Networked Politics: Political Cycles and Instability under Social Influences

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Abstract

Media, opinion leaders, co-ethnics, family members, and friends influence our political decisions. The ways in which these influences affect political cycles and (in)stability has been understudied. We propose a model of a networked political economy, where agents' choices are partly determined by the opinions of the individuals with whom they are connected in a fixed influence network. The model features two types of individuals: *ideological* individuals who never change their views and who seek to influence the rest of the society; and *non-ideological* individuals who have no political allegiance and do not influence anybody, but who can be influenced by ideological individuals with whom they are connected. We show that influence networks increase political turnout and cause non-ideological individuals who are subject to antagonistic influences to keep changing their political views. This in turn increases political cycles and instability in two ways: (1) by reducing the number of stable and popular political leaders; and (2) by worsening the tradeoff between political competition and the existence of a stable leader.

We uncover a necessary and sufficient condition that characterizes all of the political technologies and network structures that guarantee political stability. This condition introduces a *preference-blind stability index*, which maps each pair of a constitution and an influence network into the maximum number of competing political leaders that a society can afford while remaining stable regardless of the extent of preference heterogeneity in its population.

Our findings have testable implications for different societies. They shed light on the network origins of political cycles in two-party systems. They also imply that individualist societies are more politically stable than collectivist societies and societies organized around ethnic groups or characterized by a high level of homophilous behavior and influences. For ethnic democracies, we quantify the exact tradeoff between political competition and stability, and show that ethnic fragmentation increases stability. This latter finding further provides a rationale for using the "divide and rule" strategy for maintaining power. Finally, we find that cliques and multi-layer cliques maximize the competition-stability tradeoff, whereas star networks, lines and rings minimize it.

Keywords : Political cycles, instability, influence networks, homophily, ethnic democracy, competition-stability tradeoff.

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1 Introduction

Recent political events including the Arab Spring, the Occupy protest movement, and the 2014 Ukrainian revolution have demonstrated the powerful role that social media and networks can play in destabilizing a society and potentially bringing about political change. Social networks have also recently proven to be an essential tool for political marketing in developed societies, as they are being used by political leaders and parties to raise political awareness, garner support, and mobilize significant shares of the population to the polls. The importance of social networks for political participation has also been demonstrated by the amount of effort that certain governments invest into restricting access to communication technologies, including telephone and Twitter communications, during elections in order to prevent the coordination of street protests. In general, the people with whom we are connected– including opinion leaders, public and private media, friends, colleagues, co-ethnics, and family members– have a substantial influence on our political choices. Therefore, the impact of social networks on the extent to which the status quo is preserved or is overturned is important.

In this paper, we propose a formal analysis of the role of social networks in political cycles and (in)stability. Given the high human and economic costs of political instability, it is important to analyze the factors that make some societies more stable than others. Our analysis reveals that the network structure of a society plays a crucial role in its level of stability, in addition to its constitution and its level of political competition. We find that social networks increase political instability in two ways: (1) by reducing the number of political leaders who can govern the society in a stable manner; and (2) by increasing the tradeoff between political competition and the existence of a stable leader. We are able to quantify this tradeoff, while also showing how it depends on the prevailing constitution (or political technology) and on the network structure of a society. The findings have testable implications for the comparative political economy of countries. For instance, they reveal that collectivist societies and societies organized around ethnic groups or characterized by a high level of homophilous behavior are more prone to political instability than individualist societies. The findings also shed light on the network origins of alternation in power in two-party systems, thus addressing an important limitation of the classical model of a political economy. Furthermore, we provide a characterization of social networks that maximize or minimize the tradeoff between political competition and stability, showing, for example, the destabilizing properties of such popular networks as cliques and multi-layer cliques, and the stabilizing role of star networks, rings, and lines.

1.1 Overview of the Model

We augment the classical model of a political economy by incorporating social networks. Within the classical framework, a *political economy* is a human society endowed with a constitution and a collection of political leaders or parties. Each leader has a distinct political platform, and therefore promotes a different vision of how the society should be run. Citizens then form preferences over leaders. A leader is said to be *unpopular* or *unstable* if his platform is less preferred by a constitutional majority than that of another leader.[2]

The main question addressed within the classical framework is that of the existence of a stable leader. This important problem has been considered by a wide range of scholars, at least starting with Black (1948). Their different analyses, which uncover structural conditions for the existence of a stable leader, have provided important insights into the reasons why some societies are more politically stable than others.

[2] This notion of stability is classic (see, for example, Black (1948), Varian (1992, page 424)).

We consider the problem of the existence of a stable leader in a context where each individual's political views might be influenced by the opinions of other individuals.[3] We assume the existence of a weighted directed influence network on which we impose no particular structure.[4] The model features two types of individuals: *ideological* individuals (such as leftists and rightists in certain democracies like the United States), who have strong political views that they never change and who seek to influence the rest of the society; and *non-ideological* individuals who have no political allegiance and do not exert any influence on their neighbors, but can be influenced by ideological individuals with whom they are connected. During an election, a non-ideological individual is a citizen who abstains unless he is persuaded by an ideological individual to cast a ballot in favor of the leader or the party that he supports. The neutrality of a non-ideological person may arise from the fact that he does not know the political leaders well enough to discriminate among them, from the fact that he is clearly not interested in politics, or from the fact that he likes a particular aspect of each leader but does not like any leader in all aspects. Within our framework, however, neutrality only applies locally, as an individual might have strong preferences over a pair of politicians while being neutral over a different pair.

Our simple model of influence is inspired by Acemoglu et al. (2013). A non-ideological individual can only be influenced by his ideological neighbors. If all his ideological neighbors have the same political view, then he will follow it. However, if his ideological neighbors have opposing views, his ultimate decision will reflect the cumulative weight of the links that connect him with the proponents of each view. In other words, a non-ideological individual fully internalizes the possibly opposing opinions among his influencers, and this endows him with what can be characterized as a "fuzzy opinion" of each leader (see Zaddeh (1965, 1971) for a first formalization of the notion of fuzzy language and preferences). Such an individual therefore favors a leader x over a competitor y to a degree determined by the cumulative weight of his ideological influencers who strictly prefer x over y, and favors y over x to a possibly different degree. In the context of elections, this fuzziness translates into the frequency with which he favors one leader over a competitor. For instance, depending on the relative weight of his influencers who hold a particular view, he might vote for x against y in two-thirds of all of the electoral contests between the two, and for y against x in one-third of the contests. Our approach to modelling influence characterizes non-ideological individuals essentially as *swing voters*. The behavior of such individuals is qualitatively close to that of "regular" agents in Acemoglu et al. (2013) as the latter regularly alternate their beliefs if they are connected with "stubborn" agents who hold opposing views. This approach differs from models in which an individual adopts a particular view if this view is held by the majority of his neighbors (see, for example, Granovetter (1978)).

In order to study the influence of networks on political stability, we extend the classical notion of stability to our context. First, we introduce a new network-based measure of the likelihood of political instability associated with each leader. This measure provides a description of the relative (un)popularity of competing politicians. Building on this measure, we introduce the notion of the *fuzzy equilibrium set*, which describes the probabilistic stability of each leader against each of his competitors. It can be used to compute the

[3] See, for example, Lazarsfeld, Berelson and Gaudet (1944), Katz and Lazarsfeld (1955), and Kearns et al. (2009) for empirical evidence on political influence. Acemoglu, Hassan and Tahoun (2014) also show that, during Egypt's Arab Spring, the number of protesters in Tahrir Square was increasing with the number of tweets generated by Egyptian Twitter users to mobilize masses. Also, a paramount chief from Kono district in Sierra Leone, answering a question about whether he was able to exert any influence on people's voting decisions in elections, said: "if I say left they go left, if I say right they go right" (Acemoglu, Reed and Robinson (2013). There is a broad empirical literature showing the effect of networks on opinion formation.

[4] The weight of an influence link may have several interpretations. It might be a measure of the extent to which an individual trusts another individual with whom he is connected (see Acemoglu et al. (2013)), the level of persuasion that an opinion leader, a village chief, or a television channel exerts on an uninformed individual; or the amount of money spent on advertising by a political party to win the support of a non-ideological individual.

probability of a politician being defeated by a competitor in a pairwise election, or the frequency at which political parties alternate or rotate in power in a two party-system like that in the United States.

We also define the *certainty equilibrium set*, which is the set of political leaders who are stable against each of the other leaders with a probability equal to one. Obviously, the politicians in this set are those whose leadership is sought after by the society, especially given the high desirability of political stability. A society is said to be *politically stable* if its certainty equilibrium set is never empty regardless of the preferences of the people and their social influences.

1.2 Overview of the Results

We have *descriptive*, *predictive*, and *prescriptive* findings. In order to determine the causes of political instability in a society, we examine the economic properties of its *certainty equilibrium set*. We rationalize this set by showing that each of its elements is maximal with respect to a binary relation that generalizes the majority relation used in Black (1948) (Theorem 1). We also show that the certainty equilibrium set can be empty. This possibility implies that a society can be thrust into an endless cycle of leadership unpopularity and instability. We find that political instability is determined by the constitution, the network structure, and the level of political competition within the society. A comparative statics analysis implies that, holding the constitution and the number of competing leaders constant, increasing the number of social connections increases the level of political turnout, which in turn increases the likelihood of political change and instability. This increases the probability of an initially stable leader becoming unstable due to network influences, and thus reduces the size of the certainty equilibrium set. Social networks therefore refine the certainty equilibrium set (Theorem 2). Although this refinement might be viewed as a positive property of networks if one cares about the uniqueness of the equilibrium, it might also lead to a complete social destabilization by eradicating leaders who can govern the society in a stable manner, as is illustrated in Examples 1 and 2. These examples show that a peaceful society can easily slip into a cycle of political instability due to a change in the structure of its social network.

The high desirability of political stability forces us to identify conditions on the structure of a political economy under which stability is guaranteed. Put another way, what are the forms of constitution, the number of competing political leaders, and the network structures that guarantee political stability in a society regardless of the extent of diversity in the political views of its people? Our analysis leads to a necessary and sufficient condition under which political stability is always achieved (Theorem 3). This result introduces a *preference-blind index of political stability*, a function that maps each pair of a constitution and a social network into the maximum level of political competition (measured by the number of competing politicians or parties) that a society can afford while remaining politically stable. Conversely, this condition characterizes or prescribes, for each desired or exogenous level of political competition, all of the pairs of constitutions and network structures that guarantee political stability regardless of the diversity and dynamics of the opinions in the population.

It follows that, if the constitution, the network structure, and the level of political competition in a society are known, then our characterization result can predict whether or not the society will always be stable. Importantly, we do not need to know the (future) political preferences of the people to make this prediction. If only the constitution and the network structure of the society are known, then our analysis prescribes the number of competing political leaders or parties that should not be exceeded for the society to remain stable regardless of the extent of preference diversity in the population. Conversely, for societies that have traditions regarding the number of political parties, our characterization result permits the identification of all of the network structures and forms of constitution that ensure political stability. This identification might be computationally challenging, but it is not impossible within our framework.

Our main characterization result can also be viewed as quantifying the tradeoff between political competition and political stability. It shows that this tradeoff depends on the network structure and on the political technology or constitution of a society. The analysis reveals that a high level of social connection and a large number of competing leaders are two major threats to political stability. Similar to the finding that shows that increasing the number of social connections reduces the certainty equilibrium set (Theorem 2), we conduct another comparative statics analysis that shows that increasing the number of social connections in a society decreases its preference-blind index of political stability (Theorem 4), and thus worsens the tradeoff between political competition and the existence of a stable leader. As is explained in the next section, we compute this tradeoff for several types of societies, while at the same time highlighting the testable implications of our findings.

1.3 Some Applications and Testable Implications

We present some applications and testable implications of our model, and feature some popular networks that we find maximize or minimize political instability.

Political cycles in two-party systems. Several countries have a system in which two major parties dominate the political arena. The ways in which social networks affect political cycles in two-party systems has been understudied. Our findings imply that, in the absence of social networks, change or alternation in power is not possible, with the incumbent party remaining in power forever. Of course, this is a highly implausible outcome, as alternation is observed in reality. We find the presence of influence networks to be a source of political cycles in two-party systems (Proposition 1). Cycles arise from the fluctuating behavior of non-ideological voters who are subject to antagonistic influences. Our model further allows one to compute the frequency with which parties alternate in power, and this depends on the structure of the prevailing influence network. The analysis addresses an important limitation of the classical model of a political economy, which implies that it takes at least three parties to create a political cycle.

Collectivist societies, ethnic democracies, and homophilous influences. Our analysis also has implications for how social interactions affect political (in)stability in collectivist versus individualist societies. The amount of interaction that exists among people varies widely across societies. Some societies are characterized by a high prevalence of loneliness and individualism, whereas others, such as ethnic societies, are characterized by a high level of interaction among members of the same ethnic group (see, for example, Greif (1994), Rothwell (2010), Barth (1969)). Our analysis implies that collectivist or ethnic societies are less likely to be politically stable than individualist societies (Theorem 5).

In particular, in ethnic societies in which individuals have equal voting rights, we quantify the exact tradeoff between political competition and stability, and also show how this tradeoff depends on the number of ethnic groups. Our working assumption is that an individual can only be influenced by co-ethnics (see Greif (1994) for a justification of this assumption). An ethnic group is also viewed as a group of people who have similar characteristics and exhibit homophilous behavior, thus making it easier for them to influence each other.[5] We find that, under certain natural conditions, a society is politically stable if and only if

[5] McPherson, Smith-Lovin and Cook (2001) argue that *homophily*–the tendency to associate with individuals with similar characteristics–"limits people's world in a way that has powerful implications for the information they receive, the attitudes they form, and the interactions they experience." They identify homophily in race and ethnicity as leading to the "strongest divides" in society, which is consistent with Currarini, Jackson and Pin (2008) who show strong homoliphy in race in a sample of high-school students in the United States.

the number of its competing political leaders or parties does not exceed $\frac{2p-r}{n-q} - 1$, where p is the number of ethnic or homophilous groups, r is the number of individuals who are lonely and so are not subject to any ethnic influence, n is the total size of the population, and q ($q > \frac{n}{2}$) is the number of votes required to pass a decision (Theorem 6). Lonely individuals can be regarded as individuals who are emancipated from their ethnic group and therefore cannot be influenced by co-ethnics. It is clear that, as the number of such individuals increases, there is a relaxation in the tradeoff between political stability and the number of competing political leaders. In particular, if everybody is free from ethnic influences (that is, $r = p = n$), then the maximum number of political leaders that a society can afford while remaining stable is $\frac{n}{n-q}$ (the mysterious Peleg number) minus 1, which is clearly greater than $\frac{2p-r}{n-q} - 1$.

The analysis additionally shows that splitting a fixed population to create a larger number of minor ethnic groups (that is, increasing p) will result in more political stability, as population fragmentation increases the stability index and therefore relaxes the competition-stability tradeoff. In other words, countries with many minor ethnic groups like Cameroon are more likely to be stable than countries that have only two major ethnic groups like Rwanda or Burundi. This conclusion is especially true if the number of competing political parties or leaders is much smaller than the number of ethnic groups. In fact, if the number of political parties equals or exceeds the number of ethnic groups, then our analysis implies that political instability is likely because, in that case, the number of parties is greater than the political stability index (Theorem 3). Interestingly, it follows from empirical data that countries in which the number of parties generally exceeds the number of ethnic groups are countries that have only a small number of ethnic groups like Rwanda. Although Cameroon, for instance, has over 250 minor ethnic groups, it has only four or five major political parties, which might explain its relative stability. Importantly, our prediction that ethnic fragmentation makes society safer is consistent with empirical research on the effect of fractionalization on internal conflicts (Collier and Hoeffler (1998)).

Instability-maximizing influence networks. Our analysis shows that certain networks maximize political instability by maximizing the tradeoff between political competition and the existence of a stable leader. Such networks include cliques and multi-layer cliques. Multi-layer cliques reflect the structure of a hierarchical organization in which individuals in each layer influence those in lower-level layers without the inverse being true. We show that, under such networks, the index of political stability is two, which implies that societies organized as a multi-layer cliques are highly prone to political instability.

Another interpretation of the finding is that a society organized as a multi-layer clique experiences a high prevalence of alternation in power, even if there are only two political parties. This finding is surprising, as it holds for any democratic rule, including rules that are known to be strongly biased toward upholding the status quo, such as rules close to the unanimity rule. Under the closest rule to the uninamity rule ($q = n-1$) in particular, the index of political stability is n under the empty network, which means that there exists a political leader who will stay in power forever if the number of competing leaders does not exceed $n-1$. The fact that the stability index suddenly drops from $n-1$ to two in the presence of a clique or a multi-layer clique shows the powerful influence of this network structure on political cycles. Within our framework, cycles are induced by the changing opinions of non-ideological individuals who are subject to the influence of opposing ideological views.

Instability-minimizing influence networks. Our analysis also identifies networks that minimize political instability by minimizing the competition-stability tradeoff in a democracy. We find that directed stars, rings, and lines have this desirable property. A directed star, for instance, depicts a communication system in which there is only one major source of propaganda–the hub. This network structure therefore limits the propagation of opposing political views, which ensures a certain level of stability. This finding might explain why certain regimes maintain heavy control over private media in order to limit competition with government-owned media. For instance, following the transition to competitive democracy in the 1990s in most African countries, in each of these countries, there usually was only one television or radio channel offering national coverage; this channel was owned and used by the ruling party to influence the political views of the people. Media liberalization has been extremely slow, perhaps reflecting the desire of political rulers to avoid instability in power.

1.4 Plan of the Paper

This paper is organized as follows. Section 2 discusses the contributions of our analysis to the related literature. Section 3 introduces the concept of a networked political economy and presents our simple model of influence within networks. Section 4 formalizes the notion of political stability and presents the main findings. Section 5 shows applications to familiar societies and political systems, and also identifies networks that maximize or minimize the competition-stability tradeoff in democracies. Section 6 concludes. All the proofs are presented in an appendix.

2 Contributions to the Related Literature

Our study bridges the literature on formal political economy and the literature on social networks. These two areas are extremely broad and have developed separately. The literature on political stability has developed at least since Condorcet (1785), who showed that a stable political leader may not exist in a society governed by the majority rule. Subsequent studies uncovered the conditions under which a stable leader exists. Black (1948) shows that political stability is guaranteed if voters have single-peaked preferences, a condition later generalized by Dummett and Farquharson (1961). Peleg (1978) considers a more general preference domain, and proves a necessary and sufficient condition on the maximum number of competing leaders for political stability to be guaranteed. This paper mainly focuses on democratic societies in which people have identical voting rights. His work is extended to societies with more general constitutional arrangements by Nakamura (1979). By showing that political stability is intimately related to the constitutional arrangement and the level of plurality of a society, the studies of Peleg and Nakamura have had an acknowledged impact on the positive political economy literature (Austen-Smith and Banks (1999)) and have inspired a flurry of influential studies on this topic.[6]

Our paper differs from these prior studies by incorporating social networks into the classical model of a political economy. Therefore, in addition to generalizing most of the key results obtained in these works, we articulate new findings on how constitutional arrangements and the geometry of influence networks determine political cycles and instability. In particular, if the influence network is the empty network, we obtain the results of the classical model. We also introduce new equilibrium concepts–the *fuzzy equilibrium set* and the *certainty equilibrium set*–to gauge the risk of political instability in a society and measure the relative (un)popularity of leaders. Furthermore, we define a new index of political stability, which shows that the maximum level of political competition that a society can afford while remaining politically stable is a function of its constitution and its network structure. Conversely, this index also provides a complete

[6] These papers are too numerous to cite, but see, for example, Moulin (1981), Schofield (1984, 1985), Van Roozendaal (1992), Banks and Duggan (2000), Konishi (1996), Schwartz (2007), Suzumura (2009), and the references therein.

structural characterization of constitutions and network structures that guarantee political stability in a society given a fixed number of competing political leaders.

Our model of a networked political economy also predicts political cycles better than the classical model. For instance, within the framework of the classical model, alternation in power is not possible in a two-party system, which is not a realistic prediction in light of the empirical evidence on political cycles in countries like the United States. Contrary to the classical model, our model not only shows that political alternation is possible in such systems, but also predicts the frequency at which political parties alternate in power under a given network structure. The analysis establishes influence networks as the source of such outcomes.

The incorporation of networks into the classical model also leads to new analyses. For instance, comparative statics exercises reveal that the certainty equilibrium set is smaller with a higher number of social connections. Similarly, the index of political stability is a decreasing function of the number of social connections, which implies that less connected societies enjoy a greater level of political stability. Furthermore, we show that some popular networks, such as star networks, rings, and lines, maximize the index of political stability, and therefore behave like the empty network in terms of minimizing the tradeoff between political competition and the existence of a stable leader. When the level of political competition is not too high, these networks refine the certainty equilibrium set without rendering it empty, which is a positive property. Applications allow one to understand why individualist societies are more politically stable than ethnic societies. To our knowledge, none of these results or applications have been obtained in prior studies.

As already mentioned, our simple approach to modelling influence is closely related to the model of Acemoglu et al. (2013), but we also differ in some respects. Their model involves two types of agents: *stubborn* agents, who never change their views, and *regular* agents, who update their views according to information that they receive from their neighbors. Stubborn agents seek to influence the rest of the society, and are never influenced by other agents. This study finds that when stubborn agents differ in their opinions, the influence process never leads to a consensus among the regular agents. In our model, stubborn and regular agents are *ideological* and *non-ideological* individuals, respectively. Like in the model of Acemoglu et al. (2013), only ideological individuals influence their non-ideological neighbors, and they cannot be influenced. However, our approach differs from theirs in that non-ideological individuals in our model do not influence their non-ideological neighbors. This assumption is justified within our framework, as non-ideological individuals are essentially either neutral or ignorant, and so gain nothing from influencing their neighbors. Our assumption, however, qualitatively leads to the same conclusion as theirs if the graph underlying the social network, viewed as a binary relation, is transitive. Our model also differs from theirs in that ours is static, though the probability with which a non-ideological individual follows an ideological neighbor can be viewed as being endogenously determined by a "monotonic" dynamic process.[7] Despite these small differences, we essentially obtain results that are qualitatively similar in that non-ideological individuals who are connected to ideological individuals with opposing views keep changing their political views. In the context of elections, such behavioral fluctuations translate into a probabilistic voting behavior. Our model also does not assume that any non-ideological individual is (in)directly connected with an ideological individual, unlike their model. It follows that certain non-ideological individuals can abstain in equilibrium during the election day.

The use of weighted directed graphs to model social and economic interactions is standard in the economic literature. Following studies that have used this class of networks to study the diffusion of ideas, technology, and economic shocks (see, for example, Jackson and Yariv (2011), Acemoglu et al. (2012), Acemoglu

[7] Any monotonic dynamic process results in the probability with which a non-ideological individual follows an ideological neighbor being a non-decreasing function of the initial weight of the influence link between the two.

et al. (2013)), we assume that networks are exogenous in our analysis. Obviously, this is an appealing assumption, since it allows one to study the distinct effects of all the possible network structures within the class of weighted directed networks. Our analysis indeed shows that the architecture of the influence network significantly affects the existence and the identity of stable political leaders, and has prescriptive implications for the design of networks that maximize or minimize the competition-stability tradeoff.

Our simple model of influence shares some features with existing models, but it also differs from these models in significant respects. An early contribution to the study of diffusion is that of Granovetter (1978), who introduced the threshold model. In his model, a complete network is assumed, and an agent chooses one of two possible actions if the number of his neighbors who have taken that action exceeds a certain threshold. Other models, surveyed in Jackson and Yariv (2011), generally assume a distribution of connections in the population and a payoff function that depends on an individual's and his neighbors' choice of a certain behavior. Pongou (2009a) and Pongou and Serrano (2009, 2013) define the contagion potential of a network, which is a contagion index that assumes that an agent who exogenously or endogenously receives a piece of information or is infected by a disease transmits it to his neighbors.

Our influence model differs from these prior models in that, in choosing between two actions (or two competing political leaders in our case), a non-ideological individual connected with ideological individuals fully internalizes the possibly opposing views of his influencers, which endows him with what can be viewed as a "fuzzy opinion" of each leader (Zaddeh, 1971, 1965). This fuzziness translates into the frequency with which he chooses one leader over another. It follows that non-ideological individuals can be viewed as "swing voters" in elections. In fact, in a two-party system, for instance, our model predicts that alternation in power is rendered possible only by non-ideological voters who are connected with influencers with opposing views.

Although some studies have recognized the influence of social networks on political opinions and choices (e.g., Lazarsfeld, Berelson, and Gaudet (1944), Katz and Lazarsfeld (1955), Kearns et al. (2009)), their goal was not to formally analyze how the structure of the network affects political cycles and (in)stability. A recent and interesting study by Galeotti and Mattozzi (2011) shows that communication networks among voters affect parties' incentives to disclose information about their candidates, and increase political polarization. Our focus and scope are completely different, just like our analytical framework.

Another distinctive feature of our study is that we derive testable implications for a wide range of societies. In addition to shedding light on the network origins of political cycles in two-party societies as already mentioned, our analysis also allows one to understand why some societies are more politically unstable than others. Interestingly, the finding according to which collectivist societies and societies organized around ethnic networks are less politically stable than individualist societies is consistent with empirical observation. Within collectivist societies, ethnic fragmentation decreases instability, which is consistent with the findings of other studies (see, for example, Collier and Hoeffler (1998) and Bates (1999)). These studies argue that diverse societies are less cohesive, and therefore it is more difficult to form a viable multi-ethnic coalition of rebels to fight the status quo. Our analysis also has implications for the design of networks that minimize political instability. The fact that networks like stars, rings, and lines, which are extremely popular in the literature and have been found to possess some appealing properties (e.g., Jackson and Wolinsky (1996), Bala and Goyal (2000), etc.), minimize political instability proves that instability-minimizing organizations are not hard to design.

3 A Networked Political Economy

A *networked political economy* is a human society $N = \{1, 2, ..., n\}$ endowed with a political technology or constitution f, a network $\mathcal{R}$ through which members of N influence the political opinions of other members, and a finite set of political leaders or social alternatives A. Each individual $i \in N$ has a preference relation $\succeq_i$ over A. We formalize these concepts below.

3.1 Political Leaders

A *political leader* is a social alternative that may be imposed on the society. We view a political leader as promoting a distinct political platform. We assume that the set A of political leaders is finite and contains at least two elements. The number of leaders measures the level of plurality and political competition in the society.

3.2 Political Technology

A *political technology* is a distribution of political power among the different subgroups of the society. It is formalized as a function f which maps each subgroup S of the society into either 1 or 0. Furthermore $f(S) = 1$ means that the members of S have the power to change the status quo to a new social alternative, and $f(S) = 0$ means that S does not have such a power. We denote by 2^N the set of all the subsets of N, and by W the set of all the elements of 2^N such that $f(S) = 1$. We impose the following natural conditions on W:

1. For any subgroups S and T such that $S \subset T$, if $S \in W$, then $T \in W$.

2. For any subgroup S, if $S \in W$, then $N \setminus S \notin W$.

3. $W \neq \emptyset$.

Each subgroup in W is called a majority or a winning coalition. Condition (1) means that the enlargement of a winning coalition by adding new individuals results in another winning coalition. Condition (2) means that the complementary set of a winning coalition is a losing coalition. This condition is important since it allows one to prevent the obvious political instability that arises from two non-overlapping winning coalitions having totally opposing views. Condition (3) is a natural decisiveness condition.

A political technology as defined above may have several interpretations. It may be viewed as formalizing the constitution of a society (e.g., Peleg (1978, 1984), Baron and Ferejohn (1989), Winter (1996), Barberà and Jackson (2004), Brams (2008), Acemoglu, Egorov and Sonin (2012), Ray and Vohra (2014)). In such a case, it provides a formal description of *de jure* or constitutional power. It can also be viewed as formalizing *de facto* power, which is the power to change the status quo through means other than the constitution, such as a military coup or a political revolt. In this case, $f(S) = 1$ means that any military coup or political revolt initiated by the individuals in S against the current political regime, as illustrated by the 2011 Arab Spring and the 2014 Ukrainian Revolution, will succeed regardless of the actions taken by the individuals in $N \setminus S$. It follows that a political technology is a formal description of the function that produces political change in a society, be it constitutional or non-constitutional, peaceful or violent, or civilian or military.

3.3 Political Preferences: Ideological and Non-Ideological Individuals

Each individual $i \in N$ has a preference relation represented by a binary relation $\succeq_i$ on the set of political leaders A. We assume that each preference relation $\succeq_i$ is:

- reflexive: for any $x \in A$, $x \succeq_i x$;
- transitive: for any $x, y, z \in A$, if $x \succeq_i y$ and $y \succeq_i z$, then $x \succeq_i z$; and
- complete: for any $x, y \in A$, $x \succeq_i y$ or $y \succeq_i x$ or both.

The asymmetric and symmetric components of $\succeq_i$, denoted respectively by $\succ_i$ and $\simeq_i$, are defined as follows:

- For any $x, y \in A$, $x \succ_i y$ if $x \succeq_i y$ and *not*$(y \succeq_i x)$.
- For any $x, y \in A$, $x \simeq_i y$ if $x \succeq_i y$ and $y \succeq_i x$.

A preference profile is denoted by $(\succeq_i)_{i \in N}$, and is sometimes denoted by R. We denote by $\mathcal{U}$ the set of all the preference relations on A, and by $\mathcal{U}^N$ the set of preference profiles.

Let x and y be two political leaders, and i an individual. If $x \succ_i y$ or $y \succ_i x$, we say that i is *ideological* over the pair $\{x, y\}$. However, if $x \simeq_i y$, we say that i is *indifferent* , *neutral*, or *non-ideological* over the pair $\{x, y\}$. An individual may be neutral because he has perfect knowledge of the political alternatives and values them equally, or because he does not know the alternatives very well and therefore cannot evaluate them. The assumption that certain individuals might be non-ideological is realistic, and is supported by the high level of abstention that is generally observed in real-life elections.[8]

3.4 Influence Networks

An *influence network* is a collection of weighted directed links between the individuals that form the society. An influence network is formalized as $\mathcal{R} = (g, (p^i)_{i \in N})$, where g is a binary relation on N recording directed links, and p^i is a distribution of weight over the links of individual i. We formalize these concepts below.

Let i and j be two individuals. If i and j are connected by a link directed from i to j, we say that "j is linked to i" or that i is an *in-neighbor* of j and j an *out-neighbor* of i, and it means that i can influence the political choice of j. This implies that j may act according to the political preference of i over a pair of competing leaders if j is neutral and i is not. If i and j are connected by a link directed from i to j and another link directed from j to i, it follows that i and j can influence each other. If i and j are not connected at all, it follows that neither can influence the other. In particular, if i has no connection at all, which means that i is an isolated member of the society, i cannot be influenced and cannot influence anybody else. For any individual i, we denote by $g(i)$ the set of individuals with whom i is linked, and for any set of individuals S, $g(S) = \bigcup_{i \in S} g(i)$ is the set of individuals with whom the members of S are linked. Therefore the set $g(i)$ is the set of influencers of i and the set $g(S)$ is the set of influencers of individuals in S.

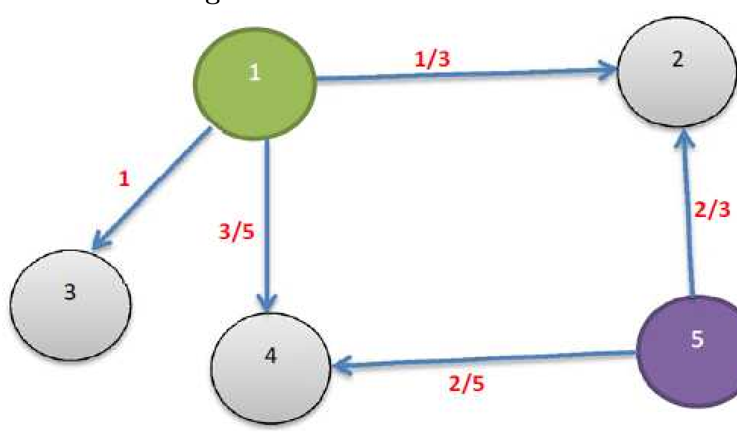
It follows from our definition of *influence* that only non-ideological individuals can be influenced. Non-ideological individuals cannot influence their out-neighbors and similarly, ideological individuals cannot be influenced by their in-neighbors.

[8] According to the American Presidency Project, voter turnout was only 58.23% of the voting-age population in the 2008 U.S. presidential election, which was the highest participation rate since the 1970s. This implies that over 41% of the voting-age population abstained.

For each individual i, the distribution of weight $p^i = (p^i(j))_{j \in g(i)}$ measures the amount of influence that each influencer j of i has on the latter. We normalize p^i so that $\sum_{j \in g(i)} p^i(j) = 1$ if $g(i)$ is not empty, and we interpret $p^i(j)$ as the probability that i will follow the preference of j if i is neutral and j is not and all the other influencers of i hold a view opposite to that of j.

In general, the influencers of an individual i may have opposing political views. We compute the probability with which i supports one option against another option as a function of the weighted number of his influencers who support each option. To be precise, let x and y be two political leaders, and $(\succeq_i)_{i \in N}$ be a preference profile. The society N can be partitioned into three sets: the set of individuals who prefer x over y, denoted N_{xy}; the set of individuals who prefer y over x, denoted N_{yx}; and the set of individuals who are neutral, denoted $N_{(xy)}$.

Figure 1: **An influence network**



Individuals 1 and 5 have opposing views over two politicians x and y, whereas the other individuals are neutral. For instance, individuals 2 and 5 follow individual 1 with probabilities 1/3 and 2/3, respectively, whereas individual 3 follows 1 with a probability of 1

Denote respectively by p^i_{xy}, p^i_{yx} and $p^i_{(yx)}$ the probabilities with which i chooses x over y, chooses y over x, and abstains between the two. These probabilities are set out hereunder.

- If $i \in N_{xy}$, then $p^i_{xy} = 1$, $p^i_{yx} = 0$ and $p^i_{(yx)} = 0$.
- If $i \in N_{yx}$, then $p^i_{xy} = 0$, $p^i_{yx} = 1$ and $p^i_{(yx)} = 0$.
- If $i \in N_{(xy)}$, then denote by $g(i) \cap N_{xy}$ the set of i's influencers who prefer x over y, by $g(i) \cap N_{yx}$ the set of i's influencers who prefer y over x, and by $g(i) \cap N_{(xy)}$ the set of i's influencers who are indifferent. Let $\mu^i_{xy} = \sum_{j \in g(i) \cap N_{xy}} p^i(j)$, $\mu^i_{yx} = \sum_{j \in g(i) \cap N_{yx}} p^i(j)$ and $\mu^i_{(yx)} = \sum_{j \in g(i) \cap N_{(yx)}} p^i(j)$ be their respective weights. The computation of p^i_{xy}, p^i_{yx} and $p^i_{(yx)}$ is set out hereunder.

Define by $\mathcal{X}_T$ the indicator function of property T:

$$\mathcal{X}_{[T]} = \begin{cases} 1 \text{ if } T \text{ is satisfied} \\ \\ 0 \text{ if not.} \end{cases}$$

The probabilities p^i_{xy}, p^i_{yx} and $p^i_{(yx)}$ are computed as follows:

$$p^i_{xy} = \begin{cases} \frac{\mu^i_{xy}\mathcal{X}_{[g(i)\cap N_{xy}\neq\emptyset]}}{\mu^i_{xy}\mathcal{X}_{[g(i)\cap N_{xy}\neq\emptyset]}+\mu^i_{yx}\mathcal{X}_{[g(i)\cap N_{yx}\neq\emptyset]}} \text{ if } g(i)\cap N_{xy}\neq\emptyset \text{ or } g(i)\cap N_{yx}\neq\emptyset \\ \\ 0 \text{ if } g(i)\cap N_{xy} = g(i)\cap N_{yx} = \emptyset. \end{cases}$$

$$p^i_{yx} = \begin{cases} \frac{\mu^i_{yx}\mathcal{X}_{[g(i)\cap N_{xy}\neq\emptyset]}}{\mu^i_{xy}\mathcal{X}_{[g(i)\cap N_{xy}\neq\emptyset]}+\mu^i_{yx}\mathcal{X}_{[g(i)\cap N_{yx}\neq\emptyset]}} \text{ if } g(i)\cap N_{xy}\neq\emptyset \text{ or } g(i)\cap N_{yx}\neq\emptyset \\ \\ 0 \text{ if } g(i)\cap N_{xy} = g(i)\cap N_{yx} = \emptyset. \end{cases}$$

$$p^i_{(xy)} = 1 - \mathcal{X}_{[g(i)\cap[N_{xy}\cup N_{yx}]=g(i)]}.$$

It follows that, if i is indifferent between x and y, i will choose x (resp. y) with a probability that reflects the cumulative weight that his influencers who support x over y (resp. y over x) have relative to the total weight of all his influencers who have an ideological view. It is a natural model of influence which has the flavor of the model developed by Acemoglu et al. (2013). In particular, our model, like their model, implies that a non-ideological individual connected with influencers who have opposing views will continue changing his political views. In our framework, when p^i_{xy} and p^i_{yx} are both strictly positive for a non-ideological individual i, i is best viewed as having a "fuzzy" behavior (Zadeh, 1965, 1971)).

In the context of elections, behavioral fuzziness within our framework translates into probabilistic voting, which implies that non-ideological individuals who are subject to opposing influences are swing voters. Indeed, assuming that multiple elections involving two political parties x and y are organized, if for any individual i, p^i_{xy} and p^i_{yx} are strictly positive, it means that i will vote for x in some elections and for y in other elections, p^i_{xy} and p^i_{yx} being respectively the proportions of elections in which i favors x over y and y over x.

We note that we can derive p^i_{xy} and p^i_{yx} by analyzing the average behavior of individual i in a dynamic model of influence by appropriately modifying the model proposed by Acemoglu et al. (2013) to fit within our framework. In this case, p^i_{xy} , for instance, can be derived as the first moment of a random variable describing the long-run political behavior of individual i. If i is ideological over $\{x, y\}$, or if i is non-ideological over $\{x, y\}$ but is influenced by ideological individuals who hold identical political views, our static model yields the same prediction as Acemoglu et al. (2013). But if i is non-ideological over $\{x, y\}$ and is influenced by ideological individuals who have opposing views over $\{x, y\}$, our actual calculation of p^i_{xy} may differ from theirs, even though our "qualitative" conclusion regarding the behavior of individual i is identical.

4 Political Instability

The analysis of political (in)stability and how it is affected by social networks is the main purpose of this paper. Classically, a political leader is said to be *unstable* or *unpopular* if he is less preferred by a majority of the people than another leader (Black, 1948).[9] In the presence of social networks, we have seen that non-ideological individuals support a leader with a probability that is smaller than one in general, and therefore

[9] In real-lile politics, the instability or unpopularity of a leader manifests itself in several ways. It shows through low approval ratings, or it might take the form of a peaceful street protest or a violent demonstration. We do not single out any of these forms of instability.

that the stability of political leaders is not certain, especially if non-ideological individuals are needed to form winning coalitions. In what follows, we introduce the notion of the *instability matrix*, which determines the probability with which each leader will be destabilized by each of his competitors in a political economy. This notion is used to introduce two network-based concepts of equilibrium that generalize the classic concept used in Black (1948).

Let $\mathcal{E} = (N, A, W, \mathcal{R}, (\succeq_i)_{i \in N})$ be a political economy, and $x, y \in A$ two political leaders. Let $(N_{xy}, N_{(xy)}, N_{yx})$ be an opinion profile, and denote by 3^N the set of all the opinion profiles. The probability with which x dominates or destabilizes y is given by :

$$P(x,y) = \sum_{\substack{S_1 \in W \\ (S_1,S_2,S_3) \in 3^N}} \prod_{i \in S_1} p^i_{xy} \prod_{i \in S_2} p^i_{(xy)} \prod_{i \in S_3} p^i_{yx}$$

The *instability matrix* of the economy $\mathcal{E}$ is given by the $|A| \times |A|$ matrix:

$$\mathcal{I}(\mathcal{E}) = (P(y,x))_{(x,y) \in A \times A}$$

The probability $P(y,x)$ also captures the frequency with which leader x loses against leader y in a paired competition as described in Black (1948). In the absence of social influences (that is, if $\mathcal{R} = (g, (p^i)_{i \in N})$ is such that g is empty), one can show that $P(x,y)$ is either 1 or 0, which implies that, in a two-party system for instance, the party in power (the status quo) never loses against the opposition, which is not consistent with reality. It follows that, by inducing swing voters, social influence networks are a major source of political cycles in such systems.

We define below two equilibrium concepts based on the notion of the instability matrix. The first is the *fuzzy equilibrium set*, which captures the relative stability of political leaders against each other. The second is the *certainty equilibrium set*, which is the set of leaders who are stable against each of the other leaders with a probability of 1. Leaders in the latter set are those whose leadership is sought after, as they cannot be unpopular.

Definition 1 *Let $\mathcal{E} = (N, A, W, \mathcal{R}, (\succeq_i)_{i \in N})$ be a political economy.*

1) The fuzzy equilibrium set of $\mathcal{E}$, denoted $\mathcal{F}(\mathcal{E})$, is the set $\mathcal{F}(\mathcal{E}) = \{((x,y), 1 - P(y,x)) : x, y \in A\}$.[10]

2) The certainty equilibrium set of $\mathcal{E}$, denoted $\mathcal{C}(\mathcal{E})$, is the set $\mathcal{C}(\mathcal{E}) = \{x : P(y,x) = 0$ for all $y \in A\}$.

In a political economy $\mathcal{E}$, $((x,y), 1 - P(y,x)) \in \mathcal{F}(\mathcal{E})$ means that in a paired electoral contest between x and y where x is the status quo and y is the challenger, x will not lose and therefore will stay in power with probability $1 - P(y,x)$. In other words, $((x,y), 1 - P(y,x)) \in \mathcal{F}(\mathcal{E})$ means that the incumbent leader x is stable against the challenger y with probability $1 - P(y,x)$. A society is said to be politically stable if its certainty equilibrium set is never empty regardless of the political preferences of its people.

Definition 2 *A society $\mathcal{S} = (N, A, W, \mathcal{R})$ is said to be politically stable if for any preference profile $(\succeq_i)_{i \in N} \in \mathcal{U}^N$, $\mathcal{C}(N, A, W, \mathcal{R}, (\succeq_i)_{i \in N}) \neq \varnothing$.*

We illustrate these equilibrium concepts through the following example.

[10] The fuzzy equilibrium set is indeed a mathematically fuzzy set. By definition, a fuzzy set is a pair (X, u) where X is a collection of objects and u a function that maps each element a of X into an element $u(a)$ belonging to the interval $[0, 1]$ (Zadeh, 1965)); $u(a)$ measures the grade of membership of a into (X, u). We note that the fuzzy equilibrium set $\mathcal{F}(\mathcal{E})$ of an economy $\mathcal{E}$ is a fuzzy set (X, u) where $X = A \times A$ and $u(x,y) = 1 - P(y,x)$ for each (x,y) in X.

Example 1 *Consider a political economy* $\mathcal{E} = (N, A, W, \mathcal{R}, (\succeq_i)_{i \in N})$ *where* $N = \{1,2,3,4,5\}$, $A = \{x, y\}$, $(\succeq_i) = (xy, (xy), (xy), (xy), yx)$, W *is the majority rule (a coalition is winning if and only if it contains at least three individuals), and* $\mathcal{R} = (g, (P^i)_{i \in N})$ *the influence network given by:*

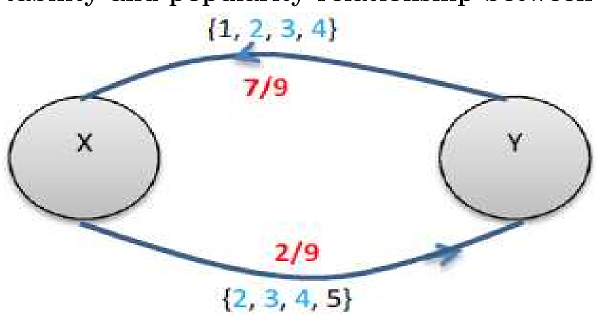
$g = \{(2,1), (3,1), (4,1), (4,5), (2,5)\}$ *(which yields* $g(1) = \emptyset$, $g(2) = \{1,5\}$, $g(3) = \{1\}$, $g(4) = \{1,5\}$ *and* $g(5) = \emptyset$*) and* $p^2(1) = \frac{1}{3}$ *and* $p^2(5) = \frac{2}{3}$, $p^4(1) = \frac{3}{5}$, $p^4(5) = \frac{2}{5}$ *and* $p^3(1) = 1$. *The network is depicted by Figure 1 in Section 3.4.*

To compute the probability with which a leader beats his competitor, we first need to list all the possible opinion profiles that lead to a victory of either leader over the other. This list is provided in Table 1 below:

Table 1			
1	$(123, A, B)$ *with* $A \cup B = 45$	***9***	$(245, A, B)$ *with* $A \cup B = 13$
2	$(124, A, B)$ *with* $A \cup B = 35$	***10***	$(345, A, B)$ *with* $A \cup B = 12$
3	$(125, A, B)$ *with* $A \cup B = 34$	***11***	$(1234, A, B)$ *with* $A \cup B = 5$
4	$(134, A, B)$ *with* $A \cup B = 25$	***12***	$(1235, A, B)$ *with* $A \cup B = 4$
5	$(135, A, B)$ *with* $A \cup B = 24$	***13***	$(1245, A, B)$ *with* $A \cup B = 3$
6	$(145, A, B)$ *with* $A \cup B = 23$	***14***	$(1345, A, B)$ *with* $A \cup B = 2$
7	$(234, A, B)$ *with* $A \cup B = 15$	***15***	$(2345, A, B)$ *with* $A \cup B = 1$
8	$(235, A, B)$ *with* $A \cup B = 14$	***16***	$(12345, \emptyset, \emptyset)$

The profile $(123, A, B)$*, for instance, means that, if individuals* 1, 2 *and* 3 *support* x *over* y*, the individuals in* A *abstain, and the individuals in* B *support* y *over* x*, then* x *will win and* y *will lose. Similarly, if individuals* 1, 2 *and* 3 *support* y *over* x*, the individuals in* A *abstain, and the individuals in* B *support* x *over* y*, then* y *will win and* x *will lose.*

Figure 2: **Stability and popularity relationship between x and y**



x beats y with probability 7/9 and y beats x with probability 2/9. This political cycle is induced by the non-ideological individuals 2, 3 and 4 under the influence of the ideological individuals 1 and 5.

The following table provides the probability with which each individual favors one leader over the other:

Table 2					
i	1	2	3	4	5
p^i_{xy}	1	$\frac{1}{3}$	1	$\frac{3}{5}$	0
$p^i_{(xy)}$	0	0	0	0	0
p^i_{yx}	0	$\frac{2}{3}$	0	$\frac{2}{5}$	1

We first compute the probability with which x beats y, its complementary being the probability with which y beats x. In order to do this, we compute the probability with which x beats y under each of the opinion profiles listed in Table 1. We note that, if $\mathcal{S} = (S_1, S_2, S_3)$ is an opinion profile where $1 \notin S_1$, then the probability with which x beats y under $\mathcal{S}$ is $P_{xy}(\mathcal{S}) = 0$. Likewise, if $\mathcal{S} = (S_1, S_2, S_3)$ is such that $5 \notin S_3$, then $P_{xy}(\mathcal{S}) = 0$. For further illustration, we show below how we compute the probability with which x beats y under opinion profiles 1 and 2 in Table 1:

$$\begin{aligned} P_{xy}(123, A, B) &= P_{xy}(123, 45, \emptyset) + P_{xy}(123, 4, 5) + P_{xy}(123, 5, 4) + P_{xy}(123, \emptyset, 45) \\ &= 0 + 0 + 0 + \tfrac{1}{3} \cdot \tfrac{2}{5} = \tfrac{2}{15} \end{aligned}$$

$$\begin{aligned} P_{xy}(124, A, B) &= P_{xy}(124, 35, \emptyset) + P_{xy}(124, 3, 5) + P_{xy}(124, 5, 3) + P_{xy}(124, \emptyset, 35) \\ &= 0 + 0 + 0 + 0 = 0 \end{aligned}$$

The probability with which x beats y under each of the 16 configurations listed in Table 1 is presented below:

$\mathcal{S}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$P_{xy}(\mathcal{S})$	$\frac{2}{15}$	0	0	$\frac{2}{5}$	0	0	$\frac{2}{45}$	0	0	0	$\frac{1}{5}$	0	0	0	0	0

The probability with which x beats y is obtained by summing $P_{xy}(\mathcal{S})$ over all of the opinion profiles $\mathcal{S}$, which is $\frac{7}{9}$. One can also check, following the same steps as above, that the probability with which y beats x is $\frac{2}{9}$. It follows that the instability matrix is:

$$\mathcal{I}(\mathcal{E}) = \begin{pmatrix} 0 & \frac{2}{9} \\ \frac{7}{9} & 0 \end{pmatrix}$$

The fuzzy equilibrium set is given by:

$$\mathcal{F}(\mathcal{E}) = \{((x, y), \frac{7}{9}), ((y, x), \frac{2}{9})\}$$

The certainty equilibrium set is:

$$\mathcal{C}(\mathcal{E}) = \varnothing.$$

This example shows that, if x and y are two political parties, they will alternate in power following a political cycle such that x will be in power $\frac{7}{9}$ of the times and x will be in power $\frac{2}{9}$ of the times.

4.1 Influence Networks Reduce the Number of Stable Political Leaders

In this section, we study the effect of influence networks on the number and identity of stable political leaders. We show that the presence of influence networks reduces the number of stable leaders by increasing political turnout, and may even thrust a society into a cycle of instability. To illustrate this point, consider the following example in which we examine the existence of a stable leader in a networked economy and contrast the outcome to the corresponding economy with an empty network.

Example 2 *Consider a political economy $\mathcal{E} = (N, A, W, \mathcal{R}, (\succeq_i)_{i \in N})$ where $N = \{1, 2, 3\}$, $A = \{x, y, z\}$, W is the majority rule (i.e., $W = \{S \subseteq N : |S| \geq 2\}$), $\mathcal{R} = (g, (P^i))$ is such that g is a star network as depicted by Figure 3 below ($g = \{(2, 3), (2, 1)\}$; $P^1(2) = P^3(2) = 1$ and $P^i(j) = 0$ for any other pair (i, j)), and $(\succeq_i)_{i \in N}$ is the preference profile described as follows: $1 : xyz$, $2 : yzx$, and $3 : (zx)y$.*

Figure 3 : **A star influence network**

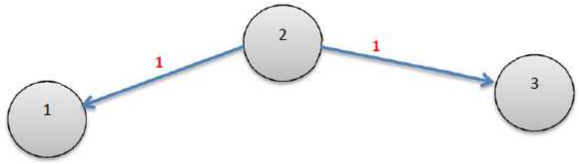
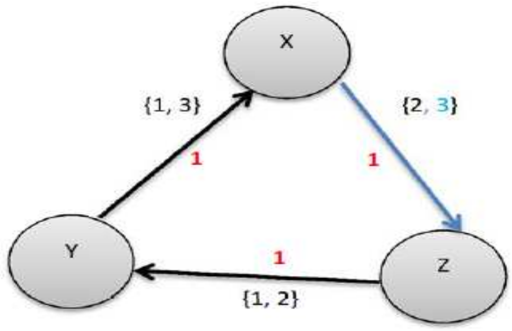


Figure 4 : **Stability and popularity relationship between x, y and z**



The influence network depicted in Figure 3 leads to a complete destabilization of the society where each leader is unpopular. In the absence of influence links, there would be one stable leader–x. In the presence of social influences, leader x is destabilized by leader z due to the influence of individual 2 on the political choice of 3 who is neutral between x and z.

It follows from these individual preferences that everybody is ideological, except individual 3 *who, over the pair* $\{x, z\}$*, is not. The popularity relationship among politicians is depicted in Figure 3. It shows that a majority strictly prefers* x *over* y *with a probability equal to 1; similarly,* y *is preferred over* z *by another majority, and* z *is preferred over* x *by another majority. On the other hand, the unpopularity of* x *against* z *is the result of the influence exerted by* 2 *over* 3*. In the absence of the existing influence network,* x *would be the only stable leader. But the prevailing influence network leads to the destabilization of* x*, thus inducing the economy to enter into a situation of political instability in which all leaders are unpopular.*

We note that, in the absence of social influences, the number of individuals expressing a strong opinion over the pair $\{x, z\}$ *would be two, compared to three under the network depicted in Figure 3. It follows that the presence of social influences increases political turnout, and increases the probability of a cycle as a consequence.*

Examples 1 and 2 show that influence networks increase political instability, possibly leading to a complete destabilization of the society. Prior to formalizing this insight, it is important to provide a rationalization of the certainty equilibrium set by proving that each leader in this set is a maximal element of a domination relation which we define below:

Definition 3 *Let* $\mathcal{E} = (N, A, W, \mathcal{R}, (\succeq_i)_{i \in N})$ *be a political economy. Define on* A *the binary relation* $\succ$ *as* *follows. For any $x, y \in A$, $y \succ x$ if there exist two coalitions S and T such that:*

$$\begin{cases} T \subseteq g(S) \text{ and } S \cup T \in W, \\ y \succ_i x \text{ for all } i \in S \text{ and} \\ x \simeq_i y \text{ for all } i \in T \end{cases}$$

According to the binary relation $\succ$, a leader is dominated if there exists a subgroup of ideological individuals who strongly prefer another leader and who, thanks to their influences, can persuade certain non-ideological individuals to join in forming a majority against the disliked leader. We show below that each stable leader is a maximal element of the binary relation $\succ$, a result that rationalizes the certainty equilibrium set.

Theorem 1 *Let $\mathcal{E} = (N, A, W, \mathcal{R}, (\succeq_i)_{i \in N})$ be a networked political economy and $x \in A$ a political leader. $x \in \mathcal{C}(\mathcal{E})$ if and only if x is maximal with respect to the domination relation $\succ$.*

Theorem 1 not only rationalizes the certainty equilibrium set, but it also proves that this equilibrium concept is a generalization of the classic equilibrium notion used in Black (1948). In fact, if we assume that $\mathcal{R}$ is the empty network, then our equilibrium concept coincides with Black's. We also note that Black considers only the majority rule, whereas we consider a much larger class of political technologies.

We now conduct a comparative statics exercise, generalizing the insights of Examples 1 and 2. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are two influence networks, we say that the set of links of $\mathcal{R}_2$ includes the links of $\mathcal{R}_1$, denoted $\mathcal{R}_1 \subseteq \mathcal{R}_2$, if, for all $i \in N$, $g_1(i) \subseteq g_2(i)$. The following result shows that, holding the constitution and the preferences constant, increasing the number of links within an influence network refines the certainty equilibrium set, and thus decreases the number of political leaders who can rule the society in a stable manner.

Theorem 2 *Let $\mathcal{E}^1 = (N, A, W, \mathcal{R}_1, (\succeq_i)_{i \in N})$ and $\mathcal{E}^2 = (N, A, W, \mathcal{R}_2, (\succeq_i)_{i \in N})$ be two networked political economies such that $\mathcal{R}_1 \subseteq \mathcal{R}_2$. Then $\mathcal{C}(\mathcal{E}^2) \subseteq \mathcal{C}(\mathcal{E}^1)$. This inclusion may be strict.*

Theorem 2 establishes the fact that influence networks refine the set of stable politicians. While this refinement might be regarded as a positive result if one cares about equilibrium uniqueness, it should also be noted that it might lead to a complete destabilization of the society, as is illustrated in Example 2, which is a negative property of influence networks.

4.2 Influence Networks Worsen the Tradeoff between Political Competition and Stability: A Preference-Blind Index of Political Stability

In Section 4.1, we analyzed political stability under the assumption that all the defining parameters of a political economy, including people's preferences, are known. In most situations, however, preferences are not known. Yet, the analysis of political stability even in those circumstances is still very important. Can we tell whether a society runs a risk of political instability if we only know its constitution, its number of competing political leaders or parties, and its network structure? Our goal in this section is to answer this question.

Our analysis reveals a tradeoff between political competition and political stability, and shows that social networks worsen this tradeoff, even undermining the stabilizing property of certain well-known constitutions.

The analysis is also useful for the design of constitutions and network structures that guarantee political stability in a plural society, and has implications for the optimal level of political competition.

To measure the competition-stability tradeoff, we introduce a *preference-blind index of political stability*, which is a function ν that maps each pair of a constitution W and a network $\mathcal{R}$ into a natural number $\nu(W,\mathcal{R})-1$ representing the maximum number of political leaders that a society can afford for a stable leader to exist regardless of the extent of preference heterogeneity.

Conversely, given an exogenous number of competing political leaders or parties p, our index provides a full structural characterization of the constitutions W and networks $\mathcal{R}$ that ensure the existence of a stable leader regardless of the extent of preference heterogeneity (i.e., $\nu(W,\mathcal{R}) = p+1$).

In order to define our stability index, we first need to define the notion of a social *circuit*, which, intuitively, is a collection of population subgroups which, by their power and social influences, can thrust a society into an endless cycle of political instability if they are endowed with certain preferences and if the level of political competition is sufficiently high.

Definition 4 *Let W and $\mathcal{R}$ be respectively the constitution and the network structure of a society N. A circuit of $(W,\mathcal{R})$ is a set $\hat{S} = \{(S_1,T_1),(S_2,T_2)...,(S_k,T_k)\}$ such that for any $t = 1,2,..k$:*

(i) $S_t \subset N$, $T_t \subset g(S_t)$.

(ii) $S_t \cup T_t \in W$ and $S_t \cap T_t = \emptyset$.

(iii) $[\cap_{s=1}^{k}(S_s \cup T_s)] \cap S_t = \emptyset$.

We denote by $\xi(W,\mathcal{R})$ the set of all the circuits of $(W,\mathcal{R})$.

We now define our preference-blind index of political stability.

Definition 5 *The stability index is a function ν that maps each pair of a constitution and a network $(W,\mathcal{R})$ into the number $\nu(W,\mathcal{R})$ defined as:*

$$\nu(W,\mathcal{R}) \quad = \begin{cases} +\infty \text{ if } \xi(W,\mathcal{R}) = \emptyset \\ \min\{\left|\hat{S}\right|, \hat{S} \in \xi_G\} \text{ if } \xi(W,\mathcal{R}) \neq \emptyset \end{cases}$$

We prove that a society is politically stable if and only if its number of competing political leaders is smaller than its preference-blind stability index.

Theorem 3 *Let $\mathcal{S} = (N,A,W,\mathcal{R})$ be a networked society. $\mathcal{S}$ is politically stable if and only if $\nu(W,\mathcal{R}) > |A|$.*

Although the proof of this result is quite involved, its intuition is simple. As noted earlier, a circuit is a collection of population subgroups that can destabilize a society by holding highly opposing political preferences. Therefore, if a society has no circuit (that is, $\xi(W,\mathcal{R}) = \emptyset$), then obviously it cannot be destabilized regardless of the number of political leaders, which is why $\nu(W,\mathcal{R}) = +\infty$. If it has a circuit, then let $\hat{S} = \{(S_1,T_1),(S_2,T_2)...,(S_k,T_k)\}$ be the smallest circuit. We then show that, if there are at least k political leaders, one can endow the members of the population subgroups in the circuit $\hat{S}$ with preferences so as to create a situation in which, for each political leader, there is always a winning coalition that will favor another leader with a positive probability.

When the network $\mathcal{R}$ is empty, we can show the index $\nu(W, \mathcal{R})$ is equal to the well-known Nakamura number. In general, however, the index is sgnificantly different from that number for nonempty networks, as we will see later.

A straightforward implication of Theorem 3 is that a political system should be designed so as to limit the number of competing politicians to a maximum of $\nu(W, \mathcal{R})-1$ if we know W and $\mathcal{R}$. Conversely, if one knows that the number of politicians will never exceed a certain number p (e.g., in ethnic societies, the number of political leaders generally reflects the number of major ethnic groups) and if the network structure of the society $\mathcal{R}$ is known, then one can design the constitution W so that $\nu(W, \mathcal{R}) = p + 1$. The knowledge of $\mathcal{R}$, however, may be difficult to obtain as networks are often dynamic and change over time. This means that instability can suddenly erupt in a stable society as a result of a change in its network structure, as is illustrated by the crucial role played by social media in the recent Arab Spring (see Acemoglu, Hassan and Tahoun (2014) for Egypt; they show that the number of protesters in Tahrir Square increased with the number of tweets sent by Twitter users for street mobilization. Note that the fact that Hosni Moubarak, who was a stable leader prior to the street mobilization became unstable afterwards is consistent with the prediction of Theorem 2).

The following result is a comparative statics exercise which shows that increasing the number of links within a social network reduces the political stability index of a society, and thus worsens the tradeoff between political competition and stability.

Theorem 4 *Let W be the constitution of a society, and $\mathcal{R}_1$ and $\mathcal{R}_2$ be two influence networks such that $\mathcal{R}_1 \subseteq \mathcal{R}_2$. Then $\nu(W, \mathcal{R}_2) \leq \nu(W, \mathcal{R}_1)$. This inequality may be strict.*

It follows from Theorems 2 and 4 that influence networks increase political instability in two ways. First, they reduce the number of political leaders who can govern the society in a stable manner, even sometimes leading to a complete destabilization. Second, they increase the tradeoff between political competition and the existence of a stable political leader. The next section will present some testable implications of these findings for familiar political systems and societies.

5 Applications and Testable Implications

In this section, we apply our findings to examine how social networks affect political cycles and instability in familiar political systems. We develop three applications. The first application considers two-party systems. The second application compares individualist and collectivist societies, with a particular focus on ethnic societies or societies characterized by a high level of homophilous behavior and influences. The third application identifies particular networks that maximize or minimize the level of conflict between political competition and stability in a democracy. This latter application is especially useful for the design of organizations that seek to minimize conflicts.

5.1 Political Cycles in Two-party Systems

Several countries have a political system in which two major parties dominate the political arena, such as the Democratic Party and the Republican Party in the United States. Our analysis has implications for how political parties alternate in power in such a system in the absence or presence of social networks. The result below implies that, in two-party systems, alternation in power is only possible in the presence of influence networks.

Proposition 1 *Let N be a society, W its constitution, and $\mathcal{R}$ its influence network. Then, $\nu(W, \mathcal{R}) \geq 3$ if $\mathcal{R} = \mathcal{R}_0$ ($\mathcal{R}_0$ is the empty network) and $\nu(W, \mathcal{R}) \geq 2$ if $\mathcal{R} \neq \mathcal{R}_0$. The last inequality may be binding.*

Proposition 1 sheds light on the network origins of political cycles in two-party systems. Essentially, it says that, in the absence of social networks through which ideological individuals can influence the non-ideological individuals with whom they are connected, it takes at least three political parties to induce a change in power. This implies that, in the absence of social networks, the incumbent party remains in power forever in a two-party system. Needless to say, this prediction is highly implausible in light of real-life politics. The fact that political cycles are observed in several two-party systems is easily explained by our analysis. Within our framework, the presence of influence networks increases political turnout by mobilizing non-ideological individuals to the polls, as is shown in Example 1. These non-ideological voters cast a probabilistic vote in favor of each party depending on the relative amount of influence exerted on them by their supporters. These floating votes induce alternation in power in turn, leading to political cycles.

In our framework, non-ideological individuals who are connected with ideological individuals with opposing opinions can be characterized as "swing voters", as their voting behavior cannot be predicted with certainty. As is already noted, a similar behavior characterizes "regular" agents connected with "stubborn" agents with conflicting views in Acemoglu et al. (2013). The knowledge of the exact structure of the influence network allows for the computation of the frequency with which a non-ideological individual under conflicting influences votes in favor of a party. It also allows to predict the frequency with which parties take turns in power as a result of fluctuating votes, as is shown in Example 1.

5.2 Collectivist Societies, Ethnic Democracies, and Homophily

Our second application compares political stability in individualist and collectivist societies. In particular, we determine a closed form solution for the exact tradeoff between political competition and the existence of a stable leader in ethnic democracies, showing how this tradeoff depends on the number of ethnic groups.

5.2.1 Individualist versus Collectivist Societies

The amount of interaction that exists among people varies a great deal across societies. Some societies are characterized by a high level of individualism, whereas others are characterized by a high level of interaction among their members (e.g., Greif (1994), Rothwell (2010), Barth (1969)). Although the economic consequences of social interactions have been widely studied, the ways in which social interactions affect political cycles and instability have been understudied. We examine this question in this section, analyzing political stability across three distinct types of societies corresponding to distinct levels of social interaction: (1) individualist societies, in which no connections exist among people; (2) completely connected societies, in which any two individuals are connected; and (3) societies falling between these two extreme types.

An *individualist* or lonely society is characterized by the empty influence network $\mathcal{R}_0$. We have shown in Proposition 1 that the stability index of such a society is at least three, regardless of the prevailing political technology or constitution.

A *completely connected* society is characterized by the complete influence network, denoted $\mathcal{C}$. A society falling between an individualist society and a completely connected society is characterized by an influence network $\mathcal{R}$ whose graph strictly includes that of $\mathcal{R}_0$ and is strictly included in the graph of the complete network $\mathcal{C}$. The following result shows that such a society is more prone to political instability than an individualist society, but is less prone to instability than a completely connected society, unless the prevailing political technology is the unanimity rule ($W = \{N\}$) or is close to the unanimity rule ($W = \{N, N\backslash\{i\}\}$ for some individual i), in which case all three types of society are equally stable.

Theorem 5 *Let W be the constitution of a society N, and $\delta = |\{i \in N : N\backslash\{i\} \in W\}|$. Let $\mathcal{R}$ be a network such that $\mathcal{R}_0 \subsetneq \mathcal{R} \subsetneq \mathcal{C}$.*

1) If $\delta \leq 1$, then $\nu(W, \mathcal{R}_0) = \nu(W, \mathcal{R}) = \nu(W, \mathcal{C}) = +\infty$.

2) If $\delta \geq 2$, then $\nu(W, \mathcal{R}_0) \geq \nu(W, \mathcal{R}) \geq \nu(W, \mathcal{C}) = 2$. These inequalities may be strict.

It follows from Theorem 5 (item 1) that, when the political technology is the unanimity rule or is close to the unaminity rule, a stable leader exists regardless of the network structure and the level of political competition in the society. In this case, networks do not play a particular role in the stability of the society. However, they still determine the identity of stable and unstable leaders if preferences are known. And although they generally increase the probability of a leader becoming unstable as implied by Theorem 2, they do not eradicate stable leaders. Theorems 2 and 5 (item 1) therefore imply that, under the unanimity rule or a near-unanimity rule, networks refine the certainty equilibrium set without emptying it regardless of the extent of preference heterogeneity in the population.

If the political technology is different from the unanimity rule and the near-unanimity rule, Theorem 5 (item 2) implies that individualist societies are more politically stable than collectivist societies in general. In this case, the stability index of a completely connected society is equal to two, which implies that the society is prone to political instability regardless of the level of political competition. Note that this value is strictly smaller than the stability index of an individualist society (which is at least equal to three), precisely implying that individualist societies can afford a higher level of political competition than collectivist societies while remaining stable It also implies that, for a given level of political competition, individualist societies are more likely to be stable than collectivist societies. This finding might partly explain why Western countries are more politically stable than less developed countries, as Greif (1994) observes that the former are more individualistic.

5.2.2 Ethnic Democracies and Homophilous Influences

In this section, we focus on ethnic societies governed by a democratic rule. These societies are a particular kind of collectivist societies that is highly prevalent in the developing world. They are characterized by a high level of homophilous behavior and influences. Greif (1994) argues that such societies have a "segregated" social structure, which is consistent with the argument of McPherson, Smith-Lovin and Cook (2001) that homophily in race and ethnicity leads to the "strongest divides" in society. Such divides obviously limit cross-group influences, as people are mostly "involved in the lives of other members of their group", as noted by Greif (1994). We analyze how ethnic divides and factions affect the tradeoff between political competition and the existence of a stable leader under a democratic constitution.

Our formalization of a democratic constitution follows the principle that all citizens have equal political rights and therefore have equal influence on social decisions. It follows that only the number, not the identity, of people who support a particular social alternative matters in imposing that alternative on the rest of the society. In order words, a democratic constitution gives decision-making power only to sufficiently large groups, which are groups that are at least as large as an exogenous threshold (see, for example, Peleg (1978), Jackson and Barberà (2004)). This threshold might vary depending on decision types. In order to be made, some decisions require the approval of more than half of the population, whereas others require the support of at least two-thirds of the population, and still others require the support of the entire population.

Formally, let W be a democratic constitution, and let q be the minimum number of votes that are needed to pass a decision under W. For simplicity, we denote such a decision rule by q instead of W. A coalition S is winning if and only if $|S| \geq q$. The third defining property of a political technology, which states that two winning coalitions should always overlap in order to avoid obvious political instability, implies that q should be greater than half of the population ($\frac{n}{2} < q \leq n$).

Let $\mathcal{R} = (g, (p^i)_{i \in N})$ be the social network. We assume that the binary relation g is symmetric and transitive. This effectively implies that the society is segregated, having a certain number of groups that we view as *ethnic groups*. These groups are technically the components of the network $\mathcal{R}$ (a component is a maximal subset of g such that any two elements are directly or indirectly connected). Assume that there are p groups N_1, N_2, ..., N_p ($p \geq 2$)[11], of which r are isolated or have only one individual member.[12] Isolated individuals might be viewed as emancipated individuals who are not subject to the influence of their ethnic group.

Denote by $\lceil x \rceil$ the smallest integer larger than or equal to a real number x, and by $\lfloor x \rfloor$ the largest integer smaller than or equal to x. The following theorem presents the exact tradeoff between political competition and stability in a democratic society organized around ethnic groups.

Theorem 6 *The preference-blind political stability index of a democratic society under an ethnic network $\mathcal{R}$ such as described above is:*

$$\nu(q, \mathcal{R}) = \begin{cases} 2 \text{ if } q \leq n - p + \lfloor \frac{r}{2} \rfloor \\ \lceil \frac{2p-r}{n-q} \rceil \text{ if } q > n - p + \lfloor \frac{r}{2} \rfloor \end{cases}$$

Several practical implications follow from Theorem 6. One implication is that increasing *individualism* or decreasing *communitarianism* by splitting a fixed population to create a larger number of minor ethnic groups relaxes the tradeoff between political competition and the existence of a stable political leader. In a situation of extreme individualism, which corresponds to an empty ethnic network (that is, $\mathcal{R} = \mathcal{R}_0$), all individuals are isolated and so $p = n = r$ and $q > n - p + \lfloor \frac{r}{2} \rfloor$. Theorem 6 then implies that $\nu(q, \mathcal{R}) = \nu(q, \mathcal{R}_0) = \lceil \frac{n}{n-q} \rceil$, which is the well-known Peleg number (see Peleg (1978)). Remark that $\lceil \frac{n}{n-q} \rceil > \lceil \frac{2p-r}{n-q} \rceil$ if $p < n$ and $q < n$. It follows that increasing individualism under a political technology different from the unanimity rule leads to more stability.

In general, Theorem 6 shows how different social structures (captured by the variable p) lead to different political-stability outcomes. It shows, for instance, that a country with only two extended families or two ethnic groups like Rwanda ($p = 2$) is less likely to be politically stable than a country that is organized around nuclear families. Indeed, imagine a country of 1 million people organized around nuclear families, with each family having only two adults who can vote or express a political opinion. Then $p = \lfloor \frac{n}{2} \rfloor = 500,000$. If that country has two ethnic groups instead, then $p = 2$. As $500,000 > 2$, it follows that the stability index of the country organized around nuclear families is greater than that of the country that has two ethnic groups, especially if the political technology q is different from the unanimity rule ($q < n$). The lesson to be learned is that, when political influence is minimal, political instability is less likely. In other words, increasing

[11] The case of $p = 1$ has been covered in the preceding section, as this case corresponds to a completely connected society.

[12] A partitioning of the population as is assumed here leads to what is popularly known as the "islands model" in the literature on networks. This formalization is consistent with the description of collectivist societies provided by Greif (1994).

fragmentation results in greater stability. This prediction is consistent with other studies that have shown that ethnic fractionalization makes societies safer (see, for example, Collier and Hoeffler (1998) and Bates (1999)). The main argument in this literature is that, because more ethnically heterogeneous societies are less cohesive, it is more difficult to form a viable multi-ethnic coalition of rebels to fight the status quo.

Theorem 6 further shows how a decision rule q can be chosen to minimize political instability given a social structure p. It implies that, if the society is likely to be polarized (which corresponds to a small number of ethnic groups or factions p), then q should be high for stability to prevail. In particular, if $q = n$ (which is the unanimity rule), stability always prevails as $\nu(q, \mathcal{R}) = +\infty$. In general, the closed form solution provided by Theorem 6 implies that, for a given or desired level of the competition-stability tradeoff $\nu(q, \mathcal{R})$, one can always find the value of q that matches that of an exogenous social structure p.

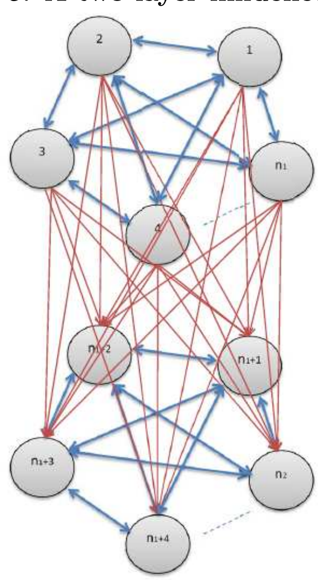
5.3 Social Networks that Maximize or Minimize the Tradeoff between Political Competition and Stability in Democracies

In this section, we identify certain network structures that maximize or minimize the tradeoff between political competition and stability. It is a useful analysis, as it has implications for the design of organizations and political systems that promote stability. We find that cliques and multi-layer cliques minimize the stability index, whereas star networks, rings, and lines maximize it.

5.3.1 Instability-Maximizing Social Networks: Cliques and Multi-layer Cliques

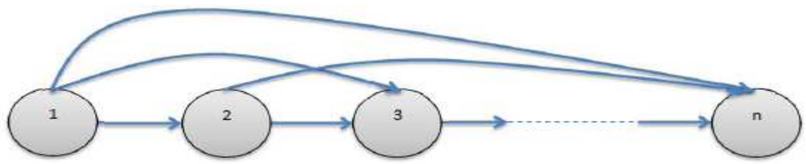
We show that cliques and multi-layer cliques minimize the index of political stability, and thus maximize the level of conflict between political competition and the existence of a stable leader. These network structures are modelled as weak orderings. They have one or multiple layers of influence in which individuals in each layer of the hierarchy influence those in lower-level layers, without the inverse being true. Individuals belonging to the same layer mutually influence each other (see Figures 5 and 6).

Figure 5: A two-layer influence clique



Formally, we say that a network $\mathcal{R} = (g, (P^i)_{i\in N})$ is a multi-layer clique if g is a weak ordering on the society N. Assume that the symmetric component of g has p equivalence classes. We know that these classes can be ordered as a sequence $(N_1, N_2, ..., N_p)$ where $\bigcup_{t=1}^{p} N_t = N$. The interpretation is that individuals in the top layer N_1 can influence each other and all the individuals in lower-level layers, and individuals in the second layer N_2 can influence each other and individuals in the lower-level layers, and so on. If $\mathcal{R}$ has only one layer, we say that $\mathcal{R}$ is a clique.

Figure 6: An n-layer influence clique with one individual in each layer



Cliques and multi-layer cliques maximize the spread of conflicting influences, which in turn maximizes the tradeoff between political competition and stability.

We show that the political stability index of a society connected by a multi-layer clique is two, which implies that societies under such networks are highly prone to political instability.

Proposition 2 *Let N be a democratic society endowed with a political technology $q < n$ and connected by a multi-layer clique $\mathcal{R}$. Then $\nu(q, \mathcal{R}) = 2$.*

Another interpretation of the result is that a multi-layer clique positively affects political cycles, inducing alternation or rotation in power with a strictly positive probability, even if there are only two political parties. The result is surprising, given the fact that it holds for any democratic constitution, including rules that are known to be strongly biased toward the status quo, such as rules that are close to the unanimity rule. Under the closest rule to the uninamity rule ($q = n - 1$) in particular, the index of political stability is n under the empty network, which means that there exists a political leader who will stay in power forever if the number of competing leaders ia at most $n - 1$. The fact that the stability index suddenly drops from $n - 1$ to two in the presence of a clique shows the powerful influence of this network structure on political cycles. Within our framework, cycles are induced by the changing opinions of non-ideological individuals who are subject to the influence of ideological individuals who have opposing views.

5.3.2 Instability-Minimizing Social Networks: Stars, Rings, and Lines

Although social networks increase the likelihood of political instability in general, in this section, we show that certain networks like directed star networks, rings, and lines minimize the tradeoff between political competition and the existence of a stable leader in democratic societies.

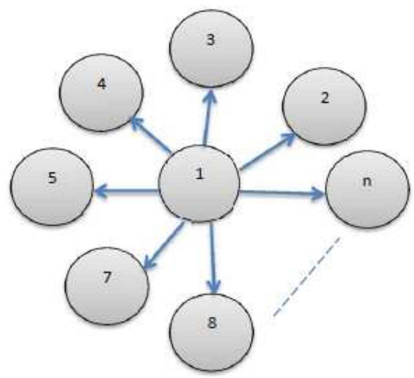
Star Influence Networks A star influence network is a network in which one individual influences all the other individuals (Figure 7). It captures social influence in a context in which there is only one opinion leader, or one television or radio channel that spreads propaganda.

Star networks are prevalent in real-world politics. For instance, answering a question asked by Acemoglu, Reed and Robinson (2013) about whether he had any influence on the political decisions of his people in an election, a paramount chief in Sierra Leone said: "if I say left they go left, if I say right they go right." We show how such star networks affect political stability, even in situations where peripheral agents might not follow the political opinion of the hub.

Formally, a star influence network is a network $\mathcal{R} = (g, (P^i)_{i \in N})$ such that there exists an individual $i_0 \in N$ such that for any individual $j \in N$:

$$g(j) = \begin{cases} N \setminus \{i_0\} \text{ if } j = i_0 \\ \emptyset \text{ if } j \neq i_0 \end{cases}.$$

Figure 7: A star influence network



Individual 1 is the only source of propaganda, which limits opposing political influences. This leads to the minimization of the tradeoff between political competition and stability.

We show below that a star influence network maximizes the index of political stability in a democratic society.

Proposition 3 *Let N be a democratic society endowed with a political technology q and a star influence network $\mathcal{R}$. Then $\nu(q, \mathcal{R}) = \lceil \frac{n}{n-q} \rceil$.*

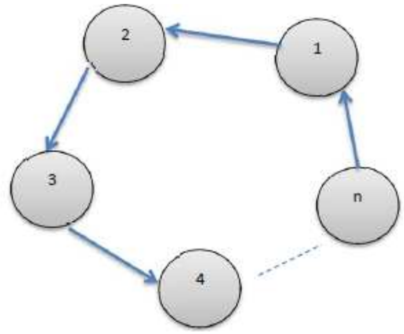
This result implies that star influence networks behave much like the empty network in terms of minimizing the tradeoff between political competition and stability. It follows that political alternation is less likely when non-ideological individuals are not subject to conflicting influences than when political competition is supported by multiple sources of propaganda.

Ring Influence Networks A ring influence network is a directed network in which each individual may influence one individual and may be influenced by one other individual (Figure 8).

In a ring, each individaul has one out-neighbor and one in-neighbor who is different from his out-neighbor. A ring influence network is formalized as a network $\mathcal{R} = (g, (P^i)_{i \in N})$ where for all $j \in N$:

$$g(j) = \begin{cases} \{j+1\} \text{ if } j \neq n \\ \{1\} \text{ if } j = n \end{cases}.$$

Figure 8: A ring influence network



Like star influence networks, rings minimize the tradeoff between political competition and stability by minimizing the spread of conflicting influences.

We show below that rings maximize the index of political stability in democratic societies.

Proposition 4 *Let N be a democratic society endowed with a political technology q and a ring influence network $\mathcal{R}$. Then $\nu(q, \mathcal{R}) = \lceil \frac{n}{n-q} \rceil$.*

This finding implies that rings behave like star networks in terms of minimizing the tradeoff between political competition and the existence of a stable leader. They do so by limiting the spread of conflicting influences.

Line Influence Networks A line influence network is a network in which each individual influences one individual and is influenced by one other individual, except one individual who influences but cannot be influenced, and another individual who may be influenced but cannot influence anybody (Figure 9).

A line influence network is formalized as a network $\mathcal{R} = (g, (P^i)_{i \in N})$ where for all $j \in N$:

$$g(j) = \begin{cases} \{j+1\} \text{ if } j \neq n \\ \emptyset \text{ if } j = n \end{cases}.$$

Figure 9: A line influence network

Like star and ring influence networks, line influence networks minimize the tradeoff between political *competition and political stability by minimizing the spread of conflicting influences.*

We show that line influence networks maximize the index of political stability in democratic societies.

Proposition 5 *Let N be a democratic society endowed with a political technology q and a line influence network $\mathcal{R}$. Then $\nu(q, \mathcal{R}) = \lceil \frac{n}{n-q} \rceil$.*

This result implies that lines, like stars and rings, minimize the tradeoff between political competition and the existence of a stable leader by limiting the spread of opposing political views.

5.3.3 How do Star, Ring, and Line Influence Networks Differ From the Empty Network?

To complete this section, it is important to draw the difference between star, ring, and line influence networks on the one hand and the empty network on the other hand in how they affect political cycles and instability. We have seen that these different classes of networks behave similarly in terms of minimizing the tradeoff between political competition and the existence of a stable political leader. However, they also differ in a significant respect. Unlike the empty network, stars, rings, and lines might significantly refine the *certainty equilibrium set*. However, they do not cause this set to be empty, unless it can be empty at a preference profile in the absence of networks. It follows that, when the number of competing politicians is bounded above by the index $\nu(q, \mathcal{R})$, these networks reduce the set of politicians who can rule the society in a stable manner without eradicating them. This can be viewed as a positive property of these networks.

6 Conclusion

We have analyzed the effects of social influence networks on political cycles and instability. The analysis has shown that influence networks increase political turnout, and cause non-ideological individuals who are subject to antagonistic influences to continue floating their political views. This in turn increases the likelihood of political cycles and instability in two ways: (1) by reducing the number of leaders who can rule the society in a stable manner; and (2) by increasing the tradeoff between political competition and the existence of a stable leader. We show that influence networks can even eradicate the existence of stable leaders, leading to a complete social destabilization.

Our main characterization of stable societies introduces a *preference-blind index of political stability*. A higher value of this index indicates a greater level of stability. It maps each pair of a constitution (or political technology) and a social network into the maximum number of competing politicians that a society can afford while continuing to maintain its stability. This finding has important implications for the design of stable societies. First, it prescribes the number of competing political parties or leaders that should not be exceeded in a society with a known constitution and network structure. Second, for societies that have laws or traditions regarding the number of political parties, this result characterizes all of the constitutions and network structures under which political stability is guaranteed regardless of the extent of conflicting political views in the population.

Applications of the findings provide insight into why some societies are more politically stable than others. In particular, we find that collectivist societies and societies organized around ethnic groups or characterized by a high level of homophilous behavior are more prone to political instability than individualist societies. We quantify the exact tradeoff between political competition and stability in ethnic democracies, showing how it depends on the number of ethnic groups. In particular, we find that more fragmented societies are more stable in general. However, in societies where the number of political parties reflects the number of ethnic groups, the stability threshold is easily exceeded, thus increasing the risk of political instability.

The analysis also sheds light on the network origins of political cycles in two-party systems. In such systems, the status quo remains in power forever in the absence of influence networks. The presence of influence networks, however, increases electoral turnout and political turnover by mobilizing non-ideological individuals to the polls. If influenced by ideological individuals with opposing views, these non-ideological individuals cast a probabilistic vote, which consists of voting in favor of a particular leader with a frequency that reflects the amount of influence exerted on them by the supporters of that leader. Their votes therefore fluctuate over time, inducing political alternation in power. Interestingly, our model of a networked political economy addresses an important limitation of the classical model of a political economy. This model predicts that there will be no change in power in a two-party system, a highly implausible prediction in light of real-life political outcomes. Our analysis shows that the incorporation of influence networks into the classical model yields more realistic predictions. Importantly, we also find that if the network structure of a two-party society is known, one can compute the frequency with which parties take turns in power.

We further identify popular network structures that maximize or minimize the tradeoff between political competition and stability. We find that cliques and multi-layer cliques maximize the competition-stability tradeoff, whereas star, ring, and line networks minimize it. It follows that, when the level of political competition is not too high, these latter networks refine the set of stable politicians but do not eradicate them, which is a positive finding.

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Appendix

Proof of Theorem 1

Sufficiency

Assume that $x \notin \mathcal{C}(\mathcal{E})$. Then $P_x > 0$ and there exits $y \in A$ such that $P(y,x) > 0$. This implies that there exists $(S_1, S_2, S_3) \in 3^N$ such that $S_1 \in W$ and $P_{(S_1,S_2,S_3)}(y,x) = \prod_{i \in S_1} p^i_{yx} \prod_{i \in S_2} p^i_{(xy)} \prod_{i \in S_3} p^i_{xy} > 0$.

We will show that $S_1 \subseteq N_{yx} \cup g(N_{yx})$. Let $i \in S_1$. Suppose by contradiction that $i \notin N_{yx} \cup g(N_{yx})$.

- If $i \in N_{xy}$, then $p^i_{yx} = 0$ and $P_{(S_1,S_2,S_3)}(y,x) = 0$, which is a contradiction.

- If $i \in N_{(yx)}$ and $i \notin g(N_{yx})$, then $g(i) \cap g(N_{yx}) = \emptyset$, which implies that $p^i_{yx} = 0$, and this is a contradiction.

- If $i \in N_{(yx)}$ and $g(i) = \emptyset$, $p^i_{yx} = 0$ and this is a contradiction.

Therefore, $i \notin N_{yx} \cup g(N_{yx})$ implies $i \notin S_1$, implying $S_1 \subseteq N_{yx} \cup g(N_{yx})$.

We also have $S_1 \subseteq N_{yx} \cup [g(N_{yx}) \setminus N_{xy}]$. Indeed if $i \in S_1$, $i \notin N_{yx}$ and $i \in g(N_{yx}) \setminus N_{xy}$, then $i \in N_{xy}$ and $p^i_{yx} = 0$, which is a contradiction.

$S_1 \subseteq N_{yx} \cup [g(N_{yx}) \setminus N_{xy}]$ implies that $N_{yx} \cup [g(N_{yx}) \setminus N_{xy}] \in W$ since $S_1 \in W$.

Now let $S = N_{yx} \cup \{i \in g(N_{yx}) \setminus N_{xy} : y \succ_i x\}$ and $T = \{i \in g(N_{yx}) \setminus N_{xy} : y \simeq_i x\}$.

It follows from what precedes that: $T \subseteq g(S)$, $S \cup T \in W$, $y \succ_i x$ for all $i \in S$, and $x \simeq_i y$ for all $i \in T$.

Necessity

Conversely, assume that there exist two coalitions S and T such that : $T \subseteq g(S)$, $S \cup T \in W$, $y \succ_i x$ for all $i \in S$, and $x \simeq_i y$ for all $i \in T$. Let us show that $x \notin \mathcal{C}(\mathcal{E})$. It suffices to show that $P(y,x) > 0$.

Let $S_1 = S \cup T$. By definition, we have :

$$P(y,x) = \sum_{\substack{S_1 \in W \\ (S_1,S_2,S_3) \in 3^N}} \prod_{i \in S_1} p^i_{yx} \prod_{i \in S_2} p^i_{(xy)} \prod_{i \in S_3} p^i_{xy}$$

$y \succ_i x$ for all $i \in S$ implies that $S \subseteq N_{yx}$. Furthermore, $T \subseteq g(S)$ and ($x \simeq_i y$ for all $i \in T$) imply that $S \cup T \subseteq N_{yx} \cup [g(N_{yx}) \setminus N_{xy}]$. It follows that $N_{yx} \cup [g(N_{yx}) \setminus N_{xy}] \in W$. Now let:

$$\begin{cases} S_1 = N_{yx} \cup [g(N_{yx}) \setminus N_{xy}] \\ S_2 = \{i \in N : i \in N_{(yx)} \text{ and } (g(i) = \emptyset \text{ or } g(i) \subseteq N_{(xy)})\} \\ S_3 = N_{xy} \cup [(N_{(xy)} \cap g(N_{xy})) \setminus (N_{(xy)} \cap g(N_{yx})] \end{cases}$$

It is clear that (S_1, S_2, S_3) is an opinion profile of N and that $S_1 \in W$. We have:

$\prod_{i \in S_1} p^i_{yx} = \prod_{i \in N_{yx}} p^i_{yx} \prod_{i \in g(N_{yx}) \setminus N_{xy}} p^i_{yx} = \prod_{i \in g(N_{yx}) \setminus N_{xy}} p^i_{yx} > 0$ because for all $i \in g(N_{yx}) \setminus N_{xy}$, $p^i_{yx} > 0$

$\prod_{i \in S_2} p^i_{(yx)} = 1$ because $p^i_{(yx)} = 1$ if $i \in N_{(yx)}$, and $g(i) = \emptyset$ and $p^i_{(yx)} = 1$ if $i \in N_{(yx)}$ and $g(i) \subseteq N_{(xy)}$).

$\prod_{i \in S_3} p^i_{xy} = \prod_{i \in N_{xy}} p^i_{xy} \prod_{i \in [(N_{(xy)} \cap g(N_{xy})) \setminus (N_{(xy)} \cap g(N_{yx})]} p^i_{xy} = 1 \times \prod_{i \in [(N_{(xy)} \cap g(N_{xy})) \setminus (N_{(xy)} \cap g(N_{yx})]} p^i_{xy} > 0$ because for all $i \in [(N_{(xy)} \cap g(N_{xy})) \setminus (N_{(xy)} \cap g(N_{yx})]$, $p^i_{xy} > 0$.

Finally, $P(y,x) = \sum_{\substack{S_1 \in W \\ (S_1,S_2,S_3) \in 3^N}} \prod_{i \in S_1} p^i_{ux} \prod_{i \in S_2} p^i_{(xu)} \prod_{i \in S_3} p^i_{xu} \geq \prod_{i \in S} p^i_{yx} \prod_{i \in S_2} p^i_{(xy)} \prod_{i \in S_3} p^i_{xy} > 0$ and thus $x \notin \mathcal{C}(\mathcal{E})$. ■

Proof of Theorem 2

Let $\mathcal{E}^1$=$(N, A, W, \mathcal{R}_1, (\succeq_i)_{i\in N})$ and $\mathcal{E}^2$=$(N, A, W, \mathcal{R}_2, (\succeq_i)_{i\in N})$ be two networked political economies such that $\mathcal{R}_1 \subseteq \mathcal{R}_2$. If $x \notin \mathcal{C}(\mathcal{E}^1)$, then thanks to Theorem 1, there exist $y \in A$ such that $y \succ x$ in $\mathcal{E}^1$. This implies that there exist two coalitions S and T such that :

$$\begin{cases} T \subseteq g_1(S) \text{ and } S \cup T \in W, \\ y \succ_i x \text{ for all } i \in S \text{ and} \\ x \simeq_i y \text{ for all } i \in T \end{cases}$$

Since $g_1(S) \subseteq g_2(S)$, it follows that $T \subseteq g_2(S)$ and $S \cup T \in W$. Therefore, $y \succ x$ in $\mathcal{E}^2$, and we conclude that, thanks to Theorem 1, that $x \notin \mathcal{C}(\mathcal{E}^2)$.

Example 2 proves that the inclusion may be strict. ■

Proof of Theorem 3

The following definition and lemma due to Suzumura (1976) and Lahiri (2002) will be needed in the proof of Theorem 3. The definition is that of the extension of a binary relation.

Definition 6 *Let R be a binary relation defined on a finite set A.*

1) A cycle of R is a sequence $(x_1, x_2, ..., x_q)$ of distinct elements of B such that:

a) $x_1 \leq_R x_2 \leq_R ... \leq_R x_q \leq_R x_1$; and

b) there exists $c \in \{1, 2, ..., q\} : x_c <_R x_{c+1}$.

2) A binary relation R' defined on A is said to extend R if: $x \leq_R y$ implies $x \leq_{R'} y$ and $x <_R y$ implies $x <_{R'} y$.

The following lemma states that a binary relation on a finite set can be extended to an ordering if and only if it does not have a cycle.

Lemma 1 *A binary relation R defined on a finite set A can be extended to an ordering on A if and only if R does not have a cycle.*

The **proof of Theorem 3** is below.

Sufficiency

Assume that a networked society $\mathcal{S} = (N, A, W, \mathcal{R})$ is not stable, and let R be a preference profile such that $\mathcal{C}((N, A, W, \mathcal{R}, R) = \emptyset$. Then the dominance relation has a cycle $a_1, a_2, ..., a_p \in A$ of pairwise distinct politicians such that: a_2 dominates a_1, a_3 dominates a_2, ..., a_p dominates a_{p-1} and a_1 dominates a_p. Thus there exist $2p$ coalitions S_1, T_1, S_2, T_2, ..., S_p, $T_p \subset N$ such that for any $t \in \{1, 2, ..., p\}$, $T_t \subseteq g(S_t)$, $S_t \cup T_t \in W$, $S_t \subseteq N_{a_t a_{t-1}}$ and $T_t \subseteq N_{(a_t a_{t-1})}$.

Consider the sequence $\hat{S} = \{(S_1, T_1), (S_2, T_2) ..., (S_p, T_p)\}$. The following statements are correct.

(i) : $S_t \subseteq N$ and $T_t \subseteq g(S_t)$.

(ii) : $S_t \cup T_t \in W$, and since $S_t \subseteq N_{a_t a_{t-1}}$ and $T_t \subseteq N_{(a_t a_{t-1})}$, it follows that $S_t \cap T_t = \emptyset$.

(iii) : If an individual $i \in [\cap_{s=1}^{k} (S_s \cup T_s)] \cap S_t$, then for all $s \in \{1, 2, ..., p\}$, it is the case that $i \in S_s \cup T_s$ and $i \in S_t$. The preferences being transitive, $i \in S_s \cup T_s$ for all s implies that $a_{t-1} \succeq_i a_t$. However, $i \in S_t$ implies that $a_t \succ_i a_{t-1}$ and this is a contradiction. Thus $[\cap_{s=1}^{k} (S_s \cup T_s)] \cap S_t = \emptyset$.

The sequence $\hat{S} = \{(S_1, T_1), (S_2, T_2) ..., (S_p, T_p)\}$ is therefore a circuit. Since $a_1, a_2, ...$ and a_p are pairwise distinct, we deduce that $|A| \geq p$, implying that $\nu(W, \mathcal{R}) \leq |A|$.

Necessity

Conversely, let us assume that $\nu(W, \mathcal{R}) = p \leq |A|$. Let $a_1, a_2, ..., a_p$ be p distinct politicians and $\hat{S} = \{(S_1, T_1), (S_2, T_2) ..., (S_p, T_p)\} \in \xi(W, \mathcal{R})$. Let $B = \{a_1, a_2, ..., a_p\}$. For every $k \in N$, we define a binary relation L^k on B as follows: For any $s \in \{1, 2, ..., p\}$:

1) if $k \in S_s$, we suppose $a_{s-1} <_{L^k} a_s$;
2) if $k \in T_s$, we suppose $a_s \simeq_{L^k} a_{s-1}$.

For any $x, y \in B$, $x \leq_{L^k} y$ means that either $a_{s-1} <_{L^k} a_s$ or $a_s \simeq_{L^k} a_{s-1}$.

First let us show that there is no cycle for $\leq_{L^k}$, that is, there is no sequence $x = (x_1, x_2, ..., x_q)$ of distinct elements of B such that: (a) $x_1 \leq_{L^k} x_2 \leq_{L^k} ... \leq_{L^k} x_q \leq_{L^k} x_1$ and (b) $\exists c \in \{1, 2, ..., q\} : x_c <_{L^k} x_{c+1}$.

Assume on the contrary that a sequence x satisfying (a) and (b) exist. Without loss of generality, we can assume that $x_1 = a_1$. Then, by the definition of $\leq_{L^k}$, necessarily $x_2 = a_2$ and thus $k \in S_2 \cup T_2$. Likewise, $x_3 = a_3$ and thus $k \in S_3 \cup T_3$, and iterating, we get $x_q = a_q$ implying $k \in S_q \cup T_q$ with $p = q$. This means that $k \in \cap_{s=1}^{p} (S_s \cup T_s)$. But thanks to (b), $x_c <_{L^k} x_{c+1}$, and we have $k \in S_c$, which implies $k \in [\cap_{s=1}^{p} (S_s \cup T_s)] \cap S_c$. This is a contradiction.

Now, since $\leq_{L^k}$ does not have a cycle, by Lemma 1, there exists an ordering $\leq_{R^k}$ on B that extends $\leq_{L^k}$.

If $k \notin \cup_{s=1}^{k} (S_s \cup T_s)$, we associate with k the ordering $x_1 \leq_{R^k} x_2 \leq_{R^k} ... \leq_{R^k} x_q \leq_{R^k} x_1$.

Finally, for all $k \in N$, we extend R^k (which is an ordering on B) to an ordering on A by considering: $\forall a \in B, \forall b \in A \backslash B$, $a >_{R^k} b$. We have therefore defined a profile of orderings on A.

We now show that the certainty equilibrium set with respect to the profile just constructed is empty.

By construction, $S_t \subset N_{a_t a_{t-1}}$ and $T_t \subset N_{(a_t a_{t-1})} \cap g(S_t)$. Since $\hat{S}$ is a circuit, $S_t \cup T_t \in W$. This implies that $a_{t-1} \notin \mathcal{C}(\mathcal{E})$. Thus for all $a \in B$, $a \notin \mathcal{C}(\mathcal{E})$. Furthermore, every $a \in A \backslash B$ is Pareto-dominated, which implies that $a \notin \mathcal{C}(\mathcal{E})$. ■

Proof of Theorem 4

Let $(W, \mathcal{R}_1)$ and $(W, \mathcal{R}_2)$ be such that $\mathcal{R}_1 \subseteq \mathcal{R}_2$ and $\hat{S} = \{(S_1, T_1), (S_2, T_2) ..., (S_k, T_k)\}$ a circuit of $\nu(W, \mathcal{R}_1)$. The following statements are true:

(i) $S_t \subset N$, $T_t \subset g_1(S_t) \subseteq g_2(S_t)$ since $\mathcal{R}_1 \subseteq \mathcal{R}_2$.

(ii) $S_t \cup T_t \in W$ and $S_t \cap T_t = \emptyset$.

(iii) $[\cap_{s=1}^{k} (S_s \cup T_s)] \cap S_t = \emptyset$.

Thus, $\mathcal{R}_1 \subseteq \mathcal{R}_2$ implies that a circuit of $\nu(W, \mathcal{R}_1)$ is also a circuit of $\nu(W, \mathcal{R}_2)$. This in turn implies that $\nu(W, \mathcal{R}_2) \leq \nu(W, \mathcal{R}_1)$.

To show that this inequality may be strict, consider that $\mathcal{R}_1$ is the empty network and $\mathcal{R}_2$ is multi-layer clique. We show in Proposition 1 that $\nu(W, \mathcal{R}_1) \geq 3$ and in Proposition 2 that $\nu(W, \mathcal{R}_2) = 2$, implying that this inequality may be strict. ■

Proof of Proposition 1

If $\mathcal{R} = \mathcal{R}_0$, a circuit of $(W, \mathcal{R})$ is reduced to a set $\bar{S} = \{S_1, S_2, ..., S_k\}$ such that $S_s \in W$ for any $s = 1, 2, ..., k$ and $\cap_{s=1}^{k} S_s = \emptyset$. Our political stability index then becomes:

$$\nu(W, \mathcal{R}_0) = \begin{cases} +\infty \text{ if } \xi(W, \mathcal{R}_0) = \emptyset \\ \min\left\{\left|\bar{S}\right|, \bar{S} \in \xi(W, \mathcal{R}_0)\right\} \text{ if } \xi(W, \mathcal{R}_0) \neq \emptyset \end{cases}.$$

Suppose by contradiction that $\nu(W, \mathcal{R}) < 3$. Then $\nu(W, \mathcal{R}) = 2$ given that $\nu(W, \mathcal{R})$ is always strictly greater than 1. This implies that there exists a circuit $\bar{S} = \{S_1, S_2\}$ such that $S_1, S_2 \in W$ and $S_1 \cap S_2 = \emptyset$, which is a contradiction given that two winning coalitions always overlap. We conclude that $\nu(W, \mathcal{R}) \geq 3$.

If $\mathcal{R} \neq \mathcal{R}_0$, since $\nu(W, \mathcal{R}) > 1$ by definition, it follows that $\nu(W, \mathcal{R}) \geq 2$. To show that this inequality may be binding, see the proof of part 2) of Theorem 5. ■

Proof of Theorem 5

1) It follows from Theorem 4 that $\nu(W, \mathcal{R}_0) \geq \nu(W, \mathcal{R}) \geq \nu(W, \mathcal{C})$. To show the equalities, it therefore suffices to show that $\nu(W, \mathcal{R}_0) = \nu(W, \mathcal{R}) = \nu(W, \mathcal{C}) = +\infty$. Assume that $\delta = 0$, then $W = \{N\}$. Let $\hat{S} = \{(S_1, T_1), (S_2, T_2) ..., (S_k, T_k)\} \in \xi(W, \mathcal{R})$. Then, for any $t \in \{1, 2, ..., k\}$, $S_t \cup T_t = N$ and $\cap_{t=1}^{k} (S_t \cup T_t) = N$. Therefore, we have $[\cap_{t=1}^{k} (S_t \cup T_t)] \cap S_1 = S_1 \neq \emptyset$, a contradiction. Hence $\xi(W, \mathcal{R}) = \emptyset$ and $\nu(W, \mathcal{C}) = +\infty$.

Assume that $\delta = 1$. Then there exists an individual $i \in N$ such that $W = \{N, N \backslash \{i\}\}$. Let $\hat{S} = \{(S_1, T_1), (S_2, T_2) ..., (S_k, T_k)\} \in \xi_G$. Then for any $t \in \{1, 2, ..., k\}$, $S_t \cup T_t \in \{N, N \backslash \{i\}\}$, and thus $\cap_{t=1}^{k} (S_t \cup T_t) \supset N \backslash \{i\}$. But for any $s \in \{1, 2, ..., k\}$, $\cap_{t=1}^{k} (S_t \cup T_t) \cap S_s = \emptyset$. Therefore, for any $s \in \{1, 2, ..., k\}$, $i \in S_s$, which is a contradiction since $\cap_{s=1}^{k} S_s = \emptyset$. Hence, $\xi_G = \emptyset$ and $\nu(W, \mathcal{C}) = +\infty$.

2) Consider that $\delta \geq 2$. Again, the inequalities follow from Theorem 4. We need to show that $\nu(W, \mathcal{C}) = 2$. Since $\delta \geq 2$, there exist two individuals i and j such that $N \backslash \{i\} \in W$ and $N \backslash \{j\} \in W$. Let $S_1 = \{i\}$, $S_2 = \{j\}$, $T_1 = N \backslash \{i, j\}$ and $T_2 = N \backslash \{i, j\}$. Then, $S_1 \cup T_1 = N \backslash \{j\} \in W$, and $S_2 \cup T_2 = N \backslash \{i\} \in W$. Furthermore, $(S_1 \cup T_1) \cap (S_2 \cup T_2) \cap S_1 = (N \backslash \{j\}) \cap (N \backslash \{i\}) \cap \{i\} = \emptyset$ and $(S_1 \cup T_1) \cap (S_2 \cup T_2) \cap S_2 = \emptyset$. Thus $\hat{S} = \{(S_1, T_1), (S_2, T_2)\} \in \xi_G$, implying that $\nu(W, \mathcal{C}) = 2$.

To see that the inequalities might be strict, it suffices to find a constitution W and a network $\mathcal{R}$ such that $\nu W, \mathcal{R}_0) > \nu(W, \mathcal{R}) > \nu(W, \mathcal{C})$. Consider a society of 90 individuals governed by a rule W under which a coalition is winning if it contains at least 88 individuals. Individuals are connected by a network $\mathcal{R}$ whose graph has three components, with each component being a symmetric and transitive relation. It can be shown (see Theorem 6) that $\nu(W, \mathcal{R}_0) = 45$ and $\nu(W, \mathcal{R}) = 3$. Since $\nu(W, \mathcal{C}) = 2$, it immediately follows that $\nu W, \mathcal{R}_0) > \nu(W, \mathcal{R}) > \nu(W, \mathcal{C})$. ■

Proof of Theorem 6

The following lemma is useful for the proof of Theorem 6.

Lemma 2 *Let A_1, A_2, ... and A_t be t subsets of a set M of α elements each. If $|M| = \beta$, then $|A_1 \cap A_2 \cap ... \cap A_t| \geq t\alpha - (t-1)\beta$.*

Let $p \in \mathbb{N}^*$ be a natural number, $a_1, a_2, ..., a_p \in A$ p politicians, $S_1, S_2, ..., S_p \subset N$ p coalitions, and $s \in \{0, 1, 2, ..., p\}$. For any non-negative integer m, we write : $a_{s+mp} = a_s$ and $S_{s+mp} = S_s$. Particularly, $a_{p+1} = a_1$, $a_0 = a_p$, $S_{p+1} = S_1$, $S_0 = S_p$.

The proof of **Theorem 6** is now ready:

Without loss of generality, write :

$$\begin{cases} N_1 = \{k_1\},\ N_2 = \{k_2\},\ ...,\ N_r = \{k_r\} \text{ and} \\ N_{r+1} = \{i_1, i_{u+1}\} \cup N^0_{r+1},\ ...,\ N_{r+u} = \{i_u, i_{2u}\} \cup N^0_{r+u} \text{ with } r + u = p \end{cases}$$

Each of the components N_1, ...and N_r is isolated, whereas each of the components N_{r+1}, ... and $N_{r+u} = N_p$ has at least two individuals. Let $L = N^0_{r+1} \cup N^0_{r+2} \cup ... \cup N^0_{r+u}$. Remark that:

$$l = |L| = n - (2p - r).$$

If $q \le u + l + \lfloor \frac{r}{2} \rfloor$, it is obvious that $N \smallsetminus L$ can be partionned into two disjoints sets S_1 and S_2 of cardinality $u + \lfloor \frac{r}{2} \rfloor$ each such that $S_1 \cap L \neq \emptyset$ and $S_2 \cap L \neq \emptyset$, where $\lfloor x \rfloor$ is the greatest integer smaller than x. Thus, $\{(S_1, L), (S_2, L)\}$ is a circuit, implying that $\nu_{\mathcal{R}}(G) = 2 = \lceil \frac{2p-r}{n-q} \rceil$.

In the sequel we assume that $q > u + l + \lfloor \frac{r}{2} \rfloor$. Let $v = q - l > u + \lfloor \frac{r}{2} \rfloor$.

1) Let us show that $\nu(W, \mathcal{R}) \le \lceil \frac{2p-r}{n-q} \rceil$**.**

It suffices to construct a circuit of $(W, \mathcal{R})$ of length $\lceil \frac{2p-r}{n-q} \rceil$.

Order the individuals in $N \setminus L$ as follows: $i_1 i_2 ... i_u k_1 k_2 ... k_r i_{u+1} ... i_{2u}$. For simplicity, we rename these individuals as: $j_1 j_2 ... j_u j_{u+1} ... j_{2u+r-1} j_{2u+r}$ where:

$$\begin{cases} j_q = i_q \text{ for all } q \in \{1, 2...u\} \\ j_q = k_{q-u} \text{ for all } q \in \{u+1, u+2, ..., u+r\} \\ j_q = i_{q-r} \text{ for all } q \in \{u+r+1, u+r+2, ..., 2u+r\} \end{cases}.$$

Note that if $r = 0$, then the ranking is simply $i_1 i_2 ... i_u i_{u+1} ... i_{2u}$ and the relabelling gives $j_1 j_2 ... j_u j_{u+1} ... j_{2u-1} j_{2u}$ where $2u = p$.

Now consider the following sequence of coalitions:

$S_1 = S_{1 \to v}$, $S_2 = S_{v \to 2v}$, ..., $S_k = S_{(k-1)v \to kv}$, where the coalitions $S_{tv \to (t+1)v}$ are constructed as follows:

$$S_{1 \to v} = \{j_1, j_2, ... j_v\},\ S_{v \to 2v} = \{j_{v+1}, j_{v+2}, ..., j_{2v}\},\ ...,$$
$$S_{tv \to (t+1)v} = \{j_{tv+1}, j_{tv+2}, ... j_{(t+1)v}\} \text{ with } j_{(2u+r)+1} = j_1,\ j_{(2u+r)+2} = j_2, ...$$

and more generally

$$j_{(2u+r)t+s} = j_s \text{ for all integer } t \text{ and } 0 \le s \le 2u + r \text{ with } j_0 = j_{2u+r}.$$

One can remark that for all integer $m > 1$, if $S_{1 \to v} \cap S_{v \to 2v} \cap ... \cap S_{(m-1)v \to mv}$ is not empty) we have:

$$|S_{1 \to v} \cap S_{v \to 2v} \cap ... \cap S_{(m-1)v \to mv}| = mv - (2u + r)(m - 1).$$

For example:

$$S_{1\to v} \cap S_{v\to 2v} = \{j_1, j_2, ...j_\delta\}$$

where

$$\delta = v - ((2u+r) - v) = 2v - (2u+r).$$
$$S_{1\to v} \cap S_{v\to 2v} \cap S_{2v\to 3v} = \{j_1, j_2, ...j_\lambda\}$$

with

$$\lambda = v - ((2u+r) - \delta) = 3v - 2(2u+r) \text{ (note that } j_{2v+1} = j_{[2v-(2u+r)]+1} = i_{\delta+1})$$

and so on.

Now let $w = \lceil \frac{2u+r}{(2u+r)-v} \rceil$ and consider $\mathcal{S} = \{(S_1, T_1), (S_2, T_2), ..., (S_w, T_w)\}$ defined by :

$$S_1 = S_{1\to v}, \ ... \ , S_w = S_{(w-1)v\to wv} \text{ and } T_1 = ... = T_w = L.$$

We will show that $\mathcal{S}$ is a circuit of $(W, \mathcal{R})$. Let $t \in \{1, 2, ..., w\}$. We have the following:

(i) $S_t \subset N$, and by construction, $S_t \cap N_k \neq \emptyset$ for all $k = r+1, r+2, ..., r+u$, which implies that $g(S_t) = \cup_{k=r+1}^{k=r+u} N_k$. Therefore, $T_t \subseteq g(S_t)$.

(ii) By construction, $S_t \cap T_t \subset S_t \cap L = \emptyset$, and S_t has exactly $tv - (t-1)v = v$ individuals, and thus the cardinality of $S_t \cup T_t$ is $v + l = q$. Therefore, $S_t \cup T_t \in W$.

(iii) Now let us show that $[\bigcap_{s=1}^{w} (S_s \cup T_s)] \cap S_t = \emptyset$. We have:

$$\begin{aligned}
[\bigcap_{s=1}^{w} (S_s \cup T_s)] \cap S_t &= [\bigcap_{s=1}^{w} (S_s \cup L)] \cap S_t \\
&= [(\bigcap_{s=1}^{w} S_s) \cup L)] \cap S_t \\
&= (\bigcap_{s=1}^{w} S_s) \cup (S_t \cap L).
\end{aligned}$$

Since $S_t \cap L = \emptyset$, it remains to show that $\bigcap_{s=1}^{w} S_s = \emptyset$. But thanks to the remark above, $|\bigcap_{s=1}^{w} S_s| = wv - (2u+r)(w-1)$, and by the definition of w, $\frac{2u+r}{(2u+r)-v} \leq w$, that is, $wv - (2u+r)(w-1) \leq 0$, which implies that $\bigcap_{s=1}^{w} S_s = \emptyset$.

$\mathcal{S}$ is therefore a circuit of $(W, \mathcal{R})$ with cardinality $w = \lceil \frac{2u+r}{(2u+r)-v} \rceil = \lceil \frac{2p-r}{n-q} \rceil$ ($(2u+r) - v = 2u + r + l - q = n - q$).

2) We now show that $\nu(W, \mathcal{R}) \geq \lceil \frac{2p-r}{n-q} \rceil$.

Let $w = \lceil \frac{2p-r}{n-q} \rceil$. We will show that for all circuit of length m, we have $m \geq w$. Let $\mathcal{C}_1 = \{(L_1, M_1), (L_2, M_2), ..., (L_m, M_m)\}$ be such a circuit of $(W, \mathcal{R})$.

Assume on the contrary that $m < w$. We assume that for all $t = 1, 2, ..., m$, $|L_t \cup M_t| = q$. Let $H = (L_1 \cup M_1) \cap (L_2 \cup M_2) \cap ... \cap (L_m \cup M_m)$ and $\mathcal{C}_2 = \{(S_1, H), (S_2, H), ..., (S_m, H)\}$ where $S_1 = L_1 \cup (M_1 \setminus H)$, $S_2 = L_2 \cup (M_2 \setminus H)$, ..., and $S_m = L_m \cup (M_3 \setminus H)$. Since $\mathcal{C}_1$ is a circuit, $\mathcal{C}_2 = \{(S_1, H), (S_2, H), ..., (S_m, H)\}$ is also a circuit of $(W, \mathcal{R})$.

Let us first show that $h = |H| \leq n - (2p - r)$.

Since $|N_t| = 1$ for all $t = 1, 2, ..., r$, $H \cap N_t = \emptyset$ for all $t = 1, 2, ..., r$. We will distinguish two cases.

First case. Assume that for any $t \in \{1, 2, ..., m\}$ and $k \in \{r+1, r+2, ..., p\}$, $S_t \cap N_k \neq \emptyset$. Consider an individual $i_k^t \in S_t \cap N_k$. We claim that the following is impossible:

$$\text{For any } k \in \{r+1, r+2, ..., p\},\ i_k^1 = i_k^2 = ... = i_k^m.$$

Indeed, if there exists $k \in \{r+1, r+2, ..., p\}$ such that $i_k^1 = i_k^2 = ... = i_k^m$, then $i_k^t \in S_1 \cap S_2 \cap ... \cap S_m$, which is a contradiction, as $\mathcal{C}_2$ is a circuit.

Therefore, for all $k \in \{r+1, r+2, ..., p\}$, $\{i_k^1, i_k^2, ..., i_k^m\}$ has at least two distinct elements; that is:

$$|N_k \cap (\bigcup_{t=1}^{m} S_t)| \geq 2.$$

Therefore,

$$|(\bigcup_{k=r+1}^{k=p} N_k) \cap (\bigcup_{t=1}^{m} S_t)| = |\bigcup_{k=r+1}^{p} (N_k \cap (\bigcup_{t=1}^{m} S_t))| \geq 2(p-r).$$

But we know that:

$$H \subset (N \setminus \bigcup_{k=1}^{k=r} N_k) \setminus [\bigcup_{k=r+1}^{p} (N_k \cap (\bigcup_{t=1}^{m} S_t))].$$

Thus,

$$h = |H| \leq (n-r) - (2(p-r)) = n - (2p-r).$$

Second case. Assume that there exist $t \in \{1, 2, ..., m\}$ and $k \in \{r+1, r+2, ..., p\}$ such that $S_t \cap N_k = \emptyset$.

Let $\{N^1, N^2, ..., N^\lambda\}$ be the subset of $\{N_{r+1}, N_{r+2}, ...N_p\}$ such that there exist $t \in \{1, 2, ..., m\}$ and $\mu \in \{1, 2, ..., \lambda\}$ with $S_t \cap N^\mu = \emptyset$.

The other $(p-r-\lambda)$ coalitions then satisfy:

$$\text{For any } t \in \{1, 2, ..., m\} \text{ and } N^\mu \notin \{N^1, N^2, ..., N^\lambda\}, S_t \cap N^\mu \neq \emptyset.$$

Note that since $H \subset g(S_t) = \bigcup_{k: S_t \cap N_k \neq \emptyset} N_k$ for all t, if $S_t \cap N_k = \emptyset$, then $H \cap N_k = \emptyset$. This observation yields $H \cap N^\mu = \emptyset$ for all $\mu \in \{1, 2, ..., \lambda\}$. H is therefore a subset of:

$$[(N \setminus \bigcup_{k=1}^{k=r} N_k) \setminus (\bigcup_{\mu=1}^{\lambda} N^\mu)] \setminus \bigcup_{\mu=1}^{p-r-\lambda} (N^\mu \cap (\bigcup_{t=1}^{m} S_t)).$$

We can proceed as in the first case to prove that $|N^\mu \cap (\bigcup_{t=1}^{m} S_t)| \geq 2$ for all $\mu = 1, 2, ..., p-r-\lambda$, implying that $|\bigcup_{\mu=1}^{p-r-\lambda} (N^\mu \cap (\bigcup_{t=1}^{m} S_t))| \geq 2(p-r-\lambda)$.

Hence, $H \subset [(N \setminus \bigcup_{k=1}^{k=r} N_k) \setminus (\bigcup_{\mu=1}^{\lambda} N^\mu)] \setminus \bigcup_{\mu=1}^{p-r-\lambda} (N^\mu \cap (\bigcup_{t=1}^{m} S_t))$ implies $h \leq [(n-r) - 2\lambda] - 2(p-r-\lambda) = n - (2p-r)$.

Now, thanks to Lemma 2, for each $S_t \subset N \setminus H$, $|S_t| = q - h$, we have :

$$\begin{aligned} |(\bigcap_{t=1}^{m} S_t)| &\geq m(q-h)-(m-1)(n-h \\ &= mq-mh-mn+mh+n-h \\ &= n+mq-mn-h \\ &= [n-m(n-q)]-h \end{aligned}$$

We assumed at the beginning that $m < \lceil \frac{2p-r}{n-q} \rceil$, this implies $m < \frac{2p-r}{n-q}$ by the definition of $\lceil x \rceil$). But,

$$\begin{aligned} m < \tfrac{2p-r}{n-q} &\Rightarrow 2p-r > m(n-q) \\ &\Rightarrow n-(2p-r) < n-m(n-q) \\ &\Rightarrow [n-(2p-r)]-h < [n-m(n-q)]-h \\ &\Rightarrow [n-m(n-q)]-h > 0 \text{ since } [n-(2p-r)]-h > 0 \\ &\Rightarrow |(\bigcap_{t=1}^{m} S_t)| > 0 \\ &\Rightarrow \bigcap_{t=1}^{m} S_t \neq \emptyset \end{aligned}$$

This is a contradiction since $\mathcal{C}_2 = \{(S_1, H), (S_2, H), ..., (S_m, H)\}$ is a circuit of $(W, \mathcal{R})$.

Finally, for any circuit of length m, $m \geq \lceil \frac{2p-r}{n-q} \rceil$, and thus $\nu(W, \mathcal{R}) \geq \lceil \frac{2p-r}{n-q} \rceil$. ■

Proof of Proposition 2

To show that $\nu(W, \mathcal{R}) = 2$, we will prove the existence of a circuit of length 2. We distinguish two cases.

First case. Assume that $|N_1| \geq 2$ and let i and j be two individuals in N_1. Consider $\mathcal{C} = \{(S_1, H), (S_2, H)\}$ where : $S_1 = \{i\}$, $S_2 = \{j\}$ and H any subset of $N \setminus \{i, j\}$ of cardinality $q-1$. It is obvious that $\mathcal{C}$ is a circuit of $(W, \mathcal{R})$.

Second case. If $|N_1| = 1$ and let $N_1 = \{i\}$. Consider any $j \in N_2$ and let $S_1 = \{i\}$, $S_2 = \{j\}$ and H be any subset of $N \setminus \{i, j\}$ of cardinality $q-1$. It is obvious that $\mathcal{C} = \{(S_1, H), (S_2, H)\}$ is a circuit of $(W, \mathcal{R})$. ■

Proof of Proposition 3

Call a family any collection of coalitions $C = \{C_1, C_2, ..., C_r\}$ such that for any $t = 1, 2, ..., r$, $|C_t| \geq q$ and $\bigcap_{t=1}^{r} C_t = \emptyset$. It is known that $\lceil \frac{n}{n-q} \rceil = \min\{|C| : C \text{ is a family}\}$. We already know from Theorem 6 that $\nu(q, \mathcal{R}) \leq \lceil \frac{n}{n-q} \rceil$. We are now going to prove that $\lceil \frac{n}{n-q} \rceil \leq \nu(q, \mathcal{R})$.

Let $\hat{S} = \{(S_1, T_1), (S_2, T_2) ..., (S_k, T_r)\}$ be a *circuit* of $(q, \mathcal{R})$. Consider the family $C = \{C_1, C_2, ..., C_r\}$ where for any $t = 1, 2, ..r$, $C_t = S_t \cup T_t$. Since $\hat{S}$ is a circuit, $S_t \cup T_t \in W$ implies that $|C_t| \geq q$. If $\bigcap_{t=1}^{r} C_t \neq \emptyset$, then there exists $i \in N$ such that $i \in C_t$ for all $t = 1, 2, ..k$. But by the definition of the network, if $j \neq i_0$, then $g(j) = \emptyset$. Thus, $i \in \bigcap_{t=1}^{r} C_t$ implies that $i \in \bigcap_{t=1}^{r} S_t$ and $i \in [\bigcap_{t=1 s=1}^{rr} (S_s \cup T_s)] \cap S_t$, which is a contradiction. Hence, $C = \{C_1, C_2, ..., C_r\}$ is a family and therefore the set $\{|C| : C \text{ is a Peleg's family}\}$ includes the set $\{|\hat{S}| : \hat{S} \text{ is a } \textit{circuit} \text{ of } (q, \mathcal{R})\}$ which means that $\lceil \frac{n}{n-q} \rceil \leq \nu(q, \mathcal{R})$. ■

Proof of Proposition 4

We already proved the following inequality: $\nu(q, \mathcal{R}) \leq \nu(q, \mathcal{R}_0) = \lceil \frac{n}{n-q} \rceil$. Now let us prove that $\nu(q, \mathcal{R}) \geq \lceil \frac{n}{n-q} \rceil$.

Let $\hat{S} = \{(S_1, T_1), (S_2, T_2) ..., (S_r, T_r)\}$ be a *circuit* of $(q, \mathcal{R})$. Consider the family $C = \{C_1, C_2, ..., C_r\}$ where for any $t = 1, 2, ..r$, $C_t = S_t \cup T_t$. Since $\hat{S}$ is a circuit, $S_t \cup T_t \in W$ implies that $|C_t| \geq q$. We will prove that $\bigcap_{t=1}^{r} C_t = \emptyset$. If it is not the case, then there exists $i \in N$ such that $i \in C_t$ for all $t = 1, 2, ..., r$.

- If there exists $h \in \{1, ..., r\}$ such that $h \in S_h$!!(is it $h \in S_h$ or $i \in S_h$)!!, then we get a contradiction because $\hat{S}$ being a circuit, we have $[\cap_{s=1}^{k} (S_s \cup T_s)] \cap S_t = \emptyset$.
- If for all $h \in \{1, ..., r\}$, $h \in T_h$!!(is it $h \in T_h$ or $i \in T_h$)!!, then, $\hat{S}$ being a circuit, we have $T_h \subset g(S_h)$ and thus for all $t = 1, 2, ..r$, there exists $i_h \in S_h$ such that $i \in g(i_h)$. But since $\mathcal{R}$ is a ring, for all $j \in N$,

$$g(j) = \begin{cases} j+1 \text{ if } j \neq n \\ 1 \text{ if } j = n \end{cases}$$

This implies that $i_h = i - 1$, where $i_0 = n$. It follows that $i - 1 \in S_t$ for any $t = 1, 2, ..r$, implying that $i - 1 \in [\cap_{s=1}^{r} (S_s \cup T_s)] \cap S_t$, which is a contradiction.

Hence, $C = \{C_1, C_2, ..., C_r\}$ is a family and therefore, the set $\{|C| : C \text{ is a family}\}$ includes the set $\{|\hat{S}| : \hat{S} \text{ is a } \textit{circuit} \text{ of } (q, \mathcal{R})\}$, which means that $\lceil \frac{n}{n-q} \rceil \leq \nu(q, \mathcal{R})$. ■

Proof of Proposition 5

The proof is similar to that of Proposition 4. ■