How Market Economies Come to Live and Grow on the Edge of Chaos

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HOW MARKET ECONOMIES COME TO LIVE AND GROW ON THE EDGE OF CHAOS:

AN ALTERNATIVE EXPLANATION

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Summary: In a Hayek-Friedman-Lucas world, market economies are assumed to be natural, stable, and ergodic; hence, government policies are harmful to their efficiency. We develop a nonlinear dissipative dynamic model that shows that market economies instead live on the edge of chaos. We next appeal to the theory of differential equation to show that if they do not usually dissipate the totality of the information produced by their evolution it is due to a far-off self-organized equilibrium brought about by a spontaneous phase change originating in an optimal government policy.

Keywords: Unstable manifolds, Lyapunov Spectrum, information dimension, metric entropy, edge of chaos, self-organized equilibria, endogenous growth.

I-INTRODUCTION

The Neoclassical Economic Theory is controversial due to some of its assumptions, omissions, outcome and, foremost, its unrealistic policy advices. A large number of very able economists had to break rank to point out the negative consequences of this state of affairs for social welfare ([1], [2], [3], [4], [5] [6], and others). Even students of economics have joined in to demand changes as spelled out in the manifesto of the Post-Autistic Movement (www.paecon.net/History Pae.htm). One omission and one assumption are worth particular attention. That is, the omission of feedback mechanisms such as adjustment costs and increasing returns to scale (IRTS) that have, for a long time, led economists to prefer Linear Time Invariant models, which are now known to be far from being representative sketches of real markets. Rarely though IRTS are included in present- value maximization over an infinite horizon, but economists just assume that end-point conditions will be used to discard non optimal Euler paths, and that stability will always obtain as a consequence of agents' maximizing behavior; we also know now that this is nothing but wishful thinking. On the other hand, the blind attachment to omniscient and stable markets manifests itself as an unrelenting opposition to all government policies. As no steady-state equilibrium has ever been observed in market economies, the assumption appears more as an old belief resurrected in the Hayek-Friedman-Lucas' paradigm that has yielded very negative consequences for Western economies, to that of the United States in particular, as I will make clear later on.

The purpose of this note is to show that, contrary to the unshakable belief in automatic stabilizers, market economies live on the edge of chaos. To do so, I examine a simple model in Part I that includes adjustment costs and IRTS while appealing to the Principle of the flexible accelerator to show that indeed market economies are very complex or chaotic constructs. In the second part, I focus on the attractor of the model to show that whenever a fragile stability obtains, it is mainly due to an efficient critical policy conducive to a reduction of Shannon's entropy followed by a self-organized equilibrium rather than to agents' sole maximizing behavior. Our results are discussed in Part III.

II- THE MODEL

For the present purpose, a review of the economic literature is not needed. However, readers interested in complexity in economics are referred to ref. ([7], [8], [9], and [10]). In ref [10], for example, Benhabib and Nishimura considered the complexity caused by IRTS, which led to a chaotic hyperbolic attractor. But instead of dealing with that complexity, they attempted to get rid of it by appealing to a series of side conditions unlikely to be found in real markets. Their approach to stability relies on some optimal control program to steer the system toward the stable manifold, but the control program is not specified. This is but one example of the manifestation of a strange belief in automatic stability, even in complex systems.

To challenge that belief, we begin by examining a simple model (Appendix 2), developed in greater detail in [11], which includes IRTS and adjustment costs in the sense of Treadway [12], and Lucas [13]. For tractability, we keep the same concepts and symbols used in [11]. Briefly, the model describes a dynamic context in which a single firm economy attempts to solve the problem of present-value maximization with a quasiconcave technology with n quasi-fixed factors (x (t)) and n variable factors (v (t)), where t stands for time. The production technology is strictly convex in x (t) and strictly concave in v (t) so that scale elasticity exceeds unity. In the present study, we are rather parsimonious with details, preferring to refer the reader to the Appendix 2 or [11].

For the present discussion, we will focus on the Jacobian matrix (J) shown in (A5) in Appendix 2 and on the concept of entropy. In the physical universe, entropy refers to the degradation of energy, i. e., a quantity measured in Joules per degree Kelvin. In Shannon's approach, entropy refers to the rate of degradation of information, measured in bits per unit of time (second). But critical policy, Shannon entropy, self-organized equilibrium, and endogenous growth, etc. are all connected via the theory of differential equations. We then turn to a brief reviewing that theory in so far as it may provide a more sensible and robust explanation of undissipated information (which is the source of endogenous growth) than the approach advocated by Lucas [14], Romer [15] and others.

II- 1 GENERALITIES

For a greater understanding, we begin by examining a continuous measure-preserving-theoretical dynamic system (mps): (X, Ω, μ, T), where X is the set of all possible states x constituting the phase space; Ω is a σ-algebra over X; $\mu: \Omega \rightarrow [0, 1]$ is a probability measure so that $\mu(X) = 1$ and $\mu(\varnothing) = 0$; and $T: X \rightarrow X$ is a measurable transformation which preserves the measure μ, i. e., $\forall A \in \Omega$, $\mu(T^{-1}(A)) = \mu(A)$. If the preserved measure is the Lebesgue measure, the system is volume preserving (it is closed). If, on the other hand, the system does not preserve volume it is then dissipative (it is open). However, even though dissipative systems do not preserve the Lebesgue measure they may preserve some other measure such the Sinai-Ruelle-Bowen measure.

In this study, we will not consider closed systems. We will instead focus on the economic system which is, by definition, open, nonlinear (because of feedbacks), and stochastic (due to some inevitable uncertainty). The Jacobian (A5) arises from such a model. As it is a 2n x 2n matrix, for ease of exposition, we let 2n = m, and we write (A1) in Appendix 2 as a nonlinear dissipative model whose behavior is determined by an attractor Λ. That is:

(1) $$\dot{x} = f(x; \pi),$$

where $f \, \varepsilon \, C^1(E)$, E is an open subset of $\Re^m$, and $\pi \, \varepsilon \, \Re$ is a policy set instead of a parameter set. We let Γ be the phase space, $\varphi(t, X) = \varphi_t(X)$ is the flow satisfying $\varphi_t: \Re^m \rightarrow \Re^m$ and $\varphi_t \circ \varphi_s = \varphi_{t+s}$, $\forall t, s \, \varepsilon \, \Re$, and $0 \, \varepsilon \, \Re^m$ be the equilibrium point located at the origin. We next consider two possibilities.

Case 1: The Jacobian matrix in (A5) is a square matrix with distinct eigenvalues. Suppose it has k negative eigenvalues and m – k positive eigenvalues. Then $\exists$ stable subspaces E^s of dimension k, E^u of dimension m-k, and eigenvectors v such that: $E^s = \text{span}\{v_1, v_2, \ldots, v_k\}$ and $E^u = \text{span}\{v_{k+1}, \ldots, v_m\}$, and $\Re^m = E^s \oplus E^u$. Then $\exists$ a differentiable stable manifold M^s tangent to E^s at 0, and an unstable differentiable manifold M^u tangent to E^u at 0. Therefore, $\forall t \geq 0$, the flow $\varphi_t(M^s) \subset M^s$ for $\forall x_0 \, \varepsilon \, M^s$ such that $\lim_{t \rightarrow \infty} \varphi_t(x_0) = 0$. Similarly, for $\forall t \leq 0$, $\varphi_t(M^u) \subset M^u$ for $\forall x_0 \, \varepsilon \, M^u$ such that $\text{limit}_{t \rightarrow -\infty} \varphi_t(x_0) = 0$, where x_0 stands for initial conditions. It then follows that orbits are compressed on M^s and are stretched on M^u. The attractor Λ of (1) is therefore a hyperbolic fixed-point at 0.

We hasten to stress that the discussion about hyperbolic attractors is going to be brief because hyperbolic attractors are rather rare in nature, being mainly the toys of mathematicians. It suffices to bear in mind that in complex systems with hyperbolic attractors, M^s and M^u usually intersect transversally at 0. It is now well known such transversal intersections produce an infinite number of homoclinic orbits that accumulate at 0, resulting in homoclinic tangles and a horseshoe map at sufficiently high iterates of the discrete map. These attractors are referred to as Axiom A attractors (see Smale [16]. This means that motion in Γ becomes more and more complicated as π increases, because volume contracts in the direction of M^s and stretches in the direction of M^u causing Λ to become chaotic with sensitivity to initial conditions (STIC). These attractors are called "strange" (see Appendix 1). They do, however, have a very useful property. They are robust to noise, but because they are not found in nature, we will say no more about them at this juncture.

Case 2: If the Jacobian matrix in (A5) has pure imaginary eigenvalues in the form of $\sigma_j = a_j + i\, b_j$, then the generalized eigenvectors are $w_j = u_j + i\, v_j$, and we suppose that the Jacobian has an additional center subspace E^c. Similarly to the above development, we then have: $E^s = \text{span}\{u_j, v_j | \, a_j < 0\}$, $E^u = \text{span}\{u_j, v_j \,|\, a_j > 0\}$, and $E^c = \text{span}\{u_j, v_j \,|\, a_j = 0\}$ such that $\Re^m = E^s \oplus E^u \oplus E^c$. There are then k eigenvalues with negative real part, g eigenvalues with positive real part, and $h = (m - k - g)$ eigenvalues with zero real part. It follows that there exist differentiable manifolds M^s, M^u and M^c. The attractor of (1) is therefore nonhyperbolic, meaning that the differentiable manifolds intersect tangentially at 0.

We hasten to add here that center manifolds are more difficult to study. These difficulties seem to center on the tangency configuration of M^c and on the fact that generally center manifolds are not unique. As research on that topic is ongoing, I will not dwell on M^c; anyhow, its dynamics will not add much to the present purpose. It suffices to note that the most important characteristics of M^c, namely, the Lyapunov dimension and the metric entropy (see Appendix

1) are both zero. However, what will simplify our task is to assume instead that all three manifolds are invariant and unique, and to focus mainly on M^s and M^u.

As already noted, volume contracts on M^s and expands on M^u, making the nonhyperbolic attractor chaotic. Nonhyperbolic attractors share many characteristics with Axiom A attractors. For example, for some policy π_i, one may observe a cascade of orbits of period 2, 4, 8,… and on until the Feigenbaum limit. Beyond that point, i. e., for a more stringent policy π_{i+1} (similar to an increase in the control parameter value), the system becomes more complex. The main difference with Axiom A attractors is that, beyond the Feigenbaum limit, nonhyperbolic attractors become more sensitive to noise. The interesting thing is that for a new (critical) policy, π^*, a phase change just occurs. If previously policy π_i was operational, we may have had bifurcations; if so, they were always preceded by phase changes as well. Exactly the same thing happens after the implementation of policy π^*; a phase change occurs, and a self-organized equilibrium follows. There could be many critical policies: π^*_1, π^*_2, etc., and the same scenario would occur at each. I will return to the phenomenon of self-organization in the next section. For now, let us focus on the notions of information dimension and metric entropy. But beforehand, a brief review of the connection between the two might further increase understanding.

II-3 Information Dimension and Metric Entropy

We begin by defining a partition $\alpha \stackrel{\text{def}}{=} \{\alpha_i | i = 1, 2, 3,\ldots.\}$ of Γ, which is a collection of nonintersecting and nonempty measurable sets that cover Γ such that Γ: $\alpha_i \cap \alpha_j$, $\forall\ i \neq j$, and $\Gamma = \cup_{i=1} \alpha_i$. Then the *Kolmogorov-Sinai* (K-S) entropy for a continuous system (given the partition α) is $H(\alpha, T) = -\sum_i \mu(\alpha_i) \log(\mu(\alpha_i))$. For a discrete system it is $H_m(\alpha, T) = (1/m) H[\alpha \vee T^{-1}, \ldots, \alpha \vee T^{-(m-1)}]$, where m is the number of successive measurements made during a time interval Δt. $H(\alpha, T) = \lim_{m\to\infty} H_m(\alpha, T)$. The K-S entropy = $\sup_\alpha \{H(\alpha, T)\}$ (see, ([17], [18])

The connection between the system and information is as follows: Each $x \in X$ produces a string of messages $\{m_1, \ldots, m_m\}$ from the system, considered as the source; hence, its outputs are the strings. If the existing measure is taken as a probability density, then there is a probability distribution over the strings. $H(\alpha, T)$ becomes the *Shannon entropy* measuring the average information of the message. The K-S entropy can then be interpreted as the highest mean information that the system produces per step, given the coding. And $S_{K\text{-}S} = \sup_\alpha \{H(\alpha, T)$ is a measure of the highest mean information that it produces per step. Put more simply, the K-S entropy governs the maximum capacity of information generated by a dynamical system; hence, it is a measure of the amount of order and randomness associated with that dynamical system. Frigg [18] has proven that the K-S entropy is equivalent to a generalized version of Shannon's information –theoretic entropy under certain plausible assumptions. And finally, the K-S entropy is also known as metric entropy. Thus, in what follows I will consider Shannon entropy as a measure of unpredictability of informational content, and I will use K-S entropy and metric entropy interchangeably.

A positive K-S entropy implies that, relative to some coding, the behavior of the system is unpredictable, because the K-S entropy also measures (given α) the average exponential divergence of solutions as time goes to infinity; this also means unpredictable behavior. On the other hand, the Lyapunov characteristic exponent (LCE) of x also measure the average exponential divergence of solutions originating close to x. Hence, positive LCEs indicate that solutions diverge exponentially on the average in some directions. This allows Persin, through a theorem of the same name [19], to assert that under certain assumptions, $S_{K\text{-}S} = \sum LCE^{+}$; that is, the sum of the positive LCEs.

As for entropy, there exist many definitions of dimension, namely, capacity dimension, topological dimension, embedding, correlation dimension, etc. However, here we are interested in the information dimension and in the metric entropy. As already indicated, volume shrinks to zero on M^s, where the dynamics is predictable. If we were to calculate the information dimension (D_{KY}) (see definition below) on M^s, it would be zero. Since the dynamics is predictable, no new information may be revealed by adding to what is already known. The metric entropy is therefore zero also. However, things are quite different on M^u. The dynamics is chaotic. For a while at least, bounded volume expands while dissipating the information generated by the evolution of the system. What is then the information dimension on M^u? It is the so-called *Lyapunov dimension*, determined by the D_{KY} index. *We then say that the Lyapunov dimension is equal to the information dimension*. And, according to Persin [19], the metric entropy is computed as the sum of the positive LCEs (λ^{+}) of the Lyapunov Spectrum.

The definition of information dimension here is consistent with the Shannon entropy, except that the number of balls of radius r needed to cover a set is replaced by the number of discrete states that the system can be in, while P_i (.) is the probability that the system is in state i. The amount of information in a sequence of m measurements performed during a time interval Δt is S_i (m) of length m. Then, t*he information dimension* is:

(2) $$I_m = - \Sigma_i P_i (S_i(m) \log P_i (S_i(m)).$$

As shown in Appendix 1, *the metric entropy* over all α partitions is:

(3) $$h_\alpha \stackrel{\text{def}}{=} \sup_\alpha [\Sigma_i P_i (S_i(m) \log P_i (S_i(m))] / m \Delta t.$$

If $\Gamma \varepsilon \Re^m$, there are then *m* LCEs (λ). The *Lyapunov spectrum* of our nonhyperbolic Λ is the set:

(4) $$\{ \lambda_1^{+}, \lambda_2^{+}, \dots, \lambda_g^{+}, \lambda_1^{0}, \lambda_2^{0}, \dots, \lambda_h^{0}, \lambda_1^{-}, \lambda_2^{-}, \dots, \lambda_k^{-} \};$$

It is recalled that the Jacobian matrix has g positive λs and h zero λs, and k negative λs. Here, as the relationship between dimensions and the λs is not as straightforward as in the case of simple attractors, Kaplan and Yorke ([20], [21]) offer a very useful conjecture, termed Lyapunov dim (D_{KY}):

(5) $$D_{KY} = \text{the position of } \lambda_j + \Sigma_i^{g+h} \lambda_i / | \lambda_1^{k}| ,$$

where the position of λ_j is the numeral of the ordinal index of the last $\lambda \geq 0$ in the spectrum; the next term the numerator is the value sum of all $\lambda \geq 0$. For greater clarity, consider an example. If the spectrum of a 4-D dynamical system is {0.6, 0.2, 0, -12}, the D_{KY} is 3 + (0.6 + 0.2 + 0) / 12 = 3.66. The $D_{KY} = D_{Im}$ and it is considered more robust to increases in degrees of freedom in Γ than the conjecture proposed by Mori and Fujisaka [22].

The evolution of points in Γ is determined by the λs. Persin has shown that the metric entropy of the chaotic attractor Λ is:

(6) $$E_m \stackrel{\text{def}}{=} \Sigma_i \lambda_i^+.$$

It can then be seen that on the stable manifold, the dynamics is predictable. The LCEs associated with M^s are all negative. Hence, its D_{KY} and its E_m are both zero. They are also both zero on M^c as the frequencies of limit circles are known. In this regard, the situation on M^u becomes that of the whole attractor. Consequently, the information dimension of Λ is given by (5) while its positive metric entropy is given by (6).

II-4 Self-organization in Economics

The concept of self-organization refers to the sort of spontaneous temporary stability that emerges from a complex dynamic system at some critical π^*. It affects all the components of the system and may last for quite a while as long as long as the critical value of the parameters does not change. The concept of self-organization is not really easy to explain formally or mathematically, and it has its critics. However, it is hard to deny as it is regularly observed in disciplines such as physics, chemistry, traffic flow, human society, cybernetics, biology, economics, etc. Lee Smolin [23] claims that it occurs even in astrophysics. To that effect he observes that in the Orion arm of the Milky Way Galaxy, there is a continuous flow of energy in and out, a low temperature, ingredients such as carbon, oxygen, hydrogen, nitrogen, organic elements, etc., and some sort of out-of-equilibrium calm due to the attainment of some critical value. Under these conditions, a local order had spontaneously emerged amid a sea of chaos. That local order has lasted long enough to give rise to conscious beings capable of understanding them.

In chaotic physical systems, self-organization arises when a system, on its own, renders a number of degrees of freedom ineffective, following the right conditions and a phase change. This means that the system rejects some thermodynamic entropy to the environment. In economics, a similar phenomenon occurs when we have a critical policy π^*. A critical policy is of course more than just monetary and fiscal policies. It also includes the legal system which is a limiting factor of the phase space, efficient competitive rules, conflict resolution mechanism, surveillance and fair enforcement of competitive rules, optimal financialization (too much is harmful to growth), research and development, etc. As far as we can judge, there is no mathematical procedure to determine policy π^*; it is arrived at by trial and error; but once it is operational, the system neutralizes a number of degree of freedom, thereby eliminates the same number of positive LCEs. In other words, the confidence generated by the optimal policy is just enough to allow economic agents to neutralize the variables that are causing instability, allowing thus system (1) to settle down into a temporary out-of-equilibrium limit point (0'); for convenience, we suppose that the limit point is located at the intersection of two manifolds:

$$(7) \qquad 0' \varepsilon\, \Re^r \varepsilon\, M^s \cap M^u ,$$

where (r ≤ m.

As already observed above, self-organization is an observed phenomenon for which there is no agreed upon explanation beside the implementation of a critical policy, which in turn produces a phase change. However, it seems to me at least that in a chaotic dissipative system with positive and negative feedbacks, there is an inflow and outflow of information. Then, $0' \varepsilon\, \Re^r$ is then an accumulation point (called a stock in control engineering parlance).

We may now ask whether or not system (1) has more to do with thermodynamic entropy? It must be remembered that system (1) is a sketch of a real economy, which nevertheless contains some of the main characteristics of a real economy, such as adjustment costs, IRTS, and productivity changes, etc. In addition, it is located in the physical environment. These characteristics connect it to the environment via the supply function. For example, adjustment costs tend to increase the thermodynamic entropy of the environment in the sense of Georgescu-Roegen and Herman Daly, while both IRTS and productivity increases tend to reduce it somewhat. That is the connection. Beside, system (1) is considered a 'source' of information, which is assessable on the unstable manifold. Now imagine two initial conditions x_{01} and x_{02} separated by a finite distance d in E^s. After a time, the two points arrived in M^s, where d is reduced to a single point of zero volume. But, we have assumed that M^s intersects M^u at 0'. The slightest change to $\pi_i \neq \pi^*$ may destroy the temporary equilibrium, and the behavior of the system would immediately tend to align itself with the dominant eigenvector of (A5) and would jump to M^u. That is why complex systems are more likely to become chaotic again when we move away from the critical value of the parameters or away from the critical policy in the present case. On M^u, on the other hand, two initial points separated by an infinitesimal distance d at t_0 tend to diverge exponentially after a time t_1. The separation d that could not be assessed at t_0 can be at t_1 even though information is being dissipated.

Recalling that $f \in C^1(E)$, where $E\ \varepsilon\ \Re^m$. Thus, (1) defines a dynamic system φ (t, x) on E. The function φ (t, x): $\Re \rightarrow E$ defines a solution curve or an orbit of (1) through the point x in E as a motion along the curve. If the equations of the system are symmetric under the reflection: $(x_1, x_2, \ldots x_m) \rightarrow (-x_1, -x_2, \ldots -x_m)$, it has periodic orbits as well as their images under the reflection. Trajectories circulating between the manifolds form complicated loops. That is, they stretch on M^u and shrink on M^s forming homoclinic loop bifurcations as policies change. In more than two dimensions, there is an infinite number periodic orbits of long period that bifurcate from the homoclinic loop as policies change beyond the critical value, indicating the presence of a bounded nonhyperbolic invariant set containing both periodic and aperiodic orbits, called 'homoclinic explosion'. I will not say more about homoclinic loop bifurcations as my interest is in information and entropy.

Even though the information dimension is zero on both M^s and M^c, some information generated by the source is preserved. The evolution of the system produces information and it must be remembered that within the economic system, there is learning and adaptation. Before reaching the temporary equilibrium, M^u was dissipating a given amount of information. After neutralizing a given number of degree of freedom, information lost is reduced, then the metric entropy is also less at $0' \varepsilon \Re^r$. Then despite the necessary fluctuations about $0' \varepsilon \Re^r$, due mainly to noise, some information is conserved, and the "stock" increases, even though $0' \in \Re^r$ cannot be computed with precision due to noise. *We believe that conserved information is the source of endogenous growth å la Romer* [23].

III-DISCUSSION

We began by underlying the concerns of heterodox economists and students with some of the assumptions of the Neo-classical Economic Theory, and with some nonsensical explanations and policy advices derived there from. Together, they give rise to a number of negative outcomes for social welfare. The Hayek-Friedman-Lucas paradigm provides a classic example of this. The paradigm posits that government policy is always harmful to market efficiency. This is not to deny the existence of inefficient and politically tinted policies. But as demonstrated in Appendix 2, the economic system is, by its very nature, chaotic. Left on its own, it will quickly collapse. *The only way to achieve some sort of temporary fluctuating stability is through self-organization for which an efficient policy is indispensable*. Ideology aside, it is rather obvious that the Gam-St-Germain Depository Act of 1982, the abrogation of the Glass-Steagall Act of 1999, the Commodity Modernization Act of 2000, and a certain posture derived from neo-liberalism have all combined to send the American and European economies into a tail spin from which they might never recover if an optimal policy is not implemented.

We made a rather brief recapitulation of the basics of dynamic stochastic systems to show that no new information is forthcoming in stable dynamics. If one could access the stable or the center subspaces of the phase space of a chaotic dynamical system, one would be able to make fundamental statements as the flow on both E^s and E^c is ergodic. However, no such statement is allowed as regards flows on E^u nor about the whole attractor as the main characteristics of the latter are determined by those of the unstable manifold. It then follows that system (1) is non ergodic, as it is the case for all nonlinear dissipative dynamical systems. In such systems, ergodicity gives way to path dependence. In that connection, as have observed Keynes, Davidson [24] and others ([25], [26]), today's data sets from discipline such biology and economics are no guide for the future as they are not stationary, and stationarity is a necessary condition for ergodicity at least in the Gibbsian approach. Therefore, Keynes was right on this while Samuelson [27] was misleading when he advised economists to accept the ergodic hypothesis if they want to move economics away from the realm of history. The same remark applies to Lucas who has been a proponent of ergodicity in economics via the concept of rational expectations. Thus, faulty assumptions have over the years clouded our understanding of the stability of the financial system, of risk management, of the dynamics of economic inequality, and our ability to make fundamental statements derived from causes and effects.

Appendix 1

Terms and Definitions

Definition 1: Quasi-concavity. A function f (x), x $\varepsilon\ \Re^m$, is quasiconcave if for any x_1, $x_2\ \varepsilon\ \Re^m$, we have: $f[\alpha x_1 + (1-\alpha) x_2] \geq \min [f (x_1, f (x_2)], \forall\alpha\ \varepsilon\ (0, 1)$.

Definition 2: Increasing Returns to Scale (IRTS). For a function f: $\Re^m \rightarrow \Re$, where f is continuous on $\Re_+$, C^2 on $\Re_{++}$, f (0) = 0, IRTS is given as: $f (\alpha x) > \alpha f (x), \forall\alpha > 1, \forall x\ \varepsilon\ \Re^m$.

Definition 3. Lyapunov Characteristic Exponents (LCE): Given a phase space Γ and two close points in Γ, denoted l_0, l_t, where l_0 is the infinitesimal separation of the two points at time t_0 and l_t is the final separation at time $t_1 > t_0$. The rate of separation of the two trajectories is measured as $| l_1 | \approx | l_0 |\ e^{\lambda t}$, where λ is the LCE. Statistical mechanics is interested in the limit at infinite time, which defines the maximum λ as the normal exponent in the limit as time goes to infinity. The maximum limit is then:

$$\lambda = \text{limit}_{t\rightarrow\infty} \text{ limit}_{l_0 \rightarrow 0} (1/t) \ln | l_1 (t) / l_0 (t_0) |$$

$\lambda < 0$ indicates contraction on the attractor; $\lambda = 0$ indicates limit cycle, and $\lambda > 0$ points to a chaotic regime. Thus, λ measures the sensibility of a dynamic system to small changes in initial conditions (STIC).

Definition 4. Lyapunov Spectrum: For a dynamic system in $\Re^m$, there are m λs. If they are arranged in descending order: $\lambda_1 \geq \lambda_2 \geq, \ldots, \geq \lambda_m$, then the Lyapunov spectrum is the set $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$.

Definition 5. The Kaplan-Yorke Dimension: If $\lambda_1, \lambda_{2,}, \ldots, \lambda_m$ are the LCEs of a dynamic system in $\Re^m$ and j is the index of the smallest $\lambda \geq 0$, then the Kaplan-Yorke dimension D_{KY} is:

$$D_{KY} = j + (\lambda_1 + \lambda_2, \ldots, \lambda_j) / | \lambda_{j+1} | .$$

Attracting Set. A close invariant set C^1 of E $\varepsilon\ \Re^m$, where E is an open subset of $\Re^m$, is an attracting set of a dynamical system in $\Re^m$ if there is neighborhood U of C such that for all x ε U, the flow $\phi_t (x)\ \varepsilon$ U for all $t \geq 0$, and $\phi_t (x) \rightarrow C$ as $t \rightarrow \infty$. Hence, the attractor (Λ) of the dynamical system is an attracting set.

Strange Attractor. An attractor Λ is called "strange" if it is a fractal. The term strange refers to motions on irregular geometric configurations in the phase space (Γ). The dimension of Λ is given by the D_{KY}. If the D_{KY} is a non integer, Λ has fractal structure.

Chaotic Attractor. For Λ to be chaotic, it suffices that $\lambda_1 > 0$. Then Λ contains a countable set of periodic orbits of arbitrarily large periods, an uncountable set of bounded aperiodic orbits, and possibly a dense orbit.

Thermodynamic Entropy (the Boltzmann's Approach): In a close measure preserving dynamical system, there exist macro states $Ш_k$, k = 1, 2, … containing micro-states m_j, where j > k. Thus, Γ can be partitioned in overlapping regions so that each region corresponds to a macro-state. And there is a measure μ on Γ and on all $Ш_k$, but more than one m can correspond to a single $Ш_k$. The entropy $S = k_B \ln \Psi$, where $k_B = 1.380658 \times 10^{-23}$ J / K and $\Psi$is the number of micro-states contained in a given Ш. Boltzmann assumes that the entropy of $Ш_k$, i. e., $S (Ш_k) < S (Ш_{k+1}), \forall k$. Thus the entropy is maximal on the final $Ш_k$. The Boltzmann's entropy is therefore a statement of the Second Law of thermodynamics, which posits that the entropy of a closed system cannot decrease as time goes forward; but that entropy is a quantity measured in joules per degree Kelvin.

Definition 6. Metric Entropy. Consider a sequence S_i (m) of m successive measurements made during a time interval Δt, and let $\mathcal{P}$ (S_i (m)) be the probability of the sequence, normalized so that $\sum \mathcal{P}_i (S_i (m)) = 1$. The amount of information contained in S_i (m) is:

$$I_m = - \sum_i \mathcal{P}_i (S_i (m)) \ln \mathcal{P}_i (S_i (m)).$$

Taking the maximum value over all partitions α, the metric entropy is the rate of dissipation of information per unit of time, measured in bits per second. In predictable dynamics, new measurements do not add to what is already known. However, if the dynamics is chaotic, new measurements indicate the rate at which the available information is being dissipated. Thus, the metric entropy is:

$$h_\alpha = \sup_\alpha I_m / (m \Delta t), \text{ where sup is taken over all partitions } \alpha$$

The Kaplan-Yorke conjecture defines a quantity called the Lyapunov dimension that is equal to the information dimension. Hence, the metric entropy is:

$$h_\alpha = \sum^k_i \lambda_i^+.$$

Therefore, a chaotic attractor is an attractor with a positive metric entropy, i. e., the sum of λ^+.

APPENDIX 2: An Economic Model

Preliminaries

The firm produces an output q, sold at a price p, using inputs x (t) and v (t). Variable input's price p_j given by competitive markets and assumed to remain constant for all time. Markets for x (t), however, are imperfect. Thus, quasi-fixed factor costs vary as $c_i (\dot{x} (t))$, where the dot refers to time differentiation, and $c_i (x (t))$ is the sum of purchase and internal adjustment costs. In this framework, $c_i (0) = 0$, $c'_i (\dot{x} (t)) > 0$, and $c''_i (\dot{x} (t)) > 0$, where the prime denotes first derivative and second derivatives of the cost function, respectively.

The technology of the firm is q = f [x (t), v (t)], where $x \in \Re^n_+$, $v \in \Re^n_+$, and f belongs to the class of quasi-concave homogeneous functions of degree r >1. Thus, f: $\Re^{2n}_+ \rightarrow \Re_+$ is continuous on $\Re^{2n}_+$, C^2 on $\Re^{2n}_{++}$, f (0) = 0, and the constant rate of investment is ρ.

Thus, integrating from 0 to ∞, we have:

(A1) $$V(.) = \int \exp (-\rho\, t) \left\{ f [\mathbf{x} (t), \mathbf{v} (t)] - \sum_{i \in n} c_i (x_i (t) - \sum_{j \in n} p_j\, v_j (t) \right\} dt,$$
$$\mathbf{s}.\ t.\ x_0 > 0,\ v_0 > 0,$$

so that V (.) has a maximum for some $x_i (t) > 0$, $v_j (t) > 0$, $\forall t$.

Positing $f_{(.)}$ as first partial derivative, we verify the 2n Euler conditions:

i) $f_{xi} (.) - \rho\, c'_j (\dot{x} (t)) + c''_j (\dot{x}_i (t))\, \ddot{x}_i (t) \neq 0,\ i \in n$;

ii) $f_{vj} (.) - p_j = 0$

iii) $\lim_{t \rightarrow \infty} \exp (- \rho t)\, c'_j (\dot{x} (t)) = 0$, where iii) is endpoint conditions.

To these, we add the scale elasticity conditions:

iv) $0 < \eta_v = [f_v(.)\, v / \mathbf{f}(.) < 1,\ 0 < \eta_{vj} < 1,\ \forall j\ \varepsilon\ n;$

v) $\eta_x = f_x(.)\, x / \mathbf{f}(.) > 1$, but $0 < \eta_{xi} < 1,\ \forall i\ \varepsilon\ n$; but

vi) $\eta = \eta_v + \eta_x > 1,$

where η is the scale elasticity. Obviously, the case of $\eta < 1$ derives from the concavity assumption for which solutions are well characterized. However, by (i) and (vi), it is not the case.

We now posit the symbol $f_{(.)\,(.)}$ for matrices of second-order partial derivatives. Hence, the Hessian matrices are:

(A2)
$$H = \left(\begin{array}{c|c} \mathbf{f}_{vj\,v} & \mathbf{f}_{vjxs} \\ \hline \mathbf{f}_{xi\,vs} & \mathbf{f}_{xi\,xs} \end{array}\right) = \left(\begin{array}{c|c} \mathbf{H}_1 & \mathbf{H}_2 \\ \hline \mathbf{H}_3 & \mathbf{H}_4 \end{array}\right),\ s\ \varepsilon\ n.$$

H_1 is negative definite, hence variable input levels is found from ii) in terms of v_j and p_j. If that were the case, one could then substitute in i) to yield a stationary solution as $x^*(p_j, \rho)$ satisfying:

(A3)
$$f_{xi}\,[\mathbf{x}^*(p_j; \rho), \mathbf{v}\,(x^*; p_j] - \rho\, c_i''(.) = 0,\quad i\ \varepsilon\ n,\ \ \ddot{x} \to 0.$$

Implying that one could select paths of quasi-fixed factors so as to maximize V (.) for initial conditions x_0, v_0. As it turns out, condition vi) precludes that procedure. Conditions i) and ii) predict a 2n parameter family while initial conditions give only n stocks. Normally, end-point conditions would allow the elimination of non optimal paths, but they cannot be identified.

To see why, we then look for some topological equivalence near the origin of the system via the transformation $\dot{x}(t) = y(t)$ given a 2n first-order system in x (t) and y (t). Expanding about the stationary point, we have:

(A4)
$$\dot{x}(t) = y(t)$$
$$\dot{y}(t) = \rho\, y(t) - c^{-1}\mathbf{A}\, dx,$$

where $c = \text{diag}\,(c_1'', c_2'', \ldots, c_n'')$, and $\mathbf{A} = [\mathbf{H}_4 - \mathbf{H}_3\mathbf{H}_1^{-1}\mathbf{H}_2]_{(\mathbf{x},\,\mathbf{v})}$.

The Jacobian of system (4) is then:

(A5)
$$\mathbf{J}(\mathbf{0}) = \left(\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hdashline \rho\,\mathbf{I} & -\,\mathbf{c}^{-1}\mathbf{A} \end{array}\right)_{(\mathbf{0})},$$

where I is the n x n identity matrix, and $\mathbf{J}(.)$ is of order (2n x 2n). As Tr (J) > 0, det (J) < 0, the neoclassical economist would conclude by (A5) that the solution of (1) is not well characterized, and that the instability depicted by (A5) is attributable to condition vi). However, IRTS is ubiquitous in aggregate economic data. How can that be since economic systems do not systematically explode?

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