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Stress Testing and Modeling of Rating Migration under the Vasicek Model Framework *

- Empirical approaches and technical implementation

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Abstract

Under the Vasicek asymptotic single risk factor model, stress testing based on rating transition probability involves three components: the unconditional rating transition matrix, asset correlations, and stress testing factor models for systematic downgrade (including default) risk. Conditional transition probability for stress testing given systematic risk factors can be derived accordingly. In this paper, we extend Miu and Ozdemir's work ([14]) on stress testing under this transition probability framework by assuming different asset correlation and different stress testing factor model for each non-default rating. We propose two Vasicek models for each non-default rating, one with a single latent factor for rating level asset correlation, and another multifactor Vasicek model with a latent effect for systematic downgrade risk. Both models can be fitted effectively by using, for example, the SAS non-linear mixed procedure. Analytical formulas for conditional transition probabilities are derived. Modeling downgrade risk rather than default risk addresses the issue of low default counts for high quality ratings. As an illustration, we model the transition probabilities for a corporate portfolio. Portfolio default risk and credit loss under stress scenarios are derived accordingly. Results show, stress-testing models developed in this way demonstrate desired sensitivity to risk factors, which is generally expected.

Keywords: Stress testing, systematic risk, asset correlation, rating migration, Vasicek model, bootstrap aggregation

1. Introduction

Stress testing is important for financial institutions either for regulatory requirements or for internal capital allocation ([1], [3], [7], [18]). In practice, stress testing focuses on systematic risk, with shocks originating from the market or macroeconomic factors ([5], [7], [19]).

Let $\{R_i \mid 1 \le i \le k\}$ denote a rating system with k ratings, with lower indexes i indicating lower default risk. Thus R_1 is the best quality rating and R_k is the worst rating, i.e., the default rating.

For a credit portfolio, stress testing can be implemented through modeling the conditional transitional probabilities under systematic risk ([2], [14]). The transition probabilities for an entity with a non-default rating R_i are assumed to be governed by a latent random variable z_i, called the firm's normalized asset value, which splits into two parts as:

$$z_i = s_i\sqrt{\rho_i} + \varepsilon_i\sqrt{1-\rho_i}, \; 0 \le \rho_i \le 1, \; s_i \sim N(0,1), \; \varepsilon_i \sim N(0,1)$$

where s_i represents the systematic risk (i.e., the common risk to all entities in the rating), while ε_i represents the idiosyncratic risk. The constant ρ_i is called the asset correlation. It is assumed that

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there exist threshold values $\{b_{ij}\}$ such that a firm's rating migrates from R_i to R_j or worse (called downgrade (including default) risk) when z_i falls below the threshold value $b_{i(k-j+1)}$.

Modeling for stress testing purposes for a credit portfolio under this framework involves:

(a) Determining the threshold values $\{b_{ij}\}$ (or equivalently, the unconditional transition probabilities).
(b) Estimating the asset correlation ρ_i for each non-default rating.
(c) Modeling the downgrade risk by a multifactor model for each non-default rating.
(d) Deriving conditional transition probabilities given scenario risk factors thus assessing portfolio level credit loss.

While threshold values $\{b_{ij}\}$ can be estimated by using historical point-in-time migration matrices (see section 2), the estimation of asset correlations and modeling of conditional downgrade risk by factor models are more challenging. Miu and Ozdemir ([14]) propose approaches to deriving the conditional transition probabilities based on a factor model for the systematic risk *s:*

$$s = f(x_1, x_2, ..., x_m) + e, \quad e \sim N(0, \sigma_e^{\ 2}) \qquad (1.2)$$

by assuming the same systematic risk and same asset correlation for all non-default ratings. Miu and Ozdemir also show how parameters of model (1.2) and those from (a)-(c), including threshold values $\{b_{ij}\}$, can be estimated simultaneously in one single process of likelihood maximization.

We extend Miu and Ozdemir's work to a more granular rating level, by assuming different asset correlation ρ_i and different systematic risk s_i for each non-default rating. Specifically, for a non-default rating R_i, let $d_i(s_i)$ denote the downgrade probability given systematic risk s_i. We proposed two Vasicek models, each with a latent random effect, for stress testing purposes:

$$d_i(s_i) = \Phi(a_{i0} + a_{i1}s_i), \ s_i \sim N(0,1) \qquad (1.3)$$

$$d_i(s_i) = \Phi(a_{i0} + a_{i1}x_1 + a_{i2}x_s + a_{im}x_m + e_i), \quad e_i \sim N(0, \sigma^2(e_i)) \qquad (1.4)$$

where Φ denotes the standard normal cumulative distribution, and $x_1, x_2, ..., x_m$ are scenario or macro risk factors. As shown in later sections, asset correlation ρ_i and conditional transition probabilities can be derived accordingly from these two stress testing models, given the threshold values $\{b_{ij}\}$ (see Lemma 3.1 and Theorem 3.3).

The advantages for the proposed approaches are the following:

(1) Asset correlations and stress testing factor models are differentiated between non-default ratings, achieving desired risk sensitivity for the stress testing models under stress scenarios.
(2) Similar to the results by Miu and Ozdemir ([14]), analytical formulas for conditional transition probabilities are derived.
(3) Parameters in (a)-(c) are estimated separately, expert judgements and adjustments for threshold values $\{b_{ij}\}$ and asset correlations are made possible.
(4) Modeling downgrade probability rather than default risk addresses the issue caused by low default counts for high quality ratings.

Models (1.3)-(1.4) can be fitted effectively by using, for example, SAS non-linear mixed procedure ([21]), assuming a binomial distribution for the event count given the event probability. We will propose a two-step fitting procedure in section 3.3 for training the model (1.4): first by a master model for all non-default ratings targeting the portfolio default risk, and then calibrating this master model to rating level for each non-default rating, targeting the downgrade risk.

The paper is organized as follows: We review in section 2 the Vasicek asymptotic single risk factor model (ASRF) for modeling of rating migration. In section 3, we propose the stress testing models (1.3)-(1.4), and derive the analytical formulas for conditional transition probabilities. Parameter estimation methodologies, including the bootstrap aggregation technique (called bagging, for addressing the time series serial correlation), are reviewed in section 4. In section 5, we validate the proposed approaches by building stress testing models for a US corporate portfolio. Portfolio credit loss and default risk on stress scenarios are assessed accordingly.

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2. Rating Migration under the Vasicek Asymptotic Single Risk Factor Model Framework

2.1. The Vasicek Asymptotic Single Risk Factor Model

Under the Vasicek asymptotic single risk factor model ([2], [9], [11], [12], [13], [14], [20]), default risk for an entity is driven by a latent variable *z,* the normalized asset value of the entity. A default event occurs in horizon if this normalized asset value falls below a threshold value, called default point. For a group of risk homogenous entities, z splits into two parts:

$$z = s\sqrt{\rho} + \varepsilon\sqrt{1-\rho},\ 0 \le \rho \le 1,\ s \sim N(0,1),\ \varepsilon \sim N(0,1) \qquad (2.1)$$

where s represents the systematic risk (i.e., the common risk), common to all entities in the group, while ε represents the idiosyncratic risk. The constant ρ is called the asset correlation of the group.

2.2. The Unconditional Rating Migration Matrix

We assume that entities in the same rating are risk homogeneous. Thus model (2.1) applies.

Given a non-default rating R_i, we assume that there exist k threshold values

$$b_{i1} < b_{i2} < \ldots < b_{i(k-1)} < b_{ik} (= +\infty) \qquad (2.2)$$

such that an entity will migrate to rating R_j or worse in horizon if z falls below $b_{i(k-j+1)}$, i.e., $z < b_{i(k-j+1)}$.

Denote by p_{ij} the unconditional transition probability of migrating from R_i to R_j, and d_{ij} the unconditional transition probability from R_i to R_j or worse. This means $p_{ik} (= d_{ik})$ is the unconditional default probability for rating R_i. The following propositions hold and can be found in ([2], [14]).

Proposition 2.1. (a) If $j = 1,\ d_{ij} = 1$.

(b) If $j > 1,\ d_{ij} = P(z < b_{i(k-j+1)}) = \Phi(b_{i(k-j+1)})$

Proposition 2.2 (a) $p_{ik} = d_{ik} = \Phi(b_{i1})$ and $p_{i1} = 1 - \Phi(b_{i(k-1)})$

(b) If $1 < j < k$ then

$$p_{ij} = \Phi(b_{i(k-j+1)}) - \Phi(b_{i(k-j)}) = d_{ij} - d_{i(j+1)}.$$

Consequently, given the unconditional transition matrix $T = (p_{ij})$, the threshold values $\{b_{ij}\}$ in (2.2) can be determined sequentially, first b_{i1} by Proposition 2.2 (a), then b_{i2} by Proposition 2.2 (b) and Proposition 2.1 (b), and so on.

2.3. Calibration of the Unconditional Rating Migration Matrix

The unconditional transition probabilities $\{p_{ij}\}$ can be estimated using the historical point-in-time migration matrices. This is because:

$$\begin{aligned} d_{ij} &= P(z < b_{i(k-j+1)}) = E_s(P(z < b_{i(k-j+1)} \mid s)) \\ \Rightarrow p_{ij} &= d_{ij} - d_{i(j+1)} \\ &= E_s[P(z < b_{i(k-j+1)} \mid s) - P(z < b_{i(k-j)} \mid s))] \\ &= E_s[P(b_{i(k-j)} \le z < b_{i(k-j+1)} \mid s)] \end{aligned} \qquad (2.3)$$

where $E_s()$ denotes the expectation with respect to *s*. Since

$$P(b_{i(k-j)} \le z < b_{i(k-j+1)} \mid s)$$

is the point-in-time transition probability of migrating from R_i to R_j given the systematic risk *s*, we conclude that the unconditional transition probability p_{ij} can be estimated by taking the average of the historical transition rate of moving from rating R_i to rating R_j.

This average migration matrix, estimated from historical point-in-time transition matrices as above, are usually subjected to experts' reviews. Adjustments may be required before it is used to derive the threshold values $\{b_{ij}\}$. In general, the following rules are imposed:

(a) Transition probabilities p_{ij} have to be floored at a positive number to ensue that the threshold values $\{b_{ij}\}$ are different for a given rating R_i.

(b) The unconditional default probability p_{ik} is an increasing function of *i*, i.e., better quality ratings have lower default probabilities

(c) Given a risk rating R_i, the transition probability p_{ij} is a decreasing (not necessarily strict) function for the distance $|i - j|$ between *i* and *j*, where $i \neq j$. This means an entity is more likely to migrate to a closer rating than a farther away rating in the same direction.

2.4. Conditional Rating Migration Given Systematic Risk

Given model (2.1), let $p_{ij}(s)$ denote the transition probability of moving from R_i to R_j conditional on systematic risk *s*, and $d_{ij}(s)$ the transition probability of moving R_i to R_j or worse, conditional on systematic risk *s*. The two propositions below for $d_{ij}(s)$ and $p_{ij}(s)$ follow similarly as Propositions 2.1 and 2.2 via model (2.1), and can be found in ([2], [14]).

Proposition 2.3 (a) If $j=1$, $d_{ij}(s)=1$.

(b) If $j>1$, $d_{ij}(s)=\Phi[(b_{i(k-j+1)}-s\sqrt{\rho})/\sqrt{1-\rho}\,]$

Proposition 2.4. (a) $p_{ik}(s)=d_{ik}(s)=\Phi[(b_{i1}-s\sqrt{\rho})/\sqrt{1-\rho}\,]$

(b) $p_{i1}(s)=1-\Phi[(b_{i(k-1)}-s\sqrt{\rho})/\sqrt{1-\rho}\,]$

(c) If $1<j<k$ then

$$p_{ij}(s)=\Phi[(b_{i(k-j+1)}-s\sqrt{\rho})/\sqrt{1-\rho}\,]-\Phi[(b_{i(k-j)}-s\sqrt{\rho})/\sqrt{1-\rho}\,]=d_{ij}(s)-d_{i(j+1)}(s)$$

3. Stress Testing Models

3.1 The Vasicek Models

For simplicity, we denote the downgrade probability $d_{i(i+1)}(s)$ for a non-default rating R_i by $d_i(s)$. Given a non-default rating R_i, we propose two Vasicek models for stress testing purposes:

$$d_i(s)=\Phi(a_0+a_1 s),\ s\sim N(0,1) \qquad (3.1)$$

$$d_i(s)=\Phi(a_0+a_1x_1+a_2x_s+a_mx_m+e),\ \ e\sim N(0,\sigma_e^{\ 2}) \qquad (3.2)$$

where $x_1, x_2, ..., x_m$ are macro variables or market factors in horizon, subjected to an appropriate transformation by Φ^{-1} when necessary. The latent random residual *e* is independent of $x_1, x_2, ..., x_m$. With the model (3.2), we are required to estimate parameters a_0, a_1, a_2, ..., a_m and σ_e. Note that, models (3.1) and (3.2) are at rating level. We are required to fit models (3.1) and (3.2) for each non-default rating independently.

With the rating level Vasicek model (3.1), the asset correlation ρ can be calculated as in the next lemma below.

Lemma 3.1 ([17]) (a) $E_s[\Phi(a_0+a_1s)]=\Phi(a_0/\sqrt{1+a_1^{\ 2}}\,)$, where $s\sim N(0,1)$

(b) ([22]) $\rho_i=a_1^{\ 2}/(1+a_1^{\ 2})$ under model (3.1).

3.2 Conditional Transition Probabilities Given Factor Model (3.2)

Given model (3.2) for a rating R_i, let

$$u=a_0+\sum_{i=1}^{m} a_i x_i$$

Denote by $d_i(x_1, x_2, ..., x_m)$ the downgrade probability for the rating R_i given the macro condition $x_1, x_2, ..., x_m$. By model (3.2) and Lemma (3.1) (b), we have the following proposition:

Proposition 3.2. ([22]) $d_i(x_1, x_2, ..., x_m) = E(\Phi(u+e) | x_1, x_2, ..., x_m) = \Phi(u / \sqrt{1+\sigma_e^2})$

Let $b_i = b_{i(k-i)}$, the threshold value corresponding to the downgrade probability for rating R_i. Given threshold values $\{b_{ij}\}$ and the market condition $x_1, x_2, ..., x_m$, the conditional migration probabilities can be derived (using models (3.1)-(3.2)) as in the theorem below. We get similar but slightly different from the results by Miu and Ozdemir ([14]).

Theorem 3.3. The following statements hold under model (3.2):

(a) $p_{i1}(x_1, x_2, ..., x_m) = 1 - \Phi[u / \sqrt{1+\sigma_e^2} + (b_{i(k-1)} - b_i) / \sqrt{(1-\rho)(1+\sigma_e^2)}\ ]$

(b) $p_{ik}(x_1, x_2, ..., x_m) = \Phi[u / \sqrt{1+\sigma_e^2} + (b_{i1} - b_i) / \sqrt{(1-\rho)(1+\sigma_e^2)}\ ]$,.

(c) If $1 < j < k$ then

$$p_{ij}(x_1, x_2, ..., x_m) = \Phi[u / \sqrt{1+\sigma_e^2} + (b_{i(k-j+1)} - b_i) / \sqrt{(1-\rho)(1+\sigma_e^2)}\ ]$$
$$- \Phi[u / \sqrt{1+\sigma_e^2} + (b_{i(k-j)} - b_i) / \sqrt{(1-\rho)(1+\sigma_e^2)}\ ]$$

Proof. By model (3.2), and Proposition 2.3 (b), we have

$$(b_i - s\sqrt{\rho}) / \sqrt{1-\rho} = u + e$$
$$\Rightarrow -s\sqrt{\rho} / \sqrt{1-\rho} = u + e - b_i / \sqrt{1-\rho}$$

We just prove the statements (b) and (c), the proof for (a) is similar. By Proposition 2.4 (a), we have

$$p_{ik}(s) = \Phi[(b_{i1} - s\sqrt{\rho}) / \sqrt{1-\rho}\ ]$$
$$= \Phi[u + e + (b_{i1} - b_i) / \sqrt{1-\rho}\ ]$$
$$\Rightarrow p_{ik}(x_1, x_2, ..., x_m) = \Phi[u / \sqrt{1+o_e^2} + (b_{i1} - b_i) / \sqrt{(1-\rho)(1+\sigma_e^2)}\ ]$$

by Lemma 3.1 (a). This proves statement (b).

For statement (c), we have

$$p_{ij}(s) = \Phi[(b_{i(k-j+1)} - s\sqrt{\rho}) / \sqrt{1-\rho}\ ] - \Phi[(b_{i(k-j)} - s\sqrt{\rho}) / \sqrt{1-\rho}\ ]$$
$$= \Phi[u + e + (b_{i(k-j+1)} - b_i) / \sqrt{1-\rho}\ ] - \Phi[u + e + (b_{i(k-j)} - b_i) / \sqrt{1-\rho}\ ]$$

by Proposition 2.4 (c) for $1 < j < k$.

$$\Rightarrow p_{jk}(x_1, x_2, ..., x_m)$$
$$= \Phi[u / \sqrt{1+o_e^2} + (b_{i(k-j+1)} - b_i) / \sqrt{(1-\rho)(1+\sigma_e^2)}\ ] - \Phi[u / \sqrt{1+o_e^2} + (b_{i(k-j)} - b_i) / \sqrt{(1-\rho)(1+\sigma_e^2)}\ ]$$

by Lemma 3.1 (a). This proves statement (c) □

3.3 Two-Step Fitting Procedure for the Multifactor Vasicek Model (3.2)

To simplify the model fitting work at rating level for model (3.2), and ensure weights are fairly allocated to risk drivers with respect to portfolio default risk, we propose the following two-step fitting procedure for model (3.2):

(i) Let $p_{all}(s)$ denote the portfolio level default probability given systematic risk *s*.
First, fit a model over the portfolio for all non-default ratings, targeting the portfolio level default probability $p_{all}(s)$:

$$p_{all}(s) = \Phi(a_0 + \sum_{i=1}^{m} a_i x_i + e) = \Phi(w + e),\ e \sim N(0, \sigma_e^{\ 2}) \qquad (3.3)$$

where *w* sums up the fixed effects, and *e* is the model residual.

(ii) Next, calibrate the above model to each non-default rating R_i targeting the downgrade probability as:

$$d_i(s) = \Phi(c_i w + e_i), \qquad e_i \sim N(0, \sigma_{e_i}^{\ 2}) \qquad (3.4)$$

where e_i denotes the model residual.

Master model (3.3) captures the sensitivity with respect to the portfolio level default risk, ensuring fair risk weights are allocated among risk factors. In general, default is barely observed for high quality ratings, a model targeting default risk has to be fitted on portfolio level. With model (3.4), the master model is calibrated back to the rating level for each non-default rating, targeting the downgrade risk.

4 Parameter Estimation
4.1 Binomial Likelihood Approaches

Let $S = \{(x_{1i}, x_{2i}, ..., x_{mi}, k_i, n_i\}$, $i = 1, 2, ..., N$, be a time series sample, where $x_1, x_2, ..., x_m$ are market or macroeconomic variables, and n_i, k_i are respectively the numbers of entities and numbers of downgrades in one-year horizon at time index *i*. Given the downgrade event probability

$$p(s) = \Phi(u + ce),\ e \sim N(0,1)$$

where *u* is a deterministic linear function of $x_1, x_2, ..., x_m$, the likelihood of observing *k* downgrades for a non-default rating with *n* entities is:

$$\binom{n}{k} p(s)^k (1 - p(s))^{n-k}$$

Its expected value with respect to random factor *e* gives the unconditional likelihood:

$$\begin{aligned} bin(k,n) &= \binom{n}{k} \int_{-\infty}^{+\infty} p(s)^k (1 - p(s))^{n-k} \phi(e) de \\ &= \binom{n}{k} \int_{-\infty}^{+\infty} \Phi^k (u + e)(1 - \Phi(u + e))^{n-k} \phi(e) de \end{aligned}$$

where $\phi(\,)$ denotes the standard normal density. The negative natural logarithmic likelihood is given by the sum ([6], [10], [14]):

$$-\log L = -\sum_{i=1}^{N} \ln(bin(k_i, n_i))$$

With the maximum likelihood parameter estimation approach, we are required to estimate the model parameters for models (3.1) and (3.2) by minimizing -log L. SAS non-linear mixed procedure (NLMIXED, [21]) provides a tool for fitting this type of models, while maximizing the binomial likelihood.

4.2 The Serial Correlation and Bootstrap Aggregation Methodologies

A model describes the joint distribution between the target and explanatory variables. Given a modeling sample, independence between data points is generally expected. However, serial correlation for a times series sample is in general significant. This causes an issue for parameter estimation ([15], [16, pp.159-175]).

Instead of fitting the models (3.1) and (3.2) directly on the time series sample, we propose a bootstrap approach, assuming that the time series variables are stationary. This approach is analogous to the bagging (bootstrap aggregation) technique ([4], [8]):

(a) Generate B (a sufficiently large number, for example, 200) bootstrap samples using the original time series. Each bootstrap sample is of the same size as the original sample, and is created by randomly sampling from the original sample with replacement. Block sampling may be required when variables used are not evaluated at the same time.
(b) For each bootstrap sample, fit a model of form (3.1).
(c) Calculate the average of each parameter over all bootstrapped models, and select the model with parameters that are the closest to these parameter averages.

For model (3.2), we follow the two-step fitting procedure proposed in section 3.3, and sequentially fit the models (3.3) and (3.4) using the bootstrap technique proposed as above.

5 An Empirical Example: Stress Testing for a Corporate Portfolio

In this section, we validate our proposed approaches to stress testing a corporate portfolio. The sample is created synthetically for a US corporate portfolio. The sample includes the portfolio quarterly data covering the years between 2001 and 2013 in one-year horizon: the first one-year observation period starts at the beginning of the 1st quarter 2001 and ends at 4-th quarter 2001, the second one-year observation period starts at the beginning of the 2nd quarter of 2001 and ends at 1st quarter 2002, and so on. The last one-year observation period starts at the beginning of the 1st quarter 2013 and ends at 4-th quarter 2013.

This is a low default portfolio, with an average portfolio default rate below 1%. There are 21 ratings, with the first rating 1 as the best; and the last rating 21 as the default rating.

Our stress testing for the portfolio follows the steps as proposed in section 1.

(a) Determining the threshold values $\{b_{ij}\}$

Using all the historical point-in-time migration matrices, we calculate the average migration matrix, and take it as the preliminary unconditional transition matrix $T = (p_{ij})$. Minor adjustments are made following the rules (a) -(c) as proposed in section 2.3.

(b) Estimating the asset correlation ρ_i for each non-default rating

We estimate asset correlation for each non-default rating by using Lemma 3.1 (b) via model (3.1). Model (3.1) is fitted using the bootstrap technique as proposed in section 4.2. The two tables below show the estimated asset correlations for all 20 non-default ratings: the asset correlation is over 30% for ratings between 17-20, is about 20% for ratings 1-2 and 16, and is around 10% for ratings 3-15.

Table A. Sample asset correlation for non-default ratings 1-10

RTG	1	2	3	4	5	6	7	8	9	10
Asset Corr	0.20	0.21	0.07	0.13	0.07	0.11	0.11	0.09	0.07	0.09

Table B. Sample asset correlation for non-default ratings 11-20

RTG	11	12	13	14	15	16	17	18	19	20
Asset Corr	0.13	0.08	0.10	0.09	0.12	0.19	0.30	0.58	0.64	0.49

(c) Fitting the multifactor model (3.2) for each non-default rating

The risk factors we select are the following:

(1) US Growth GDP
(2) US Unemployment Rate
(3) US Government 10-year bond yield
(4) US 30-year BBB corporate bond credit spread

We follow the two-step fitting procedure proposed in section 3.3 and the bootstrap technique proposed in section 4.2: First fit a master model of the form (3.3) for all non-default ratings targeting portfolio level default risk, using the bootstrap technique; then calibrate the selected master model (3.3) to model (3.4) for each non-default rating, targeting downgrade risk and using the bootstrap technique.

(d) Deriving conditional transition probabilities and assessing portfolio credit loss

We derive the conditional transition probabilities by using Theorem 3.3. Conditional portfolio level default rate and loss are calculated respectively as:

$$p = \sum_{i=1}^{20} p_i n_i / n, \quad loss = \sum_{i=1}^{20} p_i (EAD_i)(LGD_i) n_i$$

where for a given rating R_i, the numbers n_i, p_i are respectively the size and the model predicted probability of default, and EAD_i, LGD_i are the exposure at default (EAD) and loss given default, and n is the portfolio size.

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The table below shows the results for model projected portfolio loss and model predicted portfolio default rate.

The 1st column indicates the horizon (one-year) end quarter, the 2nd column records the realized portfolio default rate at the end quarter, and the 3rd column is the model predicted portfolio default rate at the end quarter. We also calculate the conditional 95-percentile upper bound for the predicted portfolio default rate, assuming default count follows a binomial distribution with the event probability given by the predicted portfolio level default probability. The last column shows the model projected portfolio scenario loss, reported as a percentage of the portfolio total EAD.

As shown in the table below, both the model predicted and realized portfolio default rates peak at 2009.3. The projected loss peaks at 2009.2 rather than 2009.3. Note that, projected loss is not necessarily 100% concordant to the predicted default probability due to the LGD factor. The loss rate is generally higher for the entities that get hit in the first round of market shocks. As market moves further into the downturn period, more entities, including those with better risk profiles, start to default, resulting in a relatively higher portfolio default rate but slightly lower loss rate.

Table C. Projected portfolio default rate and loss

	Portfolio Default Rate			Projected
Quarter	Actual	Predicted	95-Percentile	Portfolio Loss
2006.2	0.29%	0.37%	0.74%	0.29%
2006.4	1.15%	0.77%	1.28%	0.33%
2007.2	0.12%	0.60%	1.09%	0.32%
2007.4	0.12%	0.69%	1.17%	0.32%
2008.2	0.33%	0.82%	1.31%	0.34%
2008.4	1.42%	1.26%	1.86%	0.48%
2009.2	2.52%	2.72%	3.58%	0.96%
2009.3	2.73%	2.94%	3.89%	0.89%
2009.4	2.58%	2.30%	3.12%	0.56%
2010.2	1.74%	1.38%	2.09%	0.32%
2010.4	0.96%	1.28%	1.83%	0.25%
2011.2	0.63%	1.38%	1.98%	0.28%
2011.4	0.30%	1.10%	1.57%	0.25%
2012.2	0.30%	1.00%	1.44%	0.21%
2012.4	0.40%	0.93%	1.32%	0.20%
2013.2	0.36%	0.88%	1.26%	0.19%

Conclusion. Risk sensitivity is a key measure for a stress-testing model. In this paper, we differentiate asset correlations between non-default ratings, and fit stress testing factor models at the rating level, targeting the downgrade risk. We thus achieve desired risk sensitivity and robustness for the stress testing models under stress scenarios. By targeting the downgrade risk rather than default risk, we address the issue of low default counts for high quality ratings. The proposed models can be fitted effectively by using, for example, SAS non-linear mixed procedure. Bootstrap aggregation technique is used to address the serial correlation issue for the time series sample. We believe the proposed approaches provide a step-by-step, effective, and practical tool for practitioners in the fields of financial stress testing.

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