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**GAUGE FIELD THEORY OF MARKET DYNAMICS:
TOWARD A SOLUTION OF THE “MAN VS. MEN” DILEMMA**

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The current economics and psychology are developed within the Newtonian tradition in physics from both conceptual and instrumental perspectives. This paper aims to integrate economics and cognitive science by applying gauge field theory of modern theoretical physics. Many controversies between normative theories and behavioral theories are characterized by the “man vs. men” dilemma. Gauge potential and gauge field strength is constructed at both the man-level and the men-level in order to satisfy the principle of gauge invariance. To maintain the Lagrangian density function invariant, the gauge transformations of the first kind and the second kind are performed at the man-level and the men-level, respectively. The market dynamics is modeled by the logic of electrodynamics. The interactions of the market and individual participants are formulated by the logic of electromagnetic coupling. In establishing the market dynamic equations, individual utility function serves as gauge function and efficiency provides gauge freedom.

Keywords: bounded rationality; economic rational man; electrodynamics; gauge theory; market dynamics; cognitive field.

1. INTRODUCTION

1.1. *Why make theoretical physics involved*

There is a great and long standing tradition in modern economics, which is well characterized by Simon (1979), “The social sciences have been accustomed to look for models in the most spectacular successes of the natural sciences.” In order to make economics an “objective” science, Samuelson (1938) suggests “dropping off the last vestiges of the utility analysis”, because utilities are not observable. Friedman (1953) claims, “In short, positive economics is, or can be, an ‘objective’ science, in precisely the same sense as any of the physical sciences.” Nevertheless, Simon (1979) states, “In economics, it has been common enough to admire Newtonian mechanics (or, as we have seen, the Law of Falling Bodies), and to search for economic equivalent of the laws of motion.” To this observation, I share (Yang 2012). Then, Simon continues, “But this is not the only model for a science, and it seems, indeed, not to be the right one for our purposes.” To this view, so far, I agree. My concern is about what alternative models that can advance current economics by keeping, but not pushing away from its great long standing tradition with physics. Such a new model should be capable of preserving the conceptual framework and systematized knowledge of the current economics. This alternative, which I see as the most promising and with the least opportunity cost, is modern theoretical physics.

Simon suggests a number of directions for alternative scientific models, such as biology or information processing. At the end of his (1984), Simon refers his new direction for new microeconomic dynamics to “the direction of micro level investigation proposed by Behavioralism.” To this approach, I have certain reservations. On the one hand, I share Simon’s vision (1984) about the development and advancement of cognitive psychology from experimental as well as methodological perspectives in recent decades. Our earlier work in human reasoning is also along this line (e.g., Yang & Johnson-Laird, 2000_{a, b}, 2001; Yang, Braine & O’Brien 1998). Indeed, cognitive psychology is a close neighbor field to economics. On the other hand, I would have to admit that in higher order cognition research (particularly in psychology of reasoning and decision making that is closely related to micro economic dynamics), the empirical methods usually use verbal tasks, and the

observations are commonly based on what is called the “Yes/No” type experimentation by von Neumann (Penrose 2004). At this point, quantum theory has many systematized modeling tools available, which are different in many features, and in principle, from current information processing or computational approaches.

In modern theoretical physics, non-observables such as virtual particles or vector potentials are regular contents to study. In this regard, I stand firmly with Marshall’s view (1890; 1938), “In economics, those results whose reasons are known or those reasons whose consequence are known, in general are not most important. Invisible thing is more valuable than ‘visible thing’, to study”. The gauge potential and gauge transformations applied in this paper are not directly observable in physics. The alternative direction committed in the present paper is to integrate economics and cognitive science by taking the many conceptual and instrumental advantages from modern theoretical physics.

Perhaps the most powerful engine in the development of modern physics is the spirit and continuous effort in seeking a unified account of the physical world. It is well known that Einstein spent many years in pursuing a unified field theory of different forces. In addition, the successful standard model of particle physics is also a significant case. In economics, we have the distinction of normative theories and positive theories (Keynes 1936), descriptive theories, and psychological theories. These are the assets of systematized knowledge we have about economic world. Consequently, we must take the controversies and debates between competing approaches as scientific heritages. In order to construct a minimum model to unify different theoretical accounts, we need to look into their interactions and to establish the principle of invariance. In general, the gauge theory of modern physics is about interactions. The gauge field is the field of interactions.

1.2. Topics, Scopes, and the Organization

In economics, the economic rational man is a central concept which implies a global symmetry of market participants. This approach should be admitted by those schools of economic thought which focus on aggregative phenomenon. In behavioral economics, the concept of businessmen is studied under the maps of bounded rationality (Kahneman 2003),

which should be supported by some local symmetry across different individual market participants. A great deal of research effort, though often implicitly, has been made to provide a unified account of the two approaches; however, several difficult issues still remain. These issues are recognized as not only conceptual but also technical (Barberis 2013), which I refer to the “man vs. men” dilemma. In other words, economic rational man and bounded rational businessmen are two approaches studying market dynamics. The ‘man vs. men’ dilemma reflects the controversy between normative theories and behavioral theories.

The present paper aims to provide an integrated account of both the man-approach and the men-approach. To serve this purpose, we acquire theories and modeling tools from three other disciplines. First, cognitive science will be integrated with economics, namely, to take into account the mental energy and cognitive effort when modeling the market dynamics. Second, theoretical physics, particularly electrodynamics, will be used as a logical frame to lead inferences in the model development. Third, the gauge field theory will be applied as a modeling formalism.

The purpose of applying gauge theory to market dynamic analysis, by the principle of gauge invariance, is to preserve the form of market dynamic density function and dynamic equations invariant. A gauge field consists of two layers: the gauge potentials and the field strength. The gauge transformations are necessary to make the two layers consistent in form when differential operators are applied. The market dynamics is analyzed by constructing the market potentials and market field strength at both the global and the local levels, referred to the man-approach and the men-approach (by taking into account individual differences), respectively. Electrodynamics admits only one source charge – the electric charge and it produces two interacting fields: the electric field and the magnetic field. The market dynamics adopts only the market charge which produces two interacting fields: the market field and the cognitive field. Accordingly, only one gauge field is needed for single-charge systems, and the market dynamics shares with electrodynamics the same gauge symmetry of the $U(1)$ group. To construe the market dynamics into this picture at the man level and at the men level, the conceptual and instrumental tools from theoretical

physics are introduced and applied. The results show that the market global symmetry at the man level requires the gauge transformation of the first kind, and the market local symmetry at the men level requires the gauge transformation of the second kind. Toward a gauge theoretic unified account, interactions of the man and men are modeled following the logic of electromagnetic coupling.

The rest of this paper is organized as follows. Section 2 provides necessary conceptual and instrumental preparations for Section 3, which applies the gauge transformation of the second kind to the market dynamics at the ‘men’ level. Section 4 provides a more complete and detailed introduction for gauge theory, and addresses a number of related issues. Section 5 applies the gauge transformation of the first kind to the market dynamics at the ‘man’ level. Section 6 discusses the interactions of the man and men by the logic of electromagnetic coupling. Section 7 is a content summary of Sections 2 to 5. Section 8 provides concluding remarks. The paper ends at an envoi, which shows the lesson we should learn from Hendrik A. Lorentz (Huang 2007). The contents of this paper cross several domains, certain efforts are made to make the paper conceptually and instrumentally as self-sufficient and self-contained as possible. The map of main content structures of this paper is given in the Appendix.

2. CONSTRUE COGNITION INTO ECONOMICS WITH MODERN PHYSICS AS THE LOGICAL FRAMEWORK

2.1. Quantum Invisible Hand and the Market Version of the Uncertainty Principle

We start by getting an intuitive sense about why the market is quantum. Unlike current experimental economics, we ontologically admit that human economy itself should be seen as an experiment on the greatest scale our civilization has ever run. Accordingly, consider the financial market as a typical case; each individual market participant must be seen as an active market observer. When they are making an effort to look into the future market, they are not only concerned with what happened before on record, but are also motivated by economic rationality. They wish to observe what other participants have observed. To

be able to observe what others have observed would be a great advantage when investing in the market.

Let α_β be observer α 's observation on the observation of β , and β_α be observer β 's observation on the observation of α . By economic rationality, each observer is trying to be the final observer. Therefore, α_β and β_α are two uncertainties: if one becomes more accurate, then the other would become less accurate. Hence, the order of the observations matters. In formula, we may have

$$\alpha_\beta\beta_\alpha - \beta_\alpha\alpha_\beta \neq 0 \quad (\text{or } [\alpha_\beta, \beta_\alpha] \neq 0).$$

This shows that the observation α_β and the observation β_α are not commutative. The mathematical framework for quantum theory is Hilbert space consisting of wave functions, which are not directly observable, plus operators representing observables. These wave functions are possible solutions of a Schrodinger equation. As the vectors of Hilbert space, wave functions follow the law of superposition but not the law of addition (because the spin of particles). Accordingly, the formula above shows that the order of operators matters; i.e., the two observation operators are non-commutative because of the disturbance in the observations. It can be seen as a modified market version of Heisenberg's uncertainty principle in quantum mechanics. It can also be seen as one of the invisible hands behind the market. From the insight in Noether's theorem (Lee 1988), the modern market needs this invisible hand to keep the participants in symmetric positions in order to maintain a sustainable market. When the non-commutative relation is established in a domain, the domain is quantized (in quantum physics, this is done by introducing a new operator: $H(x, p) = ih$).

2.2. Classical Market Dynamics and Virtual Price

In classical consumption theory, the intersection of the demand curve and the supply curve is called the market clearance point. While such an equilibrium point is ideal, in practice, it provides a standard common sense on the market: one would not intend to sell a regular table for two thousand dollars or expect to buy it for two dollars. This at least makes the market a conservative system. However, by economic rationality, people do

intend to sell high and buy low. Think of the financial market as an example. For a given theoretical clearance point, the potential buyer, who wishes to buy for low, needs to think of a price level to start a bid. This thinking process, however, costs a certain amount of mental energy and cognitive effort. In other words, the cognition has to work on the market. This market-work determines the potential energy, which is denoted by V .

After the initial bid prices are proposed, the process of price negotiation (which may or may not be observable) between potential traders begins. The negotiation process creates the kinetic energy, E . Here we have the Lagrangian $L = E - V$, and Hamiltonian $H = E + V$, each of which must hold invariant in form for a dynamic system. These two forms are logically equivalent; nevertheless, intuitively, a hot market would mean high H but low L . Here, the key point is that during a V -process and an E -process, the prices are running virtual prices. The real price used in classical economics is counted as the price that cuts the deal. While the virtual price creates a pair of potential traders, the real price annihilates a pair of traders (i.e., the deal is done with a real cutting price) in quantum field theoretical terms. As it is known, the Schrodinger Equation can be seen as a quantum theoretic version of the Hamilton equation:

$$i\hbar \frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \varphi}{\partial x^2} + V\varphi,$$

where \hbar indicates that the energy carried is discrete. On the right side of the equation, the first term is for the kinetic energy (i.e., squared momentum), and the second term for the potential energy. On the market, the energy carried by price is discrete, and the market dynamics are sensitive to the energy level. For example, assume the price for a computer was \$2000. A potential consumer really liked this computer but had in mind a virtual price of \$1700. This virtual price carries much of the mental energy of the consumer but they would not get it by paying only \$1700. The current consumption theory assumes that the consumers have no control over the price. This is the limitation for market analysis within the classical Newtonian tradition. Nevertheless, market dynamics are interested in some deeper analysis. Between a virtual price of \$1700 and the actual price of \$2000, there might be many mental (or say, cognitively quantum) fluctuations with different amplitudes

underlying the market, which are not directly observable. Eventually, the underlying cognitive field may cause price displacement.

Maxwell equations are a classical framework, which is discussed in Section 6. The relationship between the Maxwell's framework and the Schrodinger Equation can be established through the so-called second quantization. Notice that this is just a technical treatment, which does not change the market dynamics that we will discuss. The gauge transformation is a focus of this paper. For the Schrodinger equation, the gauge transformation is necessary (Huang 2007). On the right side of the Schrodinger equation, the first term is the squared momentum. The momentum is treated as an operator in quantum mechanics, which is defined by $p = i\hbar\partial$. In order to characterize the electromagnetic coupling, it must make a gauge transformation of the momentum operator: $p \rightarrow p - \frac{q}{c}A$, and substitute the classical derivative by the covariate derivative such that $D = \partial + \frac{iq}{\hbar c}A$, where A is a gauge function. Thus, the Schrodinger equation is gauge invariant because it is supported by a gauge theory. The reason why I provide the above description is that in earlier research (e.g., Mirowski 1989; Rose 2005) applying Hamiltonian approach to economics (to push it into the best end), gauge theoretic issues are not addressed. However, to study market dynamics, the Lagrangian approach, rather than Hamiltonian approach, is more appropriate. This is due to the fact that the efficiency issue in economics can be formulated by the minima solution of an extremal problem in variation theory. In this sense, as some physicists used to say, nature is an economist.

In economics, the notion of Pareto efficiency can be characterized by the notion of geodesics in physics (Yang 2012). This is an example of variation problem. In market dynamics, for a trade to have the minimum transaction cost (including time and negotiation process) is to estimate the minima of a variation problem. In general, there can be many different ways to achieve the expected production or welfare state from given scarce resource. These paths all share the same start and end points so it is called the family of functions. The efficient allocation can then be modeled by what is called the least action in physics. An action S is the integral of a given Lagrangian density function Φ , which is a

functional (i.e., a function of functions). Consider a curve between two points on a plane.

An infinitesimal element of the curve has length $= \sqrt{dx^2 + dy^2}$; then we have

$$\sqrt{dx^2 + dy^2} = dx \sqrt{\frac{dx^2}{dx^2} + \frac{dy^2}{dx^2}} = dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx \sqrt{1 + y'^2}.$$

Let $\Phi(y) = \sqrt{1 + y'^2}$, we can see that it is a functional $\Phi(x, y, y')$. In established dynamic equations, the only independent variable x used to be substituted by the time variable t . The study of the variation problem in market dynamics requires maintaining the form of Lagrangian density function invariant, which as we will see from later sections is not a trivial issue. In order to satisfy this requirement, the gauge transformations and covariate derivatives must be introduced.

2.3. Budget Potential and Proper Cost

The notion of a budget can be analyzed in terms of four-vectors from relativistic perspectives. In consumption theory, the notion of a budget set is two-fold: it is a set of bundles that must be understood in monetary terms. The Walrasian competitive budget (Samuelson 1948) is given by

$$\{p_1x_1, \dots, p_ix_i, \dots, p_nx_n, I\},$$

where p_i denotes the price of a bundle x_i , and I is the wealth level. The sum of p_ix_i must be less than or equal to I by the affordability constraint,

$$I - \sum_{i=1}^n p_ix_i \geq 0.$$

It is assumed that the consumer has no effect on price making. We may let $n = 3$ in the context below without the loss of generality. In Minkowskian space-time, the main structure is the interval defined as:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

with gauge, $g_{\mu\nu} = (+, -, -, -)$. The interval format is invariant. When comparing the two formulas above, the budget set shares a similar internal structure of the interval, which is called the Lorentz type (Penrose 2004). The four-momentum vector is given by $\rho = \gamma m_0 u$, where γ is the Lorentz factor and u is the four-velocity vector:

$$u = \left(\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right),$$

where t is the absolute time and τ is called the proper time defined by $-d\tau^2 = ds^2$. If we think about the possible margin of the budget, one shall look at the components of the four-velocity vector and you can find that it needs to have the proper time τ , which should be understood as the proper cost for the consumer in the present context. This is one of the key concepts proposed in the current discussion.

In the definition of the interval introduced above, c can be seen as the speed of money, which is an invariant (Yang 2012). Here is a very intuitive explanation of the notion: spin, which is a quantum theoretic intrinsic property. Suppose John was hungry; if you gave him a burger, he would take it. If you offered him a second burger, he might say yes or no; if you provided him a third burger, in case that he is full, he would say no. In this sense, we say the burger has two directions to go. In the same sense, money is different from any other economic things. No matter how much money was given to John, he would take it; money has only one direction to go. At this point, we say, money is spin 1 but other economic things are spin $\frac{1}{2}$. The time dimension t can be treated as the money dimension.

To push the notion of the Walrasian budget further, we must see that the consumption theory has to be relativistic. This is the reason why the Walrasian formulation of budget is of $n + 1$ dimensions. However, the proper cost, τ , is different from the given price. By special relativity, the proper time is the clock time that may vary from one inertial system to another. Taking into account private and environmental factors, proper costs may vary from one bundle to another bundle for the same individual and may vary from one individual to another individual. The relation between proper time and the coordinate time is given by the Lorentz transformation:

$$\tau = t \sqrt{1 - \frac{v^2}{c^2}}.$$

This shows that the proper time and the velocity of a system are inversely proportional: the faster the system moves, the slower its proper time. By this logic, we can observe a common phenomenon: the higher the proper cost for a consumer, the slower (meaning less active) the consumer behaves on the market, and vice versa.

Although a consumer may not affect the actual price, he or she may consider different plans. Thus, going beyond the constraint in physics, we may treat proper cost as a local (meaning for each individual consumer) independent variable with parameters of personal factors, including the perceived information from commercials and promotions. In Section 3, we will see that budgeting reflects the market potential, which has a gauge freedom.

In Section 4, we will also explain that in order to study the gauge field of budget potentials across different consumers, a gauge transformation function is needed. In order to study the covariate margin, we need to introduce the covariate derivative acting on the gauge field. Notice that budgeting is different from taking an action; in physics terms, it is massless like a virtual photon. Consider the momentum space at a consumer point; all the market activities of a given consumer can be treated as the events happening in a tangent space. Its structure is a budget cone from which we can envisage the family of all possible budget rays at a consumer point. Since a budget, like a virtual photon, is massless ($m_0 = 0$), its four-momentum is a null vector on the surface of this budget momentum cone. Within this cone, the time-like world-lines stand for personal affordability, which reflects a kind of historical causality. Keep in mind that here the four-momentum vector is related to the proper cost τ for an individual consumer. Meanwhile, each consumer has his or her own budget cone as the structure of the personalized momentum tangent (cotangent) space (Penrose 2004).

For each individual consumer, the budget cone may have a different horizontal/vertical scalar ratio (see the Figures 1A and 1B below). Imagine the phase θ between the side-surface of a budget cone and the horizontal axes. We say the smaller the phase θ the greater the proper affordability (and vice versa), given the proper cost of an individual consumer. This idea is applied in Section 4 where the gauge transformation is formulated.

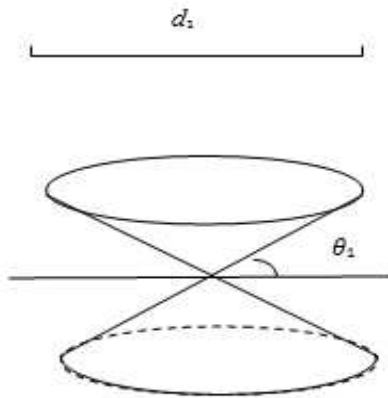


Figure 1A. Rich Cone with
Smaller Proper Cost τ .

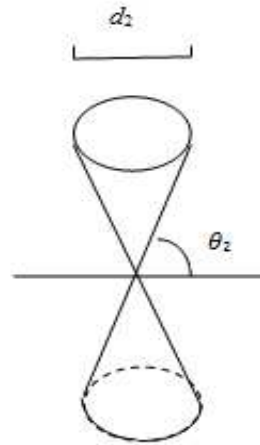


Figure 1B. Poor Cone with
Greater Proper Cost τ .

2.4. Hesitation and the Wavefunction

On the market, hesitation is a common phenomenon, which costs a great deal of mental energy and cognitive effort. When a potential consumer is interested in some commodity, the hesitation can be characterized by a shilly-shally process between state φ_1 (to-buy) and state φ_2 (not-to-buy). In a thought experiment, if an observation operated to measure this process, any superposition state, $\varphi = C_1\varphi_1 + C_2\varphi_2$ (where C_1 and C_2 are any given complex numbers) may be detected. In quantum theory, the states follow the superposition law but not the summation law. By Dirac's bra-ket formalism, a state is denoted by $|\varphi\rangle$. How do we observe such a state? Quantum theory is about experimentation, which works like a projector. We can run an experiment by using a price Φ as the stimuli to test the interested state, written as $\langle\Phi|\varphi\rangle$. Thus, we can have a projection function, denoted by $\varphi(\Phi)$, which is called the wave function. One can imagine putting a slide on the projector, by turning the knob (changing the angle θ) on the projector to adjust the focal length Φ , you can see a range of foci of the projection. As a result, the wavefunction is measured by an amplitude and is characterized by a complex number C , which has an exponential format on the complex plain: $Re^{i\theta}$. Here, θ is the potential with the gauge freedom. As we know,

$\{e^{i\theta(x)}\}$ is a compact unitary group $U(1)$ (e.g., consider all the possible rotations on the unit circle of the complex plane).

3. COGNITIVE MARKET FIELD AND ELECTROMAGNETIC LOGIC

3.1. *Market charge, Demand-supply, and Electron-positron*

The market functions as a place for people to trade. Once on the market, a potential trader, either a buyer or seller, who must be motivated or stay charged, behaves like an electric charge. In this sense, we say a trader's intention carries some market charge, which is counted as the source for market dynamics. In the current microeconomics, the consumer's Walrasian (market) demand function is defined by a single-valued demand correspondence $x(p, w)$ with certain constraints such as the Walras' law and homogeneous of degree zero. If we push to ask what a demand (or a supply) is, the common sense answer would be: someone intends to buy (or sell) something for a given price. Market demand (or supply) reflects a relation between the potential trader and bundles of commodities. A demand can be real or virtual. We define the notion of demand by introducing a two-component unit: when the demand is virtual we have:

$$D_{virtual} = [intention, property].$$

When the demand is real we have:

$$D_{real} = [action, property].$$

Accelerations of demand and supply are both sensitive to the price (analogous to a photon). Here, the notion of *market charge* is postulated as a fundamental construct, analogous to the electric charge (e^- for electron or e^+ for positron). The intention component is responsible for carrying the market charge to buy (with a negative sign) or to sell (with a positive sign).

The electron has another intrinsic quantum property: spin. It can spin in two directions: up or down. By this logic, the property-component in a virtual demand (supply) does not spin itself but is being spun. Consider the hesitation phenomenon; during the shilly-shally process (analogous to the so-called zigzag process for the electron by Penrose, 2004) between to-buy and not-to-buy, the property would be experiencing a process of being-bought or not-being-bought. A more intuitive example can be seen in the job market.

Consider someone who just had an interview and was told that the decision will be made in two weeks. During this two-week period, the person could not help from speculating if he would be hired or not. The taxonomy of the constructs (and their physical counterparts) is shown in the table below.

Table I
The Taxonomy of Related Constructs

Marketing charge	Potential participant	Action charged	Property/Goods/service		Market particles	Analogy to physics	Electric charge
			Spun up ↑	Spun down ↓			
m ⁻	Potential buyer	Buying	Bought ↑	Anti-bought ↓	Demand	Electron	e ⁻
m ⁺	Potential seller	Selling	Sold ↑	Anti-sold ↓	Supply	Positron	e ⁺
Neutral			(Sensitive to price)		Price	Photon	Neutral

3.2. Cognitive Field by the Magnetic Logic

Though it would not be difficult for social scientists to permit the cognitive charge, we will follow physics to only admit the market charge introduced above as a source for market dynamics. The market charge must automatically produce a cognitive field by the logic of electromagnetism. This can happen in two ways. First, during the shilly-shally process, the consumer might change the strength degree of his intention and the frequency in hesitating between Yes (to-buy) and No (not-to-buy). This creates the abnormal cognitive (mostly perceptual) moment. In physics this is called the abnormal magnetic moment.

Second, and more significantly, a moving market charge yields the market (displacement) current with some certain intensity, which in turn produces a cognitive field as the market counterpart of the magnetic field (by the logic of electromagnetism). During the hesitation period, the potential consumer could not help himself or herself from

thinking ‘around’ his or her intention to buy. This cognitive effort may be more intensive or less intensive, it may cause more or less mental energy, it may involve more complex or less complex contents, and it may take a longer or shorter time.

The market current can be treated as the source to produce the corresponding cognitive field, which in turn can yield two cognitive poles (analogous to the magnetic field with the magnetic poles: the North and the South). We may consider this cognitive field as a rationality field, which involves reasoning toward two logic poles (true or false) or decision making with preferential poles (prefer one choice to another). It is not hard to imagine that the interaction between the market current and the corresponding cognitive poles are sensitive to price in monetary terms.

The cognitive potential will be introduced next. Its effect on the consumer’s trading intentions can be seen as consistent with the Aharonov-Bohm effect in physics, which provides empirical evidence about how the magnetic potential affects the behavior of electrically charged particles. The global effect of cognitive potentials is obvious, which we will discuss in Section 5.5; indeed, that is partly what makes the demand and supply curves being curved.

3.3. Market-Cognitive Potentials and Their Gauge Freedoms

Perhaps the easiest way to understand the notion of potential and its gauge freedom is to think about a function $f(x)$, whose primitive function is

$$F(x) = \int f(x)dx + C.$$

This indefinite integral is the potential with the gauge freedom of a constant C , due to fact that $d(C) = 0$. Gauge theory makes the clear distinction between the field of potentials and the field of strengths. A strength field can be obtained by applying the appropriate differential operations on the related potential field, and vice versa by performing appropriate integral operations. The current margin analysis can be seen as a quasi-example at this point, which is familiar to economists and is one of the key technologies used in economic modeling. One of the purposes of this paper is to take full advantage of gauge

theory to develop a method of covariate margin analysis connecting different individual consumers.

The market potential is a scalar potential φ , which is a one-dimensional function. The strength field of φ is given by its gradient $\nabla\varphi$, which has any given constant C as the gauge freedom because we have,

$$\nabla C = 0, \text{ and } \nabla\varphi = \nabla(\varphi + C).$$

The cognitive potential $A = (Ax, Ay, Az)$, is a vector potential as the market current J is characterized by a three-dimension vector. The strength of A is given by its curl $\nabla \times A$, which has a gauge freedom of $\nabla\psi$, where ψ is any scalar function, because we have:

$$\nabla \times (\nabla\psi) = 0, \text{ and } \nabla \times (A + \nabla\psi) = \nabla \times A.$$

Putting the two potentials together, (φ, A) forms a four-vector potential $A_\mu = (\varphi, A)$, where $\mu = 0, 1, 2, 3$. Let us call A_μ the cognitive-market potential analogous to the electromagnetic potential.

The vector potential A_μ has different names as per different theoretical perspectives in physics. It is called the Maxwell field because from A_μ we can calculate the field strengths, which satisfy Maxwell's equations, which we will explain shortly. It is called the (virtual) photon field in quantum electrodynamics from the perspective of momentum, as explained in Section 2.3. In the market dynamic context, A_μ serves as the budgeting potential involving the consumer's proper cost and is responsible for interactions between potential demand and potential supply. In the field theoretic terms, the running virtual price can create a pair of virtual demand and virtual supply. It is called the gauge potential field not only because it carries gauge freedoms, but also because it is responsible for constructing gauge transformations as well as covariate derivatives, which we will show in Section 4. Here, the crucial point is that A_μ is not a simple combination of φ and A . In order to determine the field strength, the market scalar potential φ and the cognitive vector potential A have to interact with each other.

3.4. Margin as Field Strength Tensor, Interactions, and the Lagrangian Invariant

Given the gauge potential $A_\mu = (\varphi, A)$, the field strength tensor $F_{\mu\nu}$ is defined by a 4-by-4 matrix, which contains only six independent components instead of 16; this is because $F_{\mu\nu}$ is a completely anti-symmetric second-rank tensor written as

$$F_{\mu\nu} = \left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \right), \text{ or briefly, } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Of the 6 independent components ($F_{23}, -F_{13}, F_{12}$) are the three components of $(\nabla \times A)$ and transform as an axial vector, where A is the three-vector (A_x, A_y, A_z) . The other three independent components ($F_{14}, -F_{24}, F_{34}$) are equivalent to

$$i \left(\nabla\varphi + \frac{1}{c} \frac{\partial A}{\partial t} \right),$$

which is a polar vector. Accordingly, by the electromagnetic logic, the market field strength can be defined as

$$E = -\nabla\varphi - \frac{1}{c} \frac{\partial A}{\partial t},$$

and the cognitive field strength is defined by $B = (\nabla \times A)$. Notice that $F_{\mu\nu}$ is obtained by applying differential operators to A_μ , so it belongs to the same category of velocity as the notion of margin as economics does! By this logic, it hints us to treat the margin as a strength tensor. In physics, the strength tensor $F_{\mu\nu}$ (with constructs E and B) are empirically observable. Similarly, if we assume the consumers are economically rational, then their behaviors should be predictable by the margin strength tensor. In Section 4, we will treat the margin strength as an observable operator in the wave function. As many authors (e.g., Wu & Yang 1975) pointed out, the gauge potential is corresponding to the affine connection in general relativity. Accordingly, we will also discuss why the different individual consumers are connected by gauge transformation between their budget potentials.

In order to help us better understand what the above equations mean in our context, here is a quote from Schwartz (1972, with minor connecting modifications) that will provide helpful insights,

“Hence in any given coordinate system it would appear as though there independent vectorial fields at each point in space. Only when we transform between them do the two fields become mixed together. Nevertheless, one important observation is that $\nabla\phi$ and $\left(\frac{1}{c}\frac{\partial A}{\partial t}\right)$ are inseparable! Component by component they always occur together, and so a particle must experience their combination as though it were one type of force. This result embodies all of Faraday’s famous law and is even more general”.

What we can learn from Schwartz’s words is that market intention and cognitive effort always go together, and they interact with each other to behave as one force: the cognitive market force. The gauge potential $A_\mu = (\phi, A)$, serves as the field for market-cognition interaction to occur.

The strength field tensor $F_{\mu\nu}$ is constructed with the complete anti-symmetric structure in order to make the form of Lagrangian density invariant (under Lorentz transformation); we have

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$Tr(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{2}i(E^2 - B^2)\psi .$$

Here, \mathcal{L} must be a scalar. Indeed, Lagrangian density is what we shall call the margin density in economics. Notice that the structure in $F_{\mu\nu}$ involves applying the differential operators to the market and cognitive potentials as well as their interactions. So, in the field theory, a state is generally represented by (ϕ, ∂_μ) . All the discussions so far can be seen as only referring to the internal space at a single point, or say, about an individual consumer’s personal account. If we move from one consumer’s budgeting potential to another, as shown in Sections 2.4 and 3.3, we need to make a potential transformation:

$$\psi \rightarrow \psi' = e^{i\theta}\psi.$$

Accordingly, in order to keep the form of the Lagrangian density L invariant, we would expect:

$$\partial_\mu\psi \rightarrow e^{i\theta}\partial_\mu\psi.$$

However, the classical derivative does not work this way. Recall the notion of the wave function we introduced in 2.4, its amplitude is measured by a unit complex number $e^{i\theta}$. When taking into account the individual differences of the consumer's budget potential, the θ is no longer a constant but becomes a phase function $\theta(x)$; consequently, we have

$$\partial_\mu(e^{i\theta}\psi) = (e^{i\theta}i\partial_\mu\theta)\psi + e^{i\theta}\partial_\mu\psi.$$

On the right side of the equation, the first term is not what we would expect to be there. To solve this problem, we need to introduce new mathematical tools such as the gauge transformations and the covariate derivative.

4. GAUGE FIELD THEORETIC MODELING

4.1. Principle of Gauge Invariance

The principle of gauge invariance requires that the form of Lagrangian density be invariant under the gauge transformation. In terms of field theory, a field on a set of consumers, such as a scalar field, means that there is a scalar assigned to each and every consumer, which is a one-dimensional function $f(x)$.

In classical economics, as long as the notion of the economic rational man is concerned, there is an underlying assumption: when a gauge transformation

$$\psi \rightarrow \psi' = e^{i\theta}\psi$$

is applied, θ is a constant at every consumer point x . This is called the gauge transformation of the first kind, which implies the global gauge symmetry of all the consumers. Section 5 provides more detailed discussions about this idea.

In behavioral economics, as long as the notion of businessmen is concerned, the phase θ becomes a phase function $\theta(x)$. In order to establish the local gauge symmetry, we need to introduce the gauge transformation of the second kind and additionally a new field A_μ , called the gauge field or vector potential. The covariate derivative can then be introduced to act on the gauge field accordingly to maintain the form of the Lagrangian density function invariant. The idea is to add some gauge freedom to the classical derivative such that the new derivative can serve as the derivative potential field. The idea is to introduce

a covariate derivative (i.e., the covariate margin can be used in economic analysis) field to accommodate the phase function at each individual point.

The gauge transformation (function) from one local budget potential to another is given below

$$A'_\mu = A_\mu - \frac{1}{g} \partial_\mu \theta.$$

This transformation can be rewritten as

$$\frac{1}{g} \partial_\mu \theta = A_\mu - A'_\mu.$$

The term on the left hand side of this equation is the gauge adjustment made to connect the budget potential of one individual to that of another individual. This is the reason why the gauge field can be compared with the affine connection in differential geometry (Wu & Yang 1975). This gauge transformation enables us to introduce a new type of derivative that is covariate, meaning

$$\begin{aligned} \psi &\rightarrow \psi' = e^{i\theta} \psi, \\ D_\mu e^{i\theta} \psi &\rightarrow e^{i\theta} D_\mu \psi, \end{aligned}$$

where the covariate derivative, D_μ is defined as

$$D_\mu = \partial_\mu + igA_\mu.$$

To be convinced by this new (not new to physicists) differential technique, the necessary derivation is provided below:

$$\begin{aligned} D_\mu e^{i\theta} \psi &= (\partial_\mu + igA'_\mu) e^{i\theta} \psi \\ &= \partial_\mu (e^{i\theta} \psi) + ig(A_\mu - \frac{1}{g} \partial_\mu \theta) e^{i\theta} \psi \\ &= \partial_\mu (e^{i\theta} \psi) + igA_\mu e^{i\theta} \psi - ig \frac{1}{g} (\partial_\mu \theta) e^{i\theta} \psi \\ &= (\partial_\mu e^{i\theta}) \psi + e^{i\theta} \partial_\mu \psi + igA_\mu e^{i\theta} \psi - i(\partial_\mu \theta) e^{i\theta} \psi \\ &= ie^{i\theta} (\partial_\mu \theta) \psi + e^{i\theta} (\partial_\mu + igA_\mu) \psi - i(\partial_\mu \theta) e^{i\theta} \psi \\ &= e^{i\theta} (\partial_\mu + igA_\mu) \psi \\ &= e^{i\theta} D_\mu \psi. \end{aligned}$$

The above derivation is given step by step. This derivation is usually not fully spelled out in textbooks; nevertheless, by all means it is a key effort worth making for interested social scientists who would like to familiarize themselves with how gauge theory works.

What are gauge transformations about and why are they necessary? Here is a general comment. To study a dynamic system, there are three layers that need to be considered consistently. The first layer is a vector potential or the state (a wavefunction). The second layer is about the strength field or the rate of change. This involves performing various kinds of differential operations, such as the classical derivative, gradient, divergence, or curl. Together the first and the second layers are called a gauge field. The third layer involves establishing dynamic equations, such as the Lagrangian density function, which is a functional (i.e., a function of functions). By the gauge invariance principle, the form of the Lagrangian density function must be kept unchanged after the gauge transformations made in the gauge field. This requires that the gauge transformations at the first layer and at the second layer be consistent. Keep in mind that for a dynamic system, a state Φ can be transformed to another state Φ' over time by a gauge function A at the first layer: $\Phi \rightarrow \Phi' = A\Phi$. In case that A is a constant, the classical derivative works well at the second level due to that fact that $\partial A\Phi = A\partial\Phi$, which is consistent in form with the transformation at the first layer. In case A is a function as Φ does with respect to the same variable x , the classical derivative does not work anymore because $\partial A\Phi = A\partial\Phi + \Phi\partial A$, of which the second term at the right side makes it inconsistent in form with that at the first layer. Thus, in this case we need to introduce a gauge field and to construct a new covariate derivative, in order to maintain the form of the Lagrangian unchanged at the third layer.

4.2. An Intuitive Explanation

The mathematical model of gauge theory is the fiber bundle theory in differential geometry (Healey 2007), which goes beyond the scope of the present paper. K. Huang (2007), nevertheless, provides an intuitive and simplified picture showing the relation of fiber bundle theory and gauge theory, which is helpful to understand how gauge transformations work. Imagine that each space-time point (which represents a consumer)

is surrounded by a ring and has a fiber attached. The set of all the fibers is called the fiber bundle. This structure can well characterize the gauge field. Accordingly, the gauge transformation can be considered in one classical case and two quantum cases below.

In the classical case, at each space-time point, the associated ring and the attached fiber are not connected. The vector potential can freely and smoothly move up and down along the fiber, and the ring stays at rest (being unaware of the change in potential). In the classical electromagnetic theory, a charged particle interacts with an electric field or magnetic field directly. In this sense, as said by K. Huang, the particle does not know the gauge potential. Thus, the vector potential (i.e., the gauge potential) is not necessary for classical electromagnetism. Indeed, without vector potential, the set of Maxwell equations can be formulated even more symmetrically between electricity and magnetism (C.N. Yang 2014). A similar observation can be found in the classical consumption theory which is still current in economics. For example, the demand curve is about the relation between the total quantity of demand and the price. It has nothing to do with an individual consumer's demand and sensitivity of a particular price.

While from the perspectives of quantum theory, the charged particle interacts directly with the vector potential, and it knows about the gauge. The Aharonov-Bohm experiment shows that charged particles can be directly coupled to a gauge field. For instance, in our context, imagine a market-charged potential investor. The investor will interact with the market not only based on the observation of the current market data, but also on the forward observation that is the speculation of the market potentials. In case the investment result turns out to be expected, we say that the investment was entangled with an eigenstate. So at each space-time point (standing for an individual market participant), the ring and the fiber are connected and they work coordinately. Note that by the Aharonov-Bohm experiment, it does not mean that the vector potential is observable. Gauge potential field is only responsible for interactions and it is not observable in nature. In economics, this means that the gauge potentials behind the market are still invisible hands.

There are two kinds of possible quantum cases. In one case, it looks like all the fibers merged to a single meta-fiber such that there is only one person, the economic rational man

climbs up or down. Accordingly, the ring associated with every space-time point is turning to the same phase; i.e., $\theta = C$. In other words, the quantum phase is a constant, meaning it is independent of the individual points. This property is called the global gauge symmetry. In this case, the gauge transformation of the first kind works, which will be explained in Section 5. In another case, each of the businessmen climbs up and down along his or her personal fiber. Accordingly, each associated ring may turn to a different phase; i.e., $\theta = \theta(x)$, meaning the quantum phase is a function and it is individual dependent. In this case, the gauge transformation of the second kind is needed to balance out the phase differences and to permit the local gauge symmetry, as it showed in Section 3.

4.3. Some Deeper Issues and Inferences

The following comments might be helpful to social scientists who are interested in gauge theoretic modeling. First, in the gauge transformation of the second kind and the definition of covariate derivative, there is a term " qA ", which means the electromagnetic coupling. This is why the quantum phase changes in a gauge transformation. This logic tells us that we should not take the interaction between the market charge and cognition as always ready by nature. They need to be coupled, and the way of coupling varies from one individual to another. On the market, this coupling demands individual efforts. It causes the individual differences in quantum phases. Thus, the gauge transformations are needed in order to maintain Schrodinger equation (in quantum mechanics) and the form of the Lagrangian density function (in quantum field theory) gauge invariant. We will address the issue of the electromagnetic coupling about how the economic rational man interacts with individual businessmen in Section 6.

Second, in physics, the charged particle and quantum phase are proportional. Thus, the charge can be fundamentally defined as the generator of gauge transformation. Recall that the notion of the market charge was introduced at the beginning of Section 3. It may cause some wondering at that moment, but now one would feel more comfortable with it because the market charge can be defined as the generator of gauge transformation. The intention to buy or to sell can be stronger or weaker, and so is the market charge, which may cause

proportional changes in quantum phase. If someone was just walking into a mall to relax but did not intend to buy anything, this individual would be neutral to market charge and should not take part of gauge transformation. The point is, by the insight of the Noether's theorem, the symmetry achieved by the gauge transformation preserves the conservation of the market charge.

Third, in physics, the local symmetry is more advanced than the global symmetry. In order for a system to be capable of coupling with the electromagnetic field and to establish the local symmetry, this system must possess the global gauge symmetry before the coupling; this is called the gauge principle. For example, given a constant phase change, the classical derivative works well in Schrodinger equation; hence, only the gauge transformation of the first kind is needed. Then, for a charged particle to be able to couple with the electromagnetic field, Schrodinger equation is permitted to make the gauge transformation of the second kind and to replace the classical derivative by the covariate derivative. This is the way to achieve the local gauge symmetry. In the economic context, this simply means that there is no free lunch on the market. If someone has no idea about any kind of economic efficiency, this person should not be counted as a market participant. In Sections 5 and 6, the economic rational man is treated as the personified market. Hence, the gauge principle would tell us that the bounded rational businessmen must be accompanied by the economic rational man because if there were no market, there would be no market participants. It also shows that without the concept of the economic rationality, the notion of bounded rationality would be hard to define. This is similar to the idea in the general theory of relativity: without the geodesics, it would be hard to define the curvature, geometrically speaking.

Finally, in string theory, the magnetic charge is permitted. The electric charge and magnetic charge are inversely proportional. Thus, the strong-weak duality holds: the stronger the electric charge, the weaker the magnetic charge. The magnetic charge is not granted by the standard model in physics, as it has never been observed. My approach is to stay with the standard model in my current work. It would not be difficult for cognitive science to admit the cognitive charge in studying market dynamics. Then the strong-weak

duality would become commonsense: the stronger the intention to buy something, the less one needs to think about it, and vice versa.

5. ECONOMIC RATIONAL MAN AND THE GAUGE TRANSFORMATION OF THE FIRST KIND

In this section, we briefly introduce the decision theoretic structure, and define the notion of economic rational man on this decision structure. Then, we will argue that the notion of economic rational man must be treated as the personified perfectly competitive market, and we explain how it is related to the gauge transformation of the first kind in order to work toward establishing the principle of gauge invariance.

5.1. *Decision Theoretic Structure*

The classical decision theoretic structure consists of three standard parts: the syntax, the utility semantics, and the meta-property that is called the representation theorem. The decision theoretic syntax has three layers. Layer one is a set of choices. Layer two is that each choice is followed by a set of possible outcomes. Layer three is that a possible outcome is characterized by two and only two properties: desirability and feasibility.

The corresponding decision theoretic semantics, called the utility semantics, are then built from the bottom up. The decision theoretic meaning of desirability is determined by a dollar value; the meaning of feasibility is characterized by a probability, which is determined by the normalized probability distribution on the set of possible outcomes associated with a given choice. The multiplication of the dollar value and the corresponding probability is a utility, which is the decision theoretic meaning of that outcome. Then the summation of the corresponding utilities of the set of possible outcomes returns the mathematical expectation of the given choice as its decision theoretic meaning.

The solution of a decision problem is a full scale preference order on the set of choices. It must satisfy the representation theorem: for any given choices C_i and C_j , we have

$$C_i \succ C_j, \text{ if and only if, } ME(C_i) > ME(C_j),$$

which means that C_i is preferred to C_j if, and only if, the mathematical expectation of C_i is greater than the mathematical expectation of C_j . This meta-property provides the internal symmetry between the decision theoretic syntactic components and the semantic components globally. Historically, this formulation followed the formalism of standard logic, which requires the soundness and completeness for any logic system: for any given set of premises Γ and a conclusion A , we have

$$\Gamma \vdash A, \text{ if and only if, } \Gamma \vDash A.$$

The left side denotes a proof (i.e., infer A from the premise set Γ .) Here, the proof is a purely syntactic concept in standard logic. The right side says that it is a valid argument form.

5.2. Economic Rational Man

The terms economic rationality and economic rational man (note the singular) are often used interchangeably in economic literatures. It is one of the conceptual cornerstones of the neoclassical economics. Given the above decision theoretic structure, the notion of economic rationality or the economic rational man can be defined accordingly. The economic rational man satisfies the following four properties (Rubinstein 1998; Yang 2012):

Property 1: Full knowledge. This is a syntactic requirement, which assumes the rational man knows all the syntactic components of the decision problem, including all the choices, all the possible outcomes of a choice, and how to characterize an outcome by two properties - desirability and feasibility.

Property 2: Full capacity. This is a semantic requirement, which assumes the rational man is capable of calculating the mathematical expectation for every choice from the bottom up, based on a given utility function regardless of the design complexity. This should imply the capacity of optimization.

Property 3: Full scale preference order. This is a syntactic requirement. The preference order has to be established on, but not into, the set of choices. In other words, for any two choices, it must prefer one to another and cannot prefer not to prefer.

Property 4: Fully logical. It assumes indifference between logically equivalent descriptions. Economic rationalities are not affected by how the decision theoretic components are framed.

These four properties have received countless criticisms by treating the economic rational man as a possible individual market participant. Next, we will clarify the issues of why economic rational man still stands firmly and what it really stands for.

5.3. Economic Rational Man as the Personified Market

Assume that the notion of an economic rational man is to characterize individual participants of the market. Then each participant, having the four capacities shown above, should be able to beat the market and gain all the wealth. But such an individual participant has been so far invisible to us. How many such economic rational *man(s)* are on the market? If at least two existed, who would win the competition?

Theoretically, it is difficult to apply the notion of the economic rational man in characterizing any individual participant of the market; however, there is one thing that fits this notion and may well satisfy the four properties: *the market itself*. Property 1 indicates that an ideal perfectly competitive market should be information-maximizing. At this point, the market is the container of all the information as long as it exists. Property 2 is a good characterization of the power of the invisible hand, which should be fully capable of generating prices for any potential trades and, in turn, guide the market. Property 3 emphasizes the pricing function: any product has to be priced and, in turn, all the products can be compared by taking their prices into account; that is the order of market. Property 4 says that the market cannot be fooled, cheated, or confused. In order to do so, however, the market has to be regulated and managed (e.g., bad financial items should not be repackaged to get a higher rating). Note that the notion of the economic rational man is always expressed as a *man* (not *men*) which can only be used to characterize the personified market itself. Notice Simon used the term “businessmen” under his idea about bounded rationality.

Now, we have a clearer picture. The notion of economic rationality has a well-conceptualized symmetric structure. It consists of syntax (Property 1), semantics (Property

2), meta-property (Property 3), and inference rule (Property 4). The notion of *homo economicus* has confused people for quite a long time; it should not be misused in conceptualizing individual consumers, but instead, used as a platform for individual consumers to play on.

5.4. The Economic Rational Man, Forward Observation, and Wavefunction

By the very principle of economic mechanics (Yang 2012), human economy is in nature an experiment of the largest scale that our civilization has ever run. Thus, each and every market participant should be treated accordingly as an active observer. Ordinary businessmen used to look back and forth; not only would they make forward observations into the future economy, but they would also make backward observations by taking into account the sunk cost. However, the market never goes back. The economic rational man is concerned with the efficiency, which can be characterized by the ratio of marginal gain and marginal cost if we invest ‘one more unit’ of resource. Here, by ‘one more unit’ it must mean the near future or the present. Thus, the economic rational man only does the forward observation by looking into the future economy.

The degree of disturbance in the forward observation is high. As Paul Dirac (1930), the higher the disturbance in the observation, the smaller the world can be observed. Thus, when the economic rational man is trying to observe future market, it is analogous to the situation when the quantum physicists tried to observe the particle world. At this point, a policy or an investment action can be used as the stimuli to test the future economy or market. Three perspectives are worth mentioning from quantum theoretic viewpoint. First, the result states of performing a stimuli such as a monetary policy can be that it works, it does not work, or any superposition of the two. Second, the states can be seen as a set of ‘YES/NO type experimentations by von Neumann (Penrose 2004). Third, for a sample of ‘YES/NO’ tests we can only obtain the amplitude of a wavefunction (recall Section 2.4); the square of the absolute value of this amplitude is used to predicate the probability of finding a future market action. Here, statistically speaking for an observable, the sample mean is certain but the standard deviation is uncertain. Fourth, the stimuli will affect the

future market so that it needs to be taken into account as part of the future market. Finally, the states of this system should be characterized by a wavefunction. When we say to transform from one state to another state, it means the states of the wavefunction. Recall that the meaning of the wavefunction is a complex number, which has the exponential form $Re^{i\theta}$.

5.5. Internal (Global) Symmetry and the Gauge Transformation of the First Kind

The economic rational man may switch from one view to another view by looking into the future economy. This is similar to say that the market is rotated from one angle to another angle. Mathematically, it means to transform from one basis to another basis. Accordingly, the wavefunction is transformed from one state to another state: $\varphi \rightarrow \varphi' = e^{i\theta}\varphi$. Notice the difference from the function $\theta(x)$ introduced in Section 2.4; here the θ is a constant and so is $e^{i\theta}$. The advantage in this case is that when the differential operation is applied to calculate the strength of a field, the classical derivative preserves the transformation format. Mathematically, we have

$$\begin{aligned}\varphi &\rightarrow \varphi' = e^{i\theta}\varphi, \\ \partial\varphi &\rightarrow \partial\varphi' = \partial e^{i\theta}\varphi = e^{i\theta}\partial\varphi.\end{aligned}$$

This is called *the gauge transformation of the first kind*. Here, one might wonder about two questions. It seems easier to understand the necessity of the gauge transformation of the second kind introduced earlier. But why do we call the above transformation the gauge transformation of the first kind, and why do we need it? If the market can be personified as the economic rational man, are the two the same thing? If not, how are they different? The answer is: yes and no. A gauge field consists of two layers: the field of gauge potential and the field of strength. The economic rational man should stand solely for the gauge potential, while the market by its own right can only stand for the strength field. Conceptually, this is a key distinction from gauge theoretic perspectives.

As C.N. Yang (2014) states, the concept of vector potential \mathbf{A} (originally proposed by W. Thomson) remained central in Maxwell's thinking through his life. What is the vector potential in the present context? Recall the four requirements for the economic rational

man (see Section 5.2): Full knowledge, full capacity, full scale, and fully logical. These will serve as the four components of the vector potential. Of which, (if further analysis is needed), the full capacity of generating the prices should be the best candidate to serve as the meta-market potential, and the other three requirements will serve as the components of the cognitive potential vector \mathbf{A} . Economics tells us that the market can create wealth by minimizing the transaction cost. Here, the idea is to imagine the economic rational man as a meta-market participant. It produces the prices to exchange for potential trades, which in turn can benefit the potential buyer and the potential seller.

6. INTERACTIONS OF THE MAN AND MEN

6.1. *Gauge Freedom, Economic Efficiency, and Displacement Current*

In textbooks as well as other literatures of physics, the gauge transformation of the first kind is always introduced first without much discussion. Most discussions are given to the gauge transformation of the second kind with various examples and detailed explanations of physical backgrounds. In the context of this section we will see why the gauge transformation of the first kind is not only important but necessary. To follow the logic of Maxwell and the historical causal development of Maxwell equations (C.N. Yang 2014), the following discussions will first introduce the concept of displacement in the Ampere-Maxwell law. Second, I introduce the concept of individually oriented efficiency in a parallel economic context. Then I connect the two concepts by the gauge potential and gauge freedom. The connection of gauge freedom and displacement is considered through Lorentz gauge fixing.

First, in the development of electrodynamics, Faraday's law postulated the interaction between the curl of electricity and the change of magnetism over time

$$\nabla \times E = -\frac{\partial B}{\partial t}.$$

Since the Ampere's Circuital law was lacking an account for the relation between the time rate of change of the electric strength and the curl of magnetic strength, Maxwell introduced a second term, called displacement current, on the right-hand side of Ampere's law. The resulting Ampere-Maxwell law is

$$\nabla \times B = \mu J_f + J_D,$$

where $J_D = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} = \frac{1}{c^2} \frac{\partial E}{\partial t}$. In a similar sense, we may call the right-hand side of Faraday's law the displacement magnetism. Physicists used to explain the two laws above as that the rate of change over time of E (or B) causes the curl of B (or E). Logically, the inverse statement should also hold that the curl of B (or E) has an effect on the time rate of change of E (or B). Here the hammer-throw game may be used as an everyday example: an iron ball is attached to a line (rope). An athlete spins it in a marked circle on the ground, he lets go when ball is at the height of its speed aiming for maximum distance. Conceptually, the notion of displacement by Maxwell is fundamentally important because it induced Maxwell's idea of the electromagnetic wave. Technically, both Faraday's law and the Ampere-Maxwell law are used in the derivation of the electromagnetic wave equation.

Second, in the economic context parallel to electromagnetism, people would generally agree on the existence of the interaction between market and cognition. Economists would be happy to recognize how a changing market affects cognition. When the inflation rate is high, the consumers would think that the price has increased too fast. However, some arguments would be needed to convince economists for how cognition affects the market. Often after buying something, people would think that it was too expensive, which means people felt regret that the action was not efficient. Such regretful tastes may collectively move the market down. By the same token, individual cheerful feelings after buying things would cumulatively build up and collectively move the market up. This shows that people do have the rationality of efficiency in mind. It would be more convincing to show how that account can be reasonably construed into the equation. Market changes as well; it is not steady in general. The further question is how the cognitive field collectively makes an unsteady market directional.

Third, as C.N. Yang (2014) states, "Maxwell was aware of what we now call the gauge freedom in (Maxwell) equations 1-3, namely, that the gradient of an arbitrary scalar function can be added to \mathbf{A} without changing the result", and "Already in Faraday's electrotonic state and Maxwell's vector potential, gauge freedom was an unavoidable

presence.” What is the gauge freedom in the present context? The simple answer is: individually oriented efficiency. In the present paper, we have not discussed the concept of efficiency so far, which is perhaps the most popular concept in the literature of modern economics. We always say that given the limited resource, economics is needed to study the efficient allocations of scarce resources. We also often say that economic rationality motivates market participants to pursue the maximum benefit. When it comes to the definition of efficiency, it turns out to be a micro-economic concept concerning individual decision making. Economics made it clear that decision is individually oriented. This is absolutely correct from the viewpoint of cognitive science because the human mind is individually embodied.

The concept of efficiency can be defined by the ratio of the marginal gain over the marginal cost. The concepts of gain and cost are based on the values, which vary from one individual to another (Heyne et al. 2013). Thus, the gain and the cost are scalar fields. The concept of the margin is based on the notion of differential limit. Hence, the concept of efficiency is characterized by the gradient of a scalar field.

The arbitrary individual efficiency term must economically carry a great deal of mental energy and cognitive effort; it is reasonable for us to wonder where these mental energy and cognitive effort go in the economic world? Efficiency is economic, so it must be hiding somewhere under the market. Notice that the efficiency term depends on the concept of margin, which is mathematically characterized by a derivative. In other words, each individual efficiency term points at a tangent direction. Collectively, they have the potential freedom to change the market strength in certain directions from time to time. This means possible displacements of the market. Here, as a hint, the interested readers might want to have a look at the Maxwell’s vortices model for some helpful insights (C.N. Yang 2014). This model is similar to the hammer-throw example given earlier. In physics terms, this is a process of polarization. This is consistent with our commonsense observation: collectively the individual efficiency of efforts polarizes the market field to certain directions. This process may happen implicitly and hence non-observably. As the result, the individually oriented cognitive efficiencies may reappear collectively as the (market)

displacement effect in the Ampere-Maxwell law. Here, I follow Feynman (Shadowitz 1975) treating the displacement as an effect rather than a kind of current.

In the above discussion, I connected the concept of economic efficiency to the gauge freedom, and then connected them to the displacement effect. Here we have a seemingly serious problem because by the commonly shared view in physics, there would be no relation between gauge freedom and displacement current at the curl level of analysis. The gauge freedom term is already gone in the first order curl: $\nabla \times (A + \nabla\psi) = \nabla \times A = B$, while the displacement current term only occur in the second order curl: $\nabla \times (\nabla \times A) = \nabla \times B = \mu J_f + J_D$. Keep in mind that the greatest insight of Maxwell's framework is to direct us to move from displacement effect to the electromagnetic wave. In Section 6.2 we will see that in establishing the electromagnetic wave equation, the divergence of the gauge potential must be constrained by Lorentz gauge condition, which will reconnect the gauge freedom and displacement effect.

Note that from the economic rational potential to the displacement market current, the curl operator is applied twice. The question would be: where were the efficiency energies and efforts stored during this journey? This turns out to be a false question. By the logic of the electromagnetic field, the interaction between market field and cognitive field itself should be modeled as one field: the market-cognition field, which contains market energies as well as cognitive energies.

6.2. Efficiency Matters: Gauge Transformation of Electromagnetic Field

So far we have made the electrodynamics and market dynamics highly integrated; hence, we will start to use the terms electromagnetic field and market-cognitive field interchangeably. With Maxwell's idea of electromagnetic wave in mind, here the purpose is to establish the wave equation. During this process, we will see why gauge freedom is associated with displacement effect. In the free space (i.e., the vacuum), the electromagnetic wave equation can be obtained by the superposition of the wave equations of the magnetic potential $\square^2 A = 0$ and the wave equation of the electric potential $\square^2 \varphi = 0$, where \square^2 is the d'Alembertian operator $\square^2 = \nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2}$. In order to establish these

two wave equations, it needs to fix the potential (φ, A) . By the Maxwell equations, we have $B = \nabla \times A$ and $E = -\frac{\partial A}{\partial t} - \nabla\varphi$. Now, let ψ be an arbitrary scalar function, we can have gauge transformations $A' = A + \nabla\psi$ and $\varphi' = \varphi - \frac{\partial\psi}{\partial t}$, from which we can also obtain the same B and E . Thus, from $\{(A', \varphi')\}$ to (B, E) is a many-to-one mapping. In order to obtain a uniquely fixed potential (A', φ') from (B, E) , it needs to put certain bound conditions. This is called gauge fixing. The Lorentz gauge condition is to put the constraint on the divergence of A' such that $\nabla \cdot A' = -\mu\varepsilon \frac{\partial\varphi'}{\partial t}$. Then we can obtain the two homogeneous wave equations

$$\begin{aligned}\nabla^2 A - \mu\varepsilon \frac{\partial^2 A}{\partial t^2} &= \square^2 A = 0, \\ \nabla^2 \varphi - \mu\varepsilon \frac{\partial^2 \varphi}{\partial t^2} &= \square^2 \varphi = 0.\end{aligned}$$

However, putting the Lorentz gauge condition on the divergence of A requires to also put certain constraint simultaneously on the gauge function ψ such that

$$\square^2 \psi = \nabla^2 \psi - \mu\varepsilon \frac{\partial^2 \psi}{\partial t^2} = 0.$$

Note that

$$\mu J_f + J_D = \nabla \times B = \nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A.$$

Thus, in order to achieve the wave equations needed, the Lorentz gauge condition makes displacement effect J_D and gauge function (a freedom factor) ψ connected at a deeper level involving the divergence of the magnetic potential. Recall that in our context, the value function is defined as the gauge function ψ and the efficiency is defined by its gradient $\nabla\psi$ in A' (and $\frac{\partial\psi}{\partial t}$ in φ'). This implies that under Lorentz gauge condition, the value function and efficiency must have certain contributions to the market displacement and market-cognitive wave, and this is consistent with the intuitive analysis in Section 6.1.

6.3. Electromagnetic Coupling and Interactions of the Man and Men

In this section, the interaction of the man (the market-cognitive potential) and men (individual market participants) will be modeled by following the logic of electromagnetic

coupling (outlined in Chapter 4, Huang, 2007). The inferences are given in five steps each involves an action characterized by an integral formula.

Step 1. Recall that in Section 2.2 we introduced the notion of action, denoted by S ; in Section 2.3, we introduced the notion of individually oriented proper cost, denoted by τ . Imagine that as a free individual, you were charged with intention to shop in a mall. You walked into the mall at point a and get out the mall at point b . You bought several things in the mall and were interested in a number of other goods. During the time spent in the mall, not only were you calculating how much money you spent in absolute cash value, but from time to time you were also cumulatively estimating the proper costs to you. One might think of the budgeting situation, how the family members' attitudes toward some purchasing, the possible discounts in the near future, whether spent too much, some items were too expensive or cheap, etc. As described in 2.3, spending the same amount of money may mean different proper costs from one individual to another. Estimating the proper cost causes one's mental energy and cognitive effort. This is the kind of action we are concerned here, which can be characterized by

$$S_{individual} = C \int_a^b d\tau.$$

In physics, the action for a free relativistic particle is simplicity itself, which is the proper time spent in going from point a to point b (Huang, 2007).

Step 2. In Section 3.4, we introduced the notion of the market strength field $F_{\mu\nu}$, and pointed out that $F_{\mu\nu}F^{\mu\nu} = F^2$ is gauge invariant. Note here the market strength field can be determined by any economic rationality potential but not from the budgeting potential of any charged individual market participants. Physics assumes that the electromagnetic field fills all the space. This assumption leads us to imagine that the market is a mall, where many stores are located and the volume of trading varies from one store to another and from time to time. Thus, the action of the market strength here is independent of any individuals. In formula, we have

$$S_{market} = C \int F^2.$$

In physics, S_{market} should be written as S_{em} (the action of electromagnetic strength field).

Step 3. From Section 6.2, we know that under the Lorentz gauge condition, an economic rational potential A can be fixed and derived by the market strength, as long as the gauge function (the gauge freedom factor) is also conditioned accordingly. Now comes the crucial point. Assume that A is the cognitive potential of the economic rational man, which fills all the price space in the mall. Some charged consumer tends to buy a list of things in the mall from the point a to the point b . During this shopping process denoted by (x) , the consumer keeps evaluating the pricing potential A . In the simplest case, the action of this interaction is given by

$$S_{interaction} = C \int_a^b dx \cdot A.$$

In physics, it means that A is evaluated at the particle's position x . The question is how to evaluate it.

Step 4. The answer is that A is evaluated by the consumer's proper cost τ . Hence, a mathematically trivial but conceptually significant trick leads to

$$S_{interaction} = C \int_a^b d\tau \frac{dx}{d\tau} \cdot A.$$

In physics, this means that the interaction energy is proportional to $\frac{dx}{d\tau} \cdot A$.

Step 5. Taking $\frac{dx}{d\tau}$ as the speed of the consumer's spending, which is proportional to the consumer's market current J , we finally have

$$Interaction\ energy = J \cdot A.$$

This is what I mean by the coupling of the man's cognitive potential and the market charge of individual men (consumers).

6.4. Free Trade and Dirac's Spinor

In Section 3.1, we defined the notion of the virtual demand (and virtual supply) as a two-component unit, in which intentions carry market charge with a negative sign, meaning to buy for the demand (or with a positive sign, meaning to sell for the supply). These were made analogous to the electron (or the position) with the negative electric charge. So far, our discussions in this paper have mostly focused on the demand because it seems more

intuitive to describe the associated hesitation, which involves a shilly-shally process between the end point of “to-buy” and the end point of “not-to-buy.” In this sense, we say the demand is being spun, and similar discussion can be applied to the supply. Perhaps the best example to imagine the spin of a supply is the stock market. An investor who bought a certain amount of stocks or options is then associated with a shilly-shally process between “to-sell” and “not-to-sell”. Thus, together we observe four possible states: the demand spinning on the “to-buy” direction, the demand spinning on the “not-to-buy” direction, and their counterparts of the supply. This two-by-two structure can be perfectly characterized by the Dirac’s spinor, $\gamma^\mu = (\gamma^1, \gamma^2, \gamma^3, \gamma^4)$, which has four components.

The Dirac equation is about free particles in a rather abstract sense. To better understand the idea of free trade in the present context concerning market dynamics, imagine a shopping mall, in which there are many stores with sales people and many potential buyers. Further imagine many potential trades flowing freely in the mall, each of which is a possible pair of some potential demand and potential supply. To explain the nature of free particles, physicists originally attempted to model a two-component entity, more specifically called a free particle (such as an electron) with two spin directions. Building on this idea, the Dirac equation also modeled the anti-particle (e.g., the positron) which has two spin directions. Thus, the Dirac spinor consists of four components, which satisfy the requirements for this investigation: the supply is treated as the anti-market-element of the demand, and each of them is associated with their own hesitation processes in two spin directions.

Studying the coupling of the Dirac equation and electromagnetic field (i.e., the cognitive-market field in our context) can be helpful to better understanding the origin of gauge theory. The original Dirac equation is in the form

$$i\gamma^\mu \partial_\mu \psi - \frac{mc}{\hbar} \psi = 0,$$

which only holds without interacting with the electromagnetic field because the derivative operator (∂_μ) is classical. When it is coupled with the electromagnetic field, we need to replace the classical ∂_μ by a covariate differential operator: $D_\mu = \partial_\mu + ieA_\mu$. Then we have

$$i\gamma^\mu (\partial_\mu + ieA_\mu) \psi - \frac{mc}{\hbar} \psi = 0,$$

where e is the electric charge and A_μ is the gauge field. Of course, the gauge field is responsible for the interaction. The lesson is that for an economic model of free trade to be coupled with the cognitive-market field, it needs to take into account individual differences case by case. This requires replacing the classical idea of marginal analysis by some covariate marginal analysis, which acts on the budget potential field concerning individual proper costs.

7. DISCUSSION: SYMMETRIES, THE CONSERVATION OF MARKET CHARGE, AND RATIONALITY CURRENT

Let us review one more time about what we went through in Section 4, but in a reverse order: from gauge transformations to the principle of gauge invariance, and then to what it implies. Gauge transformation leaves the form of Lagrangian invariant, which implies some symmetry. Indeed, physicists sometimes call that gauge transformation itself the gauge symmetry. From Section 4 we can see that in electrodynamics, the gauge transformations concerned in this paper satisfies the unitary group $\{e^{i\theta}\}$. It is the phase θ that matters. If the θ is a constant, it is independent of the space-time. In this case, the classical derivative does not cause any problem in the form of the Lagrangian. When the θ transforms from one phase to another, it leaves the form of the Lagrangian unchanged. This is called the gauge transformation of the first kind, which implies some internal symmetry that is intrinsic to the system. This internal symmetry can also be seen as a global symmetry because it treats all the space-time positions the same way. By this logic, the approach of the economic rational man is committed to the gauge transformation of the first kind. It admits the economic rationality as an internal symmetry that is intrinsic to an efficient market system, and is globally shared by all the individual market participants. In this case, the classical margin analysis works in economic modeling.

If the phase $\theta(x)$ is individual dependent (i.e., $\theta(x)$ is space-time point dependent), as explained in Sections 3 and 4, the gauge transformation itself becomes the so called gauge potential, $A_\mu = A_\mu(x)$ (Healey 2007; McMahon 2008). In this case, the gauge transformation

$$A_\mu \rightarrow A'_\mu = A_\mu - 1/g \partial_\mu \theta$$

is required so that the covariate derivative can act on A_μ to connect the budget potential of one individual to that of another. This is called the gauge transformation of the second kind, which enables the covariate marginal (derivative) operation act on the gauge field in order to maintain the form of Lagrangian used in economic or market modeling invariant. This achieves what is called the local gauge symmetry. The approach of businessmen is committed to the gauge transformation of the second kind, which allows each individual market participant to hold the bounded rationality locally without breaking a possible globally valid market-dynamic model.

The gauge symmetry is one of the fundamental principles of symmetry in physics. It determines the form of field equations. Both gauge transformations of the first and second kinds follow the principle of gauge invariance. This suggests that both the economic man approach and the businessmen approach should follow the same principle of gauge invariance. Noether's theorem allows us to derive conservation laws about the conserved market charge and the conserved market current from symmetries in the Lagrangian.

8. CONCLUSION

From modeling and theorizing perspectives, current economics and cognitive psychology have been developed largely within the Newtonian tradition. The approach presented in this paper applied the conceptual and instrumental tools from modern theoretical physics to revisit a set of core ideas in economics. The results show some new pictures, which we would miss without taking these new tools as the logic.

To provide a unified account of mainframe economics and behavioral economics is by all means a desired advancement. I generated the "man vs. men" dilemma as the target issue being solved. Market and electrodynamics are the most studied domains in economics and physics, respectively. Interestingly, the two domains share a great deal of features. This paper took the market dynamics as the scope of investigation, and used electrodynamics as the logic in its model development.

Gauge theory used in the standard model of particle physics has been well adopted as a widely applied basic language in modern physics. It enabled us to model the man-approach by the gauge transformation of the first kind, and to model the men-approach by the gauge transformation of the second kind, and this naturally led us to achieve the gauge invariance shared by the two approaches. Within the gauge theoretic framework, the clear distinctions of gauge potentials and field of strength were made at the men-level as well as at the man-level. The gauge theory also enabled this paper to relate the market field with the cognitive field. As a result, we achieved a better understanding about the interactions of the two fields, and led to the necessity of what we call the market-cognition field by the logic of Maxwell equations.

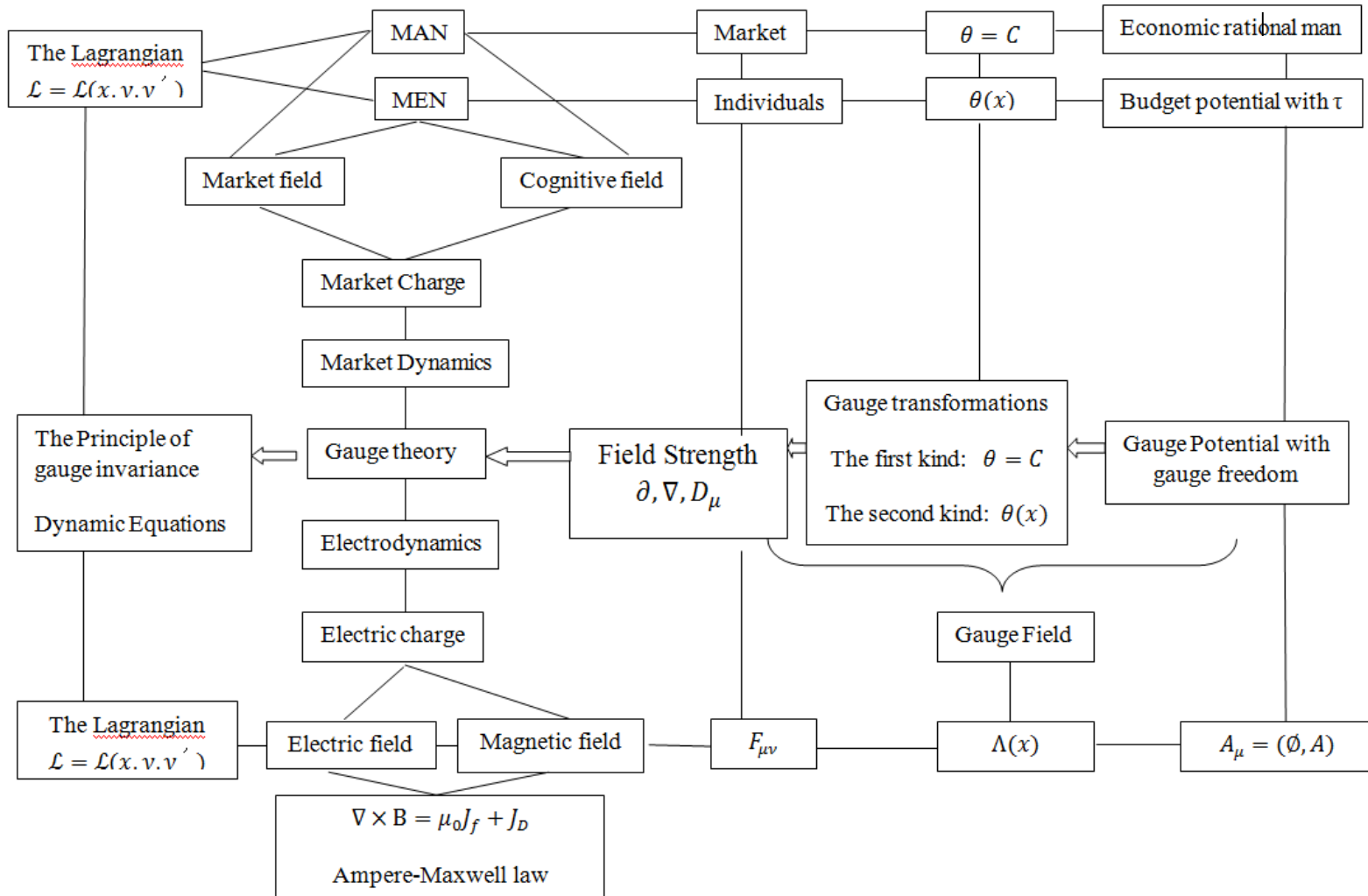
The general methodology used in this paper is a top-down strategy. I started from gauge theoretic modeling and went pretty far along this line, but I did not go down to the very roots of electrodynamics. For example, I left the concepts of the electric permittivity and the magnetic permeability of free space untouched, as I did not feel that the economics and cognitive science were ready to go that far experimentally. The work in this paper crosses several disciplines, and certain efforts were made to make the descriptions as conceptually and instrumentally self-contained as possible. The work presented in this paper is a continuation in the line of research called economic mechanics (Yang 2012), which relates to what is today called integration science.

ENVOI

In the history of modern physics, there is an ironic story (Huang 2007). Hendrik Lorentz is known for coming very close to discovering the special theory of relativity but stopping his work barely a foot away. He later confessed, “The chief cause of my failure was my clinging to the idea that only the variable t can be considered as the true time, and that the local time t' must be considered no more than an auxiliary mathematical quantity”. For this, Paul Dirac wrote, “I think he must have been held back by fears, some kind of inhabitation. He was really afraid to venture into entirely new ground, to question ideas which had been accepted from time immemorial”. I think there are two lessons we need to

learn. One is not afraid to accept modern theoretical physics, and another is not afraid to study into it.

Appendix. The Map of Main Contents



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