On a preference analysis in a group decision making

Mazurek, Jiří

School of Business Administration in Karvina

12 June 2015

On a Preference Analysis in a Group Decision Making

Jiří Mazurek
Silesian University in Opava
mazurek@opf.slu.cz

Abstract: The aim of the paper is to provide several quantitative measures concerning preference structure in a group decision making setting. These measures enable to assess group and individual discord, core preferences and outliers, or to find a consensus, where a consensus is defined as a preference with a minimum sum of distances to other preferences. Also, it is shown that a distance of a consensus to a median preference is upper bounded, which might reduce a search for a consensus significantly.

Keywords: consensus, decision making, distance, discord, geometric median, group decision making, group discord, preference, preference structure.

JEL: D71

1 Introduction

The aim of collective decision making is to undertake the best (optimal) solution to a given problem by a group of experts in a given field. Group decision making is present everywhere where a committee, a board, a council, etc., has to carry out a decision. It occurs in many areas of human action such as economics, politics, environmental protection, education, civil engineering, medicine, military, etc., but it is also present in everyday family lives. For a brief review of past, present and future of group decision making see e.g. Kameda et al. (2002).

Decision makers can express their preferences in many different ways. Given a set of feasible options (objects, alternatives, candidates, etc.), the most common preference formats include rankings of all compared options from the best to the worst, a selection of the best option only, or assigning each option its 'value' (expressed in points or marks, language variables such as 'very good', etc.).

Ranking of options or selection of the best option (without ranking the rest) is a well-known setting from the social choice theory, where individual decision makers (DMs) are called *voters*, they choose among a finite set of *candidates*, and their preferences (rankings of candidates, usually without ties) are called *votes*, *individual preference list* or *ballots*. For an introduction to the social choice theory, see e.g. (Sen 1970; Fishburn 1973; Feldman and Serrano 2006; Wulf 2006; Taylor and Pacelli 2009; or Myerson 2013). A winner of an election is found with the use of many different social choice functions or procedures such as plurality voting, Condorcet's majority rule, Borda's method of marks, Copeland method, Hare system, see e.g. Taylor and Pacelli (2009). All these methods satisfy some 'reasonable' properties, such as unrestricted domain, monotonicity, independence of irrelevant alternatives, Pareto efficiency, non-dictatorship, etc., see e.g. (Arrow 1951; Fishburn 1973; Wulf 2006 or Taylor and Pacelli 2009). But, as shown by Arrow (1951), none method satisfies all of them if there at least two decision makers and at least three alternatives (see also Gibbard–Satterthwaite theorem). Also, some social choice procedures are susceptible to voting paradoxes, such as Condorcet's paradox; see e.g. (Saari 2000; Felsenthal 2010).

While the literature on social choice theory focuses mainly on examination and comparison of social choice functions (procedures) under different conditions, the orientation of this paper is slightly different. It focuses on an evaluation of a structure of decision makers' preferences, and a relationship between this structure and a group consensus, because it is the structure of preferences that determines whether achieving a group consensus is possible (and whether this consensus is unique).

In this paper a consensus is defined as a preference which minimizes the sum of distances to all preferences provided by DMs. In this sense a consensus is an analogue to the *geometric median*, which is defined as a point in an n dimensional *Euclidean space* E^n, minimizing a sum of distances to a given (finite) set of points from E^n. However, in this study this concept is extended to all spaces endowed with a metric function, called *decision spaces*, such as a space of all permutations S_n of the order n, a space of pairwise comparison matrices, etc.

As mentioned before, DMs' preferences determine the result – the consensus. The aim of this paper is to provide several new measures and concepts enabling to evaluate a structure of decision makers' preferences, and to show how to use it in finding a consensus especially if the decision space is discrete (for example S_n).

The paper is organized as follows. In section 2 basic concepts and notation is introduced. In section 3 some theoretical properties are examined and section 4 provides numerical examples. Section 5 provides a brief discussion of well-defined and ill-defined problems and in section 6 some possible extensions to the proposed approach are discussed. Conclusions close the article.

2 Concepts and notation

In this paper it is assumed that a finite set of decision makers evaluate a finite set of alternatives and provide their (crisp and complete) preferences in such a form that these preferences can be considered elements of some metric (decision) space.

The format of preferences includes:

- rankings of n alternatives, so they can be regarded elements from a space S_n of all permutations of the order n,
- pairwise comparison matrices as proposed in the analytic hierarchy/network process, see Saaty (2001), which are elements from a space of square matrices of the order n,
- real (integer) number values assigned to each alternative, etc.

Assumption that a decision space is endowed with a suitable metric function is important, because it allows measuring distances among preferences.

DEFINITION 1: *Let μ be a function on a set X so that $\mu : X \times X \rightarrow \mathbf{R}$. Function μ is called a metric, if it satisfies the following axioms (1) to (4):*

1. $\mu(x,y) \geq 0$, *(non-negativity),*
2. $\mu(x,y)=0$ *if and only if* $x=y$, (*identity*),
3. $\mu(x,y)=\mu(y,x)$, (*symmetry*),
4. $\mu(x,z) \leq \mu(x,y)+\mu(y,z)$, (*triangular inequality*), *for all x, z, y* $\in$ *X.*

Some well-known examples of metric function include:

i) The distance between two real numbers x and y on a real axis: $\mu(x,y)=|x-y|$.

ii) Euclidean metric, $x=(x_1,...,x^n), y=(y_1,...,y_n) \in R^n$: $\mu(x,y)=\sqrt{(x_1-y_1)^2+...+(x_n-y_n)^2}$

iii) Manhattan metric, $x=(x_1,...,x^n), y=(y_1,...,y_n) \in R^n$: $\mu(x,y)=|x_1-y_1|+...+|x_n-y_n|$

iv) Kendall's tau metric defined as a number of transpositions of adjacent pairs of digits necessary to turn one permutation into other (also known as the *bubble-sort distance*).

v) The distance between matrices $A(a_{ij})$ and $B(b_{ij})$: $\mu(A,B)=\left(\sum_{i,j}\left|a_{ij}-b_{ij}\right|^{p}\right)^{1/p}$, etc.

In this study all metric functions are equivalent, in numerical section 4 metrics i) and iv) are applied.

Merriam-Webster's dictionary defines a 'consensus' as a general agreement, a unanimity of opinions, but in this paper a consensus is defined as a preference closest to a set of given preferences, see Definition 2 below. Hence, it is a preference that best describes an opinion of a group (also, it can be regarded a compromise).

DEFINITION 2: Let *DS* be a decision space, that is a space of all feasible decisions *D*. Let μ be a metric function on *DS*. Suppose a set of *n* decision makers (DMs) provide *n* (not necessarily distinct) decisions $D_i \in DS$, $i \in \{1,2,...,n\}$, which form a subspace of *DS* denoted as *DNS*. Then:

i) A decision $D_j \in DS$ for which $\sum_{i=1}^{n}\mu\left(D_j,D_i\right)$ is minimal is called a *consensus* and is denoted as D_C thereinafter.

ii) A decision $D_j \in DNS$ for which $\sum_{i=1}^{n}\mu\left(D_j,D_i\right)$ is minimal is called a *pivot* and is denoted as D_P thereinafter.

iii) An *average distance* of all $D_i \in DNS$ to a consensus (a pivot) is denoted as *AVCD* *(AVPD)*: $AVCD=\frac{\sum_{i=1}^{n}\mu\left(D_C,D_i\right)}{n}$ ($AVPD=\frac{\sum_{i=1}^{n}\mu\left(D_P,D_i\right)}{n}$).

iv) A *maximal distance* between two $D_i \in DS$ is denoted as *MAXD*.

v) A *group discord GDIS* among $D_i \in DNS$ is given as: $GDIS=\frac{AVCD}{MAXD}$.

vi) An *individual (relative) discord IDIS* (*RIDIS*) of an element $D_i \in DNS$ is given as: $IDIS_i=\sum_{j=1}^{n}\mu\left(D_i,D_j\right)$ ($RIDIS_i=\sum_{j=1}^{n}\mu\left(D_i,D_j\right)/\sum_{i=1}^{n}\sum_{j=1}^{n}\mu\left(D_i,D_j\right)$).

vii) A $D_j \in DNS$ is called a *DNS outlier* iff $\mu\left(D_i,D_C\right)>\left(1+\varepsilon\right)AVD$, $\varepsilon \geq 0$.

viii) A $D_j \in DNS$ belongs into a *DNS core* iff $\mu\left(D_i,D_C\right)\leq AVD$.

ix) A problem is called *well-defined* iff there is a unique consensus. Otherwise it is called an *ill-defined* problem.

x) A *cumulative distance function CDF* (for well-defined problems) is given as: $CDF(x)=\left|D_i \in A;\mu(D_i,D_C)\leq x, 0\leq x\leq MAXD\right|$

REMARK 1. In Definition 2i) a consensus might not be unique, in Definition 2ii) a pivot might not be unique. An individual discord *IDIS* from Definition 2vi) expresses the total distance of a given preference to all other preferences (RIDIS expresses a ratio of a discord of each individual to a group discord, respectively), thus each decision maker is assigned a degree of his/her disagreement within a group. By Definition 2vii) decision makers-outliers can be identified, and the parameter ε controls the outlier threshold distance. Outlier DMs might be excluded from a decision making process. In Definition 2ix) well-defined and ill-defined problems are introduced, as usually only one consensus is required. In Definition 2x) the (piece-wise constant) cumulative distance function enables to recognize a structure of decision makers.

3 Some relationships regarding a consensus

Intuitively, a consensus should lie somewhere 'in the middle' of decision makers' preferences, and it should be not too distant from a pivot preference, which is a 'midpoint' of all provided preferences. In this section some propositions regarding a consensus are provided.

The following proposition postulates a maximal distance between a consensus and a pivot.

PROPOSITION 1. Let $DNS \subseteq DS$ be a space of n decisions D_i. Let $D_P \in DNS$ be a pivot and let $D_C \in DS$ be a consensus. Then: $\mu(D_C, D_P) \le 2AVPD$.

Proof: From triangular inequality we have $\mu(D_C, D_P) \le \mu(D_C, D_i) + \mu(D_i, D_P)$ for all i, hence summing by all i we obtain: $n \cdot \mu(D_C, D_P) \le \sum_{i=1}^{n} \mu(D_C, D_i) + \sum_{i=1}^{n} \mu(D_i, D_P)$. Dividing by n we get $\mu(D_C, D_P) \le \frac{\sum_{i=1}^{n} \mu(D_C, D_i)}{n} + \frac{\sum_{i=1}^{n} \mu(D_i, D_P)}{n} = AVCD + AVPD \le 2AVPD$.

The following proposition restricts a maximum value of $AVCD$, when a decision space is a one dimensional (real) space.

PROPOSITION 2. Let $DNS = [a,b] \subset R$ be the decision space of n decisions D_i. Then: $AVCD \le \frac{MAXD}{2}$.

Proof:
Let's consider case of $n = 2$ and let $D_1 = a, D_2 = b$. The distance of these two decisions is maximal possible and clearly $AVCD = \frac{MAXD}{2}$ for every consensus (each number from $DNS = [a,b]$ is a consensus). By adding another preference D_3 (and then D_4, D_5, ...) $AVCD$ always decreases, so $AVCD < \frac{MAXD}{2}$ holds.

Next proposition concerns a uniqueness of a consensus in the case when a decision space is a one dimensional (real) space.

PROPOSITION 3. Let $DNS = [a,b] \subset R$ be the decision space of n decisions D_i. Let $a \le D_1 < D_2 < ... < D_n \le b$ (all decisions are distinct), and let $\mu(D_i, D_j) = |D_i - D_j|$. Then:

a) For n odd there is a unique consensus $D_C = D_{\frac{n+1}{2}}$.

b) For n even a consensus is not unique and lies in an interval: $D_C \in \left[D_{\frac{n}{2}}, D_{\frac{n}{2}+1} \right]$

Proof (by a contradiction):
a) Let's assume that $D' = D_c + \varepsilon = D_{\frac{n+1}{2}}$ is a consensus instead of D_c, where ε is a small ($\varepsilon < \left| D_{\frac{n+1}{2}} - D_{\frac{n+3}{2}} \right|$) positive number which expresses a shift from D_c. Let $R = \sum_{i=1}^{n} \mu(D_C, D_i)$ and let $S = \sum_{i=1}^{n} \mu(D', D_i)$. By substituting $D' = D_c + \varepsilon = D_{\frac{n+1}{2}}$ into S we obtain: $S = \sum_{i=1}^{n} \mu(D', D_i) = R + \frac{n+1}{2}\varepsilon - \frac{n-1}{2}\varepsilon = R + \varepsilon > R$, so D' cannot be a consensus.

Geometrically explained, when we shift a consensus from D_c to D' (to the right), the distances to all D_i on the left side from D' grow by ε, while distances to all D_i on the right side from D' decrease by ε. But after the shift there is always at least one more D_i on the left, so the overall distance (to all D_i) always grows. The proof for larger (or negative) ε is analogical.

b) The proof is analogical to a).

If the decision space is one dimensional, the geometric median is equal to the median, the result shown already in Haldane (1948), though the median is not defined unequivocally for even number of points. However, from Proposition 3 it follows that for n even every point (not only the median) between two 'middle' given points is a consensus.

The importance of Proposition 1 can be seen when considering decisions in the form of rankings of k objects (objects ordered from the 1st to the kth place). In such a framework rankings are usually treated as permutations from a space S_k of all permutation of order k, and consensus is a permutation minimizing the sum of distances to other (given) permutations. For a solution (finding a consensus) several permutation methods were proposed, such as CRM or DCM, see e.g. (Cook and Kress 1985; Cook 2006; Tavana et al. 2007; Mazurek 2011).

Main disadvantage of these methods is that they search through the whole space of S_k, but the number of permutations grows as $k!$ and the maximum distance as $\binom{k}{2}$, so for larger k (approx. $k > 10$) these methods are inapplicable. However, from Proposition 1 it is clear that a consensus cannot be too distant from a pivot, hence the searching (discrete) space can be reduced significantly (by several orders of magnitude).

4. Numerical examples

In this section several numerical examples are provided to illustrate concepts and measures introduced in section 2.

EXAMPLE 1. A 'classic' example from the social choice theory: three candidates (*a*, *b*, and *c*) are ranked by three voters in the following way. Find a consensus (a winner):

place	**voter 1**	**voter 2**	**voter 3**
1.	a	b	c
2.	b	c	a
3.	c	a	b

Solution:
All candidates are ranked equally; there is no consensus; and no winner as well. We can evaluate (Kendall's tau) distances between all preferences (columns C1 to C3): $\mu(C1, C2) = \mu(C1, C3) = \mu(C2, C3) = 2$. All preferences have the total distance to others equal to 4. Each preference is a pivot and also a consensus. Apparently, the problem is ill-defined.

EXAMPLE 2. Six decision makers (DM_1 to DM_6) rank 7 candidates (from A to G) for a given managerial position. All rankings are provided in Table 1. Find:

a) a pivot,
b) a consensus,
c) a group discord *GDIS*, *AVCD* and *AVPD*,
d) an individual discord of all DMs,
e) outliers,
f) a cumulative distance function.

Solution:
At the beginning we have to decide what metric is going to be used, as different metrics might lead to (slightly) different results. In this example Kendall's tau distance is applied as a natural metric when dealing with rankings (permutations). Also, we set $n = 6$ and $k = 7$.

Table 1. Rankings of all alternatives by all decision makers.

Position	**DM1**	**DM2**	**DM3**	**DM4**	**DM5**	**DM6**
1	A	C	C	D	A	C
2	C	A	D	A	C	D
3	D	F	A	C	F	A
4	F	D	F	B	B	F
5	B	B	B	G	D	G
6	G	E	G	F	E	E
7	E	G	E	E	G	B

a) Pivots are DM_1 and DM_3 with the sum of distances from the other D_i equal to 16.
b) The unique consensus D_C = (C,A,D,F,B,G,E). Its sum of distances to other D_i is 15, which is a minimum.
c) $AVCD = \frac{5}{2}$, $AVPD = \frac{8}{3}, MAXD = \frac{k(k-1)}{2} = 21$, $GDIS = \frac{AVCD}{MAXD} = \frac{5}{42}$.
d) $IDIS_1 = 16$, $IDIS_2 = 20$, $IDIS_3 = 16$, $IDIS_4 = 26$, $IDIS_5 = 24$, $IDIS_6 = 22$, $RIDIS_1 = \frac{16}{124} = 0.129$, $RIDIS_2 = 0.161$, $RIDIS_3 = 0.129$, $, RIDIS_4 = 0.210$, $RIDIS_5 = 0.194$, $RIDIS_6 = 0.177$. ($RIDIS_4 = 0.210$, for example, means that DM_4 is responsible for 21% of the disagreement of the group)
e) For $\varepsilon = 0$ DM_4 and DM_5 are outliers. These two DMs could be asked to revise their preferences or they could be excluded form a decision making process. The rest of DMs belongs to the core.
f) Some values of *CDF*: *CD*F(0) = 0, *CD*F(1) = 2, *CD*F(2) = 4, *CD*F(3) = 4, *CD*F(4) = 5, *CD*F(5) = 6.

From Proposition 1 we know that $\mu(D_C, D_P) \le 2AVPD$, and indeed, in this case we obtain: $\mu(D_C, D_P) = 1 \le 2AVPD = \frac{16}{3}$.

Therefore, the search for the consensus can start with each of two pivots. Then all permutations with the distance from the pivot equal to 1, 2, 3, 4 and 5 (integers smaller than 16/3) are examined, and the consensus is found. It is not necessary to examine all possible permutations from S_7 space (5040 permutations), as it suffices to scrutinize only 343 permutations for each pivot. (The distribution of permutation distances can be found e. g. in Margolius (2001))

Table 2. Distances among all alternatives from Example 2.

	DM1	**DM2**	**DM3**	**DM4**	**DM5**	**DM6**
DM1	0	3	2	4	3	4
DM2	3	0	3	7	2	5
DM3	2	3	0	4	5	2
DM4	4	7	4	0	7	4
DM5	3	2	5	7	0	7
DM6	4	5	2	4	7	0

EXAMPLE 3. Six decision makers (DM_1 to DM_6) rank 7 candidates (from A to G) according to their leadership skills. All rankings are provided in Table 3. Find:

a) a pivot,
b) a consensus,
c) a group discord *GDIS*, *AVCD* and *AVPD*,
d) an individual discord of all DMs,
e) outliers,
f) a cumulative distance function.

Solution:
The problem is apparently ill-structured, as there are two groups of DMs with identical rankings, which are in an 'opposition'. One cannot expect to get a reasonable result – a consensus – under such circumstances. Nevertheless, we set $n = 6$ and $k = 7$ and proceed. The problem emerges immediately:

a) All DM_i are pivots.
b) A consensus is not unique. Actually, each permutation 'between' (A,B,C,D,E,F,G) and (D,A,C,B,G,F,E), for example (D,A,B,C,E,F,G), including the both aforementioned permutations, is a consensus with a total distance of 21 to all other permutations.
c) $AVCD = \frac{7}{2}$, $AVPD = \frac{7}{2}, MAXD = \frac{k(k-1)}{2} = 21$, $GDIS = \frac{AVCD}{MAXD} = \frac{1}{6}$.
d) $IDIS_i = 21, RIDIS_i = \frac{21}{126} = 0.167$ for all i.
e) Outliers or a core cannot be identified (there is no unique consensus).
f) The cumulative distance function requires a unique consensus. Setting (for example) D_C = DM$_1$ we obtain: $CDF([0,7)) = 3$, $CDF(7) = 6$. A sudden 'jump' in CDF values indicates a problem in DMs' preference structure.

But if one decision maker, for example DM$_1$, changes his preferences, then a (unique) consensus would be possible, and it will be equal to DM$_1$. Applying the procedure for ill-defined problems introduced in the next section would result in a removal of all DMs in one step, with no consensus (solution) found.

Table 3. Rankings of all alternatives by all decision makers.

Position	**DM1**	**DM2**	**DM3**	**DM4**	**DM5**	**DM6**
1	A	A	A	D	D	D
2	B	B	B	A	A	A
3	C	C	C	C	C	C
4	D	D	D	B	B	B
5	E	E	E	G	G	G
6	F	F	F	F	F	F
7	G	G	G	E	E	E

EXAMPLE 4. Five members of a director board discuss the optimal amount of an investment. Their proposals (preferences) are as follows (in thousands of dollars): DM$_1$ = 200, DM$_2$ = 250, DM$_3$ = 230, DM$_4$ = 310 and DM$_5$ = 190. Find a consensus.

Solution:
By application of Proposition 3 we immediately get D_C = DM$_3$ = 230. The overall distance of D_C to all DM$_i$ is 170. By a change, for example, to D' = 240 ($\varepsilon = 10$), we obtain overall distance to all D_i = 180, which is more (by $\varepsilon = 10$) than in the case of D_C = 230.
The arithmetic mean of all preference numbers is 236, but it is not the closest value to all preferences, while the median 230 is the correct result.

5 Well-defined and ill-defined problems

Problems of achieving a consensus in a group decision making may be classified as *well-defined* (with one unique consensus) or *ill-defined* (problems leading to no consensus or more than one consensus).

Analysis of preferences is able to indicate the ill-defined problems, as shown in numerical examples of the previous section. However, there is not known general method for discriminating both cases based only on an analysis of preferences, though it is known that if preferences are points in Euclidean space and no three points lie on the same line (there is no collinearity), then the consensus is unique, see Vardi and Zhang (2000).

Question arises how to handle ill-defined problems. In real-world situations, decision makers may discuss (negotiate) the problem and change their preferences in a way that enables achieving a consensus (see for example *Delphi method*). This might be case of various boards or councils, but it is not possible for example in elections, where once polling is over, nothing can be changed. In the social choice theory ill-defined problems are considered paradoxes; Example 1 in section 5 for example illustrates the so called *circular ambiguity*, which stems from intransitivity of preferences. During time various sophisticated methods were developed to avoid the problem, see e.g. Nurmi (1999).

In the context of this study a simple way to obtain a well-defined problem from an ill-defined one lies in the identification of preferences (aka decision makers) with the highest discord (outliers). Such preferences might be a subject of revision or can be excluded from a decision making process. Then, a new analysis could be performed and a unique consensus might be found. If not, a preference with the second highest discord can be changed (excluded), and the whole process is repeated until a unique consensus is found or there is no preference left (and no solution as well).

This procedure might succeed in some cases, though unsolvable (ill-defined) instances such as from Example 3 (two equally strong opposing groups) will persist (the proposed procedure excludes all preferences at the beginning as all preferences have the same discord, leaving the set of preferences empty).

6 Some possible extensions of the proposed approach

In previous sections preferences provided by decision makers were supposed to be precise and complete. But sometimes decision makers are not able to express their preferences in that manner due to lack of time, insufficient or incomplete information, limited knowledge, lack of appropriate education, prejudice, etc. In such a case, decision makers can provide uncertain preferences in many forms ranging from applications of fuzzy sets, computing with words, use of intervals, to belief and possibility theory.

The proposed analysis of preferences can be extended to comprise some forms of uncertainty:

- *Fuzzy pairwise comparisons*: One common way of expressing preferences are pairwise comparisons of objects. When uncertainty is involved in pairwise comparisons, an additive fuzzy preference relation is defined as a binary relation on a universal set X assigning each pair (x,y), $x, y \in X$, a value ρ from the interval [0,1] so that $\rho(x,y) = 1 - \rho(y,x)$ (additive reciprocity). The value $\rho(x,y) = 0.5$ denotes indifference between objects, while the value 1 means that a given object is absolutely preferred over some other object, see (Orlovsky 1978; Perny and Roubens 1998; Herrera et al. 2001). Then, all pairwise comparisons can be written in a fuzzy pairwise comparison matrix, and their distance can be evaluated with the use of an appropriate matrix norm.

- DMs can express his/her uncertain preferences with the use of triangular or trapezoidal *fuzzy numbers* with a suitable metric, see e.g. Voxman (1998). Moreover, fuzzy preferences can be incomplete too, see Herrera-Viedma et al. (2007).

- Individual preferences can be expressed also in a form of words (the so called *linguistic variables*), see e. g. (Zadeh 1975; Xu 2013), where terms such as 'good' 'very good', etc., are represented by fuzzy numbers.

- DM can express his/her opinion in a crisp, but *incomplete* form. For example when ranking ten objects a DM can be able to rank only top three, leaving the rest of the list empty. Though there are methods for the evaluation of incomplete pairwise comparisons or incomplete rankings described in literature, the approach proposed in this paper relies on a precise measurement of a distance by a suitable metric function, which cannot be achieved for incomplete preferences in general.

- The proposed approach can be extended to *multiple criteria* framework as well, with the evaluation of a distance among preferences taking place before an aggregation phase (for each criterion separately), or after the aggregation (for all criteria).

Conclusions

The aim of this paper was to introduce several measures for an analysis of decision makers' preferences in a group decision making environment, which is a rather neglected issue in the literature. By the proposed measures a disagreement of each preference (each decision maker) can be assessed, so decision makers with the highest disagreement within a group can be identified, and asked to revise their opinion. Also, analysis of preferences can reveal ill-defined problems before a (futile) search for a consensus begins. Furthermore, the distance theorem (Proposition 1) bounding the maximum distance between a pivot and a consensus was introduced. The theorem facilitates search for a consensus especially for discrete spaces (for example for permutation spaces), as it reduces the searching space significantly. A simple procedure for dealing with ill-defined problems was proposed as well, though, of course, some unsolvable cases will persist.

The proposed approach can be easily extended to cases with preferences not only in the form of ordinal rankings or real numbers, but also to cases with fuzzy preferences, fuzzy numbers or linguistic variables. Also, it can be extended to a multiple criteria environment. In such a case preferences can be analyzed for each criterion separately, or for all criteria at once after an aggregation step takes place.

The future research might focus on the problem of the existence of a unique consensus. Vardi and Zhang (2000) showed that a unique consensus (unique geometric median) is achievable if all data points (in E^n) are not collinear (no three points lie on the same line), but no general result is known at present. Furthermore, as shown in Fiedor and Mazurek (2011), with growing differences among preferences (with growing entropy) a unique consensus is less likely to be achieved. The ultimate goal towards this direction might be to find general conditions (concerning a structure of preferences) under which a unique consensus exists or cannot be achieved, respectively.

Acknowledgment
The paper was supported by the Czech Grant Agency project no. 14-02424S.

References

Arrow, K. J. (1951). *Social Choice and Individual Values*. New York: John Wiley & Sons, Inc.

Cook, W. D. (2006). Distance-Based and Ad Hoc Consensus Models in Ordinal Preference Ranking. *European Journal of Operational Research*, Nr. 172, 369-385.

Cook, W. D., Kress, M. (1985). Ordinal ranking with intensity of preference. *Management Science*, vol. 31, No. 1, 26-32.

Feldman, A. M., Serrano, R. (2006). *Welfare Economics and Social Choice Theory*, 2nd ed., New York: Springer.

Felsenthal, D. S. (2010). Review of paradoxes afflicting various voting procedures where one out of m candidates (m ≥2) must be elected. In: *Assessing Alternative Voting Procedures*, London School of Economics and Political Science, London, UK.

Fiedor, J. Mazurek, J. (2011). The Evaluation of the Entropy of Decision Makers' Preferences in Ordinal Consensus Ranking Problem. In: *Proceedings of the 29th International Conference on Mathematical Methods in Economics 2011*. Professional Publishing: Prague, 157-162.

Fishburn, P. C. (1973). *The Theory of Social Choice*. Princeton, N. J.: Princeton University Press.

Haldane, J. B. S. (1948). Note on the median of a multivariate distribution. *Biometrika* **35** (3–4), 414–417.

Herrera-Viedma, E., Alonso, S., Chiclana, F., Herrera, F. (2007). A consensus model for group decision making with incomplete fuzzy preference relations, *Fuzzy Systems, IEEE Transactions* 15 (5), 863-877.

Kameda, T., Tindale, R. S., & Davis, J. H. (2002). Cognitions, preferences, and social sharedness: Past, present, and future directions in group decision making. In S. L. Group Decision Making 51 Schneider & J. Shanteau (Eds.), *Emerging perspectives on judgment and decision research*. Cambridge, UK: Cambridge University Press.

Margolius, B. H. (2001). Permutations with Inversions. *Journal of Integer Sequences*, Vol. 4, 1-12.

Mazurek, J. (2011). The Fuzzy Permutation Model for the Ordinal Ranking Problem Solution. In: *Proceedings of the 29th International Conference on Mathematical Methods in Economics 2011* (Dlouhý, M., Skočdopolová, V., eds.). Professional Publishing, Prague, 462-467.

Myerson, R. B. (2013). Fundamentals of Social Choice Theory, *Quarterly Journal of Political Science*: Vol. 8: No. 3, 305-337.

Nurmi, H. (1999). *Voting Paradoxes and How to Deal with Them*. Berlin/Heidelberg/New York: Springer-Verlag.

Orlovsky, S. A. (1978). Decision-making with a fuzzy preference relation. *Fuzzy Sets and Systems*, Volume 1, Issue 3, 155–167.

Perny, P., Roubens, M. (1998). Fuzzy Preference Modeling. In R. Slowinski (ed.), *Fuzzy Sets in Decision Analysis, Operations Research and Statistics*. Boston/Dordrecht/London: Kluwer.

Saari, D. G. (2000). Mathematical Structure of Voting Paradoxes: I. Pairwise Votes, *E*conomic *Theory*, Vol. 15, No. 1, 1-53.

Saaty, T. L. (2001). *Fundamentals of Decision Making and Priority Theory*. Pittsburgh, Pennsylvania: RWS Publications.

Sen, A. K. (1970). Collective Choice and Social Welfare. San Francisco, Calif.: Holden-Day.

Tavana, M., LoPinto, F., Smither, J. W. (2007). A Hybrid Distance-Based Ideal-Seeking Consensus Ranking Model. *Journal of Applied Mathematics and Decision Sciences*, 18 pages.

Taylor**,** A. D.**,** Pacelli**,** A. M. (2009). *Mathematics and Politics, Strategy, Voting, Power, and Proof*, 2nd ed., Springer.

Vardi, Y., Zhang, Cun-Hui. (2000). The multivariate L_1-median and associated data depth. *Proceedings of the National Academy of Sciences of the United States of America* **97** (4), 1423–1426

Voxman, W. (1998). Some remarks on distances between fuzzy numbers. *Fuzzy Sets and Systems*, 10(1-3), 353-365.

Wulf, G. (2006). *A primer in social choice theory*. Oxford: Oxford University Press.

Zeshui Xu. (2013). *Linguistic Decision Making: Theory and Methods*, Springer Science & Business Media.

Zadeh, L. A. (1975). The concept of a linguistic variable and its application to approximate reasoning - I, *Information Sciences*, vol. 8, no. 3, 199-249,