New Fractional Dickey and Fuller Test

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***Abstract*—The aim of this paper is motivated by the following question: "If a series were best characterized by fractional process, would a researcher be able to detect that fact by using conventional Dickey-Fuller (1979) test?" To answer this question, in simple framework, we propose a new fractional Dickey-Fuller (F-DF) test, different from the test of Dolado, Gonzalo and Mayoral (2002).**

I. Introduction

The concept of fractionally integrated time series processes was originally introduced by Granger and Joyeux (1980) and Hosking (1981). Diebold and Rudebush (1991) asked the question: "If a series were best characterized by a fractional process, would a researcher be able to detect that fact by rejecting the hypothesis of unit root using the conventional Dickey-Fuller (1979) test? To study this issue Diebold and Rudebush (1991) examined the power of Dickey-Fuller tests when the data-generating process is a pure fractionally-integrated process

$$(1-L)^d y_t = u_t,\ with\ d \in \left(\frac{1}{2};\frac{3}{2}\right) \qquad (1.1)$$

or equivalently,

$$(1-L)y_t = (1-L)^{-\delta} u_t, \qquad (1.2)$$

with white noise innovation, $u_t \rightsquigarrow (0, \sigma_u^2)$ and $\delta = d - 1$. $(1-L)^d$ is the fractional difference operator defined by its Maclaurin series (by its binomial expansion if d is an integer):

$$(1-L)^d = \sum_{j=0}^{\infty} \frac{\Gamma(-d+j)}{\Gamma(-d)\Gamma(j+1)},$$

where

$$\Gamma(z) = \begin{cases} \int_0^{+\infty} s^{z-1} e^{-z} ds, & if\ z > 0 \\ \infty & if\ z = 0, \end{cases}$$

if $z < 0$, $\Gamma(z)$ is defined in terms of the above expressions and the recurrence formula $z\Gamma(z) = \Gamma(z+1)$.

By using the usual auxiliary regression model,

$$y_t = \phi y_{t-1} + \varepsilon_t,$$

in order to test the following hypotheses

$$H_0 : \phi = 1\ against\ H_1 : \phi \neq 1,$$

Diebold and Rudebush (1991) showed, by Monte Carlo simulations, that this test has quite low power and can lead to the incorrect conclusion that a time series has a unit root also when this is not true. They pointed out that a more appropriate testing procedure is needed to draw conclusions about the presence of the unit root.

In this paper, we point out that these disappointing results originate from an ill defined statistical problem. Indeed, Diebold and Rudebush gave a special attention to the parameter ϕ in testing hypothesis rather than the parameter d. However, to express the hypotheses in term of the parameter ϕ, by ignoring the parameter d, this can lead to incorrect conclusions. Since the seminal work of Dickey and Fuller (1979) on formal tests for unit roots, these tests became standard in applied time series analysis and econometrics. In recent years, an increasing effort has been made to establish reliable testing procedures to determine whether or not an observed time series is fractionally integrated. Some contributions on this topic include Dolado, Gonzalo and Mayoral (2002), Nilsen and Johansen (2010), Lobato and Velasco (2007). In particular, there has been a considerable interest in generalizing the Dickey-Fuller type test by taking into account the fractional integration order. For instance, Dolado, Gonzalo, and Mayoral [DGM] (2002) introduced a test based on an auxiliary regression for the null of unit root against the alternative of fractional integration. Further, the DGM test was refined by Lobato and Velasco [LV] (2006, 2007). The fractional Dickey-Fuller (FD-F) test considered by DGM (2002), in the basic framework, is described by the following.

Let $\{y_t\}_{t=1}^n$ denotes a fractionally integrated process whose true order of integration is d, denotes as $FI(d)$,

$$y_t = (1-L)^{-d} u_t, \qquad (1.3)$$

with white noise innovation $u_t \rightsquigarrow (0, \sigma_u^2)$ and d is any real. For the data generating process (DGP) (1.3), DGM (2002) propose to test the following hypotheses test,

$$H_0 : d = d_0\ against\ H_1 : d = d_1,\ with\ d_1 < d_0, \qquad (1.4)$$

by means of the t statistic of the coefficient of $\Delta^{d_1} y_{t-1}$ in the ordinary least squares (OLS) regression

$$\Delta^{d_0} y_t = \phi_1 \Delta^{d_1} y_{t-1} + \varepsilon_t,\ \ (t = 1, \cdots, n\ ),$$

where n denotes the sample size. LV (2006, 2007) argue that $\Delta^{d_1} y_{t-1}$ is not the best class of regression one can choose and propose another auxiliary regression model for the hypotheses test (1.4). In the case $d_0 = 1$, they propose to test (1.4) by using the following auxiliary model

$$\Delta y_t = \phi_2 z_{t-1}(d_1) + \varepsilon_t,\ \ (t = 1, \cdots, n\ ),$$

where

$$z_{t-1}(d_1) = \left(\frac{\Delta^{d_1-1} - 1}{1 - d_1}\right) \Delta y_t.$$

The DGM (2002) and LV (2007) tests present an analogy with the original Dickey-Fuller test, but can not be considered as a generalization of the familiar Dickey-Fuller test in the sense that the conventional $I(1)$ vs $I(0)$ framework is recovered (for the DGM test the conventional framework is recovered only if $d_0 = 1$ and $d_1 = 0$). Indeed, under the null and d_1 known, the t statistic in the regression model of DGM (2002) depends on fractional Brownian motion if $0 \leq d_1 < 0.5$ and $t \to N(0,1)$ if $0.5 \leq d_1 < 1$. These asymptotic distributions are different from those derived by Dickey and Fuller (1979) which depend only on standard Brownian motion. The implementation of DGM (2002) test would require tabulations of the percentiles of the functional of fractional Brownian motion, which imply that the inference on the presence of unit root would be conditional on d_1, and thus might suffer from misspecification. When d_1 is not taken to be known a priory, a pre-estimation of it is needed to implement the test. In this case, we can perform the test only if the estimator of d_1 ($\widehat{d_1}$) is sufficiently close to unity (see DGM (2002) for details). The problem is that the [DGM] and [LV] approaches are based on having a choice of two possible orders of integration of which the true order can be different from the null and alternative. In fractional case, we have a continuum of possible orders of integration which makes the simple null hypothesis against the simple alternative being invalid. For example, for the D.G.M. test we have three cases,

- The case where the true value of d is equal $d_0 = 1$,
- The case where the true value of d is equal d_1,
- The case where the true value of d is different from $d_0 = 1$ and d_1.

The third case cause serious troubles in practice. To overcome this problem, the null hypothesis and alternative must be complementary and mutually exclusive and then, we suggest to use a composite null hypothesis against the composite alternative. More precisely in this paper, we deal with a fractionally integrated, $FI(d)$, processes $\{y_t\}_{t=1}^{n}$, defined by (1.3) where the order d is any real number in $\left]\frac{1}{2}, +\infty\right[$. Under this setting, we propose to test the following hypotheses test[1,2]

$$H_0 : d \geq d_0 \ against \ H_1 : d < d_0, \tag{1.5}$$

The hypotheses test (1.5) is based on having a choice of two possible cases of which one is true. The test statistics is the same as in Dickey-Fuller test using as output $\Delta^{d_0} y_t$ instead of Δy_t and as input $\Delta^{-1+d_0} y_t$ instead of y_t, exploiting the fact that if y_t is $I(d)$ then $\Delta^{-1+d_0} y_t$ is $I(1)$ under the null $d = d_0$. If $d \geq d_0$, using the generalization of Sowell's results (1990), we propose a test based on the least favorable case $d = d_0$, to control type I error and when $d < d_0$ we show that the usual tests statistics diverges to $-\infty$, providing consistency. We call this test procedure (like DGM) the F-DF test.

The rest of this paper is organized as follows. In Section 2, we provide, in simple framework, the auxiliary regression model used to test the null and the main results on asymptotic null and alternative distribution for the testing problem (1.5). In Section 3, we show how to use the new F-DF test, in practice.

II. The Model and the New Fractional Dickey Fuller Test (F-DF Test)

To test the null, our proposal is based upon testing the statistical significance of the coefficient ϕ (or $\rho = \phi - 1$) in the following autoregression model,

$$\Delta^{-1+d_0} y_t = \phi \Delta^{-1+d_0} y_{t-1} + \varepsilon_t, \tag{2.1}$$

or equivalently

$$\Delta^{d_0} y_t = \rho \Delta^{-1+d_0} y_{t-1} + \varepsilon_t, \tag{2.2}$$

where $\rho = \phi - 1$ and $\{\varepsilon_t\}$ the residuals. The most important idea behind the choice of framework above is that if $d = d_0$, then

$$x_t = \Delta^{-1+d_0} y_t \, is \ integrated \ of \ order \ 1$$

More generally, we have:

$$x_t \ is \ integrated \ of \ order \ 1 + d - d_0,$$

with
$$\begin{cases} 1 + d - d_0 \geq 1 \ , if \ d \geq d_0 \\ 1 + d - d_0 < 1 \ , if \ d < d_0 \end{cases}$$

Before stating the main results of this article, we give some technical tools that we need for this study. Let $\eta_t = (1 - L)^{-\delta} u_t$, with $\delta \in \left]-0.5, 0.5\right]$ and u_t defined as above. Let $\sigma_S^2 = var(S_n)$, where $S_t = \sum_{j=1}^{t} \eta_j$. When $|\delta| < \frac{1}{2}$, we have (see Sowell (1990))

$$\lim_{n\to\infty} n^{-1-2\delta} \sigma_S^2 = \frac{\sigma_\varepsilon^2 \Gamma(1-2\delta)}{(1+2\delta)\Gamma(1+\delta)\Gamma(1-\delta)} \equiv \kappa_\eta^2(\delta), \tag{2.4}$$

If in addition, $E\left|u_t\right|^a < \infty$for $a \geq \max\left\{4, \frac{-8\delta}{1+2\delta}\right\}$, we have the following useful results that apply to this type of process:

$$n^{-\frac{1}{2}-\delta} \kappa_\eta^{-1}(\delta) S_{[nr]} \Rightarrow \frac{1}{\Gamma(1+\delta)} \int_0^r (r-s)^\delta d\mathbf{w}(s), \tag{2.5}$$

if $-\frac{1}{2} < \delta < \frac{1}{2}$,and

$$n^{-\frac{1}{2}-\delta} \left(\log^{-1} n\right) \kappa_\eta^{-1}(\tfrac{1}{2}) S_{[nr]} \Rightarrow \mathbf{w}_{0.5}(r), \tag{2.6}$$

if $\delta = 0.5$.

Where $\mathbf{w}(r)$is the standard Brownian motion on $[0,1]$associated with the $u_t$sequence and the symbols " $\Rightarrow$ "and " $\xrightarrow{p}$ "denote weak convergence and convergence in probability, respectively.

By noting that $d - d_0$can always be decomposed as $d - d_0 = m + \delta$, where $m \in \mathbb{N}$and $\delta \in]-0.5, 0.5]$, the asymptotic null and alternative of the Dickey-Fuller normalized bias statistic $n\widehat{\rho}_n = n\left(\widehat{\phi}_n - 1\right)$and the Dickey-Fuller t-statistic, $t_{\widehat{\rho}}$, in the model (2.2)are provided by the theorem 1.

Theorem 1. *Let* $\{y_t\}$ *be generated according DGP* (1.3)*. If regression model* (2.2) *is fitted to a sample of size* n *then, as* $n \uparrow \infty$*,*

[1] This paper was presented at ICMSAO'15 Conference, Istanbul, Turkey, 27–29 Mayl 2015.

[2] 1The special case of hypothesis testing $H0 : d \leq 1$ against $H1 : d < 1$ was presented at ICMSAO'13 Conference, Hammamet, Tunisia, 28–30 April 2013, in the paper entitled "A consistent against for unit root against fractional alternative". Expanded version of this paper forthcoming in Inderscience journal "International Journal of operational research

1) $n\widehat{\rho}_n$ *verifies that*

$$\widehat{\rho}_n = O_p(\log^{-1} n) \text{ and } (\log n)\,\widehat{\rho}_n \xrightarrow{p} -\infty, \quad (2.7)$$

if $d - d_0 = -0.5$.

$$\widehat{\rho}_n = O_p(n^{-1-2\delta}) \text{ and } n\widehat{\rho}_n \xrightarrow{p} -\infty, \quad (2.8)$$

if $-0.5 < d - d_0 < 0$.

$$\widehat{\rho}_n = O_p(n^{-1}) \text{ and } n\widehat{\rho}_n \Rightarrow \frac{\frac{1}{2}\left\{\mathbf{w}^2(1) - 1\right\}}{\int_0^1 \mathbf{w}^2(r)dr}, \quad (2.9)$$

if $d - d_0 = 0$.

$$\widehat{\rho}_n = O_p(n^{-1}) \text{ and } n\widehat{\rho}_n \Rightarrow \frac{\frac{1}{2}\mathbf{w}^2_{\delta,m+1}(1)}{\int_0^1 \mathbf{w}^2_{\delta,m+1}(r)dr}, \quad (2.10)$$

if $d - d_0 > 0$.

2) $t_{\widehat{\rho}_n}$ *verifies that*

$$t_{\widehat{\rho}_n} = O_p(n^{-0.5}\log^{-0.5} n) \text{ and } t_{\widehat{\rho}_n} \xrightarrow{p} -\infty, \quad (2.11)$$

if $d - d_0 = -0.5$.

$$t_{\widehat{\rho}_n} = O_p(n^{-\delta}) \text{ and } t_{\widehat{\rho}_n} \xrightarrow{p} -\infty, \quad (2.12)$$

if $-\frac{1}{2} < d - d_0 < 0$.

$$t_{\widehat{\rho}_n} = O_p(1) \text{ and } t_{\widehat{\rho}_n} \Rightarrow \frac{\frac{1}{2}\left\{\mathbf{w}^2(1) - 1\right\}}{\left[\int_0^1 \mathbf{w}^2(r)dr\right]^{1/2}}, \quad (2.13)$$

if $d - d_0 = 0$.

$$t_{\widehat{\rho}_n} = O_p(n^{\delta}) \text{ and } t_{\widehat{\rho}_n} \xrightarrow{p} +\infty, \quad (2.14)$$

if $0 < d - d_0 < 0.5$.

$$t_{\widehat{\rho}_n} = O_p(n^{0.5}) \text{ and } t_{\widehat{\rho}_n} \xrightarrow{p} +\infty, \quad (2.15)$$

if $d - d_0 \geq 0.5$.

where $\mathbf{w}_{\delta,m}(r)$ *is* $(m-1)$*–fold integral of* $\mathbf{w}_\delta(r)$ *recursively defined as* $\mathbf{w}_{\delta,m}(r) = \int_0^r \mathbf{w}_{\delta,m-1}(s)ds$, *with* $\mathbf{w}_{\delta,1}(r) = \mathbf{w}_\delta(r)$ *and* $\mathbf{w}(r)$ *is the standard Brownian motion.*

These properties and distributions are the generalization of those established by Sowell (1990) for the cases $-\frac{1}{2} < d-1 < 0$, $d-1 = 0$ and $0 < d-1 < \frac{1}{2}$. From (2.7) and (2.8), the rate at which $\widehat{\rho}_n = \widehat{\phi}_n - 1$ converge to zero (i.e. $\widehat{\phi}_n$ converge to 1) is slow for nonpositive values of $d - d_0$, particularly it is very slow for $-\frac{1}{2} < d - d_0 < -\frac{1}{4}$. Moreover for $-\frac{1}{2} < d - d_0 < 0$, the limiting distribution of $\widehat{\rho}_n$ has nonpositive support and then $\lim_{n\to\infty} P\left(\widehat{\phi}_n < 1\right) = 1$. From (2.9) and (2.10), $\widehat{\rho}_n$ converge to zero at the rate n, when $d \geq d_0$. The rate convergence n is faster than the usual standard rate $n^{\frac{1}{2}}$, when we deal with stationary $I(0)$ variables. Then, for $d - d_0 \geq 0$, the least squares estimate is super consistent. In the other words, if a first order autoregression model (2.1) is fitted to a sample of size n generated according an $ARFIMA(0, 1 + d - d_0, 0)$, where $1 + d - d_0$ is the order of integration of $\Delta^{-1+d_0} y_t$, then the OLS estimator, $\widehat{\phi}_n$, will not exceed 1 in probability, when $d - d_0 \geq 0$. Figure 1 and figure 2 below illustrates this fact in an obvious way.

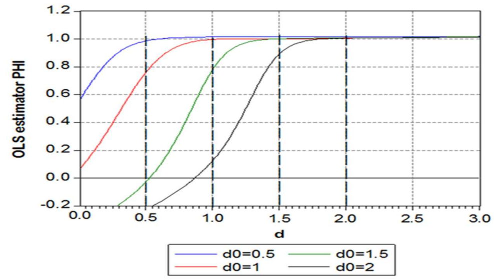

Fig. 1. Relation between the order of integration $d_0$of the process $\{y_t\}$and the OLS estimator $\widehat{\phi}_n$in the regression model $\Delta^{-1+d_0} y_t = \phi\Delta^{-1+d_0} y_{t-1} + \varepsilon_t$($d$ fixed and $d_0$varied)

Figure 1 shows that so long as $d - d_0 \geq 0$, we have $\widehat{\phi}_n = 1$, and so long as $d - d_0 < 0$, we have $\widehat{\phi} < 1$, where $\widehat{\phi}_n$ is the OLS estimator in the autoregression model (2.1).

For example, for $d_0 = 0.5$, we have,

$$\begin{cases} d - 0.5 < 0 \text{ and } \widehat{\phi}_n < 1 \text{ for } 0 \leq d < 0.5, \\ d - 0.5 \geq 0 \text{ and } \widehat{\phi}_n = 1 \text{ for } d \geq 0.5, \end{cases}$$

and for $d_0 = 2$, we have,

$$\begin{cases} d - 2 < 0 \text{ and } \widehat{\phi}_n < 1 \text{ for } 0 \leq d < 2, \\ d - 2 \geq 0 \text{ and } \widehat{\phi}_n = 1 \text{ for } d \geq 2. \end{cases}$$

Figure 2 shows that so long as $d - d_0 \geq 0$, we have $\widehat{\phi}_n = 1$, and so long as $d - d_0 < 0$, we have $\widehat{\phi} < 1$, where $\widehat{\phi}_n$ is the OLS estimator in the autoregression model (2.1).

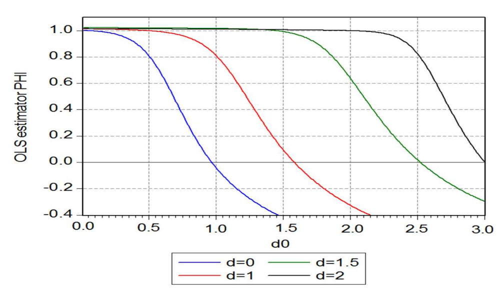

Fig. 2. Relation between the order of integration d_0 of the process $\{y_t\}$ and the OLS estimator $\widehat{\phi}_n$ in the regression model $\Delta^{-1+d_0} y_t = \phi\Delta^{-1+d_0} y_{t-1} + \varepsilon_t$ (d_0 fixed and d varied)

For example, for $d = 0.5$, we have,

$$\begin{cases} 0.5 - d_0 < 0 \text{ and } \widehat{\phi}_n < 1 \text{ for } 0 \leq d_0 < 0.5, \\ 0.5 - d_0 \geq 0 \text{ and } \widehat{\phi}_n = 1 \text{ for } d_0 \geq 0.5, \end{cases}$$

and for $d = 2$, we have,

$$\begin{cases} 2 - d_0 < 0 \text{ and } \widehat{\phi}_n < 1 \text{ for } 0 \leq d_0 < 2, \\ 2 - d_0 \geq 0 \text{ and } \widehat{\phi}_n = 1 \text{ for } d_0 \geq 2. \end{cases}$$

The figure 1 is made as follows: For a fixed sample $\{u_{1-n}, \cdots, u_0, \cdots, u_n\}$ generated from $i.i.d.(0,1)$, with $n = 1000$, samples of $ARFIMA(0, 1 + d - d_0, 0)$ processes were generated for d varying between 0 and 3 with step of 0.01

and d_0 fixed. The figure 2 is made as follows: For a fixed sample $\{u_{1-n}, \cdots, u_0, \cdots, u_n\}$ generated from $i.i.d.(0,1)$, with $n = 1000$, samples of $ARFIMA(0, 1+d-d_0, 0)$ processes were generated for d_0 varying between 0 and 3 with step of 0.01 and d fixed. For each sample $\{x_t, t = 1, \cdots, n\}$ a first order autoregression model (2.1) is fitted and estimate of ϕ are calculated. By plotting the parameter $\widehat{\phi}_n$ against the fractional parameter d one obtains the figure 1 and by plotting the parameter $\widehat{\phi}_n$ against the fractional parameter d_0 one obtains the figure 2 . A general procedure for generating a fractionally integrated series of length n is to apply for $t = 1, \cdots, n$, the formula $x_t = \sum_{j=0}^{t-1} \frac{\Gamma(d+1-d_0+j)}{\Gamma(d+1-d_0)\Gamma(j+1)} u_{t-j}$.

The relations, from one side, between $\widehat{\phi}_n$ and d and from another side between $\widehat{\phi}_n$ and d_0, highlighted by the results, (2.7), (2.8), (2.9), (2.10) and illustrated by figures 1 and 2, suggests that when we deal with degree of fractional integration test, we have,

$$\phi = 1 \Longleftrightarrow d \geq d_0 \quad \text{and} \quad \phi < 1 \Leftrightarrow d < d_0$$

In the other words, the testing problem $H_0 : \phi = 1$ against $H_1 : \phi < 1$ is equivalent to (1.5).

A. *How to use the New F-DF Test*

To use this test, we proceed as follows:

1) Estimate the parameter ρ in the regression model $\Delta^{d_0} y_t = .\rho\Delta^{-1+d_0} y_{t-1} + \varepsilon_t$. This regression model provides a more flexible and unified framework to test the null for different values of d_0, by using the same critical value.
2) The null hypothesis is rejected if $Z_i < c_i(\alpha)$, where Z_i is the usual statistic test $t_{\widehat{\rho}_n}$ or $n(\widehat{\rho}_n - 1)$.
3) The size of the test can be approximated by its asymptotic value: $\alpha = Sup_{d \geq d_0} P(Z_i < c_i(\alpha)) = P[Z_i < c_i(\alpha)/d = d_0]$.
4) The critical value $c_i(\alpha)$ can be chosen so as to achieve a predetermined size by using the usual table statistics of Dickey-Fuller.
5) To implement the test we don't need to estimate the parameter d.

III. Empirical Application

To illustrate in practice, how to use the F-DF test, we apply our procedure test to the Nelson-Plosser data set to provide a new evidence (Nelsson and Plosser (1982). The starting date is 1860 for consumer price index and industrial production; 1869 for velocity, 1871 for stock prices; 1889 for GNP deflator and money stock; 1890 for employment and unemployment rate; 1900 for bond yield, real wages and wages; and 1909 for nominal and real GNP and GNP per capita. The variables are expressed in natural logarithms. All variables exhibit an upward trend with the exception of velocity, which shows a strong downward trend and the unemployment rate, which tends to fluctuate around a constant level.

Since the empirical work by Nelson and Plosser (1982) suggests that there is strong evidence that the unit root hypothesis, for most macroeconomic time series data, cannot be rejected, two possible specifications data generating processes (DGP) are

$$y_t = (1-L)^{-d} u_t, \tag{3.1}$$

$$y_t = \beta_0 + (1-L)^{-d} u_t. \tag{3.2}$$

The theoretical framework provided in this paper, does not allow us to use DGP (3.2). At this level of theoretical framework we only use the DGP (3.1).

For DGP (3.1), we test the null for forth value of d_0, namely, 0; 0.5; 1; 1.5 and 2 by using respectively the following regression models,

$$y_t = \rho\Delta^{-1} y_{t-1} + \varepsilon_{1,t}, \qquad (model\ I)$$

$$\Delta^{0.5} y_t = \rho\Delta^{-0.5} y_{t-1} + \varepsilon_{2,t}, \qquad (model\ II)$$

$$\Delta y_t = \rho y_{t-1} + \varepsilon_{3,t}, \qquad (model\ III)$$

$$\Delta^{1.5} y_t = \rho\Delta^{0.5} y_{t-1} + \varepsilon_{4,t}, \qquad (model\ IV)$$

$$\Delta^2 y_t = \rho\Delta y_{t-1} + \varepsilon_{5,t}. \qquad (model\ V)$$

Note that all the size of the 14 United States annual macroeconomic variables of the Nelson-Plosser data used here are between $n = 80$ and $n = 129$, consequently the decision rules adopted for the testing problem (1.5), are,

$$\begin{array}{lcl} \text{reject } H_0 \text{ if } Z_1 & < & -7.9, \\ \text{reject } H_0 \text{ if } Z_2 & < & -1.95, \end{array}$$

where Z_1 and Z_2 are respectively the usual statistics test $n\widehat{\rho}$ and $\dfrac{\widehat{\rho}}{\sigma_{\widehat{\rho}}}$ and (−7.9,, −1.95) are the critical values, at size $\alpha =$ 5%, from the usual tables statistics of Dickey-Fuller (1979). The results shown in table 5 provide that

- for the model (I), all series are found to be integrated of order $d \geq 0$,
- for the model (II), all series are found to be integrated of order $d \geq 0.5$,
- for the model (III), all series are found to be integrated of order $d \geq 1$,
- for the model (IV), all series are found to be integrated of order $d < 1.5$, with exception of Industrial production and Money stock.
- for the model (V), all series are found to be integrated of order $d < 2$.

To summarize, all the variables are integrated for order d, with $1 \leq d < 1.5$, with exception of Industrial production and Money stock which have the value of integration order parameter d is greater or equal 1.5 and less than 2, i.e. $1.5 \leq d < 2$.

To reinforce these results we use the figure 2. We recall that to interpret the results of theorem 1, for a purely fractionally integrated process of order d, we use the relation between d_0 and $\widehat{\phi}_n$, where $d_0 \geq 0$ and $\widehat{\phi}_n$ is the OLS estimator in the autoregression model

$$\Delta^{-1+d_0} y_t = \phi\Delta^{-1+d_0} y_{t-1} + \varepsilon_t.$$

The same treatment is made to the Nelson-Plosser data for which orders d of integration are unknown. In the same graph we reproduce the curves of the simulated processes for $d = 1$, $d = 1.5$ and $d = 2$, the curves of the Bond yield, Nominal GNP, Real GNP, GNP per capita, Real wages, Stock prices, Unemployment, Velocity, Nominal wages, GNPdeflator, CPI, Employment. This set of variables are those for which the values of the integration parameter are between 1 and 1.5. Figure 3 shows that the curves of the empirical data are localized between the curves of the simulated processes for $d = 1$ and $d = 1.5$.

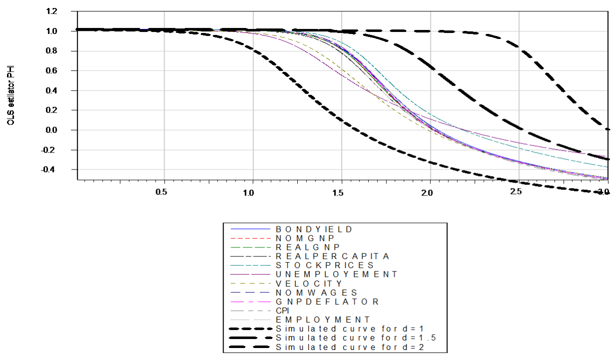

Fig. 3. Relation between d_0 and the OLS estimator $\widehat{\phi}_n$ in the regression model (2.1) for some US macroeconomic variables.

For the Industrial production and Money stock, for which the values of the integration parameter are between 1.5 and 2, Figure (9) show that the curves of the empirical variable "Money stock" is localized between the simulated curves $d = 1.5$ and $d = 2$**.** For the "Industrial production" even if the F-DF test indicates that the value of the integration order of this variable is between 1.5 and 2, the figure 4 show that the curve, represented by the relation between d_0 and the OLS estimator of the regression model (2.1), is localized between the simulated curve of $d = 1$ and $d = 1.5$.

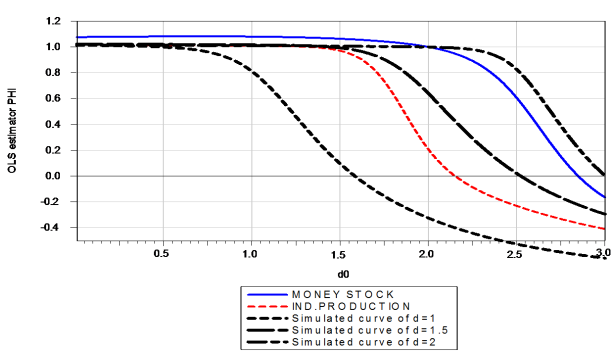

Fig. 4. Relation between d_0 and the OLS estimator $\widehat{\phi}$ in the regression model (2.1) for Money stock and Ind.production

We must not loose sight that the F-DF test was done assuming that the empirical variables are derived from data generating process $ARFIMA(0, d, 0)$. More general study is needed to achieve adequate conclusions about the integration order of the Nelson-Plosser Data, by considering more general data generating process $ARFIMA(p, d, q)$ and incorporating non zero drift and time trend in data generating process (1.3) and the auxiliary regression model (2.1).

IV. Conclusion

In this paper, we have proposed a consistent test that can distinguish between $FI(d)$ processes. The test is based on a composite null hypothesis, $H_0 : d \geq d_0$, rather than a simple one. To use this test, we proceed as follows:

1) Estimate the parameter ρ in the regression model $\Delta^{d_0} y_t = .\rho \Delta^{-1+d_0} y_{t-1} + \varepsilon_t$. This regression model provides a more flexible and unified framework to test the null for different values of d_0, by using the same critical value.
2) The null hypothesis is rejected if $Z_i < c_i(\alpha)$, where Z_i is the usual statistic test $t_{\widehat{\rho}_n}$ or $n(\widehat{\rho}_n - 1)$.
3) The size of the test can be approximated by its asymptotic value: $\alpha = Sup_{d \geq d_0} P(Z_i < c_i(\alpha)) = P[Z_i < c_i(\alpha)/d = d_0]$.
4) The critical value $c_i(\alpha)$ can be chosen so as to achieve a predetermined size by using the usual table statistics of Dickey-Fuller.
5) To implement the test we don't need to estimate the parameter d.
6) Regarding the Dickey-Pantula test, both upward and downward procedure are valid (see Dickey, Pantula 1987)

The new F-DF test is applied to the Nelson-Plosser Data. The empirical study on Nelson-Plosser Data is only made to illustrate the F-DF test. This article does not discuss the situation when there is short memory in series, of AR or MA type. This seems a very serious drawback for practical implementation of the tests. Here, we give just an indication when $y_t \sim ARFIMA(p, d, 0)$

$$A(L)\Delta^d y_t = \varepsilon_t,$$

where $A(L) = \sum_{j=0}^{p} \alpha_j L^j$, with $\alpha_0 = 1$ and the roots of $A(z) = 0$ are outside the unit circle and ε_t is defined as above. Then the fractional augmented Dickey-Fuller test, for the null hypothesis $d \geq d_0$, is based on the regression model

$$\Delta^{d_0} y_t = \rho \Delta^{-1+d_0} y_{t-1} + \sum_{j=0}^{p} \alpha_j \Delta^{d_0} y_{t-j} + \varepsilon_t.$$

More general study is needed to achieve adequate conclusions about the integration order of the Nelson-Plosser Data, by considering more general data generating process $ARFIMA(p, d, q)$ and incorporating non zero drift and time trend in data generating process (1.3) and the auxiliary regression model (2.1).

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