \geq 1 \\ \\ (45+7n_1+10n_2)t^2+4n_1t-54+16n_2t^2\ln t & if\ 0<t<1 \end{cases}$$

$$\zeta_2(t) = -\frac{1}{18t^2}\begin{cases} (45+7n_1+8n_2)t^2+4n_1t-33-11n_1-8n_2+20n_2\ln t & if\ t \geq 1 \\ \\ (45+7n_1+10n_2)t^2+4n_1t-54+16n_2t^2\ln t & if\ 0<t<1 \end{cases}$$

The equilibrium prices are the solution of the equation $\zeta(t) = 0$. To find the solutions of this equation we consider the following functions:

$$f : (0, \infty) \to R \quad and \quad g : (0, 1) \to R$$

defined by:

$$f(t) = (45+7n_1+8n_2)t^2+4n_1t-33-11n_1-8n_2-20n_2\ln t, \quad (a)$$

$$g(t) = (45+7n_1+10n_2)t^2+4n_1t-54+16n_2t^2\ln t. \quad (b) \tag{26}$$

Note that $f'(t) = 2(45+7n_1+8n_2)t+4n_1-n_2\frac{20}{t} > 0\ \forall t \geq 1$ and $n_1, n_2 \geq 0$ this means that $f(t)$ is strictly increasing in the interval $[1, \infty)$. Since $f(1) = 12$ it follows that $\zeta(t) \neq 0$ for each $t > 1$, and then there is not an equilibrium with $t > 1$.

On the other hand, since $\lim_{t\to 0} g(t) < 0$, and $g(1) > 0$, $\forall (n_1, n_2) \in \Delta$ there exists at least one equilibrium such that $t^* < 1$.

Note that $\Pi_1(\bar{t}) = \Pi_2(\bar{t})$ if and only if $\bar{t} =\approx 0.516691803$. This means that a Walrasian equilibrium is a dynamical equilibrium if and only if $\sqrt{\frac{p_2^*}{p_1^*}} = t^* = \bar{t} \approx\approx 0.516691803$. This equilibrium corresponds (approximatelly) for instance to an economy with $n_1^2 = 1$ and $n_2^e = 40$.

6.1 The dynamics

Assuming that the number n of firms remains fixed, but they can choose technology, we obtain for this example, that the evolution of the economy is given by the following dynamical system:

$$\dot{N}_1 = \phi\left(\pi_1(y_1^*(N_1)) - \pi_2(y_2^*(N_1))\right) N_1$$
$$\dot{N}_2 = -\dot{N}_1 \tag{27}$$
$$N(t_0) \in \Delta^0.$$

Hereafter, to avoid confusions we use t to symbolize time, and $p = \sqrt{\frac{p_1}{p_2}}$.

So the dynamics along an equilibrium path is given by the

$$\dot{N}_1 = N_1\phi(\Pi_1((N)) - \Pi_2((N)),\ 0 < t < 1$$

in our case

$$\dot{N}_1 = N_1\phi\left(-2 + \frac{1 - p^2(N_1)}{2p^2(N_1)\ln p(N_1)} + \frac{1}{p(N_1)}\right)$$

Note that this equation is decreasing in the interval $(0,1)$ and $\phi(\bar{t}) \approx \phi(0.516691803) = 0$.

So, if for a distribution N of the firms over the available technologies the inequality $t(N) < 0.516691803$ then $\Pi_1(N_1) - \Pi_2(N) > 0$ and the quantity of firms of type 1 increase and then this difference decrease. Contrarily, if $t(N) > 0.516691803$ then $\Pi_1(N_1) - Pi_2(N) < 0$ and the quantity of firms of type 2 increase and then this difference increase.

6.2 The transition path

Let us now characterize the trajectory of the economy of our example along a Walrasian equilibrium trajectory.

Consider a regular economy such that in time $t = t_0$

$$n(t_0) = n_0 = (n_{10}, n_{20})$$

i.e, $\mathcal{E}_{n_{10},n_{20}}$. It is possible to choose a neighborhood and a correspondence

$$\mathbf{p}^* : \Delta \cup U_{n_0} \to 2^{R_+^2},$$

such that $\mathbf{p}^*(\mathbf{n_1}, \mathbf{n_2}) = \mathbf{Eq_{n_1}}$.

Let $p^*(n_0) = (p_1^*(n_{10}), p_2^*(n_{20})) \in \mathbf{p}^*(n_{10}, n_{20})$ such that $p_0 = \sqrt{\frac{p_1^*(n_{10})}{p_2^*(n_{10})}} < 0.516691803$

From the correspondence $\mathbf{p}^*$ choose a continuous selection p^*, such that $p^*(n_0) = p^*(n_{10}, n_{20})$.

There exists a neighborhood $V_{n_0)} \subset \Delta \cap U_{n_0}$ such that for all for all $(n_1, n_2) \in V_{n_0}$, $p^*(n_1)$ verify that $p(n_1) = \sqrt{\frac{p_1^*(n_1)}{p_2^*(n_1)}} < 0.516691803$.

From the implicit function theorem $p : V_{n_0} \to U_{p_0}$ is a smooth function verifying

$$\frac{dp^*}{dn_1}(n_{10}) = p'_{n_1}(n_{10}) = \frac{-3p_0 + 4 - 16p_0 lnp_0}{-4n_1(p_0)^3 + 108p^-2_0 16(n - n_{10})}.$$

The economy $\mathcal{E}_{n_1(t_0),n_2(t_0)}$ evolves along the transition path to an stationary state, $(n_1^e, n_2^e) \in \mathcal{D}_e$ being

$$\lim_{(n_1,n_2)\to(n_1^e,n_2^e)} p^*(n_1, n_2) = p^*(n_1^e, n_2^e) \approx 0.516691803.$$

Along this transition path a economy $\mathcal{E}_{n_1}$ the share of economies of type 1 increases and the economies of type 2, decreases in equal proportion.

An economy $\mathcal{E}_{n_1}$ is singular if and only if for some p_0,

$$-4n_1(p_0)^3 + 108p_0^{-2} + 16(n - n_{10}) = 0.$$

Given that the initial economy is regular, then all economy in the neighborhood V_{n_0} is regular.

Analogously, it follows that, if the equilibrium for the initial economy, verify that $0.516691803 < p_0 < 1$ then we will observe a contrary evolution, i.e, the share of firms of type 1 decreases, and increases the share of firms of type 2.

Note that, a Walrasian equilibrium price $p^*(n_1, n_2)$ will be, at the same time, an stationary equilibrium if and only if, there exists a distribution $n = (n_1, n_2)$ for which the equalities

$$\pi_1(p^*(n_1, n_2)) = \pi_2(p^*(n_1, n_2)) \text{ and } \zeta(p^*(n_1, n_2)) = 0$$

are verified.

7 Conclusions

In this paper, we have shown that a dynamical process based on imitation leads to the economy over a trajectory of Walrasian equilibria. Since the equilibrium prices and the corresponding profits depend on the distribution of the firms over the set of available technologies, the hypothesis of rationality implies that each manager (with identical interest than the owner), try to anticipate the behavior of their competitors. This occurs because the future profits will depend on the joint choice of the managers on the set of available technologies.

One of the main questions, that we attempt to answer in this paper, is how or based in what arguments, managers, looking for more profitable investments, decide to maintain or change the actual technology under which their firms are producing. We considered that, under incomplete information, managers follow an imitation process.

As this process imitative process continues, the economy changes, more specifically, the set of Walrasian equilibria changes. The economy evolves along a transition path, which reflects the evolution of the distribution of the firms over the set of available technologies. In this process, the wealth of consumers changes, because their shares on firms change, and thereby the excess demand functions change. Consequently, the economy changes. So, the set of Walrasian equilibria depends on the distribution of the firms, and so, this distribution determines the characteristics of the economies, i.e, if they are regular or singular. The changes in the economy along the transition path, will no longer be continuous in a neighborhood of a singular price. It is for these reasons, that we consider singular economies as the thresholds of economic crises. Only singular economies have associate singular equilibria. Big changes can be expected when the distribution of the firms determines that the economy is in a neighborhood of a singular economy. Therefore, the possibility that an economic crisis happens is a structural phenomenon, that lies at the basis of the model itself, particularly in the assumed rationality of the agents.

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