Generalized Statistical Means and New Price Index Formulas, Notes on some unexplored index formulas, their interpretations and generalizations

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Generalized Statistical Means and New Price Index Formulas

Notes on some unexplored index formulas, their interpretations and
generalizations

Peter von der Lippe

The theory of (increasingly more generalized types of) statistical means can be used to create a plethora of index formulas. Some of them are new and some were indeed discussed in the past but fallen into oblivion, because their rationale was not well understood. Surprisingly many possess interesting interpretations and attractive properties that deserve being unveiled. We begin with unweighted indices with implications to what now is called "low level aggregation" and proceed to weighted index formulas that lend themselves to productive generalizations and thereby to some new formulas. It turns out that contrary to popular belief the Laspeyres and Paasche formula are not equally well justified and that some indices from the more comprehensive system of statistical means are attractive regarding their economic interpretation and how they are related to indices of "quantity" and purchasing power of money.

1. Unweighted indices (low level aggregation)

1.1. Unweighted indices and implicit quantities

Carli's index \( P^C_{0i} = \frac{1}{n} \sum_{i=1}^{n} \frac{p_i}{p_{i0}} \) was widely used in early price level measurement in the 19th century. It was considered desirable to account for the different relative "importance" of the n goods in this formula. However, difficulties with aggregating over widely different types of goods and their quantities (we cannot add kilograms of vegetables, liters of fuel, yards of cloth, hours of services etc) have led some authors to adopt the idea of reciprocal prices as "implicit quantities": obviously \( 1/p_{i0} \) is the amount of good i you get for one currency unit (say 1€, or 1$) at the base period. With such "implicit quantities" a mean of price relatives becomes a ratio of expenditures. When exactly 1€ was spent for each good in the base period, we have a basket of n goods and an expenditure (or cost for a basket) of \( E_0 = n€ \) in the base period and Carli's index \( P^C \) compares \( E_t = \sum_{i=1}^{n} p_i (p_{i0})^{-1} \) with \( E_0 = \sum_{i=1}^{n} p_{i0} (p_{i0})^{-1} = n \), because

1 Laspeyres and some other authors made extensively use of this formula, then also known as "Sauerbeck index" in these days, and of Sauerbeck's price statistics for the British foreign trade. It was only in the 20th century owing to Walsh 1901 that it became generally known that the formula actually was much older and originated from Gian Rinaldo Carli (1720 -1795).

2 A problem inherent in unit-value-indices (see below part 4)

3 Note reciprocal prices are not quantities. Just as a price is measured in currency units per unit of the quantity in question (say € per gram) so \( 1/p_{i0} \) is a gram per € expression.
It may also be interpreted as comparing weighted average prices (with reciprocal prices or implicit quantities as weights), $P^c_{0t} = E_i/n$ and $P^c_0 = E_0/n = 1$. With this (admittedly somewhat farfetched) notion of "implicit quantities" an average of price ratios (or AOR-formula) can easily be translated into a ratio of average expenditures/prices or ROA formula. Or in other words: an unweighted index may be viewed as implicitly, i.e. with reciprocal prices weighted. The same logic, if applied to an unweighted harmonic mean will lead to

$$P^H_{0t} = \left( \frac{1}{n} \sum \left( \frac{p_{it}}{p_{0i}} \right) \right)^{-1} = \frac{n}{\sum p_{0i}} = \frac{\sum p_{0i} (p_{n})^{-1}}{\sum p_{i0} (p_{n})^{-1}} = \frac{E^*}{E_0},$$

where $E^*$ and $E_0$ are expenditures with current period "implicit quantities" $1/p_{it}$. That $P^H_{0t} = (P^c_{0t})^{-1}$ means that $P^H$ is the "time antithesis" (as Fisher would have put it) of $P^C$ (and vice versa). Note that using relative prices or "price shares" (or price quotas $p/\Sigma p$) rather than reciprocal prices we also can transform a ROA type index such as Dutot's index $P^D$ into a weighted AOR index like of the $P^C$ or $P^H$ type

$$P^D_{0t} = \frac{\sum p_{it}}{\sum p_{0i}} = \frac{1}{n} \frac{\sum p_{it}}{\sum p_{0i}} = \frac{\bar{p}_t}{\bar{p}_0} \text{ as weighted (P^C-type) AOR index},$$

and

$$P^D_{0t} = \left( \frac{\sum p_{0i} (p_{n})^{-1}}{\sum p_{it} (p_{n})^{-1}} \right) = \frac{\bar{p}_t}{\bar{p}_0} \rightarrow P^D_{0t} = \left( \frac{\sum p_{0i} (p_{n})^{-1}}{\sum p_{it} (p_{n})^{-1}} \right) = \frac{\sum p_{0i} (p_{n})^{-1}}{\sum p_{it} (p_{n})^{-1}} \text{ as weighted P^H type AOR index.}^4

<table>
<thead>
<tr>
<th>ROA form</th>
<th>weights</th>
<th>AOR form</th>
</tr>
</thead>
<tbody>
<tr>
<td>ratio of average expenditures/prices* as for example $P^D$</td>
<td>relative prices $p/\Sigma p$</td>
<td>average of price ratios $[= price relatives]$ as for example $P^C$ and $P^H$</td>
</tr>
<tr>
<td></td>
<td>reciprocal prices (as implicit quantities)</td>
<td></td>
</tr>
</tbody>
</table>

* equal implicit quantities $1/n$ for each good

Obviously in (1.1) and (1.2) the "quantity" of each good is indirectly proportional to its price, so expensive goods are less, and cheaper goods are more represented in such an index. This suggests taking some average of such implicit quantity weights (leading to Young's index $P^Y$).

The formulas $P^C$ (Carli) and $P^H$ (harmonic) were mainly discussed (and rejected) because of their failing the time reversal test since $P^C_{0t} = \left( p_{0t}^{\dagger} \right)^{-1} \geq \left( p_{0t}^c \right)^{-1}$ and $P^H_{0t} = \left( p_{0t}^c \right)^{-1} \leq \left( p_{0t}^{\dagger} \right)^{-1}$.

### 1.2. Crossing of formulas and weights

A quite natural idea is to take a mean of $P^C$ and $P^H$. The geometric mean of $P^C$ and $P^H$, that is

$$P^{CSWD}_{0t} = \sqrt{P^C_{0t} P^H_{0t}} = \sqrt{\frac{1}{n} \sum p_{it} \frac{n}{p_{0i}}} = \frac{\sum p_{it}}{\sum p_{0i}},$$

is known as CSWD-index$^5$ and for approximating the time reversible index of Jevons $P^{J}_{0t} = \sqrt[n]{\prod p_{it}}$. It can easily be seen that

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$^4$ So taking a harmonic mean of the price relatives with weights $p_{it}/\Sigma p_{it}$ will again yield $P^D$. Unlike Carli's arithmetic mean index the harmonic mean $P^H$ of price relatives never gained much attention.

$^5$ So taking a harmonic mean of the price relatives with weights $p_{it}/\Sigma p_{it}$ will again yield $P^D$. Unlike Carli's arithmetic mean index the harmonic mean $P^H$ of price relatives never gained much attention.
\( P^J \) and \( P^{CSWD} \) pass the time reversal test, since \( 1/P^J_{t0} = P^J_{t0} \). In a similar manner we can also take averages of weights \( P^{CW}_{t0} = \sum p_{i0} w_i / \sum p_{t0} w_i \) where the part of weights \( w_i \) can be taken by

- \( A_i = \frac{1}{2} \left( \frac{1}{p_{i0}} + \frac{1}{p_t} \right) \) (arithmetic) giving \( P^{CW(A)}_{t0} = \sum p_{i0} A_i / \sum p_{t0} A_i = 1 + P^C_{t0} / 1 + P^C_{t0} \)

- \( H_i = \frac{2}{p_{i0} + p_t} \) (harmonic), yielding \( P^{CW(H)}_{t0} = \sum p_{i0} H_i / \sum p_{t0} H_i = \sum (p_t / (p_{i0} + p_t)) / \sum (p_{i0} / (p_{i0} + p_t)) \), and

- \( G_i = \sqrt[\alpha]{(p_{i0} p_t^{-1})} = \sqrt[\alpha]{A_i H_i} \) (geometric) \( \Rightarrow P^{CW(G)}_{t0} = \sum p_{i0} G_i / \sum p_{t0} G_i = \sum \sqrt[p_{i0}]{p_t / p_{i0}} = P^Y_{t0} \)

In all three cases both, numerator \( \sum p_i w_i \) and denominator \( \sum p_{i0} w_i \) may be viewed as expenditures, or (upon division by \( n \)) as weighted average prices. So crossed implicit weight (CW) indices can be given an ROA-interpretation in addition to an AOR interpretation as for example \( P^{CW(H)}_{t0} = \sum p_{i0} (p_t / (p_{i0} + p_t)) / \sum p_{t0} (p_{i0} / (p_{i0} + p_t)) \) is an average of price relatives \( p_t / p_{i0} \). The most interesting index is \( P^{CW(G)}_{t0} \) as it coincides with an index \( P^Y \) suggested by Allyn A. Young (1923).

### 1.3. Index of Allyn A. Young \( P^Y \)

Young proposed the following seemingly weird and unmotivated formula

\[
P^Y_{t0} = \frac{\sum p_t}{\sum p_t / p_{t0}} = \frac{\sum p_t}{\sum p_t / p_{t0}}
\]

For Young it was the "best unweighted index"; it meets both, time reversibility and linear homogeneity:

<table>
<thead>
<tr>
<th>linear homogeneity</th>
<th>time reversal test</th>
</tr>
</thead>
<tbody>
<tr>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>yes</td>
<td>Y</td>
</tr>
<tr>
<td>C, H, CSWD</td>
<td></td>
</tr>
<tr>
<td>no</td>
<td>CW(A), CW(H)</td>
</tr>
</tbody>
</table>

Given this favourable situation of \( P^Y \) the question may arise why this index did not find attention and why it obviously is completely fallen into oblivion. We think that this is due to the fact that its rationale is not well understood when it was proposed in 1923. Nobody seems to know by which underlying "logic" (of crossing inverse prices as implicit quantities) we would arrive at Young's index \( P^Y \). So some remarks to its rationale should be pertinent here: Young found that "base year weighting" in \( P^C_{t0} = \sum p_t (p_{i0})^{-1} / \sum p_{i0} (p_{i0})^{-1} \) (with weights in the way of inverse base period prices or implicit quantities) tends to "overweight rising prices", while

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5 As acronym for Carruthers, Selwood, Ward and Dalen.

6 \( P^J \) has both, an AOR and a ROA interpretation (taking geometric means of prices averages in the ROA).

7 This Young is not to confound with Arthur Young 1812, also often quoted in index theory literature. Unlike \( P^{CW(G)} \) the formulas \( P^{CW(A)} \) and \( P^{CW(H)} \) possibly rightly did not receive much attention.

8 Irving Fisher 1927; 530f later called \( P^Y \) "an ingenious anomaly, scarcely classifiable" (in the scheme of Fisher's book) and "a scientific curiosity".
\[ p_{0t}^{H} = \sum_{t=0}^{T} \frac{p_t(p_t)^{-1}}{\sum_{t=0}^{T} p_t(p_t)^{-1}}, \] tends to underweight them. Thus he was quite naturally lead to seek a compromise using the geometric mean \( \sqrt{\frac{1}{p_0p_t}} \). Later Bert Balk rediscovered Young's formula and called it Balk-Walsh (BW) index, because with explicit (quantity) weights we get Walsh's formula \( p_{0t}^{W} = \sum_{t=0}^{T} \frac{p_t \sqrt{q_0q_t}}{\sum_{t=0}^{T} p_t \sqrt{q_0q_t}} \) which bears some resemblance to Young's formula \( P^Y \) based on implicit (inverse prices) rather than explicit weights. Another rediscovery of \( P^Y \) took place when Jens Mehrhoff – in a short note he contributed to von der Lippe 2007; 45f – looking for a linear index able to approximate \( P^{CSWD} \) and thereby \( P^J \). He called it "hybrid" and later "BMW (Balk-Mehrhoff-Walsh) index", not knowing that it coincides with \( P^Y \) and he also remarked (like von Bortkiewicz and even already Young before), that \( P^Y \) not only has a ROA interpretation (indicated in (1.4) with weighted means of prices) but also an AOR interpretation as follows

\[ P_{0t}^{Y} = \frac{\sum_{t=0}^{T} \frac{\sqrt{p_0}}{\sqrt{p_t}}}{\sum_{t=0}^{T} \frac{\sqrt{p_0}}{\sqrt{p_t}}}, \quad \text{with } \sqrt{\frac{p_0}{p_t}} \text{ as somewhat awkward weights. As mentioned already Young well appreciated this property of possessing a double interpretation (AOR in 1.5 and ROA in 1.4) of his index formula when he said:}

"In a way Professor Fisher is right in holding that all true index numbers are averages of ratios. But I should prefer to say that all true index numbers are at once averages of ratios and ratios of aggregates." (Young 1923; 359).

Young also saw that his index meets the time reversal test but not the circular test, because

\[ P_{02}^{Y} = \frac{\sum \sqrt{p_2}}{\sum \sqrt{p_0}} \neq \frac{\sum_{t=0}^{T} \frac{\sqrt{p_t}}{\sqrt{p_0}} \sum_{t=0}^{T} \frac{\sqrt{p_t}}{\sqrt{p_2}}}{\sum_{t=0}^{T} \frac{\sqrt{p_0}}{\sqrt{p_t}} \sum_{t=0}^{T} \frac{\sqrt{p_0}}{\sqrt{p_2}}}, \quad \text{and finally (interesting in view of Mehrhoff's paper) Young also noted about } P^Y \ "\text{In general it will agree very closely with the geometric average" (357).}"

2. Weighted indices: Average of ratios formulas (AOR), part I:

2.1. Fisher's system and Vogt's generalized antiharmonic mean

A distinction can be made between first, second, and third order index formulas (Köves 1983; 28ff, v. d. Lippe 2007; 54). "First" means unweighted indices, "second" relates to Irving Fisher's system of AOR-formulas and "third" to "crossing" of second generation indices (for example Fisher's ideal index as result of crossing Laspeyres and Paasche). Fisher introduced

- six types of (unweighted) means (of price relatives), of which only arithmetic, harmonic and geometric means are worth being considered,\(^9\) and

- four methods of weighting (I through IV, of which he called two, II and III "hybrid") by which he arrived at the following system:\(^{10}\)

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\(^9\) The three remaining means are median, mode, and the "aggregative" index as Fisher called it (like the Dutot index \( \Sigma p_t / \Sigma p_0 \)). They are no longer of any interest.

\(^{10}\) Note that some of the combinations are identical due to inherent relations between the arithmetic and the harmonic mean: both index formulas, \( P^H \) and \( P^Y \) can be expressed in two ways, using pure and hybrid weights. It will soon be shown that all the remaining indices can be written in two different ways as well (see tab. 3).
Table 1

<table>
<thead>
<tr>
<th>means</th>
<th>methods of weighting</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I: base $p_0q_0$</td>
</tr>
<tr>
<td>arithmetic</td>
<td>$p_{0t}$ Laspeyres</td>
</tr>
<tr>
<td>harmonic</td>
<td>$p_{0t}^{HB}$ see eq. 2.12</td>
</tr>
</tbody>
</table>

The highlighting with green n orange color is meant as hint to related fields (in the same color) of table 2.

The combination gives six different "second generation" index-formulas $P^L$, $P^P$, $P^{AH}$, $P^{PA}$, $P^{HB}$, and $P^{HH}$ (of which only the indices of Laspeyres and Paasche are well known, $P^{PA}$ refers to the index of Palgrave, and the other labels are our own: HB = harmonic base, HH = harmonic hybrid, AH = arithmetic hybrid). The indices $P^{HB}$ and $P^{PA}$ are a bit more interesting than the indices $P^{AH}$ and $P^{HH}$ (which are rarely if ever seriously considered), so $P^{HB}$ and $P^{PA}$ will be discussed in more detail below (section 3).

Obviously $P_{0t}^{OH}$ is related to the so called quadratic mean (QM) as follows

\[(2.1)\quad P_{0t}^{AH} = \sum \frac{p_i}{p_0} \frac{p_iq_0}{\sum p_iq_0} = \frac{\sum (p_i^2/p_0^2)p_0q_0/p_0 \sum p_0q_0}{\sum p_iq_0/\sum p_0q_0} = \left( \frac{P_{0t}^{OM}}{P_{0t}^L} \right)^2 \]

where $P_{0t}^{QM}$ is the quadratic and $P_{0t}^L$ the arithmetic mean of price relatives with type I weights.

The system of tab. 1 can be generalized in view of

- the fact that all means, harmonic ($\bar{x}_{hi}$) geometric ($\bar{x}_0$) arithmetic ($\bar{x}$) and quadratic ($\bar{x}_{QM}$) are special cases of $\bar{x}_p(r)$, the power mean (moment mean or generalized mean) of degree r given by

\[(2.2)\quad \bar{x}_p(r) = \left[ \sum w_i x_i^p + w_2 x_2^p + \ldots + w_m x_m^p \right]^{1/r},\]

and

- the (weighted) antiharmonic mean (denoted by $\overline{H}$ to avoid confusion with AH denoting arithmetic hybrid) of x-values with weights w discussed in Vogt 1979

\[(2.3)\quad \bar{x}_H = \frac{\sum x_i^2 w_i}{\sum x_i w_i} = \left( \bar{x}_0 \right)^2 / \bar{x} \]

that is the ratio of the squared quadratic and the arithmetic mean, so $P^{AH}$ is in fact an antiharmonic mean using weights of type I. This motivates an extension of the scheme above as follows:

Table 1a

<table>
<thead>
<tr>
<th>mean</th>
<th>methods of weighting</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I: base $p_0q_0$</td>
</tr>
<tr>
<td>antiharmonic</td>
<td>$P_{0t}^{AH} = P_{0t}^{GAH}(0,1)$</td>
</tr>
</tbody>
</table>

11. In Germany Neubauer 1998 brought them into play and v. d. Lippe 2000 discussed their properties.
12. The special cases are $r = -1$ harmonic mean, $r \to 0$ geometric mean, $r = 1$ the arithmetic mean and $r = 2$ quadratic mean. According to a corollary, first proven by Cramer the power mean is a monotonous function in the parameter r such that harmonic \leq geometric \leq arithmetic \leq quadratic (equality applies when all x-values are identical). From this follows $P^{HH} < P^P < P^{AH}$ and $P^{HB} < P^P < P^{PA}$ in tab. I
13. So $P^{AH}$ is not only arithmetic hybrid but also antiharmonic (with the non-hybrid or "pure" weight system I).
the even more general concept of the \textit{generalized antiharmonic or "GAH" mean}
(again brought into play by Vogt)

\begin{equation}
P_{0t}^{\text{GAH}}(j, k) = \frac{\sum s_j \left( \frac{p_j}{p_0} \right)^{k+1}}{\sum s_j \left( \frac{p_j}{p_0} \right)^k}
\end{equation}

with expenditure shares \( s_j = \frac{p_{0j}q_j}{\sum p_{0j}q_j} \) \( (j = 0, t, \) hybrid "weights" will henceforth no longer be considered; summation takes place over \( i \) taking the place of the weights \( w_i \) above.

\( k = 1 \) is the "usual" antiharmonic mean introduced above.

and the GAH mean allows some interesting extensions of table 1.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
& \( k = -2 \) & \( k = -1 \) & \( k = 0 \) & \( k = + 1/2 \) & \( k = + 1 \) \\
\hline
weights I & \( P_{0t}^{\text{GAH}}(0, -2) \) & \( P_{0t}^{\text{IH}} \) & \( P_{0t}^{\text{GAH}}(0, -\frac{1}{2}) \) & \( P_{0t}^{1} \) & \( P_{0t}^{\text{GAH}}(0, \frac{1}{2}) \) & \( P_{0t}^{\text{AH}} \) \\
\hline
weights IV & \( P_{0t}^{\text{HH}} \) & \( p_{0t}^{p} \) & \( P_{0t}^{\text{GAH}}(t, -\frac{1}{2}) \) & \( p_{0t}^{PA} \) & \( P_{0t}^{\text{GAH}}(t, \frac{1}{2}) \) & \( P_{0t}^{\text{PA}} \) \\
\hline
\end{tabular}
\caption{Table 2}
\end{table}

In table 2 there are no longer any hybrid weights Together with two additional rows for hybrid weights (type II and III) we would get with \( k = -1 \) (orange) and \( k = 0 \) (green) the complete table 1 as a special case (subset) of table 2.

Before going into details of table 1a and 2 it may be useful to introduce certain fictitious quantities in the following table

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
& deflated & inflated \\
\hline
\( q_0 \) & \( q_0^{D} = \frac{q_0}{p_0} = \frac{q_0p_0}{p_t} \) & \( q_0^{I} = \frac{q_0p_t}{p_0} \) \\
\hline
\( q_t \) & \( q_t^{D} = \frac{q_t}{p_t} = \frac{q_tp_0}{p_0} \) & \( q_t^{I} = \frac{q_tp_0}{p_t} \) \\
\hline
\end{tabular}
\caption{Table 1b}
\end{table}

The red double arrow means \( q_t^{I} \) is the "time antithesis" (I. Fisher) of \( q_t^{D} \) (and vice versa), and so is \( q_0^{D} \) of \( q_0^{I} \). We will come back to this table 1b in section 3 in our attempt to give a meaningful interpretation to \( P_{0t}^{\text{IH}} \) and \( P_{0t}^{\text{PA}} \).

The missing formula \( P_{0t}^{\text{II}} \) in tab. 1a now is

\begin{equation}
P_{0t}^{\text{II}} = \frac{\sum p_{t}^{2} p_{0} q_{t}}{\sum p_{t} p_{0} q_{t}} = \sum p_{t} q_{t} \quad \text{and for } P_{0t}^{\text{III}} \text{ we get}
\end{equation}

\begin{equation}
P_{0t}^{\text{III}} = \frac{\sum p_{0} q_{t}^{0}}{\sum p_{t} q_{t}^{0}}
\end{equation}

which does not seem to be a useful formula. This also applies to \( P_{0t}^{\text{HH}} \) which is given by

\begin{equation}
P_{0t}^{\text{HH}} = \frac{\sum p_{0} q_{t}^{D}}{\sum p_{0} q_{t}} \quad \text{or written as a GAH (t, -2) index}
\end{equation}

\begin{equation}
P_{0t}^{\text{HH}} = \frac{\sum \left( \frac{p_{t}}{p_{0}} \right)^{-1} p_{t} q_{t}}{\sum \left( \frac{p_{t}}{p_{0}} \right)^{-2} p_{t} q_{t}}
\end{equation}

which bears some resemblance with
\[ (2.12) \quad p_{0t}^{\text{RB}} = \frac{\sum p_0 q_0 D}{\sum p_0 q_0 D}, \text{ or written as a GAH (0, -1) index } \]

\[ p_{0t}^{\text{RB}} = \frac{\sum \left( \frac{p_t}{p_0} \right)^0 p_0 q_0}{\sum \left( \frac{p_t}{p_0} \right)^{-1} p_0 q_0}. \]

From tab. 2 two remaining formulas may be of some interest:

\[
\begin{array}{|c|c|c|}
\hline
p_{0t}^{\text{GAH}}(0,-2) & \frac{\sum p_t (p_0 q_0^D/p_t)}{\sum p_0 (p_0 q_0^D/p_t)} & \text{harmonic mean weights } p_0 q_0^D \\
& \text{(deflated base period)} & \\
\hline
p_{0t}^{\text{GAH}}(t,1) & \frac{\sum p_t (p_t q_t^l/p_0)}{\sum p_0 (p_t q_t^l/p_0)} & \text{arithmetic mean weights } p_t q_t^l \\
& \text{(inflated current period)} & \\
\hline
\end{array}
\]

Table 3 (combining tables 1, 1a, and 2)

<table>
<thead>
<tr>
<th>( k = -1/2 )</th>
<th>( k = +1/2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( j = 0 )</td>
<td>( p_{0t}^{\text{GAH}}(0,-\frac{1}{2}) = \frac{\sum p_t \sqrt{q_0 q_0^D}}{\sum p_0 \sqrt{q_0 q_0^D}} )</td>
</tr>
<tr>
<td>( j = t )</td>
<td>( p_{0t}^{\text{GAH}}(t,-\frac{1}{2}) = \frac{\sum p_t \sqrt{q_t q_t^l}}{\sum p_0 \sqrt{q_t q_t^l}} )</td>
</tr>
</tbody>
</table>

The four white fields represent indices of little or no use. We presented above the relevant formulas for three of them. The remaining formula (RF) reads as follows

\[ p_{0t}^{\text{RF}} = \frac{\sum (p_0 q_t^D)}{\sum p_t (p_0 q_t^D)} \]

Note that each of the six (more or less) meaningful index functions of table 1 appears twice in the green fields. We now can offer a list of indices and their "time antitheses". Table 4 shows that we have six pairs:

<table>
<thead>
<tr>
<th>( p_{0t} )</th>
<th>( p_{0t}^{\text{L}} )</th>
<th>( p_{0t}^{\text{P}} )</th>
<th>( p_{0t}^{\text{AH}} )</th>
<th>( p_{0t}^{\text{PA}} )</th>
<th>( p_{0t}^{\text{HH}} )</th>
<th>( p_{0t}^{\text{RF}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_{0t}^{\text{RH}} )</td>
<td>( p_{0t}^{\text{RA}} )</td>
<td>( p_{0t}^{\text{PA}} )</td>
<td>( p_{0t}^{\text{PA}} )</td>
<td>( p_{0t}^{\text{PA}} )</td>
<td>( p_{0t}^{\text{PA}} )</td>
<td>cp. table 7</td>
</tr>
<tr>
<td>( p_{0t}^{\text{GAH}}(0,-2) )</td>
<td>( p_{0t}^{\text{GAH}}(t,1) )</td>
<td>( p_{0t}^{\text{PA}} )</td>
<td>( p_{0t}^{\text{PA}} )</td>
<td>( p_{0t}^{\text{PA}} )</td>
<td>( p_{0t}^{\text{PA}} )</td>
<td></td>
</tr>
</tbody>
</table>
From the above formula for \( P_{0t}^{AH} \) follows \( P_{0t}^{AH} \geq P_{0t}^{k} \) and \( P_{0t}^{HI} \leq P_{0t}^{r} \), and as \( P_{0t}^{HI} \) is the "time antithesis" of \( P_{0t}^{AH} \) the index \( \sqrt{P_{0t}^{AH}P_{0t}^{HI}} \) as well as \( \sqrt{P_{0t}^{PA}P_{0t}^{HI}} \) follow the model of Fisher's ideal index \( P_{0t}^{k} = \sqrt{P_{0t}^{PA}P_{0t}^{HI}} \). About this more below.\(^{15}\)

We now are also able to classify unweighted – i.e. weights of \( s = 1/n \) – price index formulas using the unweighted GAH formula \( \sum \left( \frac{p_t}{p_0} \right)^{k+1} / \sum \left( \frac{p_t}{p_0} \right)^{k} \) for various values of \( k \)

<table>
<thead>
<tr>
<th>Table 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
</tr>
<tr>
<td>formula</td>
</tr>
</tbody>
</table>

The Fisher like index (geometric mean of an index and its time antithesis) is now is the CSWD index \( (P^H P^H)^{1/2} \) (as the unweighted analogon to \( P^F \)).

**2.2. Power means (generalized means) and products of power means**

Power means (PM) of order \( r \) weighted with expenditure shares \( s \) allow to construct new index functions and to demonstrate interesting relationships between existing index functions. The \( r^{th} \) PM of price relatives with expenditure shares \( s_{0i} = p_0 q_{0i} / \sum p_0 q_{0i} \) is \( P_{0t}^{PM}(r, s_0) \) is given by \( P_{0t}^{PM}(r, s_0) = \left[ \sum s_{0i} \left( \frac{p_t}{p_0} \right)^{r/2} \right]^{1/r} \). For \( r = 2 \) we have \( P_{0t}^{PM}(2, s_0) = \left[ \sum s_{0i} \left( \frac{p_t}{p_0} \right)^{2} \right]^{1/2} = \sqrt{P_{0t}^{PM}} \),

and \( P_{0t}^{PM}(r, s_1) \) for \( r = -2 \) (weights \( s_{it} = p_i q_{it} / \sum p_i q_{it} \)) is \( P_{0t}^{PM}(2, s_1) = \left[ \sum s_{it} \left( \frac{p_t}{p_i} \right)^{2/2} \right]^{-1/2} = \sqrt{P_{0t}^{PM}} \). This gives rise to study products of power means,

\[
(2.13) \quad P_{0t}^{PM}(r, s_0) P_{0t}^{PM}(-r, s_t) = \left[ \sum s_{0i} \left( \frac{p_t}{p_0} \right)^{r/2} \right] \cdot \left[ \sum s_{it} \left( \frac{p_t}{p_i} \right)^{-r/2} \right] \cdot \left[ \sum s_{it} \left( \frac{p_t}{p_i} \right)^{-r/2} \right]^{-1/2},
\]

of which – as demonstrated – Fisher's ideal index \( \sqrt{P_{0t}^{PA}P_{0t}^{HI}} \) is the special case \( r = 2 \).\(^{16}\) And \( r = -2 \) gives \( P_{0t}^{PM}(-2, s_0) P_{0t}^{PM}(-(-2), s_1) = \sqrt{P_{0t}^{PA}P_{0t}^{HI}} \). Another case is \( r = 1 \)

\[
P_{0t}^{PM}(-1, s_0) P_{0t}^{PM}(1, s_0) = \sqrt{\sum p_t q_t / \sum p_t q_t} \cdot \sqrt{\sum p_t q_t / \sum p_t q_t} = \sqrt{V_{0t} / Q_{0t}} = \tilde{\Phi}_0 \] that is the "indirect" Walsh price index (ratio of value index and Walsh quantity index).\(^{17}\)

\(^{14}\) To exceed \( P^F \) (\( P^{AH} > P^F \)) or fall short of \( P^F \) is of course not attractive. By contrast to be smaller than Laspeyres makes \( P^{HI} \) attractive, and accordingly \( P^{PA} > P^F \) may be seen as an advantage of Palgrave's index.

\(^{15}\) It turns out that most of the Fisher-type indices formed with one of the eight pairs above are nonsensical.

\(^{16}\) The unweighted variant of this with \( s_0 = s_{0i} = 1/n \) for all \( i \) is of course the CSWD index.

\(^{17}\) That is the product of \( \tilde{\Phi} \) and \( \tilde{\Phi} \) in table 6.
Obviously B is the “time antithesis” of A (and B* of A*) so that AB or A*B* is a Fisher-type index. Note also that \( t \) cannot be viewed as price indices because prices \( p \) in the numerator and \( p_0 \) in the denominator are not multiplied by the same factor. The same is true for Laspeyres type.

\[ \text{More about the weighted harmonic mean index (} P_{\text{PM}}^\text{H} \text{) and Palgrave's index (} P_{\text{PA}} \text{)} \]

### 2.3. Index type: price and quantity index

The convention is that a price index is a ratio in which numerator and denominator differ with respect to prices but not quantities. The opposite applies to a quantity index: numerator and denominator are different regarding quantities, not prices. However, the divide between a price index on the one hand and a quantity index on the other becomes a bit blurred once we introduce fictitious (deflated or inflated) quantities.

### Table 6

<table>
<thead>
<tr>
<th>( A = p_{0t}^{PM}(r,s_0) )</th>
<th>( B = p_{0t}^{PM}(-r,s_i) )</th>
<th>( B^* = p_{0t}^{PM}(r,s_i) )</th>
<th>( A^* = p_{0t}^{PM}(-r,s_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \sum s_{i0} \left( \frac{p_{it}}{p_{0t}} \right)^{\gamma} ]</td>
<td>[ \sum s_{i0} \left( \frac{p_{it}}{p_{0t}} \right)^{-\gamma} ]</td>
<td>[ \sum s_{i0} \left( \frac{p_{it}}{p_{0t}} \right)^{\gamma} ]</td>
<td>[ \sum s_{i0} \left( \frac{p_{it}}{p_{0t}} \right)^{-\gamma} ]</td>
</tr>
</tbody>
</table>

| 1 | \( \sum q_0 \sqrt{p_0 p_1} / \sum q_0 p_0 \) | \( \sum q_i p_1 / \sum q_i \sqrt{p_0 p_1} \) | \( \sum q_0 p_1 \) | \( \sum q_i \sqrt{p_0 p_1} \) |
| 2 | \( \sqrt{p_{0t}^L} \) | \( \sqrt{p_{0t}^P} \) | \( \sum p_i q_0^D / \sum p_0 q_0^D \) | \( \sum p_i \sqrt{q_i q_0^D} / \sum p_0 \sqrt{q_0^D} \) |

### Table 7

<table>
<thead>
<tr>
<th>Price index (in price index formula)</th>
<th>in a quantity index type presentation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laspeyres type</td>
<td>Paasche type</td>
</tr>
<tr>
<td>( p_{0t}^{HB} = \sum p_i q_0^D / \sum p_0 q_0^D )</td>
<td>( p_{0t}^{PA} = \sum p_i q_0^D / \sum p_0 q_0^D )</td>
</tr>
<tr>
<td>( p_{0t}^{HI} = \sum p_i q_0^D / \sum p_0 q_0^D )</td>
<td>( p_{0t}^{PA} = \sum p_i q_0^D / \sum p_0 q_0^D )</td>
</tr>
<tr>
<td>( p_{0t}^{RF} = \sum p_i (q_0 p_0 / p_i) / \sum p_0 (q_0 p_0 / p_i) )</td>
<td>( p_{0t}^{PA} = \sum p_i (p_0 / p_i) / \sum p_0 (p_0 / p_i) )</td>
</tr>
</tbody>
</table>

As to the Fisher type indices we have \( p_{0t}^F = \sqrt{p_{0t}^{PA} p_{0t}^{PF}} = \frac{\sum p_i q_0^D \sum p_i q_i^D}{\sqrt{\sum p_0 q_0^D \sum p_0 q_i^D}} \) and \( p_{0t}^{AH} p_{0t}^{HH} = \frac{\sum p_i q_0^D \sum q_i p_i^D}{\sqrt{\sum p_0 q_0^D \sum q_0 p_i^D}} \).

3. Average of ratios formulas (AOR), part II: More about the weighted harmonic mean index (\( P_{\text{HB}} \)) and Palgrave's index (\( P_{\text{PA}} \))

### 3.1. Price indices with constant expenditures

The following interpretation of \( P_{\text{HB}} \) an \( P_{\text{PA}} \) was brought into play in Germany by Werner Neubauer:\footnote{I (and possibly Neubauer too) did not know that much of the following was seen already by W. Ferger.} The rationale of the two rather unfamiliar indices \( P_{\text{HB}} \) and \( P_{\text{PA}} \) can be explained...
using the fictitious quantities introduced in table 1b. The quantity \( q^D_0 \) that is \( q^D_0 = q_{i0}(p_{i0}/p_{it}) \) in (2.12) \( p_{it}^{HB} = \sum p_{0} q_{i0} / \sum p_{0} q^D_{i0} \) must be regarded as quantity you can afford under the regime of new prices \( p_{it} \) (\( q^D_0 \) applies to period \( t \)) when you keep your expenditure (instead of the quantity) of period 0 constant for each commodity \( i \), so that \( q^D_0 p_{it} = p_{i0} q_{i0} \) and therefore \( \sum q^D_0 p_{it} = \sum p_{i0} q_{i0} \). The meaning of inflation now is: the price level is rising to the extent to which we get less quantity for the same amount of money. With rising prices \( \lambda_{it} = p_{it}/p_{i0} > 1 \) follows from \( q^D_0 p_{i0} = p_{i0} q_{i0} / \lambda_{it} \) so that \( q^D_0 < q_{i0} \) holds for each commodity, and \( \sum p_{i0} q^D_0 < \sum p_{i0} q_{i0} \) for all commodities (not to confound with \( \sum p_{it} q^D_0 = \sum p_{i0} q_{i0} \).

To interpret the products \( p_{i0} q^D_0 \) and \( p_{it} q^I_{it} \) is easy (as they are equal to \( p_{i0} q_{i0} \) and \( p_{it} q_{it} \) respectively), but \( p_{i0} q^D_0 \) and \( p_{it} q^I_{it} \) does not seem to be meaningful. However, with \( \lambda > 1 \) one can conclude

\[
\text{Table 8}
\]

<table>
<thead>
<tr>
<th>fictitious vs. real quantities</th>
<th>fictitious vs. real expenditures (interpretation)</th>
<th>price index</th>
<th>the underlying perspective of the interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q^D_0 ) refers to ( t )</td>
<td>( q^D_0 &lt; q_{i0} )</td>
<td>( p_{0t}^{HB} &gt; 1 )</td>
<td>to know and make use of ( q^D_0 ) in ( t ) so that ( \sum p_{it} q^D_0 = \sum p_{i0} q_{i0} ) requires to remember in ( t ) the expenditure of ( 0 ) (( \sum p_{i0} q_{i0} ))</td>
</tr>
<tr>
<td>( q^I_{it} ) refers to ( 0 )</td>
<td>( q^I_{it} &gt; q_{it} )</td>
<td>( p_{0t}^{PA} &gt; 1 )</td>
<td>to know and make use of ( q^I_{it} ) in ( 0 ) so that ( \sum p_{it} q^I_{it} = \sum p_{i0} q_{it} ) requires to know in ( 0 ) the future expenditure of ( t ) (( \sum p_{i0} q_{i0} ))</td>
</tr>
</tbody>
</table>

To remember in \( t \) what we used to spend in \( 0 \) in order to adjust \( q_{i0} \) so that \( q^D_0 \sum q^D_0 p_{it} = \sum p_{i0} q_{i0} \) is clearly a more realistic perspective than to transform \( q_{i0} \) to \( q^I_{it} \) so that we now (in \( 0 \)) with current prices \( p_{0} \) spend the same amount of money we are going to spend in the future (with prices \( p_{it} \) and quantities \( q_{it} \)). So we have again the situation that the implicit thought experiment with \( P^{PA} \) is more doubtful that in the case of \( P^{HB} \) just like the underlying logic of \( P^{L} \) is not on an equal footing with \( P^{L} \) (as is generally believed) but clearly less stringent.

3.2. Price indices as reciprocal quantity indices

It is interesting to see what becomes apparent when we contrast \( P^{HB} \) and \( P^{PA} \) to reciprocal quantity indices of Laspeyres and Paasche:

\[
\text{Table 9}
\]

<table>
<thead>
<tr>
<th>price</th>
<th>reciprocal quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_{0t}^{HB} )</td>
<td>( (Q_{0i}^{L})^{-1} = \sum p_{i0} q_{i0} / \sum p_{i0} q_{it} )</td>
</tr>
<tr>
<td>( p_{0t}^{PA} )</td>
<td>( (Q_{0i}^{P})^{-1} = \sum p_{i0} q_{i0} / \sum p_{i0} q_{it} )</td>
</tr>
</tbody>
</table>

Numerator and denominator differ in \( P^{HB} \) by \( q^D_{i0} \neq q_{i0} \) and in \( P^{PA} \) by \( q^I_{it} \neq q_{it} \) only. So \( P^{HB} \) and \( P^{PA} \) reflect the development of quantities.

Moreover for all commodities holds

\[
\frac{q^D_{i0} / q_{i0}}{p_{i0} / p_{0t}} = 1 \quad \text{and} \quad \frac{q^I_{it} / q_{it}}{p_{it} / p_{0t}} = 1.
\]
Hence $P_{0t}^{HB}$ and $P_{0t}^{PA}$ may also be viewed as a reciprocal quantity indices using fictitious quantities. The ratios on $\frac{q_{it}^D}{q_{0t}^D} / p_{0t}^D = q_{it}^D / p_{it}^D = 1$ and equivalently $\frac{q_{it}^L}{q_{0t}^L} / p_{0t}^L = q_{it}^L / p_{it}^L = 1$ may be viewed as (ratios) of elasticities. The "inverse quantity index" interpretation of a price index $P_{0t}^{HB}$ or $P_{0t}^{PA}$ is in line with the concept of inflation (as mentioned above) according to which a price level is rising to the extent to which we get less quantity for the same amount of money. It is also in harmony with reciprocal prices as fictitious quantities in the early index theory.

Not only $P_{0t}^{HB}$ and $P_{0t}^{PA}$, also the Laspeyres and Paasche price index can be given an appearance of a reciprocal quantity index, because we have $P_{0t}^{L} = \frac{\sum p_{i} q_{0i}}{\sum p_{i} q_{it}}$ and $P_{0t}^{R} = \frac{\sum p_{i} q_{it}}{\sum p_{i} q_{0i}}$ as counterparts of $P_{0t}^{HB} = \frac{\sum p_{0i} q_{it}}{\sum p_{0i} q_{0i}}$ and $P_{0t}^{PA} = \frac{\sum p_{i} q_{0i}}{\sum p_{i} q_{it}}$.

In Ferger 1931 a correctly developed theory can be found about when to use the weighted (weights $w_i$ where $w = \sum w_i$) arithmetic mean $\bar{x} = \frac{1}{w} \sum x_i w_i$ as opposed to the harmonic mean $(\bar{x}_H)^{-1} = \frac{1}{w} \sum \frac{1}{x_i} w_i$. Ferger realized that $\bar{x}_H$ should be used when the task is to average ratios $x_i = \frac{n_i}{d_i}$ where the numerator is constant ($n_i = n \forall i$) while $\bar{x}$ is the correct average, when the denominator is constant ($d_i = d \forall i$) so that $x_i$ is a linear transformation of $n_i$. Prices are by their very nature always ratios $p_i = t/m_i = expenditure for good i (monetary transaction)/mass (quantity of good i). So a household might be aiming to procure a constant (given) quantity, say $m = q_{0i}$ (to keep base period $q_{0i}$ constant) or to maintain the (base period) expenditure constant. And to our surprise on p. 40 we found the formula of "our" index $P_{0t}^{HB}$ derived as weighted harmonic mean $P_{0t}^{HB} = \left( \frac{\sum p_{0i} q_{it}}{\sum p_{i} q_{0i}} \right)^{-1}$.

We will come back to Ferger and thus to another interpretation of $P_{0t}^{HB}$ in a short annex below.

3.3. Again Laspeyres ($P^L$) and Paasche ($P^R$) are not equally well reasoned indices

Hence $P^L$ is related to $P^{HB}$ and so is $P^R$ to $P^{PA}$. However, as noted already in sec. 3.1 to interpret $P^{PA}$ seems less straightforward than $P^{HB}$, because $q_{it}^L$ appears more fictitious than $q_{it}^D$. To argue in terms of $q_{it}^L$ (based on future quantities $q_{it}$) for a fictitious base period expenditure (as in $P^{PA}$) is much less plausible, than to introduce quantities $q_{it}^D$ for the future based on quanti-

---

19 The "classic" example is velocities $v_i = D_i/T_i = distance/time (i = 1, \ldots, m)$ of m bikers. When all m bikers have the same distance to travel, so that they differ with respect to their elapsed time $T_1, ..., T_m$ the harmonic mean $\bar{v}_H$ is to be taken; while the arithmetic mean $\bar{v}$ would be correct when the winner is the one who covered the longest distance in a given time T. So a ratio can be viewed from different perspectives, and in $P^L$ and $P^{HB}$ simply prices are viewed from different perspectives, both reasonable and legitimate.

20 This point was also (and at great length) made by Allyn Young 1921. We use the symbols t and m Young introduced, $t_i$ instead of $p_i q_i$ for the expenditure and $m_i$ instead of $q_i$ for the quantity (and t rather than $q_0$).

21 Ferger discussed in detail which approach (orientation at quantities or at expenditures) would reflect more realistically the decisions of households, and he found that constancy of quantities $q_{0i}$ should be ruled out because it would imply a zero price elasticity of demand. He explicitly derived $P^{HB}$ which he recommended for use, but he did not discuss how $P^{HB}$ is related to $P^{PA}$ and that $P^L$ would be a kind of arithmetic analogon to the harmonic $P^{HB}$. Interestingly and most noteworthy: he already saw that $P^{HB}$ vs. $P^L$ boils down to constant expenditures vs. constant quantities. So he already was aware of an interpretation Neubauer some thirty years later rediscovered.

22 Or put more precisely: Laspeyres is the better reasoned measure of inflation than Paasche.
ties $q_0$ of the past (as in $P^{HB}$). It makes sense to ask: what can we consume in the future in order to keep to the expenditure of the past period 0, however, it is much less sensible to ask what we could have consumed in the past when we spent the same amount of money we do now, that is in the future period t (aiming at spending or imagining an expenditure amounting to $p_0q_i^t = p_0q_{it}$ in the past as in $P^p$ or in $P^{PA}$ – where $P^{PA}$ is even less rational because in $P^{PA}$ we recourse in a fictitious quantity $q_i^t$ - is less realistic a problem than spending now, in t the sum $q_0^d p_i$ as we actually did in the past, when we spent $q_0 p_0$). So we have here with $P^{PA}$ and $P^{HB}$ the same asymmetric relation as with $P^p$ and $P^L$ which shows once more, that $P^p$ and $P^L$ are – contrary to a common preoccupation – not equally reasonable index functions. There is a significant difference between these two indices that can easily be seen when one compares the sequence $P_{01}^L = \frac{\sum p_0q_0}{\sum p_0q_0}$, $P_{02}^L = \frac{\sum p_2q_0}{\sum p_0q_0}$, $P_{03}^L = \frac{\sum p_3q_0}{\sum p_0q_0}$,... where subsequent indices differ only with respect to prices in the numerator, however in $P_{01}^p = \frac{\sum p_0q_1}{\sum p_0q_1}$, $P_{02}^p = \frac{\sum p_2q_2}{\sum p_0q_2}$, $P_{03}^p = \frac{\sum p_3q_3}{\sum p_0q_3}$,... also with constantly changing quantities in numerator and denominator so that the elements $P_{01}$, $P_{02}$, $P_{03}$, ... in the sequence of prices indices are not comparable among themselves (what evidently applies to $P^{HB}$ and $P^{PA}$ too).

### 3.4. Axiomatic performance of $P^{HB}$ and $P^{PA}$

Due to the relationship between arithmetic and harmonic means $P_{0t}^{HB} < P_{0t}^L$ and $P_{0t}^{PA} > P_{0t}^p$. This may be welcomed as a valuable property of $P^{HB}$ and $P^{PA}$, because it is often said that Paasche understates and Laspeyres overstates inflation as measured by a true price index: the interval $P_{0t}^{HB} < P_{0t}^{TRUE} < P_{0t}^{PA}$ is smaller than $P_{0t}^{p} < P_{0t}^{TRUE} < P_{0t}^{L}$. In this sense it also might be somewhat attractive to consider the time reversible price index $\sqrt{P_{0t}^{HB}P_{0t}^{PA}}$, built analogously to $P_{0t}^{F} = \sqrt{P_{0t}^{L}P_{0t}^{P}}$ although an economic interpretation of this index must be challenging.

Both indices, $P^{HB}$ and $P^{PA}$ are weakly monotonous, however, $P^{PA}$ is strictly monotonous only in the base period prices and $P^{HB}$ in the current period prices. Though all indices of table 1 are means of price relatives, that is they possess the mean value property $M$, only few of them, viz $P^L$ and $P^p$ are linear (= additive): Additivity (linearity) L is defined in table 10b and neither $P^{PA}$ nor $P^{HB}$ satisfy both conditions of additivity.\(^{23}\) Property $L$ is a subset of property $M$.

Linearity (L) is a valuable property because non-linearity (non-additivity) means that it cannot be simply inferred how the index will change in response to the change of individual prices (denoted by non-zero vectors $p_i^*$ or $p_0^+$ respectively). $P_{0t}^{PA}$ is not additive in current period prices and $P_{0t}^{HB}$ not additive (linear) in base period prices. In other words, there is no strict relationship between the change of individual prices $\Delta p_i = p_i^* - p_i$ = $p_i^*$ and the amount by which the index as a whole changes $\Delta P = P(p^*) - P(p)$.

\(^{23}\) Other examples of price indices which clearly are means of price relatives but nonetheless not linear are the logarithmic Laspeyres (also called geometric Laspeyres) price index $DP^L$ where $\ln(DP^L)$ is the arithmetic mean of logarithmic price relatives (weighted with expenditure shares of the base period $p_0q_0/\sum p_0q_0$), or the quadratic mean of price relatives $P^{QM}$. 

Palgrave's index $P_{0t}^{PA} = \sum_{t} \frac{p_i}{p_0} = \sum p_t q_t$ is not linear (additive) in current period prices because this would require for $p^*_t = p_t + p^+_t$ to hold that

$$P_{0t}^{PA}(p^*_t) = \left( \sum \frac{p_i q_i}{p_0} p_t + \sum \frac{p_i q_i}{p_0} p^*_t + \sum \frac{p^*_i q_i}{p_0} p^+_t + \sum \frac{p^*_i q_i}{p_0} p^+_t \right) / \left( \sum p_t q_t + \sum p^*_t q_t \right)$$

is equal to

$$P_{0t}^{PA}(p_t) = \sum \frac{p_i q_i}{p_0} p_t / \sum p_t q_t$$

and

$$P_{0t}^{PA}(p^+_t) = \sum \frac{p^*_i q_i}{p_0} p^+_t / \sum p^*_t q_t$$

\[\text{Table 10(a)}\]

<table>
<thead>
<tr>
<th>price index</th>
<th>$P_{0t}^{HB}$ (harmonic base)</th>
<th>$P_{0t}^{PA}$ (Palgrave)</th>
</tr>
</thead>
<tbody>
<tr>
<td>bounds</td>
<td>$P_{0t}^{HB} &lt; P_{0t}^{L}$</td>
<td>$P_{0t}^{P} &lt; P_{0t}^{PA}$</td>
</tr>
<tr>
<td>monotonicity*</td>
<td>weak monotonicity fulfilled by both indices, $P_{0t}^{HB}$ and $P_{0t}^{PA}$</td>
<td>$P_{0t}^{PA}$ strictly monotonic only in the base period prices, $P_{0t}^{HB}$ is strictly monotonic only in current period prices</td>
</tr>
<tr>
<td>additivity</td>
<td>not in base period prices</td>
<td>not in current period prices</td>
</tr>
<tr>
<td>aggregative consistency (AC)</td>
<td>both indices, $P_{0t}^{HB}$ and $P_{0t}^{PA}$ are aggregative consistent</td>
<td></td>
</tr>
<tr>
<td>quantity indices</td>
<td>both indices fail the factor reversal test and the indirect (implicit) quantity indices violate the proportionality axiom $\sum_i Q_i \neq \lambda$, although $Q_i = \lambda q_i \forall i$</td>
<td></td>
</tr>
</tbody>
</table>

* the reason for failing monotonicity is that $p^*_t > p_t$ affects in $P_{0t}^{PA}$ both, price relatives and weights (in $P_{0t}^{HB}$, however, only price relatives). The opposite applies to $P_{0t}^{HB}$ that meets strict monotonicity only in current period prices but not in base period prices.

\[\text{Table 10(b)}\]

<table>
<thead>
<tr>
<th>Axiom</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>weak monotonicity</td>
<td>$P(p_0, q_0, p_t, q_t) &gt; P(p_0, q_0, p_0, q_0)$ if $p_t &gt; p_0$</td>
</tr>
<tr>
<td></td>
<td>$P(p_0, q_0, p_t, q_t) &lt; P(p_0, q_0, p_0, q_0)$ if $p_t &lt; p_0$</td>
</tr>
<tr>
<td>strict monotonicity</td>
<td>a) in current period prices $P(p_0, q_0, p^<em>_t, q_t) &gt; P(p_0, q_0, p_t, q_t)$ if $p^</em>_t \geq p_t$</td>
</tr>
<tr>
<td></td>
<td>b) in base period prices $P(p^<em>_0, q_0, p_t, q_t) &lt; P(p_0, q_0, p_t, q_t)$ if $p^</em>_0 \geq p_0$</td>
</tr>
<tr>
<td>additivity (= linearity)</td>
<td>a) in current period prices $P(p_0, p^*_t) = P(p_0, p_t) + P(p_0, p^+_t) = A + B$</td>
</tr>
<tr>
<td></td>
<td>if $p^*_t = p_t + p^+<em>t$ (can be violated by $P</em>{0t}^{PA}$)</td>
</tr>
<tr>
<td></td>
<td>b) in base period prices $\frac{1}{P(p^<em>_0, p_t)} = \frac{1}{P(p_0, p_t)} + \frac{1}{P(p^</em>_0, p_t)} = \frac{1}{C} + \frac{1}{D}$</td>
</tr>
<tr>
<td></td>
<td>if $p^*_0 = p_0 + p^+<em>0$. (can be violated by $P</em>{0t}^{HB}$)</td>
</tr>
<tr>
<td>aggregative consistency (AC)</td>
<td>compilation of an index for the total aggregate (T) in one step starting with individual price relatives (r) in symbols $r_1, r_2, \ldots, r_n = P_t$</td>
</tr>
<tr>
<td></td>
<td>or in two steps via indices for sub-aggregates (S1, S2, ...) or again in symbols: $r_1, r_2, \ldots, r_n = P_{S1}, P_{S2}, \ldots = P_t$</td>
</tr>
<tr>
<td></td>
<td>will yield the same result</td>
</tr>
</tbody>
</table>
It can easily be seen that $P_{t0}^{PA}$ not even complies with linearity in the restricted case of $p_i^* = b^i$ which means that "if all prices are increased by the same amount, the index of the new prices should equal to the old index number plus the index number of the constant amount".\footnote{Pfouts 1966 p. 176 also noticed that the famous "ideal" index $P_F$ of Fisher does not satisfy additivity (in all variants) either, not even in the restricted case above $b^i = [b \cdots b]$. His paper was a sort of pamphlet against the (unmerited) prestige of the "ideal" Fisher index $P_F$, an index he thought, should be abandoned.}

A price index formula that complies with additivity (linearity) in both, base period and current period prices can also be written as $P_{t0} = \frac{aP_t}{bP_0}$, that is as ratio of scalar vector products.\footnote{von der Lippe 2007, 193.} For example $P_{t0}^{HB}$ implies $a^i = b^i = q_i^0$ [constant], in $P_{t0}^p$ we have $a^i = b^i = q_i^t$. But $P_{t0}^{HB}$ and $P_{t0}^{PA}$ though both means of price relatives are yet not additive (linear) functions in prices $p_0$ and $p_t$. Though we can write (see table 7) $P_{t0}^{HB} = \frac{\sum p_i q_i^0}{\sum p_i q_i^0}$ and $P_{t0}^{PA} = \frac{\sum p_i q_i^t}{\sum p_i q_i^t}$ this is not in line with $P_{t0} = \frac{aP_t}{bP_0}$ because unlike $a$ and $b$ the "quantity" vectors here (with elements $q_i^t = q_i(p_t/p_0)$ and $q_i^0 = q_i(p_0/p_t)$ respectively) are not independent of $p_t$ and $p_0$.

**Aside:** The presentation in table 7 is particularly useful to relate these indices to linear indices. However the forms $P_{t0}^{PA} = \frac{\sum p_i q_i^t}{\sum p_i q_i^0}$ and $P_{t0}^{HB} = \frac{\sum p_i q_i^0}{\sum p_i q_i^0}$ are better in order to demonstrate that under normal conditions of rising prices the indices exceed unity because then $q_i^t > q_i$ and $q_i^0 < q_i$.

Just like linearity or additivity (L) and the mean value (of price relatives) indices (M) and should be kept distinct so also additivity (L) aggregative consistency (AC or simply C). However, while L is a subset of M the situation with L and C is different:

- $C \cap L$ (for example $P_L$, $P^p$),
- $C - L$ ($P^{PA}, P^{HB}$) so that $P^{HB}$ and $P^{PA}$ meet C though not L
- for L-C an example is $\frac{1}{2}(P_{t0}^{L} + P_{t0}^{P})$ which is linear but violates C because the weights $1/2$ are not related to expenditure shares; and finally for
- $C \cup L$ the most prominent example is $P^F$.

In the case of $P^{HB}$ and $K = 2$ sub-aggregates AC means that when the index for the total aggregate can be compiled in one step, say as harmonic mean of price relatives, the corresponding two step procedure via compiling indices for K sub-aggregates which are subsequently aggregated (again as harmonic mean of the K sub-indices with base year expenditure weights) to the total index we should get the same result. It can easily be seen that indeed for $K = 2$ sub-aggregates, A and B, the total index $P_{t0}^{HB}$ is a harmonic mean of $P_{t0}^{HB}(A)$ and $P_{t0}^{HB}(B)$

$$
(P_{t0}^{HB})^{-1} = \left( \frac{\sum p_{A0}}{p_{At}} \frac{\sum p_{A0} q_{A0}}{\sum p_0 q_0} \right) + \left( \frac{\sum p_{B0}}{p_{Bt}} \frac{\sum p_{B0} q_{B0}}{\sum p_0 q_0} \right)
$$
\[
(P_{0t}^{\text{HB}}(A))^{-1}g_A + (P_{0t}^{\text{HB}}(B))^{-1}g_B, \quad \text{with } g_A + g_B = \frac{\sum P_{0t}q_{0i}^A}{\sum P_{0t}q_{0i}^0} + \frac{\sum P_{0t}q_{0i}^B}{\sum P_{0t}q_{0i}^0} = 1.26
\]

Other important properties of \(P_{t0}^{\text{HB}}\) and \(P_{t0}^{\text{PA}}\) concern quantity indices of the HB and PA type. A distinction has to be made between

- **direct quantity indices** \(Q_{t0}^{\text{HB}}\) gained from price indices \(P_{t0}\) by interchanging prices and quantities so that \(Q_{t0}^{\text{HB}} = \frac{\sum q_{0i}^0 p_{0i}^q q_{0i}}{\sum q_{0i}^0 p_{0i}^q q_{0i}}\) and \(Q_{t0}^{\text{PA}} = \frac{\sum q_{0i}^0 p_{0i} q_{0i}}{\sum q_{0i}^0 p_{0i} q_{0i}}\), and

- **indirect (implicit) quantity indices** resulting from dividing the value index \(V_{t0} = \Sigma p_{0q} q_{0}/\Sigma p_{0q} q_{0}\) by the respective price index \(Q_{t0}^{\text{HB}} = V_{t0}/P_{t0}^{\text{HB}}\).

Unlike the direct quantity indices \(Q_{t0}^{\text{HB}}\) and \(Q_{t0}^{\text{PA}}\), the respective implicit quantity indices, that is \(\tilde{Q}_{t0}^{\text{HB}} = V_{t0}/P_{t0}^{\text{HB}}\) and \(\tilde{Q}_{t0}^{\text{PA}} = V_{t0}/P_{t0}^{\text{PA}}\), will violate proportionality in the quantities and consequently also identity. This clearly invalidates them as deflators.\(^{27}\) The products \(A = P_{t0}^{\text{PA}}Q_{t0}^{\text{PA}}\) and \(B = P_{t0}^{\text{HB}}Q_{t0}^{\text{HB}}\) are in general unequal \((A \neq B)\) and also different from the value index \(V_{t0}\), that is the indices of the HB and PA type are not factor reversible.\(^{28}\) Lack of proportionality of \(\tilde{Q}_{t0}^{\text{HB}} \neq Q_{t0}^{\text{HB}}\) and \(\tilde{Q}_{t0}^{\text{PA}} \neq Q_{t0}^{\text{PA}}\) (and thus a fortiori lack of identity),\(^{29}\) can easily be verified as follows: if \(q_{0i}^0 = \lambda q_{0i}^0 \forall i\) then also \(Q_{t0}^{\text{PA}} = Q_{t0}^{\text{HB}} = \lambda\) and therefore the indirect quantity indices will in general be \(\tilde{Q}_{t0}^{\text{HB}} \neq \lambda\) because \(\tilde{Q}_{t0}^{\text{HB}} = V_{t0}/P_{t0}^{\text{HB}} = Q_{t0}^{\text{HB}}P_{t0}^0/P_{t0}^{\text{HB}} = \lambda\) unless \(P_{t0}^{\text{HB}} = P_{t0}^0 = P_{t0}\). The same consideration leads to \(\tilde{Q}_{t0}^{\text{PA}} \neq \lambda\). More specifically we can conclude \(\tilde{Q}_{t0}^{\text{HB}} > \lambda = \tilde{Q}_{t0}^{\text{PA}}\) because \(P^j > P^{\text{HB}}\) and \(\tilde{Q}_{t0}^{\text{PA}} < \lambda\).

It is difficult to say something about how \(\sqrt{P_{t0}^{\text{HB}}P_{t0}^{\text{PA}}} = \sqrt{\frac{\sum p_{0t} q_{0t}^A }{\sum m_{0t}} \frac{\sum p_{0t} q_{0t}^I }{\sum m_{0t}}}\) differs from \(\sqrt{P_{t0}^{\text{PA}}P_{t0}^0} = \sqrt{\frac{\sum p_{0t} q_{0t}^I }{\sum m_{0t}} \frac{\sum p_{0t} q_{0t}^D }{\sum m_{0t}}}\) because it is the structure of weights that matters and nothing definite can be said about how the structure of weights in \(P^L\) vs. \(P^{\text{HB}}\) and in \(P^P\) vs. \(P^{\text{PA}}\) differs:

\[
P_{t0}^L = \frac{\sum p_{0t} q_{0t}}{\sum p_{0t} q_{0t}} = \frac{\sum p_{0t} q_{0t}}{\sum p_{0t} q_{0t}} = A \cdot P_{t0}^{\text{HB}} = A \cdot \sum \frac{p_{0t}}{P_{t0}^0} \frac{q_{0t}}{q_{0t}}\]

\[
P_{t0}^P = \frac{\sum p_{0t} q_{0t}}{\sum p_{0t} q_{0t}} = \frac{\sum p_{0t} q_{0t}}{\sum p_{0t} q_{0t}} = B \cdot P_{t0}^{\text{PA}} = B \cdot \sum \frac{p_{0t}}{P_{t0}^0} \frac{q_{0t}}{q_{0t}}\]

It is difficult make conclusions about \(A/B\).

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26 It can easily be verified that the corresponding equation (with arithmetic means) holds for \(P^{\text{PA}}\), so that also Palgrave's index is additively consistent.

27 When all quantities remain constant \((\lambda = 1)\), then using \(P^{\text{HB}}\) as deflator would not result in \(V/P^{\text{HB}} = 1\) unless also all prices would remain constant (because then we have \(V=PH^{\text{HB}} = 1\)). Otherwise \(V/P^{\text{HB}}\) shows (counterfactually) a rising volume (and the deflator \(P^{\text{PA}}\) analogously a decreasing volume).

28 Factor reversibility (the factor reversal test) amounts to equality of the direct and the indirect quantity index. Unlike the pair \(L/P\) (Laspeyres/Paasche), the pair HB/PA does not even pass the less demanding product-test.

29 Identity is the special case \(\lambda = 1\) of proportionality. Hence these indices cannot be regarded as quantity indices in the strict sense.
4. Weighted indices: Ratios of averages formulas (ROA)

Many index functions possess both forms (formulas), an AOR form like $P_{0t}^L = \frac{\sum p_i q_{it}}{\sum p_i q_{t0}}$ with weights $\sum p_i q_{it}/\sum p_i q_{t0}$ and a ROA form $P_{0t}^C = \frac{\sum p_i q_{0t}}{\sum p_i q_{t0}}$ in which both numerator and denominator reflect expenditure or average prices if divided by a suitable quantity. We encountered indices with an AOR form and an unfamiliar ROA form with no or only an at best far fetched AOR interpretation. To give only one example for an "ROA-only" index (with no AOR interpretation) we briefly mention Drobisch's index as ratio of a sort of absolute (e.g. expressed in $) price levels $P_t$ and $P_0$ as "macro level" or (universal) "unit values"

\[ \tilde{P}_t = \frac{\sum p_i q_{it}}{\sum q_{it}} \quad \text{and} \quad \tilde{P}_0 \quad \text{defined correspondingly, we get} \]

\[ P_{0t}^{DR} = \frac{\tilde{P}_t}{\tilde{P}_0} = \frac{\sum p_i q_{it}}{\sum p_i q_{t0}} \cdot \frac{\sum q_{0t}}{\sum q_{it}} = \frac{V_{it}}{Q_{it}^0} \]

With Dutot's quantity index $Q_{0t}^0 = \sum q_{it}/Q_{t0}$.

One of the problems with $P_{0t}^{DR}$ is that the sum over all goods and services is generally not defined (the summation fails already for the simple reason that there are quite different measuring units for the quantities). So as a rule $P_{0t}^{DR}$ cannot be compiled on the level of a national economy. This infeasible $P_{0t}^{DR}$ is often called "unit value index" (UVI) and it is often confused with another indeed existing (in particular in foreign trade statistics where we have since long the habit to report quantities of exported or imported goods) UVI index, compiled with unit values (a sort of average prices) as building blocks instead of prices as building blocks. Assume the k-th aggregate can be decomposed into $J_k$ individual goods $j = 1, 2, \ldots, J_k$.

Then its unit value in $t$ is $\tilde{p}_{kt} = \frac{\sum j p_i q_{jk} q_{kt}}{\sum q_{jk} q_{kt}} = \frac{\sum p_i q_{0jk} q_{0kt}}{q_{kt0}}$ and in 0 $\tilde{p}_{k0} = \frac{\sum p_i q_{0jk} q_{0kt}}{q_{0kt0}}$, so that a Laspeyres type (L) or (more commonly in use) Paasche (P) type unit value index can be compiled as

\[ P_{0t}^{UV(L)} = \frac{\sum p_i q_{kt0}}{\sum p_i q_{kt0} q_{k0}} \neq P_{0t}^L = \frac{\sum p_i q_{jk0} q_{kt0}}{\sum q_{jk0} q_{kt0}} \quad \text{and} \quad P_{0t}^{UV(P)} = \frac{\sum p_i q_{kt0} q_{k0t}}{\sum q_{kt0} q_{k0t}} \neq P_{0t}^P = \frac{\sum p_i q_{jk0} q_{kt0}}{\sum q_{jk0} q_{kt0}} . \]

Both indices evidently differ from $P_{0t}^{DR}$. The present author has repeatedly published papers explaining the difference between unit value indices $P_{0t}^{UV(L)}$ and $P_{0t}^{UV(P)}$ and genuine price indices $P_{0t}^L$ and $P_{0t}^P$. There may be other "ROA-only" indices in addition to $P_{0t}^{DR}$, but it seems questionable that it is worthwhile discussing them. In von der Lippe (2013) we gave mention to a rather strange index formula of Lehr which is another example (in addition to $P_{0t}^{DR}$) for an "ROA-only" index.

Appendix: Ferger's index of purchasing power (PP)

According to Ferger an appropriate measure of the purchasing power (PP) of money is "not the reciprocal of an index of prices but an index of the reciprocals of prices properly

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30 This applies to indices making use of fictitious quantities. There are of course also indices that possess none of the two forms, neither AOR nor ROA, for example Fisher's ideal index $P^F$.

31 Unlike the overall sum of quantities $\sum q_{jk}$ and $\sum q_{kt}$ the k-specific $q_{kt0}$ = $q_{jk0}$ and $q_{k0}$ can be compiled.

32 It may be noted in passing that $P_{0t}^{DR}$ is able to comply with the chain test (transitivity), however, fails (what already Laspeyres noticed) identity because if $p_{it} = p_{i0} \forall i$ we have $P_{0t}^{DR} = Q_t^0/Q_{t0}$. 

---
weighted" Ferger 1936, p 266 (Ferger's emphasis). As an unweighted version of such an index of reciprocal prices \( \bar{P}^p \) he studied \( \frac{1}{n} \sum \frac{1}{p_{it}} = \left( \bar{p}_i^c \right)^{-1} = \frac{\bar{p}_0^H}{\bar{p}_i^H} \), and he demonstrated\(^{33}\) the difference between his \( \bar{P}^p \) concept and the ruling "macro" PP concept as reciprocal price index \( (P^{-1}) \), with the example \( \left( \bar{p}_0^D \right)^{-1} = \frac{\bar{p}_0}{\bar{p}_i} \). The reason why he rejected this \( P^{-1} \) approach of inverting a price index was, that he opposed the then (in the "stochastic index theory") popular idea of a hidden general driving force that makes all prices increase or decrease more or less in unison.\(^{34}\) So he thought a measure of PP should address individual prices as a sort of "micro" approach\(^{35}\) and consider expenditures rather than quantities.\(^{36}\) Ferger realized that the difference between his favoured index \( \frac{\bar{p}_0^H}{\bar{p}_i^H} \) and the \( P^{-1} \) approach using \( \left( p_0^D \right)^{-1} = \frac{\bar{p}_0}{\bar{p}_i} \)

"arises from the different weights applied by the formula despite our proposed intention of maintaining equal weights. The price index gives more weight to the price of the expensive article A by assuming constant quantities to be bought, while the purchasing power index gives more weight to the price change of the cheaper article B by assuming constant expenditure to be maintained" (268).

For a weighted index of PP (i.e. of reciprocal prices) Ferger considered expenditures \( e_0 = p_0 q_0 \), but such weights need an adjustment because what we get for \( 1\$ \) and thus for \( e_0 \$ \) depends on the prices. So weights should be \( \frac{p_0 q_0}{1} = (p_0)^2 q_0 \) and it can easily be seen that

\[
\frac{\sum \left( \frac{1}{p_i} \right) (p_0)^2 q_i}{\sum \left( \frac{1}{p_i} \right) (p_0)^2 q_i} = \frac{1}{\bar{p}_0^H} \quad \text{and} \quad \frac{\sum \left( \frac{1}{p_i} \right) (p_0)^2 q_i}{\sum \left( \frac{1}{p_i} \right) (p_0)^2 q_i} = \frac{1}{\bar{p}_0^A}.
\]

Division by \( 1/p_0 \) or \( 1/p_i \) means that expenditure is not measured in \( \$ \) but in units\(^{37}\) per \( \$ \) which comes closer to a sort of standardized and dimensionless quantity.

As an aside: For Ferger's PP index \( (P^{HB})^{-1} \) but not for the price index \( P^{HB} \) we have a time series interpretation as we had in sec. 3.3 (p. 12 above) for the price index \( P^L \) of Laspeyres because of the constant denominator \( \sum p_0 q_0 (p_0/p_i) \). In this respect the PP index \( (P^{PA})^{-1} \) is less attractive as \( (P^{HB})^{-1} \)

\( \text{just like } \) \( (P^{PA})^{-1} \) is short of \( P^{HB} \) (or \( P^B \) short of \( P^L \)). Interestingly Ferger started out with the \( P^{PP} \) idea but ended up with a \( P^{-1} \) type of

\(^{33}\) Using a numerical example, p. 267.

\(^{34}\) He also disapproved the preference (in the stochastic index theory) for unweighted indices which was based on the idea that each price relative was considered an independent representative of the same general force driving prices up or down. So Ferger was explicitly in favour of weighting with expenditure weights (we already noticed above his interesting interpretation of \( P^{HB} \) in terms of keeping expenditures rather than quantities constant).

\(^{35}\) This is of course rather absurd: we cannot speak of a purchasing power of money with respect to cheese, shoes etc. but only of a purchasing power of a specified income spent for a bundle of clearly defined purchases. Notwithstanding his idea "There is no change in the value of money except that which is result of, or rather is composed of changes in the prices of commodities"(262f) is of course correct.

\(^{36}\) "If we are measuring changes in its [the money's] buying power over several commodities, we must thus measure the changing quantities of these commodities when the same amounts of money are paid for them respectively, rather than measuring the varying total cost of the same respective quantities" (Ferger 1936, p. 268).

\(^{37}\) Interestingly Fegers symbol for \( 1/p_0 \) is \( u_0 \) (u for "unit"). We already saw that \( 1/p_0 \) or \( 1/p \) may be viewed as "implicit" quantities. And Ferger indeed made extensive reference to Bowley who stressed that PP is closely related to a quantity and so a PP index should be somehow conceptually related to a quantity index.
PP index, viz. \((P_{HB}^{\text{HB}})^{-1}\). So in practice his distinction by which he set great store that is "not the reciprocal of an index of prices but an index of the reciprocals of prices" does not seem to be a particularly fertile one.

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