Endogenous derivation and forecast of lifetime PDs

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Endogenous Derivation and Forecast of Lifetime PDs

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Abstract

This paper proposes a simple technical approach for the derivation of future (forward) point-in-time PD forecasts, with minimal data requirements. The inputs required are the current and future through-the-cycle PDs of the obligors, their last known default rates, and a measure for the systematic dependence of the obligors. Technically, the forecasts are made from within a classical asset-based credit portfolio model, just with the assumption of a suitable autoregressive process for the systematic factor. The paper discusses in detail the practical issues of implementation, in particular the parametrization alternatives.

The paper also shows how the approach can be naturally extended to low-default portfolios with volatile default rates, using Bayesian methodology. Furthermore, the expert judgments about the current macroeconomic state, although not necessary for the forecasts, can be embedded using the Bayesian technique.

The presented forward PDs can be used for the derivation of lifetime credit losses required by the new accounting standard IFRS 9. In doing so, the presented approach is endogenous, as it does not require any exogenous macroeconomic forecasts which are notoriously unreliable and often subjective.

Keywords: Prediction, Probability of Default, PD, Default Rates, Through-the-Cycle, TTC, Point-in-Time, PIT, Credit Portfolio Model, Systematic Factor, Macroeconomic Factor, Time Series, Autoregression, Bayesian Analysis, IFRS 9, Accounting, Financial Instruments, Lifetime, Expected Credit Losses

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Introduction

The newly adopted IFRS 9 accounting standard requires the estimation of probability-weighted lifetime credit losses for some categories of high-risk credit exposures. To this end, the classical risk-management-style PD and LGD metrics can be used, in the form of their projections into the future. The PDs of interest should reflect both obligor-specific and macroeconomic conditions. This seems to correspond to what is known as Point-in-Time (PIT) PDs, rather than the more usual Through-the-Cycle (TTC) PDs. The forecast of future (forward) PDs is a nontrivial task. Forecasting obligor-specific conditions (resulting in TTC PD forecasts) is quite challenging, whereas forecasting future macroeconomic conditions (necessary for PIT PDs) is often extremely difficult and unreliable, with corresponding forecast errors being particularly high.

The notion of a PIT PD is more or less clear: it corresponds to the expected default rate (DR) of an obligor in the nearest future, taking into account all available obligor-specific and macroeconomic information. On the opposite, there is an ambiguity as to what the TTC PD actually means (and how it can be estimated or verified using observable data). In this paper, the TTC PD is defined as the unconditional expectation of an obligor default rate, i.e. the expectation in a situation where the macroeconomic conditions are assumed to be unknown.

The external credit ratings and the Basel II internal credit ratings are normally technically implemented as a discretization of such TTC PDs, in order to factor out the influence of (uncertain, volatile, and cyclical) macroeconomic conditions. The corresponding models are normally calibrated to long-term averages of defaults rates and normally lack strong macroeconomic inputs. Due to their wide availability, Basel II PDs can serve as an abundant source of approximate TTC PDs.

For a portfolio of N_t obligors, we assume that their trough-the-cycle (TTC) one-year probabilities of default $PD_{i,t}^{TTC}$ are known for each obligor i, for a recent past (in the following "current") period $t = T_0$, as well as for each future forecast period of interest $t = T_f, T_f > T_0$.

We further assume that during the current period T_0 a number of the obligors $N_{T_0}^D$ defaults, resulting in the observed portfolio default rate $DR_{T_0} = N_{T_0}^D / N_{T_0}$.

The measures we are looking for are the ex-ante forecasted (forward) PIT PDs for the individual obligors for the future periods/years $PD_{i,T_f}^{PIT}, T_f > T_0$.

Asset-based Credit Portfolio Approach

A possible theoretical approach to deal with the PIT vs TTC transformations is based on the classical credit portfolio modeling, in particular on the concepts of a firm asset return and a default barrier.

In this framework, the default of an obligor i in a period t is caused by the value of the obligor's (firm's) assets falling below a certain critical (barrier) level (which is normally linked to the debt of the firm). This condition can be restated as the firm's asset return $r_{i,t}$ happening to be (in that particular period) lower than a certain critical return $R_{i,t}^D$. Thus, the probability of default of the obligor i can be specified as:

$$PD_{i,t} = P\left(r_{i,t} < R_{i,t}^D\right) \tag{1}$$

The return $r_{i,t}$ can be assumed to be normally distributed and represented as a weighted sum of a systematic (common to all obligors) return ψ_t and an idiosyncratic (obligor-specific) return $\varepsilon_{i,t}$, with these two return factors being independent of each other. Without loss of generality, after suitable transformations, the factors can be mathematically expressed as standard normal variables related as follows:

$$r_{i,t} = \psi_t\sqrt{\rho} + \varepsilon_{i,t}\sqrt{1-\rho}$$
$$\psi_t \sim N(0,1) \qquad (2)$$
$$\varepsilon_{i,t} \sim N(0,1)$$

with the weighting (correlation) coefficient $\rho \in [0; 1]$ which is the measure of systematic dependence (also often referred to as "R-squared" in credit portfolio models).

It follows that the unconditional distribution of $r_{i,t}$ (i.e. the distribution when the realization of both ψ_t and $\varepsilon_{i,t}$ is unknown) is then standard normal as well:

$$E(r_{i,t}) = 0 + 0 = 0$$
$$Var(r_{i,t}) = (\sqrt{\rho})^2 1 + (\sqrt{\rho - 1})^2 1 = 1$$
$$r_{i,t} \sim N(0,1)$$

Application to TTC/PIT PDs

The systematic factor ψ_t in the above framework can be interpreted as one single macroeconomic factor affecting all obligors in the portfolio under consideration. Then, the through-the cycle (TTC) PD, which was defined above as the expected default rate without the knowledge of the macroeconomic state, amounts for each obligor to the unconditional PD:

$$PD_{i,t}^{TTC} = P(r_{i,t} < R_{i,t}^D) = \Phi(R_{i,t}^D) \qquad (3)$$

with Φ denoting the cumulative standard normal distribution function.

The TTC PDs $PD_{i,t}^{TTC}$ can be typically assumed to be exogenously known (e.g. from an internal or external credit rating). Then, the implicit unknown barrier $R_{i,t}^D$ can be directly derived from the known TTC PD by inverting the Φ function:

$$R_{i,t}^D = \Phi^{-1}(PD_{i,t}^{TTC}) \qquad (4)$$

The point-in-time (PIT) PD assumes, on the other hand, the knowledge about the systematic factor ψ_t, in particular that it is equal to a certain value Ψ_t. This PD can thus be viewed as the conditional PD:

$$PD_{i,t}^{PIT} = P(r_{i,t} < R_{i,t}^D | \psi_t = \Psi_t) =$$

$$= P(\psi_t\sqrt{\rho} + \varepsilon_{i,t}\sqrt{1-\rho} < R_{i,t}^D | \psi_t = \Psi_t)$$

and, upon substitution from (4) :

$$PD_{i,t}^{PIT} = P\left(\Psi_t\sqrt{\rho} + \varepsilon_{i,t}\sqrt{1-\rho} < \Phi^{-1}(PD_{i,t}^{TTC})\right) =$$

$$= P\left(\varepsilon_{i,t} < \frac{\Phi^{-1}(PD_{i,t}^{TTC}) - \Psi_t\sqrt{\rho}}{\sqrt{1-\rho}}\right)$$

Thus, finally:

$$PD_{i,t}^{PIT} = \Phi\left(\frac{\Phi^{-1}(PD_{i,t}^{TTC}) - \Psi_t\sqrt{\rho}}{\sqrt{1-\rho}}\right) \qquad (5)$$

The above framework basically derives the PIT PD from the known TTC PD, ρ coefficient and known realization Ψ_t of the systematic factor. It can however be reversed to endogenously calculate the unknown macroeconomic factor:

$$\Psi_t = \left(\frac{\Phi^{-1}(PD_{i,t}^{TTC}) - \Phi^{-1}(PD_{i,t}^{PIT})\sqrt{1-\rho}}{\sqrt{\rho}} \right)$$

More realistically, for the current period T_0, on the portfolio of N_{T_0} obligors with various (known) TTC PDs, the actual PIT PDs of obligors would be unknown, but there can be observed a total number of $N_{T_0}^D$ defaults. Then, making use of:

$$\mathrm{E}(N_{T_0}^D) = \sum_{i=1}^{N_{T_0}} PD_{i,T_0}^{PIT}$$

the unknown systematic factor can be estimated such that:

$$N_{T_0}^D \equiv \sum_{i=1}^{N_t} PD_{i,t}^{PIT} = \sum_{i=1}^{N_{T_0}} \Phi\left(\frac{\Phi^{-1}(PD_{i,T_0}^{TTC}) - \widehat{\Psi}_{T_0}\sqrt{\rho}}{\sqrt{1-\rho}} \right) \tag{6}$$

where $\widehat{\Psi}_{T_0}$ denotes the estimator of the systematic/macroeconomic factor for the period T_0. For the estimator to be accurate enough, the number of defaults $N_{T_0}^D$ has to be sufficiently large (higher than 10 or even 20 as the rule of thumb).

PIT PD Forecasts

Given the above framework, the natural interpretation for the PIT PD forecast for a future period T_f with an uncertain (stochastic) macroeconomic factor ψ_{T_f} would be the statistical expectation which can be derived analogously from (5) as follows:

$$\begin{gathered} PD_{i,T_f}^{PIT} = E\left(PD_{i,T_f}^{PIT}\right) = \\ = E\left(\Phi\left(\frac{\Phi^{-1}\left(PD_{i,T_f}^{TTC}\right) - \psi_{T_f}\sqrt{\rho}}{\sqrt{1-\rho}} \right) \right) \end{gathered} \tag{7}$$

The future TTC PDs PD_{i,T_f}^{TTC} are assumed to be known. We further assume a normal distribution for the stochastic factor ψ_{T_f}, with parameters $E\left(\psi_{T_f}\right)$ and $Var\left(\psi_{T_f}\right)$ which reflect correspondingly the expectation and the uncertainty of future macroeconomic conditions. We make the following substitution:

$$x \equiv \frac{\Phi^{-1}\left(PD_{i,T_f}^{TTC}\right) - \psi_{T_f}\sqrt{\rho}}{\sqrt{1-\rho}}$$

Then:

$$E(x) = \frac{\Phi^{-1}\left(PD_{i,T_f}^{TTC}\right) - E\left(\psi_{T_f}\right)\sqrt{\rho}}{\sqrt{1-\rho}}$$

$$Var(x) = \frac{Var\left(\psi_{T_f}\right)\rho}{1-\rho}$$

and, exploiting the following property (see Appendix):

$$E(\Phi(x)) = \Phi\left(\frac{E(x)}{\sqrt{1+Var(x)}} \right)$$

$$for\ x \sim N(E(x), Var(x))$$

we finally arrive at the closed-form expression for the prediction of future PIT PDs:

$$PD_{i,T_f}^{PIT} = \Phi\left(\frac{\Phi^{-1}\left(PD_{i,T_f}^{TTC}\right) - E\left(\psi_{T_f}\right)\sqrt{\rho}}{\sqrt{1-\rho+Var\left(\psi_{T_f}\right)\rho}}\right) \quad (8)$$

Thus, given a description for the future uncertain macroeconomic factor ψ_{T_f}, in the form of its expected (predicted) value $E\left(\psi_{T_f}\right)$ and its variance (uncertainty) $Var\left(\psi_{T_f}\right)$, the portfolio modelling framework allows to easily derive the forecasted PIT PD in that period. It is important to note that not only the expected value of the macroeconomic factor affects the PIT forecast, but also its variance. In most realistic cases, the nominator $\Phi^{-1}\left(PD_{i,T_f}^{TTC}\right) - E\left(\psi_{T_f}\right)\sqrt{\rho}$ would be negative, and thus, the higher the variance of the future macroeconomic factor, the higher the PIT PD forecast.

Two partial cases of the above equation are important. First, an exactly known future macroeconomic factor might be technically described by the assumptions $E\left(\psi_{T_f}\right) = \Psi_{T_f}$ and $Var\left(\psi_{T_f}\right) = 0$. In that case, the expression (8) reduces to the simple conditional PD (as seen in (5)):

$$PD_{i,T_f}^{PIT} = \Phi\left(\frac{\Phi^{-1}\left(PD_{i,T_f}^{TTC}\right) - \Psi_{T_f}\sqrt{\rho}}{\sqrt{1-\rho}}\right)$$

Second, if we use the unconditional distribution from the classical setting, i.e. $E\left(\psi_{T_f}\right) = 0$ and $Var\left(\psi_{T_f}\right) = 1$, the expression (8) reduces to the TTC PD forecast:

$$PD_{i,T_f}^{PIT} = PD_{i,T_f}^{TTC}$$

We now return to our factor estimate $\widehat{\Psi}_{T_0}$ in (6) for the current period T_0 and assume that it is accurate enough, so that, formally, Ψ_{T_0} can be assumed to be known exactly: $\Psi_{T_0} = \widehat{\Psi}_{T_0}$. It becomes evident that, for a future forecast period T_f, $T_f > T_0$, the distribution of ψ_{T_f} conditional on Ψ_{T_0} should be such that:

$$\begin{aligned} for\ (T_f - T_0) \to 0&:\ \ \psi_{T_f}|\Psi_{T_0} \to N(\Psi_{T_0}, 0) \\ for\ (T_f - T_0) \to \infty&:\ \ \psi_{T_f}|\Psi_{T_0} \to N(0,1) \end{aligned} \quad (9)$$

If the above conversion conditions are satisfied, the PIT PD would show the "continuity" seen in practice for default rates and macroeconomic indicators, meaning that for the time shortly after T_0, the PIT PD would not differ much from the default rate seen in T_0, and the macroeconomic conditions can also be assumed to be similar. For a remote forecast period $(T_f - T_0 \to \infty)$ on the other hand, no assumptions about the macroeconomic conditions can be made, apart from their long-term distribution, producing the PIT PD forecast which equals the TTC PD forecast.

Autoregression

The credit portfolio framework (in particular as in equation (5)) is specified generally, for each and every period. There are no immediate assumptions about serial dependencies and correlations between the systematic factors ψ_{T1} and ψ_{T2} for $T1 \neq T2$.

The intuitively expected "continuity" for the ψ_t process, along with the convergence criteria for the conditional distribution of $\psi_{T_f}|\Psi_{T_0}$ in (9) can be easily satisfied through the assumption of an autoregressive process for the macroeconomic factor ψ_t.

In its simplest form, the *autoregressive order-1* process (AR(1)) is specified for a stochastic variable x_t as follows:

$$x_t = a_0 + a_1 x_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \sigma_\varepsilon) \tag{10}$$

with parameters a_0, a_1 and σ_ε.

The distributional properties of the AR(1) process are as follows. The conditional distribution of x_{T_f} given the knowledge of $x_{T_0} = X_{T_0}$, with $T_f > T_0$ can be shown (see e.g. *Mills (2000), Johnston & DiNardo (1997)*) to be normal with:

$$\begin{gathered} E\left(x_{T_f} \middle| X_{T_0}\right) = X_{T_0} {a_1}^{T_f - T_0} + \frac{a_0}{(1-a_1)}\left(1 - {a_1}^{T_f - T_0}\right) \\ Var\left(x_{T_f} \middle| X_{T_0}\right) = {\sigma_\varepsilon}^2 \frac{1 - {a_1}^{2(T_f - T_0)}}{1 - {a_1}^2} \end{gathered} \tag{11}$$

The unconditional distribution, which is also the asymptotic distribution for $T_f - T_0 \to \infty$, is:

$$\begin{gathered} E\left(x_{T_f}\right) = \frac{a_0}{(1-a_1)} \\ Var\left(x_{T_f}\right) = \frac{{\sigma_\varepsilon}^2}{1 - {a_1}^2} \end{gathered} \tag{12}$$

If we now assume the AR(1) specification (10) for the systematic factor ψ_t, the convergence criteria (9) can be achieved via a suitable parametrization restriction. In particular, for $T_f - T_0 \to \infty$:

$$\begin{gathered} E\left(\psi_{T_f}\right) \equiv 0 \Rightarrow \quad a_0 = 0 \\ Var\left(\psi_{T_f}\right) \equiv 1 \Rightarrow \quad {\sigma_\varepsilon}^2 = 1 - {a_1}^2 \\ a_1 \in [0,1) \end{gathered} \tag{13}$$

Given these restrictions, the convergence for $T_f - T_0 \to 0$ is satisfied as well, as can be seen from (11). Thus, there remains only one free parameter a_1, which also determines the convergence speed. Given the restrictions in (13), the expressions for a conditional distribution in (11), applied to the systematic factor ψ_{T_f}, simplify to:

$$\begin{gathered} E\left(\psi_{T_f} \middle| \Psi_{T_0}\right) = \Psi_{T_0} {a_1}^{T_f - T_0} \\ Var\left(\psi_{T_f} \middle| \psi_{T_0}\right) = 1 - {a_1}^{2(T_f - T_0)} \end{gathered} \tag{14}$$

Now, substituting (14) to (8), we finally obtain the forecast for the PIT PD based on the AR(1) assumption for the systematic factor:

$$PD_{i,T_f}^{PIT,AR(1)} = \Phi\left(\frac{\Phi^{-1}\left(PD_{i,T_f}^{TTC}\right) - \Psi_{T_0}\sqrt{\rho {a_1}^{2(T_f - T_0)}}}{\sqrt{1 - \rho {a_1}^{2(T_f - T_0)}}}\right) \tag{15}$$

with current systematic risk being e.g. estimated from (6): $\Psi_{T_0} = \widehat{\Psi}_{T_0}$.

Interestingly, the expression in (15) is thus identical to the simple conditional PIT PD with adjusted (decaying according to power law) ρ coefficient (compare (15) to (5)).

Another feasible alternative for the process of the systematic factor, *the autoregressive order-2* process, is generally specified for a stochastic variable x_t as follows:

$$x_t = a_0 + a_1 x_{t-1} + a_2 x_{t-2} + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \sigma_\varepsilon) \tag{16}$$

The unconditional distribution, which is also the asymptotic distribution for $T_f - T_0 \to \infty$, can be shown to be normal (see e.g. *Mills (2000), Johnston & DiNardo (1997)*) with parameters:

$$E\left(x_{T_f}\right) = \frac{a_0}{(1 - a_1 - a_2)} \tag{17}$$
$$Var\left(x_{T_f}\right) = \frac{(1 - a_2)}{(1 + a_2)} \frac{{\sigma_\varepsilon}^2}{(1 - a_2)^2 - {a_1}^2}$$

The conditional distribution, given two known observations X_{T_0} and $X_{T_{-1}}$, has no analytical expression, but can be derived iteratively as:

$$\begin{gathered} E\left(x_{T_f} | X_{T_0}, X_{T_{-1}}\right) = a_0 + a_1 E\left(x_{T_f - 1} | X_{T_0}, X_{T_{-1}}\right) + a_2 E(x_{T_f - 2} | X_{T_0}, X_{T_{-1}}) \\ E\left(x_{T_0} | X_{T_0}, X_{T_{-1}}\right) \equiv X_{T_0} \\ E\left(x_{T_{-1}} | X_{T_0}, X_{T_{-1}}\right) \equiv X_{T_{-1}} \end{gathered} \tag{18}$$

$$\begin{gathered} Var\left(x_{T_f} | X_{T_0}, X_{T_{-1}}\right) = {\sigma_\varepsilon}^2 \sum_{t=1}^{T_f - T_0} {w_t}^2 \\ w_1 = 1, w_2 = a_1 \\ w_t = a_1 w_{t-1} + a_2 w_{t-2} \; for \; t > 2 \end{gathered} \tag{19}$$

Again, adopting an AR (2) process for the systematic factor requires following parameter restrictions to satisfy the convergence criteria (9):

$$\begin{gathered} a_0 = 0 \\ {\sigma_\varepsilon}^2 = \frac{(1 + a_2)((1 - a_2)^2 - {a_1}^2)}{(1 - a_2)} \end{gathered} \tag{20}$$

Having iteratively determined the distribution parameters $E\left(\psi_{T_f} \middle| \Psi_{T_0}, \Psi_{T_{-1}}\right)$ and $Var\left(\psi_{T_f} \middle| \Psi_{T_0}, \Psi_{T_{-1}}\right)$ from (18) and (19), and taking into account (20), the PIT PD forecast becomes just:

$$PD_{i,T_f}^{PIT,AR(2)} = \Phi\left(\frac{\Phi^{-1}\left(PD_{i,T_f}^{TTC}\right) - E\left(\psi_{T_f} \middle| \Psi_{T_0}, \Psi_{T_{-1}}\right)\sqrt{\rho}}{\sqrt{1 - \rho + Var\left(\psi_{T_f} \middle| \Psi_{T_0}, \Psi_{T_{-1}}\right)\rho}}\right) \tag{21}$$

Parametrization

The equations (15) and (21) show how future PIT PDs can be forecasted within the AR(1)and AR(2) processes of the systematic parameter respectively, based on the following inputs:

- future TTC PDs PD_{i,T_f}^{TTC} of the obligors
- the systematic correlation coefficient ρ
- autoregressive coefficients a_1 (for the AR(1) case) or a_1 and a_2 (for the AR(2) case)
- the recent/current (estimated) macroeconomic risk factors $\Psi_{T_0}, \Psi_{T_{-1}}$.

We have already discussed the estimation of the latter factors from default rates (see (6)), this issue will also be discussed in the section on Bayesian estimation below. We now elaborate on how the remaining parameters can be determined in practice

The future TTC PDs were assumed to be known so far. Generally, they can be derived using current TTC PDs and standard rating transition matrices. The credit ratings used for matrix estimation should be TTC ratings. The external rating agencies typically adhere to the TTC approach, and it is also recommended for the Basel II internal ratings of banks. The future (forward) TTC PDs, derived from transition matrix multiplications, would normally show convergence when the forecast horizon T_f increases, with both obligors with currently high TTC PDs and obligors with currently low TTC PDs converging to some future mid TTC PDs. Apart from this general TTC PD convergence, for some kinds of products the dynamics of TTC PDs would also depend on the origination and maturity of a loan. E.g. defaults caused by fraudulent behavior typically occur early after the loan origination, which would be reflected in a peak of forward PIT PDs early on. On the other hand, e.g. for bullet loans, the default risk would tend to materialize later on, with a peak of forward PIT PDs probably near maturity.

A natural choice for the correlation coefficient ρ are the values used in asset-based credit portfolio models, either internal models of banks, or the implicit regulatory portfolio model from Basel II (see *BCBS (2002)*). As to the latter, the current EU regulations (CRR) prescribe following values for this coefficient (denoted in the CRR text as “R” coefficient):

Non-retail obligors (CRR article 153):

- General setting: 12% (for PD=100%) to 24% (for PD=0%)
- Financial large and non-regulated companies: as above, multiplied with a factor of 1.25
- SMEs with sales under EUR 50M: as above, with up to 4% deduction, depending on the firm’s sales

Retail obligors (CRR article 154):

- General setting: 3% (for PD=100%) to 16% (for PD=0%)
- Loans secured by real estate (residential mortgages): 15%
- Certain qualifying revolving loans (credit cards and similar): 4%

Thus, the Basel formulas generally assume some dependence of ρ on the PD (with riskier companies being less dependent on systematic factors). This is quite a questionable assumption, although implementing this dependency would not be problematic in the above framework, as the current and future TTC PDs are assumed to be known. Also, the Basel settings realistically account for the company size: the larger a company, the greater its dependency on macroeconomic conditions and thus the higher the systematic risk portion of its return (see (2)).

It is important to note that the above CRR retail settings for ρ implicitly embed penalties for longer maturities (see *BCBS (2002)*). This explains why the (long-term) residential mortgages have that much higher settings compared to (short-term or cancellable) revolving loans. This distorts the ρ as a measure for the systematic risk. For non-retail exposures, Basel II accounts for the maturities separately (in the so called “M” term), so that the “R” coefficient seems to capture well the systematic risk.

The above Basel II settings for the ρ do have the advantage of an “official” source. They were estimated empirically in the early 2000s using the default statistics collected and aggregated by national banks. For this reason, the current internal credit portfolio models of banks might be a more suitable choice as the source for ρ. Most of these models nowadays follow the asset based approach presented above, with risk decomposition into systematic and idiosyncratic parts. Normally in internal models, the ρ is assigned to obligors based on their industry and/or region. Some models also account for the company size. Generally, for non-retail exposures the models seem to rely on ρ values in the range of 10% to 40%, and for retail exposures – in the range of 0.5% to 5%.

If using the ρ setting from internal credit models, it is important to ensure the coefficient is estimated in a similar way as it is used in the above framework. In particular, data involved for the estimation should cover at least 1-2 credit cycles of default statistics and the PDs used for the estimation should be the TTC PDs. The technical routines for ρ estimation (see e.g. *Kalkbrener & Onwunta (2009)*) do not typically account for the serial correlation patterns; however, that does not affect the estimates considerably, given long enough estimation data.

As to the autoregressive coefficients a_1 and a_2, there would not be many immediate external sources, and they should be rather based on available data, expert judgements, as well as common knowledge about the macroeconomic credit cycles in the business segments involved. The forecasts from the simple AR(1) process would generally show a mean-reverting behavior, the coefficient a_1 determining the speed of reverting. The lower this coefficient, the sooner the forecasted PIT PDs converge to their forecasted TTC PDs (in the extreme case, if $a_1 = 0$, all forecasted PIT PDs are equal to the forecasted TTC PDs). The AR(1) process does not show a clear cyclicality/periodicity, in the sense that after reverting to the mean, the process is equally likely to move on in either direction. However, the periodicity can be approximated with some proxies, e.g. using the criterion of "double crossing" (see Figure 1 in Illustration section below).

Using various sources, the range of 0.6 to 0.95 seems to be the most suitable for the AR(1) a_1 coefficient. The value of 0.8 results in the average double crossing periodicity of some 10 years, which seems to be also the average length of recent credit cycles (1990-2001, 2001-2009).

Using the more complicated AR(2) process results in refraining from the simple intuitive solution for forward PDs in the AR(1) case (see (15)), but does offer some advantages. In particular, the AR(2) processes are better able to realistically capture the cyclical behavior because – apart from the mean reverting – they can show the so called momentum. With momentum, a process crossing its mean from above is more likely to move on below the mean before returning to the mean (see Figure 2). In technical/statistic terms, the AR(2) process with a suitable parametrization would also show a frequency peak in terms of its spectral density, with the peak frequency calculated as:

$$f = \frac{1}{2\pi}\cos^{-1}\left(\frac{a_1(a_2-1)}{4a_2}\right) \tag{22}$$

The periodicity (in years) is then just $1/f$. For these formulas to be valid, further parameter restrictions are necessary (which actually facilitates the parameter choice):

$$\begin{gathered} -1 < a_2 < 1 \\ a_2 - a_1 < 1 \\ a_2 + a_1 < 1 \\ 4{a_1}^2 + a_2 < 0 \end{gathered} \tag{23}$$

The values of a_1 around 1.2 to 1.4 and a_2 around -0.5 to -0.7 result in a realistic series behavior. With $a_1 = 1.3$ and $a_2 = -0.65$, the theoretical spectral periodicity according to (22) is approximately 10 years (and the "double-crossing" periodicity is about 9 years).

Low Default Portfolios and Bayesian Estimation

The above framework requires the knowledge or estimation of the last realized systematic factor Ψ_{T_0} (in the AR(1) case) or the last two factors $\Psi_{T_0}, \Psi_{T_{-1}}$ (in the AR(2) case). These generally unknown factors can in principle be estimated from the known default statistics (as shown in (6) for $\widehat{\Psi}_{T_0}$). The uncertainty of such estimates depends primarily on the number of defaults $N_{T_0}^D$ or N_{T-1}^D. With only a small number of defaults (below 10), the estimates would be unreliable and volatile.

Fortunately, the problem can be elegantly cured in the above framework. For the simple AR(1) case, instead of using the fixed estimates ($\Psi_{T_0} = \widehat{\Psi}_{T_0}$), we can just assume that Ψ_{T_0} is a random variable with a distribution $f(\psi_{T_0})$. Then, the uncertainty of the estimate can just be embedded in the above autoregressive process. In particular, if we assume $E(\psi_{T_0})$ and $Var(\psi_{T_0})$ to be the expected value and the variance of the distribution $f(\psi_{T_0})$, the distributional properties of the future systematic factors in the simple AR(1) case become:

$$
\begin{aligned}
E\left(\psi_{T_f}\right) &= E(\psi_{T_0})\, a_1{}^{T_f - T_0} \\
Var\left(\psi_{T_f}\right) &= Var(\psi_{T_0}) a_1{}^{2(T_f - T_0)} + 1 - a_1{}^{2(T_f - T_0)} = \\
&= 1 + \left(Var(\psi_{T_0}) - 1\right) a_1{}^{2(T_f - T_0)}
\end{aligned}
\tag{24}
$$

Compared to the fixed estimate Ψ_{T_0} (14), the above adjustments:

- substitute the fixed estimate with its expected value for the short-term expectation
- do not change the long-term expectation (which still converges to 0)
- inflate the short-term variance of the systematic factor
- do not change the long-term variance (which still converges to 1)

If $f(\Psi_t)$ is normal or approximately normal, the variance and expected value of ψ_{T_f} can just be directly used in (8) to arrive at the PIT PD forecasts which now account for the uncertainty in the current systematic factor Ψ_{T_0}.

Now, turning our attention to deriving the distribution $f(\psi_{T_0})$ itself, the natural choice might be a Bayesian-type modeling. In the Bayesian framework, there is an initial belief (in form of a "prior" distribution) about a parameter to estimate. That belief changes (to a "posterior" distribution) if a new evidence is encountered. We can assign (omitting the T_0 index):

- the uncertain current systematic factor Ψ as the parameter under investigation
- the unconditional distribution $N(0,1)$ as its prior distribution
- the default count observation N^D (out of N obligors) as the data evidence
- the (unknown) distribution $f(\psi|N^D)$ as the posterior distribution of the systematic factor

We also assume the knowledge of (in this case, common for all obligors) TTC PDs and ρ coefficient.

Then, the posterior distribution can be estimated using the Bayes rule:

$$
f(\psi|N^D) = \frac{f(\psi) f(N^D|\psi)}{f(N^D)}
\tag{25}
$$

with:

$$
\begin{aligned}
f(N^D) &= \int f(N^D|\psi) f(\psi) d\psi \\
f(N^D|\psi) &= B(N, N^D, PD^{PIT}|\psi) \\
PD^{PIT}|\psi &= \Phi\left(\frac{\Phi^{-1}(PD^{TTC}) - \psi\sqrt{\rho}}{\sqrt{1-\rho}}\right)
\end{aligned}
\tag{26}
$$

where B stands for the binomial distribution density.

As it turns out, the posterior distribution is quite close to normal (see Figure 3). Upon calculation, its expected value and variance can be estiamted and used for the Bayesian forecast of PIT PDs as outlined above in (24).

It is important to stress that the Bayesian approach implicitly accounts for the sample size. Holding the default rate N^D/N constant, an increasing N leads to a narrower posterior distrubution with its peak getting more extreme and approaching the systematic factor Ψ which is defined by $PD^{PIT}|\Psi = N^D/N$ (see Figure 3).

Last but not least, the above Bayesian methodology allows for an easy way to embed expert judgements about the current macroeconomic enviroment. To this end, the prior $N(0,1)$ just needs to be appropriately adjusted, e.g.to $N(-1,1)$ for a judgement of a mild economic depression.

Application for IFRS 9 Lifetime PDs

In July 2014, the final version of the new IFRS 9 accounting standard was issued by the IASB. Upon the endorsement through EU (currently ongoing) the standard is going to become the new ruleset for accounting of financial instruments. One or the most important innovations of IFRS 9, compared to previous regulation IAS 39, lies in the application of the expected lifetime credit losses for accounting of a large class of risky credit exposures.

The following details advocate the usage of the above presented methodology for the calculation of forward (future) PDs, with those PDs offering the basis for the calculation of expected lifetime credit losses:

The IFRS 9 standard (see *IFRS Foundation (2014)*) requires that:

"An entity shall measure expected credit losses of a financial instrument in a way that reflects:

(a) an unbiased and probability-weighted amount that is determined by evaluating a range of possible outcomes…

(c) reasonable and supportable information that is available without undue cost or effort at the reporting date about past events, current conditions and forecasts of future economic conditions."

IFRS 9 defines the reasonable and supportable information as "*reasonably available at the reporting date without undue cost of effort, including information about past events, current conditions and forecasts of future economic conditions*", with no need to "*undertake an exhaustive search*".

In the above framework, the probability weighting of macroeconomic outcomes is achieved through the assumption of an autoregressive process for the systematic factor, which results in the future factors being a stochastic variable covering all possible macroeconomic outcomes. The past events are implicitly accounted for in the estimation of the current/past factors $\Psi_{T_0}, \Psi_{T_{-1}}$. The forecasts are basically a technical extrapolation of these (under autoregressive model parametrization), which is a reasonable approach given that professional (expert-based) macroeconomic forecasts normally show (at best) only very slight advantages compared to technical or even naïve forecasts (such as the last known conditions). The standard even explicitly states that "*in some cases, the best reasonable and supportable information could be the unadjusted historical information*" and "*the estimate … does not require a detailed estimate for periods that are far in the future-for such periods, an entity may extrapolate projections form available, detailed information*."

The standard further reads that "*The information used shall include factors that are specific to the borrower, general economic conditions and an assessment of both the current as well as the forecast* *direction of conditions…"*. That also seems to correspond well to the presented approach, as it resorts to TTC PDs, macroeconomic factors, and autoregressive extrapolation respectively.

Last but not least, the standard states that "*possible data sources include internal credit loss experience, internal ratings…*". That is also captured through embedding the default statistics and the Basel-II ratings in the presented approach.

Illustrations

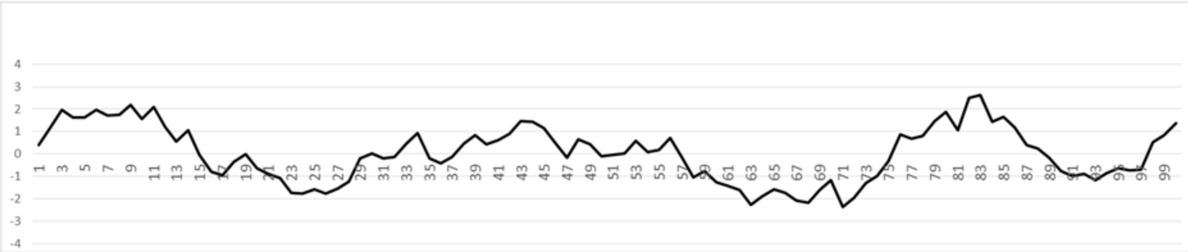

Figure 1: Simulated realizations for 100 years of systematic factor following AR(1) process with α1=0.8. Using the criterion of "double crossing", a full "cycle" begins e.g. in the year 75 and ends in the year 97.

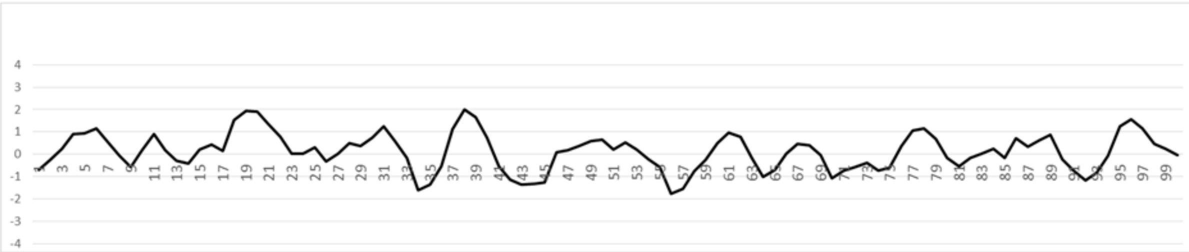

Figure 2: Simulated realizations for 100 years of systematic factor following AR(2) process with α1=1.3 and α2=-0.65.

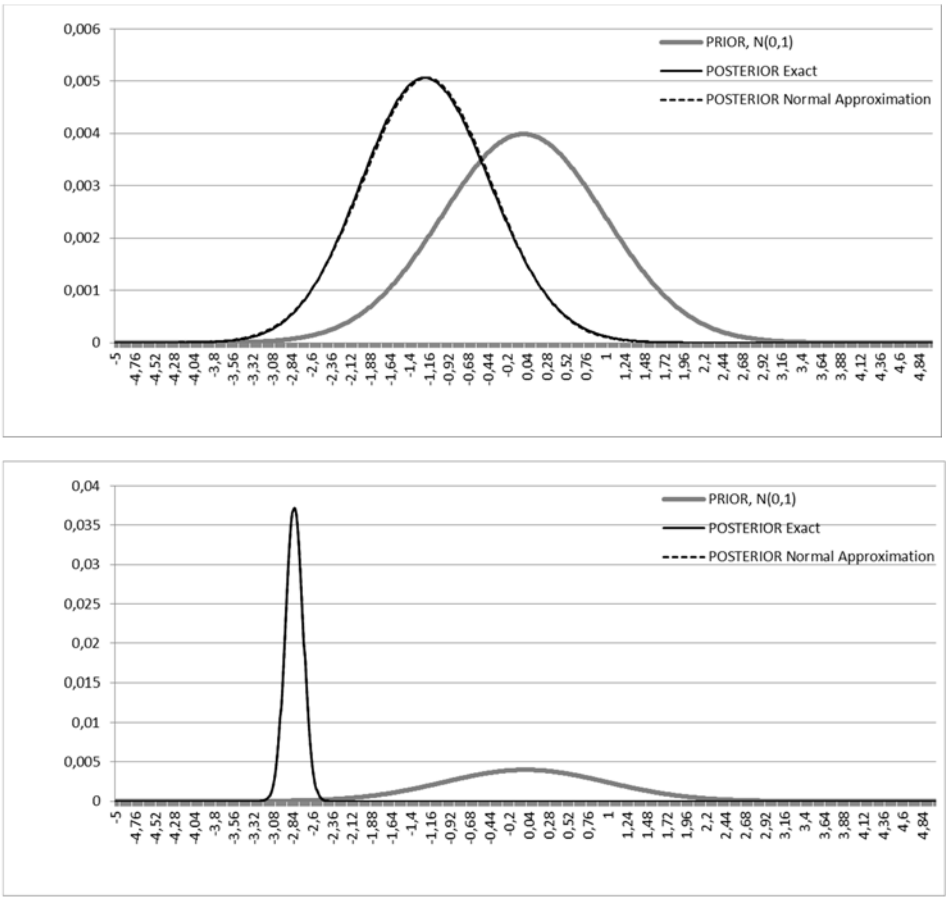

Figure 3: Bayesian prior, posterior exact, and posterior approximated distributions for the systematic factor, with $PD^{TTC} = 3\%, \rho = 15\%, N = 10\ (above)\ or\ N = 1000\ (below)$*, and observed default rate* (N^D/N) *of 20%.*

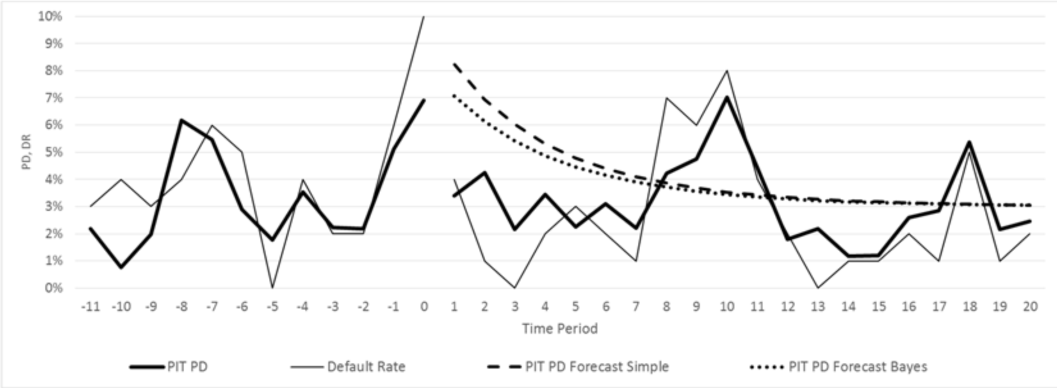

Figure 4: Simulated PIT PDs and default rates, as well as forecasted (simple and Bayes) PIT PDs (as of T=0) for: $PD^{TTC} = 3\%, \rho = 15\%, N = 100$, AR(1) process with a1=0.8.

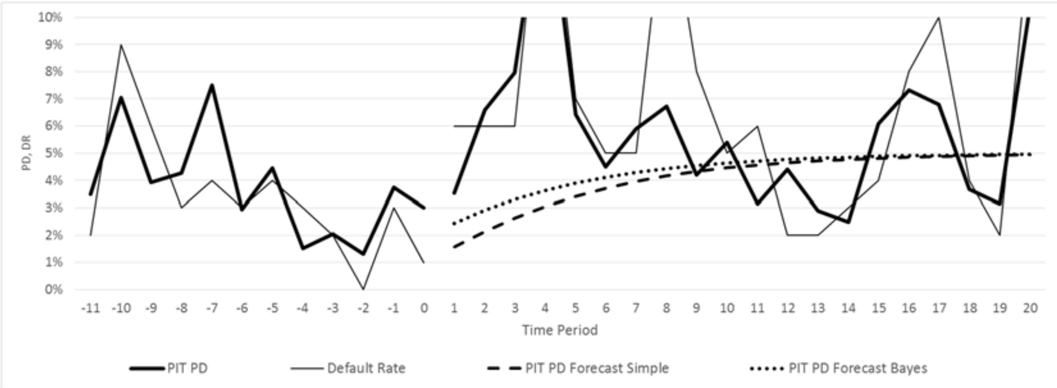

Figure 5: Simulated PIT PDs and default rates, as well as forecasted (simple and Bayes) PIT PDs (as of T=0) for: $PD^{TTC} = 5\%, \rho = 15\%, N = 100$, AR(1) process with a1=0.8.

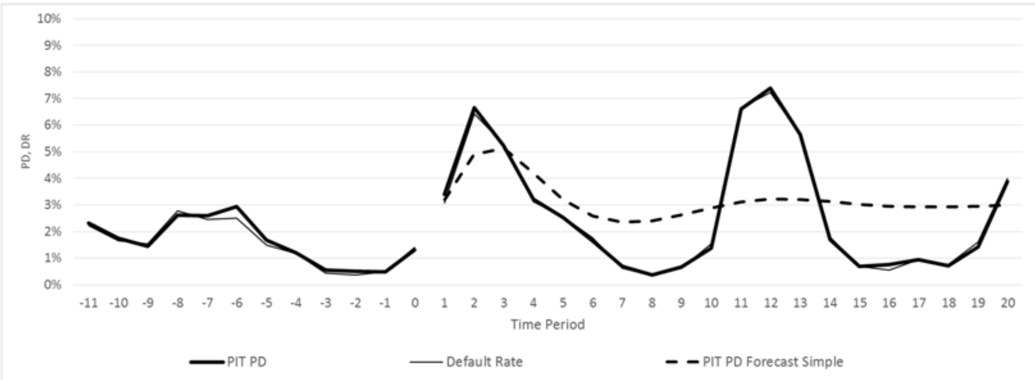

Figure 6: Simulated PIT PDs and default rates, as well as forecasted PIT PDs (as of T=0) for: $PD^{TTC} = 3\%, \rho = 15\%, N = 10{,}000$, AR(2) process with a1=1.3 and a2=-0.65.

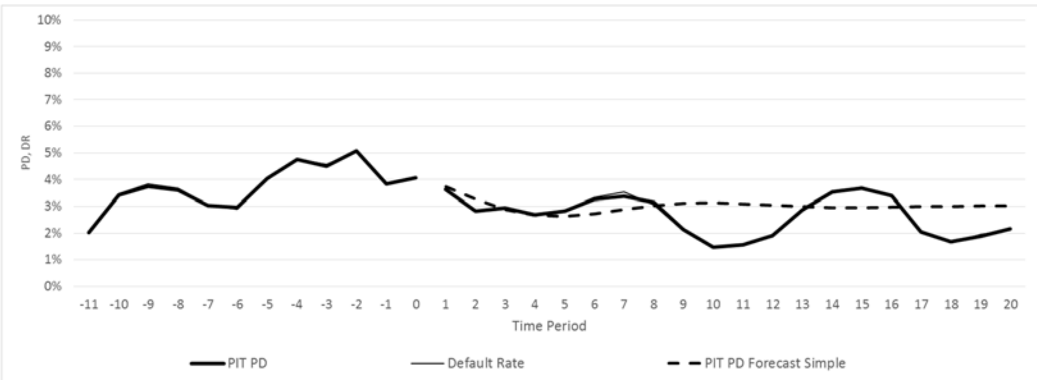

Figure 7: Simulated PIT PDs and default rates, as well as forecasted PIT PDs (as of T=0) for: $PD^{TTC} = 3\%, \rho = 3\%, N = 100{,}000$, AR(2) process with a1=1.3 and a2=-0.65.

Appendix: Expected Value of Normal Cumulative Function

For $x \sim N(\mu, \sigma)$:

$$E\big(\Phi(x)\big) = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right)$$

Proof:

$$E\big(\Phi(x)\big) = E\big(\Phi(\mu + \sigma y)\big), \qquad y \sim N(0,1)$$

$$E\big(\Phi(x)\big) = E\big(P(z \leq \mu + \sigma y)\big), \qquad z \sim N(0,1)$$

$$E\big(\Phi(x)\big) = E\big(P(z - \sigma y \leq \mu)\big) = P(z - \sigma y \leq \mu) = P\left(\frac{z - \sigma y}{\sqrt{1+\sigma^2}} \leq \frac{\mu}{\sqrt{1+\sigma^2}}\right) = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right)$$

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