Trial-Based Tournament: Rank and Earnings

Pongou, Roland and Tchantcho, Bertrand and Tedjeugang, Narcisse

University of Ottawa, École Normale Supérieure (UY1), Université de Yaoundé I

1 June 2015

Trial-Based Tournament: Rank and Earnings

Roland Pongou, Bertrand Tchantcho and Narcisse Tedjeugang[1]

This Version: June 2015

Abstract

Trial-based tournament is a widespread hiring mechanism in organizations. Upon a job opening, an applicant is tried out at the job, then swaps with another competing applicant, and so on, with each non-competing worker holding the same position across trials. The job is offered to the applicant whose trial has had the most positive effect on the organization's output. We formalize this tournament model, deriving measures of relative performance that can be used to rank workers for *each job* and assess their comparative advantage when absolute performance cannot be observed. As a second goal, we study the relationship between tournament rank and earnings as determined by marginal productivity. We show that pay is a weakly increasing function of tournament rank, and we characterize organizations for which pay strictly reflects tournament rank and vice-versa. These organizations are *linear* and *top-down biased*, and they strictly include the popular class of von Neumann-Morgenstern organizations. The analysis implies that hierarchical organizations that promote fairness in pay should not have too many layers.

Keywords: Hierarchical organizations, trial-based tournament, tournament rank, marginal productivity, top-down biased organizations.

JEL Classification: C7, M5, D03, L2, G34

[1]Pongou is at the Department of Economics of the University of Ottawa and a visiting scholar in the Department of Economics of the Massachusetts Institute of Technology; Tchantcho is at École Normale Supérieure (UY1) and the Department of Economics of the Université de Cergy Pontoise; and Tedjeugang is at École Normale Supérieure (UY1).
We thank Sean Horan, Philippe Jehiel, Michael Peters, Andrew Postlewaite, and Aloysius Siow for useful discussions and comments. We also thank participants at the 2013 CEA Annual Meeting in Montreal for useful comments. Roland Pongou acknowledges research grant from the University of Ottawa.

Correspondence can be addressed to Pongou at: rpongou@uottawa.ca or rpongou@mit.edu.

1 Introduction

Trial-based tournament is a widespread mechanism used by managers, coaches and team leaders to make decisions on hiring, firing and promotion, and to organize labor. Our goal in this paper is twofold. First, we propose a simple model of this tournament, deriving specific measures of relative performance that can be used to rank workers for each job and assess their comparative advantage when absolute performance cannot be observed. Second, we study the relationship between tournament rank and earnings as determined by marginal productivity. The analysis provides a characterization of organizations for which earnings strictly reflect tournament rank.

A trial-based tournament is a hiring model that pits workers against each other. Upon a job opening, an applicant is tried out at the job, and swaps with another competing applicant, and so on, with each non-competing employee keeping the same role across trials. The job is offered to the applicant whose trial has had the most positive effect on the organization's output. This hiring mechanism is used when it is difficult, *a priori*, to predict the performance of a worker or how well he will fit within an organization. It is used by such organizations as soccer teams, manufactures, law firms, academic institutions, and governmental institutions.

We model this hiring process as a generalized *ladder tournament*. We consider an organization with a finite number of (potential) workers and jobs. Jobs are ranked based on their importance or prestige. Workers compete for higher-ranked jobs as follows. Consider two individuals p and q who desire to fill an open position indexed r in an organization. These workers initially occupy a lower-ranked position s.[2] We say that p beats q in position r relative to position s if aggregate output increases more as a result of moving p from s to r than moving q from s to r no matter how the other members of the organization are allocated to tasks. If p beats q in each position relative to positions that are lower-ranked, we say that p is globally more influential or productive than q.

Our model has several advantages. It yields task-specific measures of relative performance that we use to rank workers for each job. These ordinal measures can therefore be used to make decisions on hiring, firing, and promotion. They can also be used to assess the comparative advantage of each worker in a job. In fact, if a worker p dominates another worker q in a job r relative to a lower-ranked job s, it implies that q dominates p in s relative to r. This implication means that if p has a comparative advantage over q in r relative to s, then q has a comparative advantage over p in s relative to r. This is true, even if p has absolute advantage over q in both r and s.

Another appeal of the ordinal measures of productivity generated by our model is that they can be used to compare workers even when their absolute productivity cannot be observed. In certain organizations like soccer teams which involve complex social interactions and externalities in the production process, it might be extremely difficult to measure the real performance of a player or his contribution to a victory or a defeat. In such organizations, measures of relative productivity are needed to compare workers. For instance, assume that p and q are two strikers. If the team always wins whenever p plays and q does not, and loses or draws when q replaces p, we will say, following our approach to evaluating relative performance, that p is a better striker than q. But

[2] If s is a job within the organization, then p and q are current employees executing identical tasks and competing for a promotion to r, and if s is a position outside of the organization, then p and q are outside candidates competing for r.

in reality, it is possible that p is not "intrinsically" more skilled than q. It may just be that p is a better teammate able to motivate other players. As a consequence, the latter perform better when they play with p than when they play with q. It may also be that other players dislike q and sabotage his performance by not passing him the ball when he is in a favorable position to score, hence making him look a bad striker. All these scenarios and others are possible, but still, we care only about a player's effect on the aggregate output, as we cannot observe his inputs. In this sense, our comparative measure of productivity is really a measure of how a worker fits nicely into a given role within an organization, or a measure of how desirable or influential a worker is in a role within a group. In general, it is impossible to observe all the social interactions going on in a real-world organization, or their consequences on individual productivity, but it is easy to observe the output of the organization. Our methodology shows how a tournament can be designed in order to compare workers only by observing aggregate output. It follows that although individual inputs may not be totally observable, pitting workers against one another provides incentives to put forth sufficient effort to outperform competitors, which alleviates or eliminates problems of moral hazard and adverse selection. This enhances the appeal of our tournament model and confirms the several advantages of relative comparisons that have been acknowledged in the literature, especially in the context of tournament theory.

Although relative comparisons have several advantages, we know from Green and Stokey (1983) that there may be significant drawbacks from throwing away cardinal information on workers' performance. For this reason, as the second goal of the paper, we study the relationship between tournament rank (a measure of relative productivity) and earnings as determined by marginal productivity (or any monotonic transformation of marginal productivity). Marginal productivity indeed contain important information on a worker's absolute performance. Our analysis can therefore be viewed as trying to reconcile ordinal and cardinal theories of worker productivity. This analysis is challenging in environments like ours, where the production process involves complex social interactions and externalities. Just as in chain productions, a worker's performance is contingent upon the ability of other workers to deliver. This reality does not only pose a technical problem as to how to measure marginal productivity, but it also complicates the analysis of the relationship between relative productivity and earnings. The analysis also shows how having a comparative advantage in a job relative to a lower-ranked job predicts a worker's pay.

Comparisons of ordinal and cardinal concepts are in line with tournament theory. Our measures of relative productivity tell if a worker is more productive or desirable than another worker, but does not tell by how much the former outperforms the latter. On the other hand, a cardinal measure of compensation determines the wage gap between two workers. By comparing ordinal and cardinal measures, one only cares about whether a higher-ranked worker in a tournament earns more. As put by Lazear (1995), the "tournament model, taken literally, implies that the wage that goes to a vice-president is independent of the amount by which he exceeds the performance of the assistant vice president in winning the job." (p. 26).

In organizations in which absolute performance is hard to measure, the Shapley value is the only available theory that measures marginal productivity.[3] It was proposed by Shapley (1953) as a way

[3] We note that several other values have been proposed in the literature, including, for instance, the Banzhaf value, the Johnson value, the Holler-Packel value, the Andjiga-Berg value, and the Owen value. Of all these other values, only

of redistributing revenue in organizations that have only two levels of participation. It has since been widely studied and generalized to fit other types of environments (see, for example, Aumann and Shapley (1974), Roth (1988), Laruelle and Valenciano (2001), Holler and Owen (2001), Leech (2003), Freixas (2005a), Serrano (2013), and the references therein). The Shapley value as a compensation scheme is justified if workers' true inputs cannot be observed, and the only observed variable is aggregate outcome like in our model. The generalization of the Shapley value by Freixas (2005a) shows that it might also be used to redistribute revenue in a context where allocation of workers to tasks is somewhat random, maybe due to exogenous factors, or due to the fact that the production function is not entirely known to the manager so that experimentation in labor allocation becomes useful to determine the optimal allocation of workers to the tasks. We adapt this generalization to our context, and examine how a worker's pay reflects his rank as determined by our tournament model.

We find that earnings as given by the Shapley value only weakly reflect tournament rank. If two workers have the same rank, then they earn the same wage. If one worker has a higher rank than the other, the former will not earn less than the latter, but they may be treated identically. So a violation of the fairness principle in compensation is possible in certain organizations.

Our main result is the characterization of a large class of organizations in which a worker's pay strictly reflects his productivity rank. Such organizations have a *linear* and *biased* structure in that they discriminate between workers only in case of promotion to the top level or in case of firing.[4] Importantly, this class of organizations includes well-known subclasses, such as organizations whose production functions are von Neumann-Morgenstern. It also includes the class of organizations with three layers in the hierarchy. We note that this class of organizations can be a viewed as a superset of the class of von Neumann-Morgenstern organizations.

Studies have noted that some workers earn more than their marginal product and others earn less (Lazear (1995), Gabaix and Landier (2008)). We wonder whether there exist other revenue sharing mechanisms that agree with our measures of relative productivity but do not necessarily reflect marginal contributions like the Shapley value. We show that there exist infinitely many transformations of the Shapley value that reflect tournament rank, and that either redistribute income away from workers at higher levels towards those below them such as in a progressive tax system, or alternatively increase the income gap across the layers of hierarchy consistent with the notion of prize spread in tournament theory. When income is redistributed to achieve more equality, this redistribution does not alter income rank, and so it does not make a less productive worker richer than a more productive worker, which is consistent with the Pigou-Dalton principle (Sen (1973)). On the other hand, the notion of prize spread in tournament theory, which is consistent with large wage increases upon promotions, arguably motivates workers at each level of the hierarchy and those at lower levels.

the Banzhaf value has been extended to a framework similar to ours by Freixas (2005b). In results not shown here, we compare our tournament ranking to the Banzhaf value and find similar results as for the Shapley value (results are available and will be included upon request). We have chosen not to include this result in the present paper for expositional purposes and to keep the paper brief. Our focus on the Shapley value is also justified by the fact that it is the most popular notion of revenue redistribution in organizations, whereas the Banzhaf value is mostly used in political institutions to measure power.

[4] As noted above, we also find that the Banzhaf value of a worker reflects his tournament rank in the class of linear and top-bottom biased organizations (again, results are available and will be included upon request).

1.1 Contribution to the Related Literature

We propose a simple model of a *ladder tournament* that relies on job trial to rank workers even when their absolute performance is hard to observe. Some, but not all, of the performance relations derived from our tournament model generalize to hierarchical organizations the well-known influence relation (Isbell (1958), Taylor and Zwicker (1999), Diffo Lambo and Moulen (2002)). Our paper provides a new application and interpretation of this notion in the context of market organizations. Also, our comparative approach to the problem of evaluating relative productivity shares similar motivations with tournament theory, which essentially consists of pitting one worker against the other while ensuring that the physical environment in which they compete is identical (Lazear (1995), Lazear and Rosen (1981)). However, our approach differs significantly in its formalization, analysis, and practical implications. In particular, in our model, workers are not promoted based on their performance in the current job as is the case in standard tournament models. Within our framework, a worker is tried out at the job he wants to get while the other competing worker is kept in the initial position, and both swap roles afterwards. It is only after these trials that one worker is promoted based on a comparison of aggregate outputs. Also, scholars worry that standard tournament setups might lead workers to cooperate less as they are competing for the same slot, which would generally hurt productivity (e.g., Dye (1984)). A distinctive feature of our comparative approach however is in minimizing the potential for such conflicts. For instance, outside candidates competing for a position of assistant professor are usually invited to campus to be vetted separately. In this case, there is no potential for conflict that would hurt the department. The same holds for most sports teams and many other types of firms.

Relative comparisons have several advantages that have been acknowledged in the literature (e.g., Lazear (1995), Ehrenberg and Bognanno (1990)). Lazear (1995) argues that "much of the essence of personnel economics depends on relative comparisons rather than on absolute ones." (p. 3). Indeed, comparisons are used in important decisions such as awarding tenure, hiring, firing, promoting, and structuring worker compensation. Several scholars also argue that relative comparisons are better than absolute comparisons in many contexts (e.g., Lazear (1995), Devaro (2006)). With respect to promotion, Lazear (1995) argues that a worker gets promoted in a firm not necessarily by being "good", but by outperforming his peers. As a result, relative comparisons can also be used to incentivize workers by inducing them to work harder to win a promotion. Most importantly, relative comparisons rather than absolute comparisons are key when inputs are not easily observable, and when productivity may be affected by common factors that workers cannot control. As Lazear (1995) puts it:

> "...it may be easier to observe relative position than it is to observe absolute position.... Second relative comparisons difference out common noise that risk-averse workers may not like. For example, two salespeople may have a very poor day, not because they did not put forth sufficient effort but because the economy was bad, a condition over which they had no control. If relative compensation is used, the effect of the economy is the same on both individuals and so the individual who performs better will still end up receiving the higher level of compensation." (p. 25-26).

However, as already acknowledged, ignoring cardinal information may have significant drawbacks

(Green and Stokey (1983)), which is why we study the relationship between our performance relation and the Shapley value, which is a cardinal measure of worker productivity. Because the Shapley value also measures worker pay, our analysis can be viewed as a contribution to the literature on the relationship between wage and productivity. A cardinal result in neoclassical production theory states that a worker earns the value of his marginal productivity in a competitive economy. As intuitive as this result is, its underlying assumptions, including the continuity and the differentiability of the production function, make its applicability to our discrete and externality-prone environment difficult. Lazear (1995) argues that while such assumptions enable rigorous analyses, they are generally made for convenience and do not always reflect the real world. In particular, the treatment of labor as a continuous variable in standard production models implies that workers are perfect substitutes, which in turn implies that different tasks or jobs in an organization are "equivalent", making it unimportant to account for their role in the determination of wage. To this effect, Lazear (1995) writes:

> "The neoclassical theory of production gives no explicit role of jobs. Firms hire labor, combine it with capital, and produce output according to some production function. Labor is treated as a continuous variable. Furthermore the description of specific tasks assigned to a given worker is no less vague. This traditional view contrasts sharply with the way that managers view the firm. Human resources managers think in terms of slots or jobs, and think of these slots or jobs as being fundamental to the organization of the firm." (p. 77)

Our framework highlights the role of jobs as advocated by Lazear (1995), whose view is also consistent with other studies showing that jobs play an important role in the determination of wage (Baker, Gibbs and Holmstrom (1994a), Gibbons and Waldman (2006)), independently of the role of human capital (Mincer (1974)). In particular, the huge within-firm wage disparities between CEOs and other workers have attracted increasing attention (e.g., Gabaix and Landier (2008)), with some scholars even acknowledging that CEOs' pay exceeds any measure of their marginal product (Lazear (1995)). In line with this literature, we show that there exist infinitely many transformations of the Shapley value that respect productivity rank, and that increase the income gap across the layers of hierarchy consistent with the notion of prize spread.

Our characterization of organizations in which tournament rank predicts earnings based on marginal productivity is new. Indeed, this characterization introduces a new class of organizations, which are linear and top-down biased. The analysis shows that in such organizations, workers who have a comparative advantage in higher-ranked jobs receive greater pay. It is an extremely large class, with the class of von Neumann-Morgenstern organizations being a proper subset. It also includes the popular class of three-layer organizations, such as academic departments where faculty members are either full, associate or assistant professors. The findings have implications for the design of organizations that promote fairness in worker compensation. It implies that fairness-promoting organizations should not have too many layers.

1.2 Plan

The remainder of this paper is structured as follows. Section 2 introduces our model of a discrete multi-task organization. In Section 3, we propose a model of tournament based worker trial, and obtain the influence relations as a rule for ranking workers in such a tournament. In Section 4, we analyze the relationship between tournament rank and earnings as measured by the Shapley value, and characterize organizations for which the two concepts are ordinally equivalent. In Section 5, we show that our wage-productivity rank-order equivalence results continue to hold under infinitely many measures of income that do not reflect marginal productivity. We also discuss and conclude our study. We collect all the proofs in Section 6 to ensure clarity and readability.

2 Hierarchical Multi-Task Organizations

In this section, we introduce the notion of a multi-task organization. A *multi-task organization* is a list (N, T, f) where N is a non-empty finite set of workers, $T = \{T_1, ..., T_j\}$ a finite collection of finite sets of tasks with cardinality $|T| = j \geq 2$, and $f : T^N \longrightarrow \mathbb{R}$ a real-valued production function defined on the set of all task profiles T^N. More precisely, for every $k = 1, ...j$, T_k is a set of identical tasks. Denote by k a representative task of T_k. We will call all tasks in T_k tasks of type k. $x = (x_1, x_2, .., x_n) \in T^N$ denotes a task profile of N where x_i is the task type of worker i. The function f maps each allocation x of workers to the tasks into a real number which measures the aggregate output $f(x)$ of the organization.

We note that for any organization (N, T, f), T_1 might be viewed as the set of positions outside of the organization, so that any individual filling such a position is not an employee within the organization. If all individuals are in outside positions, this constitutes a situation of "inaction" for the organization, the corresponding task or labor allocation is $x = (1, 1, ..., 1)$ and $f(x) = 0$. Let y be a real number. We denote by $Q(y)$ the set of all task allocations x such that $f(x) = y$.

In certain organizations, tasks or roles are ordered in degree of "importance" or "prestige." We say that such organizations are *hierarchical*. More formally, a hierarchical organization is a list (N, T, f) where type T_1 tasks are less important than type T_2 tasks, type T_2 tasks less important than type T_3 tasks, and so on. We say that T is the organization *ladder* or *hierarchy*, and its elements are the *layers* or *levels* of the hierarchy. An organization in which tasks are not ordered is said to be *non-hierarchical*.

The notion of task "importance" has an important corollary in certain hierarchical organizations. It implies that if a worker were to move from a less important task to a more important task, then aggregate output would (weakly) increase as a result. This corresponds to the important concept known as *monotonicity* in the game theory literature. We formalize it as follows. A hierarchical organisation (N, T, f) is *monotonic* if for any task profile $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ such that $x \leq y$, $f(x) \leq f(y)$. Throughout the paper, we assume that organizations are monotonic.

Another interpretation of an organization (N, T, f) is that each worker has a "fixed" role, and in a profile $x = (x_1, x_2, .., x_n)$, x_i measures the number of hours worked by a worker i. In this case, one can define $T = \{T_0, ..., T_j\}$ where $x_i \in T_0$ means that worker i has worked 0 hours or is unemployed. The task profile corresponding to inaction is therefore $x = (0, 0, ..., 0)$. In our tournament model defined in the next section, workers compete for higher-ranked positions if the first interpretation of an organization is followed, whereas in the second interpretation, workers compete for more work time within their predetermined role (as would be the case for instance in a soccer team). We mostly follow the first interpretation, but all our results are valid if the second interpretation instead is followed.

We also note that the production function f of an organization (N, T, f) may not be known *a priori*, which would justify the use of trial-based tournaments to extract information on workers' relative performance or comparative advantage. In fact, if f were known, then a manager would simply need to choose the optimal labor allocation[5] and therefore would not need to try workers. It follows that our model of a trial-based tournament in the next section assumes that the production function of an organization is not known *a priori* by its manager. Also, that model is also valid for non-monotonic organizations too. The monotonicity assumption is only needed in Section 4 where we examine how tournament rank predicts earning.

3 A Trial-Based Tournament

3.1 Model

We propose a simple model of trial-based tournament in hierarchical organizations as follows. Let r be a vacant position in a hierarchical organization (N, T, f), and p and q two candidates for that position. p and q maybe current employees employed a lower level s $(r > s > 1)$, or outside candidates, thus occupying position 1. p is first tried out at r, and then swaps with q, who is tried in turn. In practice, the trial might last a certain period of time. Each of the other workers keeps the same position across trials. The winner is the job candidate whose trial has the greatest positive effect on aggregate output no matter how labor was allocated among the other workers.

If p beats q, we say that p is locally more influential (or more productive) than q in job r relative to job s, or simply that p is more $(r, s)-$ influential than q, and we denote this by $p \succeq_{(r,s)} q$. We say that p is globally more influential than q if p is more $(r, s)-$ influential than q for any tasks s and r , and this is denoted by $p \succeq q$. Therefore, $\succeq$ yields the global ranking of the tournament.

Our comparative approach to evaluating worker productivity therefore yields relative measures of performance that allow to rank workers for each job. These relations are more formally defined below. We denote by $e^p = (0, ..., 0, 1, 0, ..., 0)$ the pth unit n-component vector.

Definition 1 *Let (N, T, f) be a hierarchical organization, and p and q two workers.*

1) Let s and r be two tasks such that $r > s$. p is said to be more $(r, s)-$ influential than q, denoted $p \succeq_{(r,s)} q$, if: $\forall x \in T^N$ such that $x_p = x_q = s$, $f(x + (r - s)e^p) \geq f(x + (r - s)e^q)$.

2) p is said to be more globally influential than q, denoted $p \succeq q$, if: $p \succeq_{(r,s)} q$ for all $s, r \in T$ such that $r > s$.

The symmetric and asymmetric components of each relation $\succeq_{(r,s)}$ are denoted by $\succ_{(r,s)}$ and $\sim_{(r,s)}$, respectively, and those of the relation $\succeq$ by $\succ$ and $\sim$, respectively.

Several features of our model of a trial-based tournament should be highlighted. According to the definition, p is more $(r, s)-$ influential than q if the output achieved by the organization

[5] Note that since the set T^N is finite, an optimal labor allocation for any function f always exists.

after moving p from s to r in any labor allocation x in which p and q are occupying s (that is, $f(x + (r - s)e^p))$ is always greater than if q instead of p is moved from s to r, which is translated by $f(x + (r - s)e^p) \geq f(x + (r - s)e^q)$.[6] Interestingly, the domination relation $\succeq_{(r,s)}$, by providing a measure of relative performance in job r relative to job s, can be used to assess the comparative advantage of each worker in each job. In fact, if a worker p dominates another worker q in a job r relative to a lower-ranked job s, it implies that q dominates p in s relative to r. In other words, if p has a comparative advantage over q in r relative to s, then q has a comparative advantage over p in s relative to r. It follows that a worker cannot have a comparative advantage over another worker in r relative to s and in s relative to r simultaneously, even if he is more competent in both positions in absolute terms.

It is also useful to remark that in a labor allocation x, the input of a worker is not measured, and it might be hard to measure in real-world organizations. We only observe the position of each worker and the organization's output $f(x)$. If a worker p occupying position s in x is moved to a position r, resulting in allocation $x + (r - s)e^p$ and aggregate output $f(x + (r - s)e^p)$, we will be unable to tell whether the marginal change in the output, which is $f(x+(r-s)e^p)-f(x)$, is attributable to the only action of p. In fact, if the marginal change in the output is positive, it may be that moving p from his old position to the new one has motivated other workers to work harder or created better social interactions, translating into higher aggregate output. Our methodology therefore acknowledges the fact that individual inputs might be very hard to measure (as also acknowledged by Lazear (1995)), and relies only on workers' positions and aggregate output to assess relative performance and comparative advantage in each job. In this sense, our approach can be used to compare workers even when their absolute productivity cannot be observed.

Another feature of our model is that it can be viewed as a mechanism to collect information on job applicants. As noted above, there might be noise in the applicants' performance or in other team members' performance, making it hard to evaluate an applicant's ability or "absolute" contribution to the organization. The use of Shapley value in Section 4, however, addresses part of this issue, as the Shapley value can be viewed as measuring absolute performance.

The following example illustrate the ordinal measures of performance given above.

Example 1 *A hierarchical organization* (N, T, f) *involves three workers 1, 2, and 3 and three task types. The range of its production function is* $\{0, 2\}$*, and the isoquants are defined as follows:* $Q(2) = \{x \in T^N : (3,1,1) \leq x$ *or* $(2,3,1) \leq x$ *or* $(2,2,1) \leq x\}$*, and* $Q(0) = \{x \in T^N : x \leq (2,2,2)$ *or* $x \leq (1,3,3)$ *or* $x \leq (2,1,3)\}$*.*

We derive the following structure of productivity rank among the workers:

$1 \sim_{(2,1)} 2$, $1 \succ_{(2,1)} 3$, $2 \sim_{(2,1)} 3$; $1 \succ_{(3,1)} 2$, $1 \succ_{(3,1)} 3$, $2 \succ_{(3,1)} 3$; $1 \sim_{(3,2)} 2$, $1 \succ_{(3,2)} 3$, $2 \sim_{(3,2)} 3$; *and* $1 \succ 2$, $1 \succ 3$, $2 \succ 3$.

For instance, we note that workers 1 and 2 have the same comparative advantage in tasks 2 relative to task 1, but globally, 1 dominates 2, and 2 dominates 3.

We also define below an ordinal measure of productivity via a fixed allocation of labor.

[6] If one assumes that there are exogenous shocks to productivity that affect aggregate output, $f(x)$ can be interpreted as the average output of the organization corresponding to the labor allocation x. Similarly, if a worker p is moved from an initial position s to a higher position r, the aggregate output of the organization is likely to fluctuate during the trial period, and in this case, $f(x + (r - s)e^p)$ is the average output of the organization during that period.

Definition 2 *Let (N,T,f) be a hierarchical organization, p and q two workers and $x \in T^N$.*

Let s and r be two tasks such that $r > s$. p is said to be more $(r,s)-$ influential than q via x, denoted $p \succeq_{(r,s,x)} q$, if: $x_p = x_q = s$ and $f(x + (r-s)e^p) \geq f(x + (r-s)e^q)$.

The symmetric and asymmetric components of each of these relations are denoted by $\succ_{(r,s,x)}$ and $\sim_{(r,s,x)}$, respectively.

The binary relations $\succeq_{(r,s,x)}$ are useful to the sources of the comparisons given by the relations $\succeq_{(r,s)}$ and $\succeq$. If a worker p is strictly more $(r,s)-$ influential than another worker q, for instance, then we can decompose the ranking $p \succ_{(r,s)} q$ by the binary relations $\succeq_{(r,s,x)}$ to identify those labor allocations x via which $p \succ_{(r,s,x)} q$ and those via which $p \sim_{(r,s,x)} q$.

3.2 Some Useful Classes of Organizations

We now introduce some popular classes of organizations. We define linear organizations, von Neumann-Morgenstern organizations, and top-down biased organizations.

A linear organization is one in which workers can be completely ranked following the trial-based tournament. The notion of linearity is formally defined below.

Definition 3 *A hierarchical organization (N,T,f) is said to be linear if the global influence relation $\succeq$ is reflexive, transitive and complete.*

An example of a linear organization is the organization defined in Example 1. Pay differences between workers can be rationalized by the notion of linearity in the sense that higher-ranked workers should earn more.

We also define the class of von Neumann-Morgenstern organizations, which are organizations that have only two layers of the hierarchy.

Definition 4 *A hierarchical organization (N,T,f) is said to be von Neumann-Morgenstern if $|T| = 2$.*

Familiar readers can remark that when $|T| = 2$ for an organization (N,T,f), the aggregate output function f can be shown to be equivalent to the notion of a transferable utility function introduced by von Neumann and Morgenstern (1944). In the standard definition, there is a function which maps a coalition into its worth. Within our framework, such a coalition is replaced by the task profile $x = (x_1, ..., x_n)$ where $x_p \in \{1, 2\}$; $x_p = 1$ means that worker p is outside of the coalition, and $x_p = 2$ means that worker p is inside.

This class of functions constitutes the cornerstone of game theory. It is also used by Shapley and Shubik (1967) to illustrate production in a wide class of economies in the real world. Although very popular, this class of functions may not provide an accurate description of certain real-world organizations, as organizations usually have several layers of hierarchy (e.g., academic departments in certain countries have three levels, where a faculty member is either an assistant, associate or full professor) or different levels of involvement (e.g., the number of working hours).

The third class of organizations is called *top-down biased*. These are organizations in which workers who are dominant at intermediate levels are also dominant at the top level or at the bottom level. We formally define this notion below.

Definition 5 *A hierarchical organization* (N, T, f) *is said to be top-down biased if for any tasks* r *and* s *such that* $r > s$*, and for any* $x \in T^N$, $\succ_{(r,s,x)} \subseteq \succ_{(k,1,x)}$ *or* $\succ_{(r,s,x)} \subseteq \succ_{(j,k,x)}$ *for some* k *such that* $1 < k < j$.

The class of top-down biased organizations is extremely large. We will show that it strictly includes the large class of von Neumann-Morgenstern organizations and the class of three-layer organizations. We will also show that top-down biased organizations effectively discriminate between workers of different qualities in terms of earnings.

4 Tournament Rank and Earnings

In this section, we study the relationship between tournament rank and earnings as determined by marginal productivity. As we argued in the introduction, marginal productivity in our environment is measured by the Shapley value (see, e.g., Shapley and Shubik (1967) for von Neumann-Morgenstern organizations). We adapt Freixas' (2005a) generalization of the Shapley value to our environment.

4.1 Definition of the Shapley Value

Let (N, T, f) be a hierarchical organization, and $\{z_1, z_2, ..., z_k\}$ the range of f. Without loss of generality, we assume $z_1 > z_2 > ... > z_k$. Denote by Q_N the set of all the bijections defined from N onto $\{1, ..., n\}$. An element of Q_N is a possible order in which workers are hired in the organization (workers move from a type T_1 task to another task type or they stay in T_1). Q_N and T^N are probabilistic spaces. The probability of a given order is assumed to be $\frac{1}{n!}$ and that of a given task profile is assumed to be $\frac{1}{j^n}$. A worker is assigned any of the j representative tasks or jobs in the organization with probability $\frac{1}{j}$. An ordered labor allocation of N is an ordered pair $R = \left(sR, x^R\right)$, where $sR \in Q_N$ and $x^R \in T^N$. In other words, an ordered labor allocation R specifies an order in which workers are hired (i.e. sR) and an allocation of workers to the tasks (i.e. x^R). The space of all ordered labor allocations $Q_N \times T^N$ is a probabilistic space, with all ordered labor allocations having an equal probability $\frac{1}{n! j^n}$.

The i-pivotal worker of R for $i = 1, 2, ..., k-1$, denoted by i-piv(R, f), is uniquely defined either as:

(a): the worker whose action in R clinches the outcome of x^R under at least the output level z_i, or

(b): the worker whose action in R clinches the outcome of x^R under at most the output level z_{i+1}.

In other words, i-piv(R, f) is the first worker in sR who satisfies one of the two following mutually exclusive conditions:

(1) independently of how tasks are allocated among all subsequent workers, the outcome will be f_h with $f_h \geq f_i$, or

(2) no matter how all subsequent workers were to change their actions, the final outcome would be no greater than f_{i+1}.

For every $i \in \{1, 2, ..., k-1\}$, denote by $\mathcal{R}^+_{ip}(f)$ the set of all the ordered labor allocations for which worker p is i-pivotal. The Shapley value of a worker p is the weighted average number of ordered labor allocations in which p is i-pivotal for every $i \in \{1, 2, ..., k-1\}$. The formal definition is below:

Definition 6 *Let (N, T, f) be a hierarchical organization, and p a worker. The Shapley value of p is:*

$$\phi_p(f) = \frac{1}{n! j^n(z_1)} \sum_{i=1}^{k-1} (z_i - z_{i+1}) \left| \left\{ R \in Q_N \times T^N : p = i\text{-}piv(R, f) \right\} \right| .$$

As an application of this definition, we compute the Shapley value of each worker in Example 2, finding that this value is $\frac{44}{27}$ for worker 1, $\frac{8}{27}$ for worker 2, and $\frac{2}{27}$ for worker 3. We note that these values agree with the productivity ranking induced by the global influence relation.

We answer the question of whether earnings as measured by the Shapley value strictly reflect productivity rank. In other words, does a more productive workers always earn more than a less productive worker? The answer is "no" in general, especially if the number of layers of hierarchy is more than three. We illustrate this point through the example below.

Example 2 *A hierarchical organization (N, T, f) involves two workers and four tasks. Its production function is defined in the table below.*

profile of task (x)	$f(x)$
$(1,1)$	0
$(2,1)$	0
$(3,1)$	0
$(4,1)$	0
$(2,1)$	0
$(1,3)$	0
$(1,4)$	0
$(2,2)$	0
$(3,2)$	1
$(4,2)$	1
$(2,3)$	0
$(2,4)$	1
$(3,3)$	1
$(4,3)$	1
$(3,4)$	1
$(4,4)$	1

We note that worker 1 is strictly more productive than worker 2, but the Shapley value is $\frac{1}{2}$ for each of them. Therefore, the Shapley sharing rule does not always reward productivity.

4.2 When do Earnings Strictly Reflect Tournament Rank?

We characterize organizations in which earnings as measured by the Shapley value strictly reflect tournament rank.

The following proposition states that equally ranked workers have the same pay.

Proposition 1 *Let (N, T, f) be a hierarchical organization, and $p, q \in N$ two workers. Then,*

$$p \sim q \Longrightarrow \phi_p(f) = \phi_q(f)$$

In order to prove our second result, we need the lemma below shown in Pongou, Tchantcho and Tedjeugang (2015). It states that a more productive worker is more frequently pivotal than a less productive worker.

Lemma 1 *Let (N, T, f) be a hierarchical organization, and $p, q \in N$ two workers. Then, $p \succeq q \Longrightarrow \left|\mathcal{R}_{ip}^{+}\right| \geq \left|\mathcal{R}_{iq}^{+}\right|$ for any $i \in \{1, \ldots, k-1\}$.*

It follows from Lemma 1 that a more productive worker cannot earn less than a less productive worker, as shown below.

Proposition 2 *Let (N, T, f) be a hierarchical organization, and $p, q \in N$ two workers. Then,*

$$p \succeq q \Longrightarrow \phi_p(f) \geq \phi_q(f).$$

The message of Proposition 2 is encouraging. However, it does not solve the problem arising from the possibility of two workers earning the same wage despite having different productivities (Example 2). Our main goal is to characterize organizations for which the performance structure and the wage structure coincide. The following result states that relative performance is adequately rewarded in top-down biased organizations.

Theorem 1 *Let (N, T, f) be a top-down biased hierarchical organization, and $p, q \in N$ two workers. Then,*

$$p \succ q \implies \phi_p(f) > \phi_q(f).$$

4.3 Equivalence between Tournament Rank and Pay Rank

In this section, we show tournament rank and pay rank coincide for linear and top-down biased organizations. Denote by $\succeq_S$ the preorder induced on the set of workers by the Shapley value. This preorder is total and ranks workers according to their pay. We have the following result:

Theorem 2 *Let (N, T, f) be a linear and top-down biased hierarchical organization. $\succeq$ and $\succeq_S$ coincide (that is, $p \succ q \iff \phi_p(f) > \phi_q(f)$ and $p \sim q \iff \phi_p(f) = \phi_q(f)$).*

We note that Theorem 2 on the equivalence between payment ranking and performance ranking has several practical implications. First, performance ranking, captured by the global influence relation, implies that higher-ranked workers occupy higher-level positions in the organization. Note that our tournament model compares the relative performance of workers in each job relative to a lower-ranked job and implies that a globally more performant worker should be placed in a higher-level position. In this sense, our tournament model differs from the traditional tournament theory which focuses only one job and the promotion of only one worker. Second, the fact that more performant workers earn more in top-down biased organizations (Theorems 1 and 2) therefore means that workers placed in higher-level positions are better treated, which is consistent with the spirit of tournament theory. Third, Theorem 2 implies that wage dispersion is possible among workers in the same layer of an organization. In fact, if the number of workers is greater than the number of layers, it is clear that at least one layer will have workers of different (relative) performance. Given that performance determines pay (Theorems 1 and 2), workers occupying the same layer may earn different salaries.

An important implication of Theorem 2 is that in all linear von Neumann-Morgenstern organizations the tournament rank and the pay rank coincide.

Proposition 3 *Let (N, T, f) be a linear von Neumann-Morgenstern organization. $\succeq$ and $\succeq_S$ coincide.*

The proof follows from the fact that all von Neumann-Morgenstern organizations are top-down biased.

We also show that tournament rank and pay rank coincide in three-layer linear organizations due to the fact that these organizations are top-down biased.

Proposition 4 *Let (N, T, f) be a three-layer linear organization. $\succeq$ and $\succeq_S$ coincide.*

As already mentioned, Propositions 2 and 3 are proved by simply showing that von Neumann-Morgenstern organizations and three-layer organizations are top-down biased. It follows that top-down biased organizations are numerous in real life. For instance, all the practical organizations studied in Shapley and Shubik (1967) are von Neumann-Morgenstern organizations, and so are top-down biased. These organizations include organizations where only labor, or both labor and capital, are used in the production process. Shapley and Shubik (1967) assume that a worker either works or does not work, or a capital is used or is not used (that is, $|T| = 2$). Their framework therefore does not acknowledge the fact that the level of involvement in the production process might differ across workers within an organization. In real-life organizations, some individuals might work full time, others part time, and others might not work (which is a three-layer organization). Our framework covers the framework of Shapley and Shubik (1967) as well as situations where workers have more flexibility in their work schedule or in the amount of work time (that is, $|T| = j \geq 2$).

The fact that certain organizations that have more than three layers are not top-down biased has practical implications for the design of organizations that promote fairness in pay. Theorem 2 implies that fair organizations should not have too many layers. Of course, there always exist top-down biased organizations of any number of layers. But since one cannot anticipate whether an organization will top-down biased if one allows it to have more than three layers, it is better to allow two or three layers to always guarantee that pay strictly reflects tournament rank, which implies fairness in compensation.

Interestingly, as noted in the introduction, Theorems 1-2 and Propositions 1-3 all hold when we replace the Shapley value by the Banzhaf value. Results are available and will be included in the present draft upon request. We have chosen not to include those results for expositional purposes. We also remark that concepts other than the Shapley value and the Banzhaf value exist in the literature. However, they have not been generalized to our framework yet, which is one reason why we do not study them. Our goal was not to generalize those other values. Our focus on the Shapley value is justified by the fact that it is the only value that is regarded as measuring marginal productivity. Our choice is therefore natural in the context of market organizations.

5 Conclusion

We have provided a simple model of trial-based tournament, a widespread mechanism used by organizations to hire and organize workers. This mechanism consists of pitting workers against each other by trying them at the job they would like to get. We derive specific measures of relative performance that allow to rank workers for each job and assess their comparative advantage when absolute performance cannot be observed and when the production process is characterized by complex and unmeasurable social interactions and externalities.

We have also studied the relationship between tournament rank and earnings as determined by marginal productivity. Marginal productivity in our environment is measured by the Shapley value (Freixas (2005a)). The Shapley value as a compensation scheme is justified if the allocation of workers to the tasks is somewhat random (Doeringer and Piore (1971), Reder (1955)), maybe due to exogenous factors, or due to the fact that the production function is not entirely known to the manager so that experimentation in labor allocation becomes useful, as is the case in most sports teams. We have found that the Shapley value only weakly reflects tournament rank. Hence, a weak violation of fairness in pay is possible. Our main result is to provide a characterization of organizations for which pay and tournament rank have the same ordinal structure. These organizations are linear and top-down biased in the sense that they discriminate between workers mostly in case of promotion to the highest level or in case of firing. This class of organizations is extremely large, and it includes important subclasses, such as the class of von Neumann-Morgenstern organizations and the class of three-layer organizations. The findings imply that organizations that promote fairness in earnings should not have too many layers.

Our analysis also extends to compensation schemes that do not reward workers based on their marginal productivity. For instance, consider a hierarchical organization (N, T, f), and the following compensation scheme of the organization's revenue:

$$w_p(f) = \frac{(\phi_p(f))^x}{\sum_{q \in N} (\phi_q(f))^x}$$

where $w_p(f)$ is the payoff to worker p under the sharing rule w, $\phi_p(f)$ is worker p's earnings measured by the Shapley value, but expressed as the fraction of the organization's total revenue that goes to p, and x a strictly positive real number. We can show that w is a rank-preserving transformation of ϕ in that if p earns more than q under the $\phi-$rule, then p earns more than q under the $w-$rule, and vice-versa. If $x = 1$, then $w_p(f) = \phi_p(f)$ for any p, which implies the same wage gap under the $w-$rule as under the Shapley compensation scheme. If $x < 1$, then the wage gap between workers is smaller under the $w-$rule than under the $\phi-$rule, with workers at the bottom of the organization ladder earning more than their marginal product, and those at the top earning less. This is consistent with the Pigou-Dalton progressive tax system in which income is transferred from workers at higher levels to those below them to achieve more equality (Sen (1973)). Some scholars argue that within organizations, such a revenue-sharing mechanism might promote harmony and more cooperative behavior (see, e.g., Lazear (1989)). On the other hand, if $x > 1$, then the wage gap between workers at different levels of the organization ladder is greater under the $w-$rule than under the $\phi-$rule, with workers at the bottom earning less than their marginal product, and those at the top earning more. This is consistent with the notion of prize spread in tournament theory, which implies large wage increases upon promotions, and which is intended to motivate workers at each level of the hierarchy and those at lower levels (Lazear (1995)).

To be more concrete, let us examine an application to Example 1. If the organization's revenue is redistributed according to the Shapley rule, worker 1 will earn about 81.5% of the total output, worker 2 14.8%, and worker 3 3.7%. However, if one wants to narrow the income gap, one could apply one of the $w-$rules above, choosing $x = \frac{1}{2}$ for instance. The output share would then become 61% for worker 1, 26% for worker 2, and 13% for worker 3, indeed narrowing the income gap between workers. If, on the contrary, one wants to increase the income gap between workers, one could choose $x = 2$, leading to the output share of 96.6% for worker 1, 3.2% for worker 2, and 0.2% for worker 3.

In general, since under the $w-$rule, x can take on any value, there are infinitely many transformations of the Shapley value that either redistribute income away from workers at higher levels towards those at lower levels, or alternatively, increase the income gap across the layers of hierarchy. Hence, under the assumptions of Theorem 2, these compensation schemes strictly reflect tournament rank.

6 Proofs

6.1 Proposition 1

Proof. If $p \sim q$, then p and q are interchangeable in any task profile, hence p and q have the same Shapley value. ∎

A few notation, definitions and preliminary results will be needed for the other proofs.

Let $R = \left(sR,\ x^R\right)$ be an ordered labor allocation. Let p be a worker. $R_p^j = (sR_p^j,\ x^{R_p^j})$ is the ordered labor allocation in which the order in which workers enter the organization coincides with that of R (i.e. $sR_p^j = sR$) and worker p as well as any worker q who enters before p occupies the exact same position as in R (i.e. $x_q^{R_p^j} = x_q^R$ if $sR\,(q) \leq sR\,(p)$) whereas workers who enter after p are assigned the highest position (i.e. $x_q^{R_p^j} = j$ if $sR\,(q) > sR\,(p)$). Similarly, $R_p^1 = \left(sR_p^1, x^{R_p^1}\right)$ is the ordered labor allocation in which the order in which workers enter the organization coincides with that of R (i.e. $sR_p^1 = sR$) and worker p as well as any worker q who enters before p occupies the exact same position as in R (i.e. $x_q^{R_p^1} = x_q^R$ if $sR\,(q) \leq sR\,(p)$) whereas workers who enter after p are assigned the lowest position (i.e. $x_q^{R_p^1} = 1$ if $sR\,(q) > sR\,(p)$). These definitions are formalized in Notation 1 below, where we also define the set of all ordered labor allocations in which a worker is pivotal.

Notation 1 *Let (N, T, f) be a hierarchical organization, $R = \left(sR,\ x^R\right)$ an ordered labor allocation, and p a worker.*

i) We denote by $R_p^j = (sR_p^j,\ x^{R_p^j})$ the ordered labor allocation defined by:

$$sR_p^j = sR, \text{ and } \forall q \in N,\ x_q^{R_p^j} = \begin{cases} x_q^R & \text{if } sR\,(q) \leq sR\,(p) \\ j & \text{if } sR\,(q) > sR\,(p)\,. \end{cases}$$

ii) We denote by $R_p^1 = \left(sR_p^1, x^{R_p^1}\right)$ the ordered labor allocation defined by:

$$sR_p^1 = sR, \text{ and for all } q \in N,\ x_q^{R_p^1} = \begin{cases} x_q^R & \text{if } sR\,(q) \leq sR\,(p) \\ 1 & \text{if } sR\,(q) > sR\,(p)\,. \end{cases}$$

iii) For any $i = 1,\ 2,\ .\ .\ .,\ k-1$, $\mathcal{R}_{ip}^+(f)$ denotes the set of all the ordered labor allocations for which worker p is i-pivotal. We pose:

$$\mathcal{R}_p\,(f) = \bigcup_{i=1}^{k-1}\ \mathcal{R}_{ip}^+\,(f)\,.$$

6.2 Proposition 2

Proof. Suppose that $p \succeq q$. It follows from Lemma 1 that $\left|\mathcal{R}_{ip}^+\right| \geq \left|\mathcal{R}_{iq}^+\right|$ for any $i \in \{1, ..., k-1\}$. But

$$\phi_p\,(f) = \frac{1}{n!j^n\,(z_1)} \sum_{i=1}^{k-1} (z_i - z_{i+1}) \left|\mathcal{R}_{ip}^+\right|$$

and

$$\phi_q(f) = \frac{1}{n!j^n(z_1)} \sum_{i=1}^{k-1} (z_i - z_{i+1}) \left|\mathcal{R}_{iq}^+\right|.$$

Given that $\left|\mathcal{R}_{ip}^+\right| \geq \left|\mathcal{R}_{iq}^+\right|$ for any $i \in \{1,...,k-1\}$, it follows that $\phi_p(f) \geq \phi_q(f)$. ■

6.3 Theorem 1

To prove Theorem 1 requires a definition and a preliminary result.

Definition 7 *Let (N, T, f) be a hierarchical organization, and $p, q \in N$ two workers.*

1) Let $x, y \in T^N$ be two task profiles.y is said to agree with x but at $\{p, q\}$ if:

$$\forall h \in N \setminus \{p, q\},\ x_h = y_h \text{ and } y_p = y_q.$$

We denote by $T_x^N(p, q)$ the set of all task profiles which agree with x but at $\{p, q\}$.

2) (N, T, f) is said to satisfy condition C_1 if for any task profile x such that $x_p = x_q = s$:

$$\begin{aligned} \forall r &\neq s, f(x + (r-s)e^p) \neq f(x + (r-s)e^q) \\ &\Longrightarrow \exists y \in T_x^N(p,q) : \left\{ \begin{array}{c} f(y + (1-s)e^p) \neq f(y + (1-s)e^q) \\ or \\ f(y + (j-s)e^p) \neq f(y + (j-s)e^q) \end{array} \right. \end{aligned}$$

We have the following straightforward lemma.

Lemma 2 *A hierarchical organization (N, T, f) is top-down biased if and only if it satisfies condition C_1.*

Proof of Theorem 1

Proof. Let (N, T, f) be a top-down biased hierarchical organization. Suppose that $p \succ q$, and let us show that $\phi_p(f) > \phi_q(f)$. Since $\forall i \in \{1,...,k-1\}$, $\left|\mathcal{R}_{ip}^+\right| \geq \left|\mathcal{R}_{iq}^+\right|$, it suffices to prove that there exists some $i \in \{1,...,k-1\}$ such that $\left|\mathcal{R}_{ip}^+\right| > \left|\mathcal{R}_{iq}^+\right|$. We will then determine an $i \in \{1,...,k-1\}$ for which $\psi_{pq} : \mathcal{R}_{iq}^+ \longrightarrow \mathcal{R}_{ip}^+$ is not surjective.

$p \succ q$ is equivalent to $p \succeq q$ and not$(q \succeq p)$ by definition. The following implication is true:

$$\text{not}(q \succeq p) \Longrightarrow \left\{ \begin{array}{l} \exists x \in T^N,\ x_p = x_q = s \quad s \in \{2,...,j\}, \\ \exists r > s, f(x + (r - x_p)e^p) = z_l > f_m = f(x) \text{ and } f(x + (r - x_q)e^q) < z_l \end{array} \right.$$

which implies

$$f(x + (r - x_q)e^q) < f(x + (r - x_p)e^p).$$

It therefore follows that

$$f(x + (r - x_q)e^q) \neq f(x + (r - x_p)e^p).$$

Under condition C_1, it follows that there exists $y \in T_x^N(p,q)$ such that:

$$f\left(y+(1-y_p)\,e^p\right) \neq f\left(y+(1-y_q)\,e^q\right) \text{ or } f\left(y+(j-y_p)\,e^p\right) \neq f\left(y+(j-y_q)\,e^q\right).$$

Without loss of generality, suppose that $f\left(y+(j-y_p)\,e^p\right) \neq f\left(y+(j-y_q)\,e^q\right)$. Given that $p \succeq q$, we have

$$f\left(y+(j-y_p)\,e^p\right) < f\left(y+(j-y_q)\,e^q\right).$$

Pose $y_p = y_q = s$. Let $r \leq s$. Then, because $p \succeq q$, we have:

$$f\left(y+(r-y_p)\,e^p\right) \geq f\left(y+(r-y_q)\,e^q\right).$$

Therefore

$$f(y+(r-y_p)\,e^p) \geq f(y+(r-y_q)\,e^q) \geq f(y+(j-y_q)\,e^q) > f(y+(j-y_p)\,e^p).$$

Pose $f(y+(j-y_q)\,e^q) = y_i$. Then $i \in \{1,...,k-1\}$. Consider $\Psi_{pq} : \mathcal{R}_{iq}^+ \longrightarrow \mathcal{R}_{ip}^+$.

Let us find $R \in R_{ip}$ such that R has no preimage through ψ_{pq}. Remark that the possible preimages of $R \in R_{ip}$ are R_{pq} and R_{pq}^0.

We will therefore find $R \in R_{ip}$ such that $R_{pq} \notin R_{iq}$ and $R_{pq}^0 \notin R_{iq}$.

Consider an ordered labor allocation $R = \left(sR; x^R\right)$ such that $sR\,(p) = n$ and $x^R = y + (r-y_p)\,e^p$. Then, $R \in R_{ip}$ because

$$f\left(x^R\right) \geq z_i$$

and

$$f(x^{R_p^j} + \left(j - x_p^{R_p^j}\right) e^p) = f(y+(j-y_p)\,e^p) < z_i$$

a) Suppose $R_{pq} \notin R_{iq}$.

In fact,

$$sR_{pq}\,(q) = n;\; x^{R_{pq}} = y + (r-y_q)\,e^q$$

and

$$x^{(R_{pq})_q^j} + \left(j - x_q^{(R_{pq})_q^j}\right) e^q = y + (j-y_q)\,e^q$$

and

$$f(y+(j-y_q)\,e^q) = z_i \geq z_i.$$

That is,

$$f\left(x^{R_{pq}}\right) \geq z_i \text{ and } f(x^{(R_{pq})_q^j} + \left(j - x_q^{(R_{pq})_q^j}\right) e^q) \geq z_i.$$

b) Suppose $R_{pq}^0 \notin R_{iq}$

In fact,

$$x^{R_{pq}^0} = y + (r-y_p)\,e^p.$$

Hence

$$f\left(x^{R_{pq}^0}\right) \geq z_i.$$

But because $r \leq s$, we have:

$$x^{\left(R^0_{pq}\right)^j_q} + \left(j - x_q^{\left(R^0_{pq}\right)^j_q}\right) e^q = y + (j - y_q)\, e^q + (r - y_p)\, e^p > y + (j - y_q)\, e^q.$$

Hence

$$f(x^{\left(R^0_{pq}\right)^j_q} + \left(j - x_q^{\left(R^0_{pq}\right)^j_q}\right) e^q) \geq z_i$$

because

$$f(y + (j - y_q)\, e^q + (r - y_p)\, e^p) \geq f(y + (j - y_q)\, e^q) = z_i.$$

We then conclude that R has no preimage through ψ_{pq}. Therefore, ψ_{pq} is not surjective, and hence, $|_i R_p| > |_i R_q|$.

Given that

$$\begin{cases} \forall j \in \{1, ..., k-1\}, |_j R_p| \geq |_j R_q| \text{ and} \\ \exists i \in \{1, ..., k-1\} : |_i R_p| > |_i R_q| \end{cases}$$

it follows that

$$\phi_p(f) = \frac{1}{n! j^n (z_1)} \sum_{i=1}^{k-1} (z_i - z_{i+1}) \left|\mathcal{R}^+_{ip}\right| > \frac{1}{n! j^n (z_1)} \sum_{i=1}^{k-1} (z_i - z_{i+1}) \left|\mathcal{R}^+_{iq}\right| = \phi_q(f).$$

■

6.4 Theorem 2

Proof. Assume that an organization is linear and top-down biased. The fact that $\succeq$ and $\succeq_S$ coincide immediately follows from Propositions 2 and 3 and from the fact that $\succeq$ is total. ■

6.5 Propositions 3 and 4

Proof. Let (N, T, f) be a linear hierarchical organization such that $|T| = 2$ or $|T| = 3$. It suffices to show (N, T, f) is top-down biased.

We will show only the case of $|T| = 3$.

Let $x \in T^N$ and $p, q \in N$ such that $x_p = x_q = s$.

Let $r \neq s$ such that

$$f(x + (r - x_p)\, e^p) \neq f(x + (r - x_q)\, e^q).$$

Find $y \in T^N_x(p, q)$ such that

$$f\left(y + (1 - y_p)\, e^p\right) \neq f\left(y + (1 - y_q)\, e^q\right) \text{ or } f\left(y + (j - y_p)\, e^p\right) \neq f\left(y + (j - y_q)\, e^q\right).$$

Case 1: Suppose that $s = 1$ and $r \in \{2, 3\}$.

- If $r = 2$, then $f\left(x + (t_2 - x_p)\, e^p\right) \neq f\left(x + (t_2 - x_q)\, e^q\right)$.

Pose $y \in T^N$ such that $y_p = y_q = 2$; $y_a = x_a$ for any $a \in N \setminus \{p, q\}$.

It follows that

$$y \in T_x^N(p,q) \text{ and } y + (1 - y_p)\, e^p = x + (t_2 - x_q)\, e^q$$

and

$$y + (1 - y_q)\, e^q = x + (t_2 - x_p)\, e^p.$$

Given that

$$f\left(x + (t_2 - x_p)\, e^p\right) \neq f\left(x + (t_2 - x_q)\, e^q\right)$$

it follows that

$$f\left(y + (1 - y_p)\, e^p\right) \neq f\left(y + (1 - y_q)\, e^q\right).$$

- If $r = 3$, then $f\left(x + (t_3 - x_p)\, e^p\right) \neq f\left(x + (t_3 - x_q)\, e^q\right)$. We can take $y = x$.

Case 2: Suppose that $s = 2$ and $r \in \{1, 3\}$.
- If $r = 1$, then $f\left(x + (1 - x_p)\, e^p\right) \neq f\left(x + (1 - x_q)\, e^q\right)$, and we take $y = x$.
- If $r = 3$, then $f\left(x + (t_3 - x_p)\, e^p\right) \neq f\left(x + (t_3 - x_q)\, e^q\right)$, and we take $y = x$.

Case 3: Suppose that $s = 3$ and $r \in \{1, 2\}$.
- If $r = 1$, then we take $y = x$.
- If $r = 2$, then pose $y \in T^N$ such that $y_p = y_q = t_2$; $y_a = x_a$ for any $a \in N \setminus \{p, q\}$.
We have

$$y \in T_x^N(p,q) \text{ and } y + (j - y_p)\, e^p = x + (t_2 - x_q)\, e^q$$

and

$$y + (j - y_q)\, e^q = x + (t_2 - x_p)\, e^p.$$

Given that

$$f\left(x + (t_2 - x_q)\, e^q\right) \neq f\left(x + (t_2 - x_p)\, e^p\right)$$

it follows that

$$f(y + (j - y_p)\, e^p) \neq f(y + (j - y_q)\, e^q).$$

We conclude that any hierarchical organization (N, T, f) such that $|T| = 3$ satisfies condition C_1. Following Lemma 2, we conclude that (N, T, f) is top-down biased. ■

References

[1] Aumann, R.J., and Shapley, L.S. (1974): Values of non-atomic games, Princeton University Press, Princeton.

[2] Baker, G.P., Gibbs, M., and Holmstrom, B. (1994): "The wage policy of a firm," *Quarterly Journal of Economics* 109(4), 921-955.

[3] DeVaro, J. (2006): "Internal promotion competitions in firms," *RAND Journal of Economics* 37(3), 521-542.

[4] Diffo Lambo, L. and Moulen, J. (2002): "Ordinal equivalence of power notions in voting games," *Theory and Decision* 53, 313-325.

[5] Doeringer, P., and Piore, M. (1971): Internal labor market and manpower analysis. Lexington, MA: D.C. Heath.

[6] Dye, R.A. (1984): "The Trouble with Tournaments," *Economic Inquiry* 22(1), 147-150.

[7] Ehrenberg, R.G., and Bognanno, M.L. (1990): "Do tournaments have incentive effects?," *The Journal of Political Economy* 98(6), 1307-1324.

[8] Freixas, J. (2005a): "The Shapley-Shubik power index for games with several levels of approval in the input and output," *Dec. Support System* 39, 185-195.

[9] Freixas, J. (2005b): "Banzhaf measures for games with several levels of approval in the input and output," *Annals of Operations Research* 137, 45-66.

[10] Gabaix, X., and Landier, A. (2008): "Why has CEO pay increased so much?" *Quarterly Journal of Economics* 123(1), 49-100.

[11] Gibbons, R., and Waldman, M. (2006): "Task-specific human capital," *American Economic Review* 94(2), 203-207.

[12] Green, J.R., and Stokey, N.L. (1983): "A comparison of tournaments and contracts," *Journal of Political Economy* 91(3), 349- 364.

[13] Holler, M.J., and Owen, G. (2001): Power indices and coalition formation. Kluwer Academic Publishers.

[14] Isbell, J.R. (1958): "A class of simple games," *Duke Math. J.* 25, 423–439.

[15] Laruelle, A. and Valenciano, F. (2001): "Shapley-Shubik and Banzhaf indices revisited," *Mathematics of Operations Research* 26, 89 - 104.

[16] Lazear, E.P. (1995): Personnel economics. Cambridge, MA and London

[17] Lazear, E.P., Rosen, S. (1981): "Rank-order tournaments as optimum labor contracts," *Journal of Political Economy* 89, 841-864.

[18] Leech, D. (2003): "Computing power indices for large voting games," *Management Science* 49(6), 831-837.

[19] Mincer, J. (1974): Schooling, experience, and earnings. New York: Columbia University Press for NBER.

[20] Pongou, R., Tchantcho, B., and Tedjeugang, N. (2015): "A ladder tournament," Mimeo.

[21] Reder, M.W. (1955): "Theory of occupational wage differentials," *American Economic Review* 45, 833-852.

[22] Roth, A. (1988): The Shapley value: Essays in honor of Lloyd S. Shapley. Cambridge University Press.

[23] Sen A.K. (1973): On Economic Inequality, Oxford: Clarendon Press.

[24] Serrano, R. (2013): "Lloyd Shapley's matching and game theory," *Scandinavian Journal of Economics* 115, 599-618.

[25] Shapley, L.S. (1953): "A Value for n-person Games," in Contributions to the Theory of Games, volume II, by H.W. Kuhn and A.W. Tucker, editors. Annals of Mathematical Studies v. 28, pp. 307–317. Princeton University Press, 1953.

[26] Shapley, L.S, and Shubik, M. (1954): "A model for evaluating the distribution of power in a commitee system," *Amer. Polit. Sci. Rev.* 48, 787-792.

[27] Shapley, L.S, and Shubik, M. (1967): "Ownership and the production function," *Quarterly Journal of Economics* 81(1), 88-111.

[28] Taylor, A.D, Zwicker, W.S., 1999. Simple games: desirability relations, trading and pseudoweightings. New Jersey, USA: Princeton University Press.

[29] von Neumann, J., and Morgenstern, O. 1944. Theory of games and economic behaviour. Princeton University Press, Princeton.