Stochastic stability in coalitional bargaining problems

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Abstract This paper examines a dynamic process of n-person coalitional bargaining problems. We study the stochastic evolution of social conventions by embedding a static bargaining setting in a dynamic process; Over time agents revise their coalitions and surplus distributions under the presence of stochastic noise which leads agents to make a suboptimal decision. Under a logit specification of choice probabilities, we find that the stability of a core allocation decreases in the wealth of the richest player, and that stochastically stable allocations are core allocations which minimize the wealth of the richest.

Keywords: Stochastic stability; Coalitions; Logit-response dynamics; Bargaining.

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1 Introduction

We examine an abstract model of dynamic processes of social and economic interactions. An economy with a finite number of agents and a finite number of states is considered. When the economy lies in a state, every agent receives a payoff flow which may depend on the state. At each time, a group of agents may form a coalition to move the economy to a new feasible state. How states can be changed over time depends on specific coalitions of agents. We are interested in which states likely arise in the process in the long run.

The setting can be viewed as a dynamic model of n-person coalitional bargaining problems. A characteristic function assigns a set of feasible allocations to each subgroup of agents. This function stylizes the potential gains from cooperation among subgroups. Over time, agents may form a coalition and move the economy to a new allocation. States of the economy stand for the partitions of agents into coalitions and the surplus allocation to each coalition of agents. A transition from one state to another requires the consent of those agents who change coalition. Given the setting and a characteristic function, we ask which coalitions will form and what will be the resulting allocation of gains to agents.

To address those questions, we study stochastic stability of such a non-cooperative model of n-person coalitional bargaining settings. As written above, a characteristic function describes the surplus available to different coalitions. A coalition will be formed when all prospective members agree on forming a coalition and how to share the surplus they generate. At any given period, the surplus is divided as per the agreed distribution. We explore which coalitions are formed and which surplus distribution is most likely observed among the agents in the long run.

This study characterizes allocations that are stochastically stable against both individual deviations and coalitional deviations. The notion of stochastic stability was introduced by Foster and Young (1990), Kandori et al. (1993) and Young (1993). Roughly speaking, it is a method to assess the robustness of equilibria by checking its resilience to stochastic shocks. We apply a version of the method proposed by Sawa (2014). Agents are assumed to be myopic payoff-maximizers. Over time, agents may decide to change coalition and consequently move the state to another. Their decisions on the change are perturbed by stochastic noise. Such perturbed decisions may trigger a deviation even from a core allocation which may lead the process to reach another one in the core. As in the literature, our approach can be viewed as performing a stability test for allocations in a coalitional bargaining problem.

To assess robustness against deviations, we examine the above stochastic dynamic process. In each period, some agents form a tentative team and randomly choose a surplus distribution which each agent in the team will weigh up shortly. In the unperturbed updating process, an agent agrees to the proposed distribution if it yields her higher payoff than the current allocation. The distribution will be accepted if all agents in the team agree. To this unperturbed process, we add stochastic noise that leads agents to make a stochastic choice. That is, each agent agrees to a proposed distribution according to the logit choice rule. Coalitional deviations sometimes occur even if not all members of the team benefit, and this occurs with a probability that declines in the total payoff deficits to team members. As a consequence, the process visits every allocations repeatedly, and predictions can be made concerning the relative amounts of time that the process spends at each. We examine the behavior of this system as the level of stochastic noise becomes small, defining the stochastically stable allocations to be those which are observed with positive frequency in the long run as noise vanishes.

We find that stochastically stable allocations are core allocations whenever the set of interior points in the core is not empty.[1] Moreover, we find the following characterizations. The stability of an allocation decreases in the wealth of the richest player, and the stochastically stable allocations are core allocations which minimize the wealth of the richest. We view this result interesting because equity consideration winds up playing an important role even with myopic payoff-maximizing players.

This study has two major contributions. Firstly, we employ a decentralized process of coalitional bargaining problems. We consider a situation where a group of agents may meet, discuss forming a coalition, and make some joint decision. This is the main contrast to extant studies which assume a central authority to which all agents submit claims. We don't discuss which is better because it depends on settings. Some settings are better modeled by a centralized model while other settings are so by a decentralized one. We think that to explore decentralized processes is equally important.

Secondly, we relax assumptions on the characteristic function. Stronger assumptions are often imposed on it. For example, Chatterjee et al. (1993) assume the strict superadditivity, under which the production of $S \cup T$ has to be strictly greater than the sum of the productions of S and T for two disjoint coalitions S, T.[2] We only assume the existence of interior points of the core. Furthermore, we consider an extended definition of the core. The typical definition of the core requires the grand coalition to be formed.[3] However, there may exist settings where it is optimal to have multiple teams instead of having the grand coalition. We relax the requirement by allowing multiple coalitions to coexist. Accordingly we define an extended notion of the core, called *dispersed-core*. A state with multiple teams is said to be in the dispersed core if any coalition, e.g. the grand coalition, does not have any incentive to deviate.

For the related literature, we briefly discuss it and delegate a more detailed discussion to Section 5.1. The related studies in the literature on stochastic stability are Agastya (1999) and Newton (2012).[4] Both papers consider a perturbed dynamic of coalitional bargaining games and characterize allocations which are stochastically stable against perturbations. One of the differences from ours is that those two papers assume a sort of central authority. Agents submit their claims to the authority, and then it decides which coalitions to be formed and how demands are rationed.

[1] The set of interior points is called *strict core* in the paper.

[2] This is a sufficient condition for the existence of the core.

[3] For example, see Definition 4.2 of Moulin (1988).

[4] Other related studies are, for example, Kandori et al. (2008), Serrano and Volij (2008), Jackson and Watts (2002), Nax and Pradelski (2014) and Newton and Sawa (2015). All examine stochastic stability in cooperative settings but they have a different focus from ours. The former two study stochastic stability of allocations in a housing economy, which is a simple exchange economy with indivisible goods. The latter three are focused on two-sided matching markets.

While, we assume a decentralized process, in which agents randomly meet and decide by themselves whether to form a coalition.

The related studies in the literature on coalitional bargaining models with rational agents are Okada (1996) and Compte and Jehiel (2010). Those studies resemble ours in that the proposer is randomly chosen.[5] One of differences is that the proposer rationally chooses her proposal in the two studies, while a proposal is randomly chosen to be assessed in ours. Despite the differences, it is interesting to see that an egalitarian outcome is favored by all of those studies and ours.

The paper is organized as follows. Section 2 introduces the coalitional bargaining model and the definitions of core and dispersed-core. Section 3 describes the dynamic of the bargaining model. In Section 4, we characterize stochastically stable allocations. Section 5 offers discussions on the extant literature, an extension of the model to incorporate heterogeneous utility functions, and differences and similarities between our results and extant solution concepts.

2 Model

We consider a non-cooperative model of multi-player coalitional bargaining situations examined by several studies, for example, Chatterjee et al. (1993). The model is given by a tuple $(N, v, \{u_i\}_{i \in N})$, where $N = \{1, \ldots, n\}$ denotes a set of players, v a characteristic function, and u_i player i's utility function. Let $\mathcal{R}$ be the class of all subsets of N. Any $J \in \mathcal{R}$ may form a team and the (monetary) surplus that such a team generates is given by the characteristic function $v \colon \mathcal{R} \to \mathbb{R}_+$ with $v(\varnothing) = 0$. Surplus $v(J)$ can be distributed to members if all members of J agree on an allocation. We assume that there exists small $\Delta > 0$ such that $v(J)/\Delta \in \mathbb{Z}$ for all $J \in \mathcal{R}$.[6]

Let $S_i = \{0, \Delta, 2\Delta, \ldots\}$ denote the set of player i's claim s_i. For a team $J \in \mathcal{R}$, an allocation of its surplus has to satisfy the following feasibility constraints:

$$\begin{cases} \sum_{i \in J} s_i \leq v(J) & \text{if } |J| \geq 2, \\ s_i = v(J) & \text{if } J = \{i\}. \end{cases} \tag{1}$$

The second constraint implies that player i earns what she can produce by her own when she does not form a team with other players. Player i gets the utility equal to $u_i(s_i)$ where $u_i : \mathbb{R} \to \mathbb{R}$ and it is concave and strictly increasing.[7] We assume that their utility functions are common, i.e. $u_i(\cdot) = u(\cdot)$. Let the set of feasible surplus distributions for team J be denoted by

$$S^J = \left\{ s_J \in \prod_{i \in J} S_i : s_J \text{ satisfies (1).} \right\}.$$

Player i will receive her reservation surplus $v(\{i\})$ if she forms a singleton team, i.e. $S^J = \{v(\{i\})\}$

[5] Another strand of studies assumes that the player who rejects an offer becomes the next player to make an offer. See Chatterjee et al. (1993), for example.

[6] The assumption guarantees that surplus can be distributed without loss for all $J \in \mathcal{R}$.

[7] We assume that surplus $v(\cdot)$ is transferable, but utility may not.

if $J = \{i\}$.

We assume that there can be more than one team, but a player can participate in exactly one team.[8] Let $\mathcal{M}$ denote the set of existing teams. Since each player participates one team, $\mathcal{M}$ must be a partition of N.[9] The space of claim profiles with $\mathcal{M}$ is defined as

$$\Omega_{\mathcal{M}} = \left\{(s, \mathcal{M})\colon s_J \in S^J \ \forall J \in \mathcal{M}\right\}.$$

Note that $(s, \mathcal{M})$ in $\Omega_{\mathcal{M}}$ satisfies the feasibility constraints for all existing teams. The space of feasible profiles is given by

$$\Omega = \bigcup_{\mathcal{M} \in \mathrm{part}(N)} \Omega_{\mathcal{M}},$$

where $\mathrm{part}(N)$ denotes the set of partitions of N.

Our model of team formation is similar to Okada (1996) and is more general than the model of Compte and Jehiel (2010), which restricts the number of teams of multiple players to at most one. We will compare the stochastically stable outcomes resulting from myopic players with those from perfectly rational players shown in Okada (1996) and Compte and Jehiel (2010).

We turn now to the notion of core stability and define the *core* and its extended notion, the *dispersed-core*. We say that coalition J *blocks* allocation $(s, \mathcal{M}) \in \Omega$ if there exists $s' \in S^J$ such that

$$s'_i > s_i \ \forall i \in J.$$

Similarly, coalition J *weakly blocks* allocation $(s, \mathcal{M}) \in \Omega$ if there exists $s' \in S^J$ such that[10]

$$s'_i \geq s_i \ \forall i \in J.$$

That $(s, \mathcal{M})$ cannot be weakly blocked by J implies that there exists at least one player in J who would be strictly worse off if J were to be formed and implement $s' \in S^J$.

Definition 2.1. *The* core *consists of the allocations with $\mathcal{M} = \{N\}$ that cannot be blocked by any coalition $J \in \mathcal{R}$. The* strict core *consists of those that cannot be weakly blocked by any $J \in \mathcal{R}$.*

Definition 2.2. *The* dispersed-core *consists of the allocations with $\mathcal{M} \in \mathrm{part}(N)$ that cannot be blocked by any $J \in \mathcal{R}$. The* strict dispersed-core *consists of those that cannot be weakly blocked by any $J \in \mathcal{R}$.*

In words, $(s, \mathcal{M})$ is a core allocation if, for all $J \in \mathcal{R}$, there is no coalitional deviation of J which strictly improves all players of J. It is strict if there is no such deviation which weakly improves all players of J. Notice that a strict core allocation $(s, \{N\})$ satisfies the set of inequalities:

$$\sum_{i \in J} s_i \geq v(J) + \Delta \quad \forall J \in \mathcal{R} \setminus \{N\}. \tag{2}$$

[8] An agent forms a singleton team when she does not form a team with others.

[9] $\mathcal{M}$ is a partition of N if $N = \cup \mathcal{M}$, and $M \cap M' = \varnothing \ \forall M, M' \in \mathcal{M}, M \neq M'$.

[10] For $J \in \mathcal{M}$, it is interpreted as follows: $J \in \mathcal{M}$ *weakly blocks* $(s, \mathcal{M})$ if there exists $s' \in S^J$ with $s' \neq s^J$ such that $s'_i \geq s_i$ for all $i \in J$. This will require a strict core allocation to be efficient; All $J \in \mathcal{M}$ distribute the surplus without loss.

A strict dispersed-core allocation has a similar interpretation. If $(s, \mathcal{M})$ is a strict dispersed-core allocation, then

$$\sum_{i \in J} s_i \geq v(J) + \Delta \quad \forall J \in \mathcal{R} \setminus \{\mathcal{M}\}. \tag{3}$$

Let $\mathscr{C}_\Delta$ denote the set of strict dispersed-core allocations given Δ. In what follows, we assume that $\mathscr{C}_\Delta$ is non-empty. Observe that $\mathscr{C}_\Delta$ will be the set of strict core allocations if there exists a strict core allocation. This is because the grand coalition will block any allocation s with $\mathcal{M} \neq \{N\}$ in such cases. Then, any $(s, \mathcal{M})$ with $\mathcal{M} \neq \{N\}$ cannot be in the dispersed-core.

Remark 2.3. Section 4.1 examines stochastic stability with Definition 2.1, while Section 4.2 does that with Definition 2.2. The former is consistent to a typical definition of the core. We consider the latter because there are settings which are modeled better with it. For example, we often face diminishing returns to a labor when the size of a workforce exceeds a certain threshold. After the threshold, the marginal output of such a workforce will be decreasing as the number of workers grows. In such settings, the grand coalition will not be optimal, and having multiple teams with an appropriate size will be optimal.[11] To better model those, we allow multiple teams to coexist and extend the definition of core to the dispersed one.

Remark 2.4. Observe that a strict core allocation is a strict $\mathcal{R}$-stable equilibrium defined in Sawa (2014). A pair $(s, \mathcal{M})$ is a strict $\mathcal{R}$*-stable equilibrium* if

$$\exists i \in J \;\text{ such that }\; s_i > s_i' \quad \forall s_J' \in S^J,\; J \in \mathcal{R}.$$

The method applied here refines the set of strict $\mathcal{R}$-stable equilibria, and thus will refine the set of strict core allocations in the present model.

Remark 2.5. A coalitional bargaining setting with a strictly convex characteristic function always has non-empty $\mathscr{C}_\Delta$ for sufficiently small Δ. A function v is *strictly convex* if[12]

$$v(J \cup J') > v(J) + v(J') - v(J \cap J') \qquad \forall\, J, J' \in \mathcal{R} \text{ with } J \cap J' \notin \{J, J'\}.$$

For sufficiently small Δ, the setting has the strict core as shown in the next example.

Example 1. Let $N = \{1, 2\}$ and $\Delta = 2$, and consider a strictly convex characteristic function $v(1) = 2$, $v(2) = 0$, $v(\{1, 2\}) = 4$. Note that $S^{\{1,2\}} = \{(4, 0), (2, 2), (0, 4)\}$. $(4, 0)$ and $(2, 2)$ are core allocations, but not strict. The issue of nonexistence is resolved by making the grid finer. For instance, for $\Delta = 1$, $(3, 1) \in S^{\{1,2\}}$ is a strict core allocation.

[11]One of the most famous such observations is IBM's development of OS/360, as depicted in Frederick P. Brooks (1995). Throwing additional programmers at the project had actually increased the time to completion. A reviewer of the book, Ray Duncan, wrote: "There is an inescapable overhead to yoking up programmers in parallel. The members of the team must "waste time" attending meetings, drafting project plans, exchanging EMAIL, negotiating interfaces, enduring performance reviews, and so on. . . . And as the team grows, there is a combinatorial explosion such that the percentage of effort devoted to communication and administration becomes larger and larger."

[12]A similar characteristic function is assumed in several studies, e.g. Okada (1996) and Okada (2011).

To conclude this section, we make a couple of definitions and an assumption which will be important for sections to come. For $(s, \mathcal{M}) \in \Omega$, let $s_{(i)}$ denote the i-th largest share in s. Let

$$s_{\min} = \min_{s' \in \mathscr{C}_\Delta} s'_{(1)}.$$

In words, $s_{\min}$ is the lowest claim of the richest player among all strict core allocations. Let $\Omega^*_\Delta = \{(s, \mathcal{M}) \in \Omega_\mathcal{M} \times \text{part}(N) : s \in \mathscr{C}_\Delta \text{ and } \sum_{i \in J} s_i = v(J)\,,\ \forall J \in \mathcal{M}\}$. This is the set of states in which players distribute surplus according to a strict core allocation.[13] Define

$$_{\min}\Omega^*_\Delta = \left\{ (s, \mathcal{M}) \in \Omega^*_\Delta : s_{(1)} = s_{\min} \right\}.$$

It is the set of allocations which minimize the wealth of the richest over all strict dispersed-core allocations.

We say that $(s, \mathcal{M})$ is the egalitarian allocation if $s_i = v(N)/n$ for all $i \in N$ and $\mathcal{M} = \{N\}$. Let $(s^E, \{N\})$ denote it. We assume that the egalitarian allocation is feasible.

Assumption 2.6. *For settings where the strict core exists, we assume that Δ is such that $(s^E, \{N\}) \in \Omega$.*

Lemma 2.7. *Suppose that the strict core exists. For all $s \in \mathscr{C}_\Delta$ such that $s \neq s^E$, $s_{(1)} - s_{(n)} \geq 2\Delta$.*

Proof. The proof is the way of contradiction. Suppose that $s \in \mathscr{C}_\Delta$ is such that $s_{(1)} - s_{(n)} = \Delta$. That $s \in \mathscr{C}_\Delta$ implies that s must be efficient, i.e. $\sum_i s_i = v(\{N\})$. If $s_{(1)} \leq v(\{N\})/n$, then $\sum_i s_i < v(\{N\})$. If $s_{(n)} \geq v(\{N\})/n$, then $\sum_i s_i > v(\{N\})$. It must be that $s_{(1)} = s_{(n)}$. Contradiction. $\square$

3 Dynamic

3.1 The Coalitional Logit Dynamic

We apply the stochastic stability approach to problems described in Section 2. In the approach, a static bargaining setting is embedded into a dynamic process in which players randomly form coalitions and jointly revise the distribution of the surplus. They make decisions based on improvements in their payoffs under the presence of stochastic shocks. We examine the limiting probability distribution over allocations as the level of stochastic shocks approaches zero.

The dynamic interaction proceeds as follows. The state of the process in period t is given by $(s^t, \mathcal{M}^t)$, where s^t denotes a profile of players' claims in period t, and $\mathcal{M}^t$ the set of the existing teams in t. At the beginning of period t, $J \in \mathcal{R}$ is randomly chosen, and then a payment proposal $s = \{s_i\}_{i \in N}$ to share the surplus $v(J)$ is randomly chosen.[14] Each player in J is asked whether she accepts or rejects proposal s.[15] If they all accept, players in J form a team and each team member $i \in J$ gets payoffs $u(s_i)$. If coalition J forms a team, then any existing team having some $i \in J$ will

[13] Note that this will be reduced to that $\Omega^*_\Delta = \{(s, \{N\}) \in \Omega : s \in \mathscr{C}_\Delta\}$ if the strict core exists.

[14] A proposal should satisfy conditions written in the next paragraph.

[15] A player accepts with probability $\Psi^\eta(s^t, s)$, which will be given later by Equation (4).

be dissolved, i.e. members other than i in such an existing team will form singleton teams. If at least one player in J rejects s, the state will remain $(s^t, \mathcal{M}^t)$.

We assume that, given s^t, $\mathcal{M}^t$ and J, a proposal s satisfies the feasibility constraint (1) and

$$\begin{aligned} s_i &= v(\{i\}) && \forall i \in M \setminus J,\ \forall M \in \mathcal{M}^t \text{ such that } M \cap J \neq \varnothing, \\ s_i &= s_i^t && \forall i \in M,\ \forall M \in \mathcal{M}^t \text{ such that } M \cap J = \varnothing. \end{aligned}$$

The first condition implies that players in teams being dissolved will form singleton teams and earn their reservation surplus. The second condition implies that forming coalition J should not affect teams which include no member of J.

We assume that players' utilities are temporarily affected by random shocks and those shocks perturb players' decisions. Following the literature pioneered by Blume (1993), those shocks are assumed to be distributed according to type I extreme value distribution. This makes players follow the logit choice rule.

To describe the logit choice rule, suppose that the current claim profile is given by s, that a randomly chosen coalition is J, and that $s'_J \in S^J$ is proposed as the surplus distribution. Let s' be such that $s'_J \subset s'$ and satisfy the above conditions of a proposal. The probability that agent i in coalition J agrees with s'_J is given by

$$\Psi_i^\eta(s, s') = \frac{\exp\left[\eta^{-1} u_i(s')\right]}{\exp\left[\eta^{-1} u_i(s')\right] + \exp\left[\eta^{-1} u_i(s)\right]}. \tag{4}$$

where $\eta \in (0, \infty)$ denotes the noise level of the logit choice rule. Note that agent i takes into account other agents' new claims, i.e. s'_J, in Equation (4). The probability that all members in J agree is given by $\prod_{i \in J} \Psi_i^\eta$.

Now, we turn to formal transition probabilities of the process. The dynamic process described above forms a Markov chain. A state of the chain consists of a surplus distribution and a set of existing teams, $(s, \mathcal{M})$.[16] Let Ω denote the set of states. A transition from $(s, \mathcal{M})$ occurs when coalition J forms a new team (if $J \notin \mathcal{M}$), or J redistributes its surplus (if $J \in \mathcal{M}$). Formally, transition $((s, \mathcal{M}), (s', \mathcal{M}'))$ is said to be feasible if the following conditions are satisfied.[17]

(i) If $\mathcal{M} \neq \mathcal{M}'$, then there exists $J \in \mathcal{M}'$ such that

$$\begin{aligned} &\{i\} \in \mathcal{M}' \quad \forall i \in M \setminus J, && \forall M \in \mathcal{M} \text{ such that } J \cap M \neq \varnothing, \\ &M \in \mathcal{M}', \text{ and } s_i = s'_i \quad \forall i \in M, && \forall M \in \mathcal{M} \text{ such that } J \cap M = \varnothing. \end{aligned}$$

(ii) $\sum_{i \in M'} s'_i \leq v(M') \quad \forall M' \in \mathcal{M}'$.

[16]Transitions depend on not only a current claim profile but also a set of existing teams. If a coalition forms a new team, it will affect other players in teams being dissolved. Which players will be affected depends on $\mathcal{M}$.

[17]For what follows, we write transition from $(s, \mathcal{M})$ to $(s', \mathcal{M}')$ as $((s, \mathcal{M}), (s', \mathcal{M}'))$.

(iii) If $\mathcal{M} = \mathcal{M}'$, then there exists $J \in \mathcal{M}'$ such that

$$s_i = s_i' \qquad\qquad \forall i \in M',\ \forall M' \in \mathcal{M}' \setminus \{J\}.$$

(iv) $s_i' = v(\{i\}) \quad \forall \{i\} \in \mathcal{M}'$.

Condition (i) is a feasibility constraint for the transition to $\mathcal{M}' \neq \mathcal{M}$. It says that when a new team J is formed, an existing team M will be dissolved if at least one player leaves M to join J. Otherwise, the existing team should remain. For the remaining teams, their distributions will be unaffected. Condition (ii) is a set of the feasibility constraints given by Equation (1) for all teams in $\mathcal{M}'$. Condition (iii) is for the case of a surplus redistribution ($\mathcal{M} = \mathcal{M}'$). It says that at most one team can redistribute the surplus in a transition.[18] Finally, Condition (iv) is the feasibility constraint for players forming singleton teams, including those players whose team is dissolved in the transition.

Let $R_{(s,\mathcal{M}),(s',\mathcal{M}')}$ be a set of coalitions potentially leading from $(s,\mathcal{M})$ to $(s',\mathcal{M}')$. It is given by

$$R_{(s,\mathcal{M}),(s',\mathcal{M}')} = \begin{cases} \{J \in \mathcal{R} :\ J \text{ satisfies (i).}\} & \text{if } \mathcal{M} \neq \mathcal{M}', \text{ and (i),(ii),(iv) are satisfied,} \\ \{J \in \mathcal{R} :\ J \text{ satisfies (iii).}\} & \text{if } \mathcal{M} = \mathcal{M}', \text{ and (ii)–(iv) are satisfied,} \\ \varnothing & \text{if some (i)–(iv) is violated.} \end{cases}$$

The last case denotes infeasible transitions.

For the perturbed process, the probability for transition $((s,\mathcal{M}),(s',\mathcal{M}')) \in \Omega \times \Omega$ is given by

$$P^{\eta}_{(s,\mathcal{M}),(s',\mathcal{M}')} = \sum_{J \in R_{(s,\mathcal{M}),(s',\mathcal{M}')}} q_J\, q_{s'}(J,s,\mathcal{M}) \prod_{i \in J} \Psi_i^{\eta}(s,s'), \tag{5}$$

where q_J denotes the probability that coalition J is chosen to revise, and $q_{s'}(J,s,\mathcal{M})$ the probability s' is chosen given J, s and $\mathcal{M}$.

For the unperturbed one, transition probabilities are obtained in the limit as η approaches zero. That is,

$$P^{0}_{(s,\mathcal{M}),(s',\mathcal{M}')} = \sum_{J \in R_{(s,\mathcal{M}),(s',\mathcal{M}')}} q_J\, q_{s'}(J,s,\mathcal{M}) \prod_{i \in J} \Psi_i^{0}(s,s'), \tag{6}$$

where[19]

$$\Psi_i^0(s,s') = \begin{cases} 0 & u_i(s) > u_i(s') \\ \alpha & u_i(s) = u_i(s') \\ 1 & u_i(s) < u_i(s'). \end{cases}$$

[18] Condition (iii) also applies to the cases that someone rejects the proposal (then, $\mathcal{M} = \mathcal{M}'$ holds). This implies that $((s,\mathcal{M}),(s,\mathcal{M}))$ is a feasible transition.

[19] An unperturbed dynamic with $\alpha = 1/2$ is the limiting dynamic as η approaches zero. Our analysis will not differ for all $\alpha \in (0,1)$ because the set of recurrent classes in the unperturbed dynamic does not differ for $\alpha \in (0,1)$.

3.2 Limiting Stationary Distributions and Stochastic Stability

The Markov chain induced by $P^{\eta}_{\cdot,\cdot}$ is irreducible and aperiodic for $\eta > 0$, and so admits a unique stationary distribution, denoted by π^{η}. The players' behavior is nicely summarized by the stationary distribution in the long-run. Let $\pi^{\eta}(\omega)$ denote the probability that π^{η} places on state $\omega \in \Omega$. Then, $\pi^{\eta}(\omega)$ represents the fraction of time in which state ω is observed over a long time horizon. It is also the probability that ω will be observed at any given time t, provided that t is sufficiently large. We say that state ω is $\mathcal{R}$-stochastically stable if the limiting stationary distribution places positive probability on ω.[20]

Definition 3.1. *State ω is $\mathcal{R}$-*stochastically stable *if* $\lim_{\eta \to 0} \pi^{\eta}(\omega) > 0$.

To characterize stochastically stable states, the unlikeliness of transitions between states plays an important role. We introduce several definitions to compute it. Given a state ω, define an ω-tree, denoted by $T(\omega)$, to be a directed graph with a unique path from any state $\omega' \in \Omega$ to ω. An edge of an ω-tree, denoted by $(\omega', \omega'') \in T(\omega)$, represents a transition from ω' to ω'' in the dynamic.

Let $\omega = (s, \mathcal{M})$ and $\omega' = (s', \mathcal{M}')$. We define the cost of transition, (ω, ω') as follows.

$$c_{\omega,\omega'} = \begin{cases} \min\limits_{J \in R_{\omega,\omega'}} \left[\sum_{i \in J} \max\{u_i(s) - u_i(s'), 0\}\right] & \text{if } R_{\omega,\omega'} \neq \varnothing, \\ \infty & \text{if } R_{\omega,\omega'} = \varnothing. \end{cases} \tag{7}$$

In words, the cost of a transition is the sum of utility losses of players revising in the transition.[21]

The next lemma shows that cost $c_{\omega,\omega'}$ is equal to the exponential rate of decay of the corresponding transition probability, $P^{\eta}_{\omega,\omega'}$.[22]

Lemma 3.2. *If $R_{\omega,\omega'} \neq \varnothing$, then*

$$-\lim_{\eta \to 0} \eta \log P^{\eta}_{\omega,\omega'} = c_{\omega,\omega'}.$$

Proof. See Sawa (2014). □

Lemma 3.2 implies that the amount of utility losses in a transition determins the unlikeliness of the transition. Let $\mathcal{T}(\omega)$ denote the set of ω-trees. The waste of a tree $T \in \mathcal{T}(\omega)$ is defined as

$$W(T) = \sum_{(\omega',\omega'') \in T} c_{\omega',\omega''}. \tag{8}$$

[20] We follow the definition of Sawa (2014). A state is $\mathcal{R}$-stochastically stable if it is stochastically stable on the limiting perturbed process in which deviation by any $J \in \mathcal{R}$ are feasible.

[21] Equation (7) shows a difference from the standard stochastic stability analysis which assumes unilateral deviations. The cost of (ω, ω') evaluates the payoff disadvantages of coalitional deviation s'_J for $J \in R_{\omega,\omega'}$ instead of individual deviations.

[22] See Chapter 12 of Sandholm (2010) for a discussion of defining transition costs this way.

The waste of a tree is the sum of the payoff losses along the tree. The stochastic potential of state ω is defined as

$$W(\omega) = \min_{T \in \mathcal{T}(\omega)} W(T).$$

As η approaches zero, the stationary distribution converges to a unique limiting stationary distribution. The next theorem offers a characterization of $\mathcal{R}$-stochastically stable states. Our main results which are presented in Section 4 will be built on Theorem 3.3.

Theorem 3.3. *A state is $\mathcal{R}$-stochastically stable if and only if it minimizes $W(\omega)$ among all states.*

Proof. See Sawa (2014). □

4 Characterization of Stochastically Stable Allocations

4.1 Core allocations with the grand coalition

We show our main results, the characterization of stochastically stable allocations for each definition of the core. In this section, we address coalitional bargaining problems where there exist core allocations with the grand coalition, which are those defined by Definition 2.1. Then, in Section 4.2, we turn to problems with dispersed-core allocations, which are defined by 2.2.

The next lemma shows that the unperturbed dynamic, i.e. the dynamic with no stochastic shock, will reach some strict core allocation with probability one.[23] This result guarantees that the set of stochastically stable allocations is a subset of the collection of strict core allocations. Then, the stochastic stability approach would allow us to select core allocations that are the most robust against perturbations.

Lemma 4.1. *Starting from $(s, \mathcal{M}) \in \Omega$ with $s \notin \mathscr{C}_\Delta$, the unperturbed dynamic induced by P^0 reaches some $(s^*, \{N\}) \in \Omega$ with $s^* \in \mathscr{C}_\Delta$ with positive probability.*

Let $\mathcal{R}_i = \{J \in \mathcal{R} : i \in J\}$, which is a set of coalitions including agent i. Let $I_\$(s) = \{i : s_i = s_{(1)}\}$ denote the set of the richest agents in $s \in \mathscr{C}_\Delta$. We define a condition:

$$\sum_{i \in J} s_i \geq v(J) + 2\Delta \qquad \forall J \in \mathcal{R}_{i_\$},\ \forall i_\$ \in I_\$(s). \tag{9}$$

We say that allocation s satisfies Condition (9) if Inequality (9) holds for all $J \in \mathcal{R}_{i_\$}$ for all $i_\$ \in I_\$(s)$. Any allocation satisfying Condition (9) is a strict core allocation. Furthermore, even if a player transfers Δ of her surplus to another in an allocation satisfying (9), the resulting allocation satisfies Inequality (2), i.e. it is still a strict core allocation.

For $\omega = (s, \{N\})$ with $s \in \mathscr{C}_\Delta$, let $R(\omega)$ denote the minimum waste for the process to escape from ω to some other $\omega' = (s', \{N\})$ with $s' \in \mathscr{C}_\Delta$.[24] We call the least-cost escape from ω a

[23] Proofs in this section are relegated to the Appendix.

[24] The formal definition of $R(\cdot)$ is provided in the Appendix. See Equation (14).

sequence of transitions from ω to another ω' with $s' \in \mathscr{C}_\Delta$ if the waste of the sequence is $R(\omega)$. Condition (9) is a key piece to identify which allocation the process will most likely reach after its departs from a strict core allocation. It is shown in the following lemma.

Lemma 4.2. *For $s \in \mathscr{C}_\Delta$, s has three properties.*

(i) *The least-cost of escaping from s is given by*

$$R(s, \{N\}) = u(s_{(1)}) - u(s_{(1)} - \Delta). \tag{10}$$

(ii) *If allocation s satisfies Condition (9) and that $s \neq s^E$, then the least-cost escape from s leads the process to $s' \in \mathscr{C}_\Delta$ where s' is either with the richest agent claiming $s_{(1)} - \Delta$ or with one fewer richest agents claiming $s_{(1)}$. If $s = s^E$, then the least-cost escape leads the process to $s' \in \mathscr{C}_\Delta$ where $s'_{(1)} = s_{\min} + \Delta$.*

(iii) *If allocation s violates Condition (9), then the least-cost escape from s leads the process to any $s' \in \mathscr{C}_\Delta$.*

Lemma 4.2 shows that the stability of a core allocation depends on the richest player. Moreover, the concavity of u implies that $u(x) - u(x - \Delta) < u(y) - u(y - \Delta)$ for all $x > y$. Then, the lemma together with the concavity suggests that the stability of a core allocation decreases in the wealth of the richest player. Our main result in this section is the following theorem. The stochastically stable allocations are core allocations which minimize the wealth of the richest player.

Theorem 4.3. *State $(s, \mathcal{M})$ is $\mathcal{R}$-stochastically stable if and only if $(s, \mathcal{M}) \in {}_{\min}\Omega^*_\Delta$.*[25]

We provide a sketch of the proof here. For each $h \in \{0, 1, \ldots\}$, we classify the strict core allocations into two sets: those with $s_{(1)} \geq s_{\max} - h\Delta$ and those with $s_{(1)} < s_{\max} - h\Delta$, where $s_{\max} = \max_{s' \in \mathscr{C}_\Delta} s'_{(1)}$.[26] Starting with $h = 0$, we exclude the set of allocations with $s_{(1)} = s_{\max}$ from stochastically stable allocations. This is done by showing that the waste for the process to move away from the set to the other set is smaller than the waste for the other way around. By the inductive step, we show that the same argument applies for all $h = \{1, 2, \ldots\}$ such that $s_{\min} < s_{\max} - h\Delta$.

We offer a corollary and an example of the theorem. The following corollary is immediate as for the egalitarian allocation s^E. The example is an application to a cost sharing problem.

Corollary 4.4. *If $s^E \in \mathscr{C}_\Delta$, then $(s^E, \{N\})$ is uniquely $\mathcal{R}$-stochastically stable.*

Example 2 (Cost sharing problem)**.** A public utility (water system) serves four consumers, and the cost structure is symmetrically given:[27]

[25] That $(s, \mathcal{M}) \in {}_{\min}\Omega^*_\Delta$ implies that $\mathcal{M} = \{N\}$.
[26] $s_{\max}$ is the highest claim of the richest player among all strict core allocations.
[27] This example is based on Example 4.2 of Moulin (1988)

cost of serving:	one consumer, alone	40
	two consumers	60
	three consumers	70
	all four consumers	80

Monetary benefits to the consumers from using the facility are

$$b_1 = 41, \qquad b_2 = 24, \qquad b_3 = 22, \qquad b_4 = 13.$$

A consumer i will agree on buying the facility if she is charged no more than b_i. Given the setting above, the characteristic function is given by $v(J) = \max\{\sum_{i \in J} b_i - c(J),\ 0\}$ where $c(J)$ is corresponding to the cost structure.

$$\begin{aligned}
&v(1) = 1, && v(2) = v(3) = v(4) = 0, \\
&v(12) = 5, && v(13) = 3, \quad v(14) = v(23) = v(24) = v(34) = 0, \\
&v(123) = 17, && v(124) = 8, \quad v(134) = 6, \quad v(234) = 0, \\
&v(1234) = 20.
\end{aligned}$$

Let $\Delta = 1$. The strict core is a relatively large subset of the set of allocations, and thus only provides very loose guidelines for allocations. $(17, 1, 1, 1)$ is a strict core allocation, and so is $(2, 8, 8, 2)$.

But this problem has a unique stochastically stable allocation, which is $(6, 6, 6, 2)$. The corresponding cost share is $(35, 18, 16, 11)$.

4.2 Core allocations with non-grand coalitions

So far, we restrict our attention to settings in which there exist strict core allocations where the grand coalition forms. In this section, we relax the assumption of strict core allocation with the grand coalition while keeping assuming that the set of strict core allocations is not empty. That is, we switch the solution concept from Definition 2.1 to 2.2 (dispersed-core).[28] Theorem 4.7 will characterize the set of stochastically stable allocations.

Recall that $\Omega^*_\Delta = \{(s, \mathcal{M}) \in \Omega_\mathcal{M} \times \text{part}(N) : s \in \mathscr{C}_\Delta \text{ and } \sum_{i \in J} s_i = v(J)\ ,\ \forall J \in \mathcal{M}\}$. In words, Ω^*_Δ in the settings here is the set of states in which the optimal set of teams form and they distribute surplus according to some strict core allocation. Also let $V_{\max} = \max_{\mathcal{M} \in \text{part}(N)} \sum_{J \in \mathcal{M}} v(J)$, and $\mathcal{M}_{\max} = \{\mathcal{M} \in \text{part}(N) : \sum_{J \in \mathcal{M}} v(J) = V_{\max}\}$.

The next lemma identifies properties of strict dispersed-core allocations. It shows that if there is some strict core allocation with $\mathcal{M} \neq \{N\}$, then such $\mathcal{M}$ is unique and maximizes the sum of surpluses generated by teams.

[28] A core allocation is always a dispersed-core allocation. Thus, using Definition 2.2 relaxes the assumption of the grand coalition.

Lemma 4.5. *Suppose that $\mathscr{C}_\Delta$ is non-empty. Then, we have that*

$$\text{(i)}\quad \sum_{J\in\mathcal{M}^*} v(J) = V_{\max} \quad \forall (s,\mathcal{M}^*)\in\Omega^*_\Delta,$$
$$\text{(ii)}\quad |\mathcal{M}_{\max}| = 1.$$

Proof. Claim (i):
The proof is the way of contradiction. Suppose that $\omega = (s,\mathcal{M}) \in \Omega^*_\Delta$ such that $\sum_{J\in\mathcal{M}} v(J) < V_{\max}$. Let $\mathcal{M}^*$ be such that $\sum_{J^*\in\mathcal{M}^*} v(J^*) = V_{\max}$. Observe that

$$\sum_{J^*\in\mathcal{M}^*} v(J^*) > \sum_{J\in\mathcal{M}}\sum_{i\in J} s_i = \sum_{J^*\in\mathcal{M}^*}\sum_{i\in J^*} s_i.$$

Then, there exists some $J^*\in\mathcal{M}^*$ such that $v(J^*) > \sum_{i\in J^*} s_i$. This contradicts that s is a strict core allocation.

Claim (ii):
The proof is the way of contradiction. Suppose that $\mathcal{M}_{\max} = \{\mathcal{M},\mathcal{M}',\dots\}$. Without loss of generality, let $(s',\mathcal{M}')\in\Omega^*_\Delta$.[29] Choose a profile of claims s which is such that $\sum_{i\in J} s_i = v(J)$ for all $J\in\mathcal{M}$. Observe that

$$\begin{aligned}
&\sum_{J\in\mathcal{M}}\sum_{i\in J} s_i = \sum_{J'\in\mathcal{M}'}\sum_{i\in J'} s'_i \\
\Leftrightarrow &\sum_{J\in\mathcal{M}\cap\mathcal{M}'}\sum_{i\in J} s_i + \sum_{J\in\mathcal{M}\setminus\mathcal{M}'}\sum_{i\in J} s_i = \sum_{J\in\mathcal{M}'\cap\mathcal{M}}\sum_{i\in J} s'_i + \sum_{J'\in\mathcal{M}'\setminus\mathcal{M}}\sum_{i\in J'} s'_i \\
\Leftrightarrow &\sum_{J\in\mathcal{M}\setminus\mathcal{M}'}\sum_{i\in J} s_i = \sum_{J'\in\mathcal{M}'\setminus\mathcal{M}}\sum_{i\in J'} s'_i.
\end{aligned} \tag{11}$$

There must exist $J\in\mathcal{M}\setminus\mathcal{M}'$ such that $v(J)\geq\sum_{i\in J} s'_i$.[30] Then, J weakly blocks s', which contradicts that s' is a strict core allocation. □

The next lemma guarantees that the unperturbed dynamic will converge to the set of strict dispersed-core allocation, Ω^*_Δ. Its proof is more involved than that of Lemma 4.1. For settings in Section 4.1, there is unique optimal team, the grand coalition. And it can weakly block any allocation with non-grand coalitions. For settings here, there exists the optimal set of teams, that is $\mathcal{M}^*$ for $(s^*,\mathcal{M}^*)\in\Omega^*_\Delta$. Those teams cannot form at once because at most one team is formed in each period. Instead it needs to be shown that each team of $\mathcal{M}^*$ will be formed sequentially and their surplus distribution will eventually become some s^* for $(s^*,\mathcal{M}^*)\in\Omega^*_\Delta$.

Lemma 4.6. *Starting from $(s,\mathcal{M})\in\Omega$ with $s\notin\mathscr{C}_\Delta$, the unperturbed dynamic induced by P^0 reaches some $(s^*,\mathcal{M}^*)\in\Omega^*_\Delta$ with positive probability.*

[29] Since $\mathscr{C}_\Delta$ is non-empty, some state with some $\mathcal{M}'\in\mathcal{M}_{\max}$ exists in Ω^*_Δ.
[30] Otherwise, $v(J) < \sum_{i\in J} s'_i$ for all $J\in\mathcal{M}\setminus\mathcal{M}'$, which contradicts the last equality in Equations (11).

Proof. We group non-equilibrium states, or $\Omega \setminus \Omega^*_\Delta$, into four cases. For Case I, we show that the process will reach some strict core allocation. For Cases II and III, we show that the process eventually falls into Case I. For Case IV, the process will reach some state of other cases.

Case I:

Suppose that the current state is $(s, \mathcal{M}^*)$ which satisfies the following conditions:[31]

$$\begin{array}{ll} \text{(i)}\ \ s \notin \mathscr{C}_\Delta, & \text{(ii)}\ \ \sum_{i\in J} s_i = v(J)\ \ \forall J \in \mathcal{M}^*, \\ \text{(iii)}\ \ \mathcal{M}^* \in \mathcal{M}_{\max}, & \text{(iv)}\ \ s_i \geq v(\{i\})\ \ \forall \{i\} \notin \mathcal{M}^*. \end{array} \tag{12}$$

Let $(s^*, \mathcal{M}^*) \in \Omega^*_\Delta$. Since $s \notin \mathscr{C}_\Delta$, there exists J such that $v(J) \geq \sum_{i\in J} s_i$. Let J form a coalition with s' which is such that $s'_i \geq s_i \geq v(\{i\})$ for all $i \in J$. Let $\{M_1, \ldots, M_k\} \subseteq \mathcal{M}^*$ be a set of dissolved teams due to J being formed. Then, there must exist some $M_j \in \{M_1, \ldots, M_k\}$ such that $\sum_{i\in M_j\cap J} s^*_i > \sum_{i\in M_j \cap J} s'_i$.[32] Note that $M_j \setminus J \neq \varnothing$.[33] Let M_j form a coalition with s'' which is such that $s''_i = v(\{i\})$ for $i \in M_j \setminus J$, and $s''_i = s'_i$ for $i \in M_j \cap J$.[34] Then, let $i \in M_j \setminus J$ form a singleton team.[35] The resulting state is such that all players of $\{M_1, \ldots, M_k\}$ form singleton teams. Let M_x form a coalition with $s^*_{M_x}$ for all $M_x \in \{M_1, \ldots, M_k\}$ in the subsequent periods.[36] The resulting state must be either some $(s^*, \mathcal{M}^*) \in \Omega^*_\Delta$ or $(s, \mathcal{M}^*)$ which satisfies the condition (12) above. Note that every dissolved team in the operations above forms again and that there are more teams whose claims are consistent to s^* after the operations. Repeat Case I until the process reaches some $(s^*, \mathcal{M}^*) \in \Omega^*_\Delta$.

Case II:

Suppose that the current state is $(s, \mathcal{M}^*)$ which satisfies (i),(iii), and (iv) and violates (ii) of Equation (12), i.e., there exists $J^* \in \mathcal{M}^*$ such that $\sum_{i\in J^*} s_i < v(J^*)$. Then, let such J^* form a coalition with s' which is such that $s'_i \geq s_i$ for all $i \in J^*$ and $\sum_{i\in J^*} s_i = v(J^*)$, i.e., J^* redistributes its surplus an efficient way.

Repeat the process above until there is no team J such that $\sum_{i\in J} s_i < v(J)$. The resulting state will either be some $(s^*, \mathcal{M}^*) \in \Omega^*_\Delta$ or be $(s, \mathcal{M}^*)$ which satisfies the conditions given by (12) of Case I.

Case III:

Suppose that the current state is $(s, \mathcal{M}) \in \Omega$ which satisfies (iv) and violates (iii) of Equation (12),

[31] The condition (iv) says that a player i earns at least her reservation surplus $v(\{i\})$.

[32] If such M_j does not exist, then J weakly blocks s^*, which contradicts that s^* is a strict core allocation.

[33] If $M_j \setminus J = \varnothing$, then $M_j \subseteq J$, which implies that $(M_j \cap J) \in \mathcal{M}^*$. Due to the condition (ii), we have that $\sum_{i\in M_j\cap J} s_i = v(M_j \cap J) = \sum_{i\in M_j\cap J} s^*_i > \sum_{i\in M_j\cap J} s'_i$, which implies that some member of $(M_j \cap J)$ will be strictly worse off by deviating from s. It contradicts that J weakly blocks s.

[34] Every player i of $M_j \setminus J$ forms a singleton team and earns their reservation surplus, $v(\{i\})$. They accept s'' with positive probability. Also note that s'' is feasible since M_j is chosen such that $\sum_{i\in M_j\cap J} s^*_i > \sum_{i\in M_j\cap J} s'_i$.

[35] Since $s''_i = v(\{i\})$ for $i \in M_j \setminus J$, player i weakly blocks s''.

[36] For example, let M_1 form a coalition in the next period, let M_2 do that in the period after next, and so forth.

i.e., $\sum_{J\in\mathcal{M}} v(J) < V_{\max}$. Let $(s^*,\mathcal{M}^*) \in \Omega^*_\Delta$. Observe that

$$\sum_{J^*\in\mathcal{M}^*} v(J^*) > \sum_{J\in\mathcal{M}}\sum_{i\in J} s_i = \sum_{J^*\in\mathcal{M}^*}\sum_{i\in J^*} s_i.$$

Then, there exists J^* such that $v(J^*) > \sum_{i\in J^*} s_i$. Choose some s'_{J^*} such that $\sum_{i\in J^*} s'_i = v(J^*)$ and $s'_i \geq s_i$ for all $i \in J^*$.[37] Let J^* form a coalition with s'_{J^*}. Let $\mathcal{M}'$ denote the resulting set of teams. Observe that if $\mathcal{M}' \neq \mathcal{M}^*$, then the resulting state must fall into Case III. For any member i of J^*, she will earn $s'_i \geq s_i$. For any member i of dissolved teams, she will earn $v(\{i\})$.

Repeat the operation above until the state becomes some $(s,\mathcal{M}^*)$. The resulting state will either be some $(s^*,\mathcal{M}^*) \in \Omega^*_\Delta$ or be $(s,\mathcal{M}^*)$ which satisfies the conditions given by (12) of Case I.

Case IV:
Suppose that the current state is $(s,\mathcal{M})$ which violates (iv), i.e,$s_i < v(\{i\})$ for some $i \in N$. For each i such that $s_i < v(\{i\})$, let i form a singleton team and earn $v(\{i\})$. The resulting state will fall either in Case I or in Case III. □

Recall that

$${}_{\min}\Omega^*_\Delta = \left\{(s,\mathcal{M}) \in \Omega^*_\Delta : s_{(1)} = s_{\min}\right\}.$$

Theorem 4.7. $\lim_{\eta\to 0} \pi^\eta({}_{\min}\Omega^*_\Delta) = 1$.[38]

Proof. We prove that the first two claims of Lemma 4.2 still hold for $(s,\mathcal{M}) \in \Omega^*_\Delta \setminus {}_{\min}\Omega^*_\Delta$. Then, with observing that $R(s,\{\mathcal{M}\}) = u(s_{\min}) - u(s_{\min}-\Delta)$ for $(s,\mathcal{M}) \in {}_{\min}\Omega^*_\Delta$, the subsequent proof is same as the 'only if' part of the proof of Theorem 4.3. Thus, we omit the subsequent proof.

Suppose that the current state is $(s,\mathcal{M}^*) \in \Omega^*_\Delta \setminus {}_{\min}\Omega^*_\Delta$. Recall that $I_\$(s) \in \{i : s_i = s_{(1)}\}$. Let $J_\$ \in \mathcal{M}^*$ denote a team which includes $i_\$ \in I_\$(s)$. Choose $i_\$ \in I_\$(s)$ such that there exists $h \in J_\$$ with that $s_h \leq s_{i_\$} - 2\Delta$.[39] Let s' be such that $s'_{i_\$} = s_{(1)} - \Delta$, $s'_h = s_h + \Delta$ for $h \in J_\$$, and $s'_i = s_i$ otherwise. Observe that s' satisfies Inequalities (3), and that the escaping-cost from $(s,\mathcal{M}^*)$ to $(s',\mathcal{M}^*)$ is given similarly to Equation (10). That is, we show that $R(s,\{\mathcal{M}^*\}) = u(s_{(1)}) - u(s_{(1)} - \Delta)$. The resulting allocation s' is either with the richest agent claiming $s_{(1)} - \Delta$ or with one fewer richest agents claiming $s_{(1)}$. This proves the first two claims of Lemma 4.2 for $(s,\mathcal{M}) \in \Omega^*_\Delta \setminus {}_{\min}\Omega^*_\Delta$. □

Example 3. Suppose that $N = \{1,2,3,4\}$ and $\Delta = 1$. The characteristic function is given as below.

$$\begin{aligned} &v(\{i\}) = 2 \ \ \forall i \in N, && v(\{1,2\}) = 20, && v(\{3,4\}) = 8,\\ &v(\{i,j\}) = 10 \ \text{ for all other } \{i,j\} \subset N, && v(\{i,j,h\}) = 15 \ \ \forall \{i,j,h\} \subset N, && v(\{N\}) = 25. \end{aligned}$$

[37] s'_i is not necessarily equal to s^*_i, i.e., s'_i may or may not be a part of a strict core allocation.

[38] Let $\pi^\eta(X)$ denote the sum of probability of states of X, i.e., $\sum_{\omega\in X} \pi^\eta(\omega)$.

[39] Such h exists. Otherwise, any transfer will result in at least one agent having surplus which amounts to at least $s_{(1)}$. Then, $s_{(1)} = s_{\min}$ and $(s,\mathcal{M}^*) \in {}_{\min}\Omega^*_\Delta$ must hold.

The strict dispersed-core allocation is such that players 1 and 2, and players 3 and 4 form teams respectively and they distribute surplus according to one of the allocations below.

$$\begin{aligned}(s_1, s_2, s_3, s_4) \in \; &\{(8,12,4,4),\;\; (9,11,4,4),\;\; (10,10,4,4),\;\; (11,9,4,4),\;\; (12,8,4,4),\;\; (9,11,3,5),\\ &(10,10,3,5),\;\; (11,9,3,5),\;\; (9,11,5,3),\;\; (10,10,5,3),\;\; (11,9,5,3)\}\end{aligned}$$

Observe that ${}_{\min}\Omega^*_\Delta = \{(10,10,4,4),(10,10,3,5),(10,10,5,3)\}$. And our theorem tells us that $\lim_{\eta\to 0} \pi^\eta({}_{\min}\Omega^*_\Delta) = 1$.

5 Discussions

5.1 Related literature

We compare our results with closely related studies here. Recall that the dynamic chooses each $J \in \mathcal{R}$ with positive probability in each period. In this sense, our model is closer to the 'random proposer' model of Okada (1996) and Compte and Jehiel (2010) rather than the 'rejector proposes' model of Chatterjee et al. (1993). As for the literature on stochastic stability, our study is closely related to Agastya (1999) and Newton (2012) both of which study stochastic stability in coalitional bargaining problems.

Model	Agents	Resulting allocation (among those in the core)
Okada (1996)	rational	Maximizing per capita (i.e. $\max_{J\subset N} v(J)/\lvert J\rvert$)
Compte and Jehiel (2010)	rational	Maximizing product of payoffs
Agastya (1999)	myopic	Minimizing payoff for the richest
Newton (2012)	myopic	Maximizing payoff for the poorest
This paper	myopic	Minimizing payoff for the richest

Table 1: Coalitional bargaining models and their resulting allocations

Table 1 summarizes the related studies on coalitional bargaining problems with rational agents or myopic ones. We offer a few general observations among them, and then we compare our model with the other studies which assume myopic agents. An intriguing finding is that despite various differences across the models all papers favor the egalitarian outcome if it is in the core (or in the strict core).[40] Even if the egalitarian outcome is not in the core, equity considerations play a significant role in all papers. In the present paper, the richest agents tend to transfer to others as much money as possible subject to Constraint (2). Similarly to it, other studies find that resulting outcomes tend to improve equity in surplus allocation. For example, surplus is equally distributed among coalition members even if it is not a grand coalition in Okada (1996).

[40] The model of Okada (1996) requires one more condition for the egalitarian outcome:

$$\frac{v(S)}{|S|} \le \frac{v(T)}{|T|} \quad \text{for all } S, T \subseteq N \text{ with } S \subseteq T.$$

An allocation maximizing the product of agents' payoffs, as in Compte and Jehiel (2010), can be also viewed as having equity considerations.

We note a few differences from the studies with rational agents. In the models with rational agents, their discount factors are assumed to be close to one. While agents are myopic in our model, that is, their discount factors are zero. Also note that the models with rational agents requires that each player has to have an equal chance of being a proposer. Then, agents have an equal bargaining power in their models. With myopic agents, the probability of being chosen as a proposed coalition does not play an important role. Our results will hold as long as the dynamic chooses each $J \in \mathcal{R}$ with positive probability. As we written, the similarity is of interest despite those major differences.

As for the studies on stochastic stability, it is interesting to see that both Agastya (1999)'s centralized approach and our decentralized approach lead to similar results. Both approaches have a common observation; agents who do better in a coalition are less reluctant to reduce their demands. Then, a transition in which the richest agent reduces her demand is the easiest way to leave a strict core allocation, as we see in Lemma 4.2. This determines the stochastically stable allocations. The difference from Newton (2012) comes from a tailored setting of Newton (2012) that agents evaluate a correlated strategy and switch to a pure strategy profile in its support. We roughly sketch the setting here. Suppose that $N = \{1, 2, 3\}$ and that strict core allocations are $s = \{1, 6, 6\}$, $s' = \{2, 5, 6\}$ and $s'' = \{2, 6, 5\}$. In our model, all are stochastically stable, since the wealth of the richest agent is 6 in all the allocations. Now, suppose that agents form a grand coalition with allocation s. If we allow correlated strategies, then agents may evaluate a correlated strategy in which they play each of s' and s'' with probability $1/2$. In this way, agents 2 and 3 can (probabilistically) share the cost of accommodating the poorest with one unit of money. Newton (2012) showed that, in his setting, the cost of the transition above is lower than the cost of transitions in pure strategies, e.g. evaluating switching from s to s'. It will be interesting to study our model with correlated strategies in future research.[41]

There are two major differences between our paper and others assuming myopic agents. The first difference is that others assume a central authority which collect agents' claims and chooses coalition(s) to be formed.[42] Ours does not assume such an authority, but assumes instead that agents randomly meet and decide whether to form a coalition by themselves. Settings where some institution gathers claims and determines the surplus allocation would be better modeled by a centralized model, while settings where decisions are made in a more decentralized fashion would be better modeled by ours.

[41] We conjecture that our result will not change so long as logit choice is assumed. The cost of transition from s to the correlated strategy is given by the sum of differences in agents 2 and 3's expected payoffs:

$$u(6) - \frac{1}{2}(u(6) + u(5)) + u(6) - \frac{1}{2}(u(6) + u(5)) = u(6) - u(5).$$

Observe that the cost above is indifferent from the cost of both transitions (s, s') and (s, s''). However, the result would differ with the probit choice and we may obtain a similar result to Newton (2012).

[42] In Newton (2012), each agent submits to the authority her claim and acceptable players to form a coalition together.

The second difference is about restrictions on characteristic function v. Agastya (1999) assumes strict convexity (see Remark 2.5), and Newton (2012) assumes super-additivity:

$$v(J \cup J') \geq v(J) + v(J') \quad \text{if } J \cap J' = \varnothing.$$

In contrast, our model does not require any restriction on v except the existence of strict core allocations, which other two papers assume as well.

5.2 Heterogeneity in players' utility functions

We discuss relaxing the assumption of the common utility function and examine stochastic stability with heterogeneity in the players' utility functions. The analysis will not significantly differ from those offered in Sections 3 and 4. Thus, we omit proofs of our results in this section.

Let player i's utility function be $u_i : \mathbb{R} \to \mathbb{R}$ which is concave, strictly increasing. Those functions may vary across players, i.e., it might be that $u_i(x) \neq u_j(x)$ for $x \in \mathbb{R}$ and $i \neq j$. All other aspects of the setup are the same as in Section 2.

Similarly to Section 4.1, we characterize stochastically stable allocations with our attention restricted to problems with strict core allocations. As for the general properties, the discussion provided in Section 3 still applies to the settings here.[43] As for the characterization, Lemma 4.1 still applies. Lemma 4.2 will apply to this setting by replacing Equation (10) with (13). Formally we have the lemma below.

Lemma 5.1. *For $s \in \mathscr{C}_\Delta$, s has three properties.*

(i) *The least-cost of escaping from s is given by*

$$R(s, \{N\}) = \min_{i \in N} u(s_i) - u(s_i - \Delta). \tag{13}$$

(ii) *If allocation s satisfies Condition (9) and that $s \notin \operatorname{argmin}_{s \in \mathscr{C}_\Delta} \min_{i \in N} u_i(s_i) - u_i(s_i - \Delta)$, then the least-cost escape from s leads the process to $s' \in \mathscr{C}_\Delta$ where s' has either properties below:*

$$\begin{aligned} &\min_{i \in N} u(s'_i) - u(s'_i - \Delta) < \min_{i \in N} u(s_i) - u(s_i - \Delta), \\ \textit{or} \qquad &|\operatorname*{argmin}_{i \in N} u(s'_i) - u(s'_i - \Delta)| = |\operatorname*{argmin}_{i \in N} u(s_i) - u(s_i - \Delta)| - 1, \end{aligned}$$

with that $\min_{i \in N} u(s'_i) - u(s'_i - \Delta) = \min_{i \in N} u(s_i) - u(s_i - \Delta)$.

(iii) *If allocation s violates Condition (9), then the least-cost escape from s leads the process to any $s' \in \mathscr{C}_\Delta$.*

Recall that $\Omega^*_\Delta = \{(s, \{N\}) : s \in \mathscr{C}_\Delta\}$. Define

$$_{\min}\Omega^{**}_\Delta = \left\{ (s, \{N\}) \in \Omega^*_\Delta : s \in \operatorname*{argmin}_{s \in \mathscr{C}_\Delta} \min_{i \in N} u_i(s_i) - u_i(s_i - \Delta) \right\}.$$

[43] Note that Equation (7) can be applied since it employs $u_i(\cdot)$ instead of $u(\cdot)$.

Then, we have the following result corresponding to Theorem 4.3.

Theorem 5.2. $\lim_{\eta \to 0} \pi^{\eta}({}_{\min}\Omega^{**}_{\Delta}) = 1$.

Example 4 (Heterogeneity in the degree of sensitivity)**.** Suppose that $N = \{1, 2, 3\}$, $\Delta = 1$ and that the characteristic function is given as below.[44]

$$v(\{i\}) = 0 \ \ \forall i \in N, \qquad v(\{i, j\}) = 2 \ \ \forall \{i, j\} \subset N, \qquad v(\{N\}) = 12.$$

The strict core is given by the set of the allocations (s_1, s_2, s_3) which satisfy that $\sum_i s_i = 12$, $s_i > 0$ for all $i \in N$, $s_i + s_j > 2$ for all $\{i, j\} \subset N$.

Also suppose that different players possess different degrees of sensitivity to monetary payoffs. That is, player i's utility function is given by $u_i(x) = u(\beta_i x)$ where $\beta_i > 0$ varies among the players while $u(\cdot)$ is common and given by $u(y) = \sqrt{y}$. Note that $u_i(\cdot)$ is concave, strictly increasing. Let $\beta_1 = \beta_2 = 1$ and $\beta_3 = 2$.

Observe that ${}_{\min}\Omega^{**}_{\Delta} = \{((3, 3, 6), \{N\})\}$. This is due to that

$$\min\{\sqrt{3} - \sqrt{2}, \sqrt{2 \cdot 6} - \sqrt{2 \cdot 5}\} > \max\{\max_{y \geq 4} \sqrt{y} - \sqrt{y - 1}, \max_{x \geq 7} \sqrt{2x} - \sqrt{2(x - 1)}\}.$$

The LHS gives the escaping cost from (3,3,6) while the RHS give the upper bound of that from other strict core allocations. Thus the division (3,3,6) is uniquely stochastically stable. Those who are more sensitive to monetary payoffs, e.g. player 3 with $\beta_3 = 2$, will obtain a larger share than other players.

5.3 Shapley value and the least core

In this section, we offer a discussion related to existing solution concepts, the Shapley value allocation and the least core. We restrict our attention to settings with the strict core and compare allocations characterized by Theorem 4.3 with other concepts. First, we define the two concepts and provide an example in which the stochastically stable allocation differs from those. Then, we discuss an extension of our model, in which the set of stochastically stable ones coincides with the least core.

The *Shapley value* specifies the surplus for agent i,

$$s_i = \sum_{J \in \mathcal{R}: i \in J} \frac{(|J| - 1)!(n - |J|)!}{n!} \left\{ v(J) - v(J \setminus \{i\}) \right\}.$$

The Shapley value is an expected surplus, where expectations are taken over all possible subgroups J which agent i might join.

For the least core, we start with a couple of definitions. Given that the current distribution is s, the minimal excess of J is $e(J, s) = v(J) - \sum_{i \in J} s_i$. Let $e(s) = \max_{J \in \mathcal{R} \setminus N} e(J, s)$. The ε-core $\mathscr{C}^{\varepsilon}_{\Delta}$ is

[44] Under the assumption of the common utility function, the division $(4, 4, 4)$ is stochastically stable.

defined as

$$\mathscr{C}_\Delta^\varepsilon = \{s : e(s) \le \varepsilon \text{ and } \sum_{i \in N} s_i = v(N) \text{ for some } (s, \mathcal{M}) \in \Omega\}.$$

Note that $\mathscr{C}_\Delta^\varepsilon \subseteq \mathscr{C}_\Delta$ if $\varepsilon < 0$. Define *the least core* LC_Δ is the intersection of all non-empty $\mathscr{C}_\Delta^\varepsilon$. It is straightforward to show that LC_Δ is $\mathscr{C}_\Delta^{\varepsilon^*}$ where $\varepsilon^* = \min_{s \in \mathscr{C}_\Delta} e(s)$.

Example 5. Let $N = \{1,2,3\}$. Characteristic function $v(\cdot)$ is given by $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$ and

$$v(\{1,2\}) = v(\{1,3\}) = 14, \qquad v(\{2,3\}) = 8, \qquad v(\{1,2,3\}) = 24.$$

Let $\Delta = 1$. The Shapley value allocation is $(10,7,7)$, and the least core is $(12,6,6)$. Although both solutions are in the strict core, they are not stochastically stable. Since $(8,8,8) \in \mathscr{C}_\Delta$, the egalitarian allocation $((8,8,8), \{N\})$ is uniquely stochastically stable.

We provide an extension of our model, where the least core allocation is uniquely stochastically stable. We impose the assumption below.

Assumption 5.3. (i) *$u_i(s_i) = s_i$ for all $i \in N$.*

(ii) *Given that the process is in $(s, \mathcal{M})$, the dynamic chooses $J \in \mathcal{R} \setminus \mathcal{M}$ to revise their actions.*

The assumption implies two differences from our original model.[45] First, utility functions are assumed to be linear in the extension. Second, any existing team may not negotiate the surplus redistribution. Players need to form a new team if they would like to revise their share. This modification is implemented by choosing some J, which is not currently formed, to revise their action in the dynamic. The description of the dynamic is given below.

The dynamic interaction proceeds as follows. Let s^t denote a profile of players' claims and $\mathcal{M}^t$ the set of the existing teams in period t. At the beginning of period t, $J \in \mathcal{R}$ such that $J \notin \mathcal{M}^t$ is randomly chosen, and then a payment proposal $s = \{s_i\}_{i \in N}$ to share the surplus $v(J)$ is randomly chosen. Proposal s given J satisfies the feasibility constraint (1). A player accepts proposal s with probability $\Psi^\eta(s^t, s)$. If all players in J accept, then they form a team, each team member $i \in J$ gets payoffs $u(s_i)$, and any existing team having some $i \in J$ will be dissolved. If at least one player in J rejects s, the state will remain $(s^t, \mathcal{M}^t)$.

Theorem 5.4. *Under Assumption 5.3, state $(s, \mathcal{M})$ is stochastically stable if and only if $s \in LC_\Delta$.*

Proof. To prove the 'only if' part, we compute $CR^*(LC_\Delta)$ followed by $R(LC_\Delta)$, and then apply the Radius-Coradius theorem. Let $s_{LC} \in LC_\Delta$ and $\omega_{LC} = (s_{LC}, \{N\})$. First, suppose that $\omega = (s, \mathcal{M}) \in \Omega$ in period t such that $\mathcal{M} \neq \{N\}$. There is positive probability that the grand coalition is formed and they will deviate to ω_{LC}. This implies that the cost of moving from ω to ω_{LC} is zero, i.e. $c_{\omega,\omega_{LC}} = 0$.

[45] There are more subtle differences which do not affect the results. For example, q_J, the probability of J being chosen must depend on $\mathcal{M}^t$ in the extension.

Next, suppose $\omega = (s, \{N\}) \in \Omega$ such that $s \notin LC_\Delta$.[46] Let $J(s) \in \mathrm{argmax}_{J \in \mathcal{R}\setminus N} e(J, s)$. The definition of the dynamic implies that the minimal cost of deviating from state $(s, \{N\})$ is given by $-e(s)$. Suppose that the state is ω in period t. There is positive probability that $J(s)$ is chosen and they will deviate to some state $\omega' = (s', \{J(s)\})$ such that $c_{\omega,\omega'} = -e(s)$ in period t, which leads to that $\omega^{t+1} = \omega'$. In $t+1$, there is positive probability that the grand coalition is formed and they will deviate to ω_{LC}. Note that $c_{\omega',\omega_{LC}} = 0$ since $s' \notin \mathscr{C}_\Delta$.

The discussion above implies that $CR^*(LC_\Delta) \leq \max_{s \notin LC_\Delta} -e(s)$. The definition of the dynamic suggests that $R(LC_\Delta) = -\varepsilon^* = \max_{s \in \mathscr{C}_\Delta} -e(s)$. From the definition of LC_Δ, it is immediate that $R(LC_\Delta) > CR^*(LC_\Delta)$.

For the 'if' part, we prove it by a way of contradiction. Suppose that $s \in LC_\Delta$ and that $\omega = (s, \{N\})$ is not stochastically stable. Let $\omega' = (s', \{N\})$ be stochastically stable. Consider ω'-tree $T^*(\omega')$ which minimizes the stochastic potential. The edge emanating from ω in $T^*(\omega')$ must cost at least $-\varepsilon^*$. Construct a new tree $T(\omega)$ by removing the edge from ω in $T^*(\omega')$. This operation reduces the waste of the tree by $-\varepsilon^*$. Then, add edges of a path $d(\omega', \omega)$ to $T^*(s')$. The discussion in the 'only if' part suggests that there exists some $d(\omega', \omega)$ such that the waste of the tree increases by at most $-e(s')$. The resulting ω-tree must have the waste weakly smaller than that of $T^*(\omega')$ This contradicts that ω' is stochastically stable, but ω is not. $\square$

A Appendix

We briefly state a version of the modified Radius-Coradius theorem of Ellison (2000).[47] Theorem A.1 below is a key tool to prove our results. We start with a couple of definitions.

A directed graph $d(\omega_1, \omega_k)$ on Ω is a *path* if $d(\omega_1, \omega_k)$ is a finite, repetition-free sequence of transitions $\{(\omega_1, \omega_2), (\omega_2, \omega_3), \ldots, (\omega_{k-1}, \omega_k)\}$ such that $\omega_i \in \Omega$ for all $i = 1, \ldots, k$. A path $d(\omega_1, \omega_k)$ is *feasible* if $R_{\omega_i \omega_{i+1}}$ is not empty for all $i = 1, \ldots, k-1$. Let $\mathcal{D}(\omega, \omega')$ be the set of all paths with initial point ω and terminal point ω'. Let the waste $W(d(\omega, \omega'))$ be the sum of transition costs of $d(\omega, \omega')$, i.e.

$$W(d(\omega, \omega')) = \sum_{(\omega_i, \omega_{i+1}) \in d(\omega, \omega')} c_{\omega_i, \omega_{i+1}}.$$

The *basin of attraction* of state ω, $B(\omega) \subseteq \Omega$, is the set of all states ω' such that there exists a revision path $d \in \mathcal{D}(\omega', \omega)$ with $W(d) = 0$. Let U denote a recurrent class and Θ_1 a set of recurrent classes, i.e. $U \in \Theta$ and $\Theta_1 \subset \Theta$.[48] Let $B(U) = \cup_{\omega \in U} B(\omega)$ and $B(\Theta_1) = \cup_{U \in \Theta_1} B(U)$. $B(U)$ denotes the set of all states ω' which have a path $d \in \mathcal{D}(\omega', \omega)$ with $W(d) = 0$ for some $\omega \in U$. We define the *radius* of Θ_1 as

$$R(\Theta_1) = \min_{U \in \Theta_1} \min_{\omega \in U} \min_{\omega' \notin B(\Theta_1)} \left\{ W(d) \,\middle|\, d \in \mathcal{D}(\omega, \omega') \right\}, \tag{14}$$

[46] This case considers both $s \notin \mathscr{C}_\Delta$ and $s \in \mathscr{C}_\Delta \setminus LC_\Delta$.

[47] We refer readers to Sawa (2014) for the theorem with coalitional behavior.

[48] Note that U is a set of states and Θ_1 is a set of sets of states.

$R(\Theta_1)$ is the minimum waste for the process to move away from the basin attraction of Θ_1.

Next, we define the modified coradius. Suppose that $\omega_1 \in U_1$ and $\omega_k \in U_k$ for some $U_1, U_k \in \Theta$. We define

$$\Theta(d(\omega_1, \omega_k)) = \{U \in \Theta \,|\, \omega_i \in U \text{ for some } 1 \leq i \leq k\} \setminus \{U_1, U_k\}.$$

$\Theta(d(\omega_1, \omega_k))$ denotes the set of the intermediate recurrent classes through path $d(\omega_1, \omega_k)$. Define the offset of path $d(\omega_1, \omega_k)$ as

$$OW\left(d(\omega_1, \omega_k)\right) = \sum_{U' \in \Theta(d(\omega_1, \omega_k))} R(U'). \tag{15}$$

For $\Theta_1 \subset \Theta$, define the *modified coradius* as

$$CR^*(\Theta_1) = \max_{\omega' \notin B(\Theta_1)} \min_{U \in \Theta_1} \min_{\omega \in U} \left\{ W(d) - OW(d) \,\middle|\, d \in \mathcal{D}\left(\omega', \omega\right) \right\}.$$

Theorem A.1 (Modified Radius-Coradius)**.** *If there exists $\Theta_1 \subset \Theta$ such that $R(\Theta_1) > CR^*(\Theta_1)$, then the limiting stationary distribution places probability one on Θ_1.*

Proofs for Section 4

Proof of Lemma 4.1. By definition, it is obvious that any strict core allocation is an absorbing state in the unperturbed dynamic. Let s^* denote an arbitrary strict core allocation. We will show that, for any $s \notin \mathscr{C}_\Delta$, the unperturbed dynamic starting from s with some $\mathcal{M}$ reaches $(s^*, \{N\})$ with positive probability.

First, suppose that the process is in $(s, \mathcal{M})$ in which allocation s is in the core, but not in the strict core. Then, there exists a coalition $J \subset N$ which weakly blocks s. Let s' be such that $s'_i = v(\{i\})$ for all $i \notin J$,[49]

$$s'_i \geq s_i \ \forall i \in J, \qquad\qquad \sum_{i \in J} s'_i = v(J).$$

Let players form coalition J and accept s'.[50] Let s'' be such that $s''_i = s'_i$ if $i \in J$ and $s''_i = v(\{i\})$ otherwise. Note that s'' is feasible for the grand coalition, i.e., $v(J) + \sum_{i \notin J} v(\{i\}) \leq v(N)$.[51] Let

[49] Such s' exists since J weakly blocks s.

[50] We mean by "let players form J and accept s" that there is positive probability that coalition J and allocation s are chosen, and players accept it. We assume that that event is realized in the dynamic.

[51] This comes from non-emptiness of $\mathscr{C}_\Delta$. For $s \in \mathscr{C}_\Delta$,

$$\sum_{i \in J} s_i \geq v(J) + \Delta, \quad \text{and} \quad s_i \geq v(\{i\}) + \Delta \quad \forall i \notin J.$$

Summing up all inequalities, we have

$$v(N) \geq \sum_{i \in N} s_i > v(J) + \sum_{i \notin J} v(\{i\}).$$

players form the grand coalition and accept s''. Then, let player $i^* \notin J$ form a singleton coalition and accept $s_i''' = v(\{i\})$ for all $i \in N$.[52] Note that the grand coalition is dissolved due to the deviation by i^*. Let players form the grand coalition again and accept $s^* \in \mathscr{C}_\Delta$.

Second, suppose that the process is in $(s, \mathcal{M})$ in which allocation s is not in the core. If no team is formed in $(s, \mathcal{M})$, let players form the grand coalition and accept s^*. Next, suppose that at least some players form a team in $(s, \mathcal{M})$. Let $\hat{s}$ be such that $\hat{s}_i = s_i$ for all $i \in N$. Due to the existence of the core, such $\hat{s}$ must be feasible for a grand coalition, i.e. $\sum_i \hat{s}_i \leq v(N)$.[53] Let players form a grand coalition and accept $\hat{s}$. If $\hat{s}$ is a core allocation, then there is positive probability the process reaches some $s^* \in \mathscr{C}_\Delta$ as shown above. If $\hat{s}$ is not a core allocation, then there exist J and s' such that J blocks $\hat{s}$. Let players form J and accept s'. Following the discussion in the previous paragraph, we can show that there is positive probability the process reaches some $(s^*, \mathcal{M})$ with $s^* \in \mathscr{C}_\Delta$. □

For the proofs of Lemma 4.2 and Theorem 4.3, recall that $s_{\min}$ denotes the lowest claim of the richest player among all strict core allocations, and $s_{\max}$ the highest one:

$$s_{\min} = \min_{s' \in \mathscr{C}_\Delta} s'_{(1)}, \qquad\qquad s_{\max} = \max_{s' \in \mathscr{C}_\Delta} s'_{(1)}.$$

Proof of Lemma 4.2. First, observe that RHS of Equation (10) gives the minimum cost of a mistake over all mistakes in allocation s because of the concavity of $u(\cdot)$.[54] We will prove that this least-cost mistake is enough for the process to switch to another strict core allocation.

First, suppose that the current allocation s satisfies Condition (9) and that $s \neq s^E$. Recall that $I_\$(s) = \{i \in N : s_i = s_{(1)}\}$, i.e. the set of the richest players. Let $i_\$ \in I_\$(s)$. Choose h such that $s_h \leq s_{(1)} - 2\Delta$.[55] We will show that a transfer of Δ from $i_\$$ to h will result in a new strict core allocation. Let s' be such that $s'_{i_\$} = s_{(1)} - \Delta$, $s'_h = s_h + \Delta$, and $s'_i = s_i$ for $i \notin \{i_\$, h\}$.[56] Observe that s' satisfies Inequalities (2) and thus is a strict core allocation. Also observe that s' is either with the richest player claiming $s_{(1)} - \Delta$ or with one fewer richest players claiming $s_{(1)}$. If $s = s^E$, then any transfer will increase some player's share by Δ to $s_{\min} + \Delta$. Then, the claim is immediate. This proves the claim (ii).

Next, suppose that Condition (9) does not hold for allocation s. Then there exist at least one richest player $i_\$ \in I_\$(s)$ and one coalition $J \in \mathcal{R}_{i_\$}$ such that $\sum_{i \in J} s_i = v(J) + \Delta$. Consider allocation s' such that $s'_{i_\$} = s_{i_\$} - \Delta$, $s'_h = s_h + \Delta$ for some $h \notin J$, and $s'_i = s_i$ for $i \notin \{i_\$, h\}$. Note that the cost to switch from $(s, \{N\})$ to $(s', \{N\})$ is given by (10). Suppose that the process starts with $(s, \{N\})$ and that the following events occur sequentially.

(i) Players form the grand coalition and accept s'. This costs $u(s_{(1)}) - u(s_{(1)} - \Delta)$.

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52 Note that $s''_{i^*} = v(\{i^*\})$. There is positive probability player i^* deviates and forms a singleton team.

53 Note that $\hat{s}$ is not necessarily a core allocation.

54 Since s is core allocation, some player's share must decrease by any deviation from s. Due to the concavity of $u(\cdot)$, the least cost of such a deviation is Δ decrease in the richest's share.

55 Lemma 2.7 guarantees that we can choose such h.

56 Obviously, the escaping-cost from $(s, \{N\})$ to $(s', \{N\})$ is given by (10).

(ii) Let $s''_J \in S^J$ be such that $s''_j = s'_j$ for all $j \in J$. Let s'' be such that $s''_J \subset s''$ and $s''_i = v(\{i\})$ for $i \notin J$. Players in J deviate and form a coalition by accepting s''. Note that the grand coalition is dissolved.

(iii) Let $\hat{s}$ be such that $\hat{s}_i = s''_i$ for all $i \in J$ and $\hat{s}_i = v(\{i\})$ otherwise.[57] Players form the grand coalition and accept $\hat{s}$. This moves the process to $(\hat{s}, \{N\})$.

(iv) Let $\tilde{i} \notin J$ and $\tilde{s}_i = v(\{i\})$ for all $i \in N$. Player i forms a singleton team $\{\tilde{i}\}$ and switches from $\hat{s}$ to $\tilde{s}$. Note that the grand coalition is dissolved by the player $\tilde{i}$'s deviation.

(v) Let $s^* \in \mathscr{C}_\Delta$. Players forms the grand coalition and accept s^*.

Observe that (ii)–(v) occur with positive probability even in the unperturbed dynamic. Once the process reaches $(s', \mathcal{M})$, it can reach any strict core allocation without cost. The least escaping-cost from s to s^* is again given by (10). This proves the claim (iii). Since we have examined all $s \in \mathscr{C}_\Delta$, we have also completed proving the claim (i). □

Proof of Theorem 4.3. First, we show the 'only if' part. Let $h \in \{0, 1, \ldots, \bar{h}\}$ where $s_{\max} - \bar{h}\Delta = s_{\min} - \Delta$. Let $U_{\mathscr{C}} = \{(s, \{N\}) \in \Omega \,|\, s \in \mathscr{C}_\Delta\}$. Define

$$U_{s_{\max}-h\Delta} = \left\{(s, \{N\}) \in U_{\mathscr{C}} \,\middle|\, s_{(1)} = s_{\max} - h\Delta\right\}$$
$$U^{\mathrm{c}}_{s_{\max}-h\Delta} = U_{\mathscr{C}} \setminus U_{s_{\max}} \setminus U_{s_{\max}-\Delta} \setminus U_{s_{\max}-2\Delta} \ldots \setminus U_{s_{\max}-h\Delta}.$$

$U_{s_{\max}-h\Delta}$ is the set of strict core allocations with the richest player claiming $s_{\max} - h\Delta$, and $U^{\mathrm{c}}_{s_{\max}-h\Delta}$ is the set of strict core allocations in which the richest player's share is at most $s_{\max} - (h+1)\Delta$. The proof is reduced to showing that $\lim_{\eta \to 0} \pi^\eta(U_{s_{\min}}) = 1$. We will prove it by induction. First, we will consider $h = 0$ and show that

$$R(U^{\mathrm{c}}_{s_{\max}}) \geq u(s_{\max} - \Delta) - u(s_{\max} - 2\Delta), \tag{16}$$
$$CR^*(U^{\mathrm{c}}_{s_{\max}}) = u(s_{\max}) - u(s_{\max} - \Delta). \tag{17}$$

Lemma 4.2 shows that the radius of state ω is given by Equation (10) and the process might fall in a basin of attraction of some strict core allocation. Together with the concavity of $u(\cdot)$, this gives $R(U^{\mathrm{c}}_{s_{\max}})$ above. Recall that CR^* is the maximum of the (modified) least escaping-costs from $U_{s_{\max}}$ to $U^{\mathrm{c}}_{s_{\max}}$ over all $\omega \in U_{s_{\max}}$. We will show that the least escaping cost from any state in $U_{s_{\max}}$ is given by (17). Choose $\omega^1 \in U_{s_{\max}}$.[58] Lemma 4.2 implies that the leas-cost escape can cause the process switching to either some $\omega' \in U^{\mathrm{c}}_{s_{\max}}$ or $\omega^2 \in U_{s_{\max}}$. In the case of switching to ω', the least escaping cost to $U^{\mathrm{c}}_{s_{\max}}$ is given by $R(\omega^1) = u(s_{\max}) - u(s_{\max} - \Delta)$, which is consistent to Equation

[57] The existence of the strict core guarantees that such $\hat{s}$ is feasible.

[58] For what follows, we let $\omega^i = (s^i, \{N\})$.

(17). Suppose the case of ω^2. Lemma 4.2 implies that ω^2 has one fewer richest players than ω^1, i.e.

$$\left|\{s_i \in s^2 : s_i = s^1_{(1)}\}\right| = \left|\{s_i \in s^1 : s_i = s^1_{(1)}\}\right| - 1.$$

According to Lemma 4.2 again, the process can further move to either some $\omega'' \in U^c_{s_{\max}}$ or $\omega^3 \in U_{s_{\max}}$ by the least-cost mistake. In the case of ω'', observe that

$$\begin{aligned} &W(d(\omega^1, \omega'')) = R(\omega^1) + R(\omega^2), \\ &OW(d(\omega^1, \omega'')) = R(\omega^2). \end{aligned}$$

Thus, $W(d(\omega^1, \omega'')) - OW(d(\omega^1, \omega'') = R(\omega^1)$. Again, it is consistent to Equation (17). For ω^3, now let us turn to a general discussion. Suppose that a sequence of the least-cost mistakes makes the process move from ω^1 to ω^2 to ... to ω^k and then to $\hat{\omega}$, where $\omega^i \in U_{s_{\max}}$ for $1 \le i \le k$ and $\hat{\omega} \in U^c_{s_{\max}}$. Lemma 4.2 guarantees that such a sequence of mistakes exists. Since the number of the richest players is finite, k must be finite. Let $d(\omega^1, \hat{\omega})$ denote a path induced by this sequence of mistakes. Its waste and offset are given by

$$W(d(\omega^1, \hat{\omega})) = \sum_{i=1}^{k} R(\omega^i), \qquad OW(d(\omega^1, \hat{\omega})) = \sum_{i=2}^{k} R(\omega^i).$$

Observe that

$$W(d(\omega^1, \hat{\omega})) - OW(d(\omega^1, \hat{\omega})) = R(\omega^1) = u(s_{\max}) - u(s_{\max} - \Delta). \tag{18}$$

Since the choice of $\omega^1 \in U_{s_{\max}}$ is arbitrary, Equation (18) implies that the modified Coradius of $U^c_{s_{\max}}$ is given by (17). The concavity of $u(\cdot)$ again implies that $R(U^c_{s_{\max}}) > CR^*(U^c_{s_{\max}})$. According to the modified Radius-Coradius theorem, $\lim_{\eta \to 0} \pi^\eta(U^c_{s_{\max}}) = 1$.

Now, we begin the main part of the induction discussion. We assume that the condition below is satisfied for $h \le \bar{h}$ and show that the same condition is satisfied for $h + 1$:[59]

$$\begin{aligned} &\text{For all } \omega \in \bigcup_{h' \le h} U_{s_{\max} - h'\Delta}, \\ &\exists \hat{\omega} \in U^c_{s_{\max} - h\Delta} \text{ and } d(\omega, \hat{\omega}) \text{ such that } W(d(\omega, \hat{\omega})) - OW(d(\omega, \hat{\omega})) = R(\omega). \end{aligned} \tag{19}$$

Note that we have shown that Condition (19) is satisfied for $h = 0$. Lemma 4.2 implies that from any $\omega^1 \in U_{s_{\max} - (h+1)\Delta}$ there exists path $d = \{(\omega^1, \omega^2), \ldots, (\omega^{k-1}, \omega^k), (\omega^k, \hat{\omega})\}$ with the following properties:

(I) $\omega^i \in U_{s_{\max} - (h+1)\Delta}$ for $1 \le i \le k$ and $\hat{\omega} \in U^c_{s_{\max} - (h+1)\Delta}$.

(II) $W((\omega^i, \omega^{i+1})) = R(\omega^i) = u(s_{\max} - (h+1)\Delta) - u(s_{\max} - (h+2)\Delta)$ for $1 \le i \le k - 1$.

[59] In a subsequent dicussion, we also show that similar equations to (16) and (17) hold if (19) is satisfied.

Observe that

$$W(d) = \sum_{i=1}^{k} R(\omega^i), \qquad\qquad OW(d) = \sum_{i=2}^{k} R(\omega^i).$$

This implies that, for all $\omega^1 \in U_{s_{\max}-(h+1)\Delta}$, there exist $\hat{\omega} \in U^{c}_{s_{\max}-h\Delta}$ and $d(\omega, \hat{\omega})$ such that

$$W(d(\omega,\hat{\omega})) - OW(d(\omega,\hat{\omega})) = R(\omega^1). \tag{20}$$

Together with Condition (19), the above observation implies that, for all $\omega^1 \in \bigcup_{h' \leq h} U_{s_{\max}-h'\Delta}$, there exists path $d(\omega^1, \hat{\omega}) = d_1 \cup d_2$ where $d_1 = \{(\omega^1, \omega^2), \ldots, (\omega^l, \omega^{l+1})\}$ is defined in Assumption (19) and $d_2 = \{(\omega^{l+1}, \omega^{l+2}), \ldots, (\omega^{l+k}, \hat{\omega})\}$ satisfies properties (I) and (II) above. Observe that[60]

$$\begin{aligned} W(d(\omega^1,\hat{\omega})) - OW(d(\omega^1,\hat{\omega})) &= W(d_1) - OW(d_1) - R(\omega^{l+1}) + W(d_2) - OW(d_2) \\ &= R(\omega^1). \end{aligned} \tag{21}$$

Equations (20) and (21) imply that Condition (19) is satisfied for $h+1$. Furthermore, these equations imply that $\lim_{\eta \to 0} \pi^{\eta}(U^{c}_{s_{\max}-(h+1)\Delta}) = 1$. To see this, observe that

$$\begin{aligned} R(U^{c}_{s_{\max}-(h+1)\Delta}) &\geq u(s_{\max} - (h+2)\Delta) - u(s_{\max} - (h+3)\Delta) \\ &> u(s_{\max} - (h+1)\Delta) - u(s_{\max} - (h+2)\Delta) \geq CR^*(U^{c}_{s_{\max}-(h+1)\Delta}). \end{aligned}$$

We continue this induction discussion until $h = \bar{h}$, and it leads us to conclude that

$$\lim_{\eta \to 0} \pi^{\eta}(U^{c}_{s_{\max}-\bar{h}\Delta}) = 1.$$

The proof of the 'only if' part is complete by observing that $U^{c}_{s_{\max}-\bar{h}\Delta} = U_{s_{\min}}$.

Next, we show the 'if' part, i.e., strict core allocations satisfying $s_{(1)} = s_{\min}$ are stochastically stable. If the egalitarian allocation s^E is in $\mathscr{C}_{\Delta}$, then $(s^E, \{N\})$ must be the unique element of ${}_{\min}\Omega^*_{\Delta}$. Then, the 'only if' part, which we proved, implies that $(s^E, \{N\})$ is stochastically stable.

Suppose that $s^E \notin \mathscr{C}_{\Delta}$.[61] Let $\omega^K \in U_{s_{\min}}$. We will show that $\omega^K = (s^K, \{N\})$ is stochastically stable. By the existence and the 'only if' part above, there exists some $\omega^1 \in U_{s_{\min}}$ that is stochastically stable. Let $T(\omega^1)$ denote a ω^1-tree minimizing the stochastic potential.

[60] To see the first equality, suppose path d along which the process moves from $\omega^1 \in U_{s_{\max}}$ to $\omega^2 \in U_{s_{\max}-\Delta}$ and then to $\omega^3 \in U^{c}_{s_{\max}-\Delta}$. Let $d_1 = (\omega^1, \omega^2)$, $d_2 = (\omega^2, \omega^3)$ and $d = d_1 \cup d_2$. Observe that

$$\begin{aligned} &W(d) = R(\omega^1) + R(\omega^2), && W(d_1) = R(\omega^1), && W(d_2) = R(\omega^2), \\ &OW(d) = R(\omega^2), && OW(d_1) = 0, && OW(d_2) = 0. \end{aligned}$$

$W(d) - OW(d)$ can be rewritten as $W(d) - OW(d) = W(d_1) - OW(d_1) - R(\omega^2) + W(d_2) - OW(d_2)$.

[61] This implies that s^E should be weakly blocked by some coalition even if it is in the core.

Let $k = 1$. For what follows, let $\omega^k = (s^k, \cdot)$. Consider the following operation.

(i) If s^k violates Condition (9), then Lemma 4.2 (iii) implies that there exists a sequence of transitions $d(\omega^k, \omega^K) = \{(\omega^k, \omega^{k+1}), \ldots, (\omega^{K-1}, \omega^K)\}$ such that $W(d(\omega^k, \omega^K)) = u(s^k_{(1)}) - u(s^k_{(1)} - \Delta)$. Construct a new tree $T(\omega^K)$ by adding edges of $d(\omega^k, \omega^K)$ to $T(\omega^k)$ and removing edges from $T(\omega^k)$ which emanates $\omega^{k+1}, \ldots, \omega^K$. Stop the operation.

(ii) If s^k satisfies Condition (9), then let $I^k_{2\Delta} = \{i \in N : s_{\min} - s^k_i \geq 2\Delta\}$. Observe that $I^k_{2\Delta} \neq \varnothing$.[62] Let j be such that $s^k_j = s_{\min}$. Choose $h \in I^k_{2\Delta}$. Let s^{k+1} be such that

$$s^{k+1}_j = s^k_j - \Delta, \qquad s^{k+1}_h = s^k_h + \Delta, \qquad s^{k+1}_i = s^k_i \quad \forall i \notin \{j, h\}.$$

Since s^k satisfies Condition (9), $s^{k+1} \in \mathscr{C}_\Delta$. And this implies that $\omega^{k+1} = (s^{k+1}, \{N\}) \in U_{s_{\min}}$. Construct a new tree $T(\omega^{k+1})$ by adding edge $d(\omega^k, \omega^{k+1})$ to $T(\omega^k)$ and removing the edge from $T(\omega^k)$ which emanates from ω^{k+1}. The resulting set of edges $T(\omega^{k+1})$ must be a ω^{k+1}-tree.

Stop if $s^{k+1} = s^K$. Otherwise, increment k by 1, i.e. $k = k + 1$, and repeat the operation above.

Observe that $\sum_{i \in I^1_{2\Delta}} s_{\min} - s^k_i$ is strictly decreasing over k in operation (ii).[63] The process will reach some k such that either s^k violates Condition (9) or $s^{k+1} = s^K$. When the operation stops, the resulting set of edges $T(\omega^K)$ must be a ω^K-tree. Observe that

$$\begin{aligned} W(\omega^K) \leq W(T(\omega^K)) &\leq W(T(\omega^1)) + \sum_{k=1}^{K-1} R(\omega^k) - \sum_{k=2}^{K} R(\omega^k) \\ &= W(T(\omega^1)) = W(\omega^1). \end{aligned}$$

This implies that ω^K must be stochastically stable. Since the choice of ω^K is arbitrary, strict core allocations satisfying $s_{(1)} = s_{\min}$ are stochastically stable. □

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[62] Recall that $s_{\min}$ is the share of the richest in s^k. Lemma 2.7 guarantees that $I^k_{2\Delta}$ is nonempty.

[63] $\sum_{i \in I^1_{2\Delta}} s_{\min} - s^k_i \geq \Delta \cdot |I^1_{2\Delta}|$ must hold. Then either $\omega^k = \omega^K$ holds or ω^k violates Condition (9) for some k.

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