Competitive Equilibrium in the Random Assignment Problem

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Abstract

This paper studies the problem of random assignment with fractional endowments. In the random assignment problem, a number of objects has to be assigned to a number of agents. Though the objects are indivisible, an assignment can be probabilistic: it can give an agent some probability of getting an object. Fractional endowments complicate the matter because the assignment has to make an agent weakly better off than his endowment. I first formulate an exchange economy that resembles the random assignment problem and prove the existence of competitive equilibrium in this economy. I then propose a pseudo-market mechanism for the random assignment problem that is based on the competitive equilibrium. This mechanism is individually rational, Pareto Optimal and justified envy-free but not incentive compatible.

Keywords: Random Assignment, Competitive Equilibrium, Mechanism Design
JEL classifications: D50, D47

1. Introduction

I consider the random assignment problem with *fractional* endowment, i.e., how to probabilistically assign a number of objects to a number of agents in an *efficient* and *fair* manner while respecting *individual rationality* constraints arising from *fractional* private endowments. My solution uses the concept of competitive equilibrium, and is related to two prominent existing results on competitive equilibrium in the assignment problem.

When each agent only has strict preferences over deterministic assignments and owns a distinct entire house, the assignment problem parallels the *housing market* model first analyzed by Shapley and Scarf (1974). In this model the houses are considered indivisible, and, given prices, each agent uses the market value of his house to buy his most-preferred affordable house. A competitive equilibrium consists of prices that clear the market and the associated allocation. Roth and Postlewaite (1977) show that for the housing market a competitive equilibrium exists and is unique. The competitive equilibrium allocation is ex-post efficient, individually rational and can be implemented truthfully through the prominent *Top Trading Cycles* algorithm (Roth, 1982).

In the environment where each agent has cardinal utility for each house and compares probabilistic assignments through Von Neumann-Morgenstern (vNM) expected utility and no agent

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owns any house, Hylland and Zeckhauser (1979) propose the *Competitive Equilibrium from Equal Incomes* (CEEI) solution. The houses are infinitely divisible and each allocation corresponds to a random assignment. The authors construct an economy in which, given prices, agents use an exogenous strictly positive income to purchase a utility-maximizing probability distribution over the houses. A competitive equilibrium consists of prices of the houses that clear the market, and the corresponding allocation. The authors show that a competitive equilibrium always exists and propose the pseudo-market mechanism for the assignment problem as follows. First, elicit cardinal utilities of the agents. Second, give agents the same incomes, simulate the market and compute the competitive equilibrium. Their pseudo-market mechanism is ex-ante efficient, envy-free, but not strategy-proof.

In my setting, the houses are also infinitely divisible, and each agent still evaluates random assignments using his vNM expected utility, but the endowment structure is more general: each agent can own probability shares of houses and there might be house shares not owned by any agent. In my solution, I first distribute the unowned houses among the agents and set up an exchange economy in which, given prices, agents use the market value of their endowments to purchase a utility maximizing probability distribution over the houses. A competitive equilibrium consists of prices of the houses that clear the market, and the corresponding allocation. My main result is that given strictly positive endowments, a competitive equilibrium exists. If endowments are not strictly positive, however, I can still guarantee the existence of an approximate competitive equilibrium.

My results generalize the existing ones on competitive equilibrium in the assignment problem, with and without divisibility. In my environment, when the endowment structure is such that each agent owns a distinct entire house, there exists a unique competitive allocation (with divisibility) whose allocation is unique and coincides with that of the unique competitive allocation with indivisibility. When no agent owns any house, the distribution of unowned houses evenly among the agents is equivalent to giving agents equal incomes, so my existence result implies the existence of CEEI.

My results also allow for the design of a pseudo-market mechanism for the assignment problem that accommodates endowment structures more general than those amenable to Top Trading Cycles and CEEI. In the pseudo-market mechanism, I first elicit from the agents their cardinal utilities over the houses. I then distribute the unowned houses evenly among the agents and compute the competitive equilibrium. The resulting allocation will be ex-ante efficient, individually rational and justified envy-free. My pseudo-market mechanism improves upon that by Hylland and Zeckhauser (1979) since it satisfies individual rationality constraints arising from private endowments, and collapses to the CEEI outcome when there is no private endowments. It also improves upon Top Trading Cycles by allowing for fractional private endowments, and collapses to the Top Trading Cycles outcome when each agent owns a distinct entire house. However, the mechanism is not strategy-proof.

My model and results have applications in environments in which agents have fractional endowments, such as on-campus housing (Athanassoglou and Sethuraman, 2010). Fractional endowment of an object can be interpreted as the probability of actually getting the object. For example, in the on-campus housing setting it could represent some sort of priority or claim a student has over a certain room. Another interpretation of fractional endowments comes from thinking of objects as actually being divisible goods. This interpretation is particularly suitable in environments where *time-sharing* is possible. For example, a computer cluster can be used by many programs at different times over the course of a day. Fractional endowment then is the amount of time in a day a program is entitled to run on the computer cluster. In contrast to the case of indivisible goods where a random assignment is viewed as a lottery over deterministic consumptions, in time-sharing applications a random assignment describes how much time an agent can consume a good.

The pseudo-market mechanism can also be viewed as a way to improve a given lottery. Suppose that a random assignment has been proposed as a solution. One can think of this proposed random assignment as fractional endowments, and apply the pseudo-market mechanism to find a new random assignment that is efficient and potentially Pareto-dominates the proposed solution. In time-sharing environments one can similarly improve upon an existing allocation.

Methodologically, I make use of an existing technique to overcome the problem of demand discontinuity: instead of requiring each agent to consume a probability distribution over the objects, I only require them to consume a sub-probability distribution over the objects (Budish et al., 2013). This relaxation gives the desirable continuity properties of demand, and also an interpretation of individual rationality arising from fractional endowments. A fixed point theorem is then used to show the existence of competitive equilibrium, at which market-clearing implies that each agent is now consuming a proper probability distribution. The allocation at competitive equilibrium must therefore be a random assignment, which can be shown to be efficient, individually rational and justified envy-free.

The rest of the paper is structured as follows. Section 2 describes the environment. Section 3 establishes the existence of competitive equilibrium and relates it to the case of indivisible goods. Section 4 addresses some of the assumptions made in earlier sections, as well as points out several extensions of the model. Section 5 proposes a mechanism based on the competitive equilibrium results. Section 6 concludes.

2. Preliminaries

2.1. The environment

There is a set of houses $\mathcal{H} = \{1, 2, ..., H\}$, one of each, to be assigned to a set of agents $\mathcal{I} = \{1, 2, ..., I\}$, where $I = H$. I denote a generic agent by i and a generic house by h. Each agent i has cardinal utility u^i_h for house h. Let $u^i = (u^i_h)_{h\in\mathcal{H}}$ denote agent i's preferences, and let $u = (u^i)_{i\in\mathcal{I}}$ denote the profile of preferences. Assume that $u^i_h > 0$ for all i, h. Let $\mathcal{U}^i$ be the set of preferences of agent i. Let $\mathcal{U} = (\mathcal{U}^i)_{i\in\mathcal{I}}$ be the set of preference profiles.

The consumption space of each agent i is the set of sub-probability distributions over the houses $Q^i = \{q^i = (q^i_1, ..., q^i_H) \in \mathbb{R}^H_+ : \sum_{h\in\mathcal{H}} q^i_h \leq 1\}$. Consuming a proper probability distribution has the obvious interpretation as getting a lottery over the houses. Consuming a sub-probability distribution can be interpreted in time-sharing contexts as the duration in which an agent can consume a good (Athanassoglou and Sethuraman, 2010); it can also be interpreted as a lottery in which there is non-zero probability of getting no house at all. This can arise in settings where there are more agents than houses. Each agent i evaluates a consumption $q^i \in Q^i$ through his "expected" utility $q^i u^i = \sum_{h\in\mathcal{H}} q^i_h u^i_h$.

Agent i's endowment of house h is denoted by e^i_h. Let $e^i = (e^i_1, ..., e^i_H)$ be agent i's endowment. I assume that $\sum_{h\in\mathcal{H}} e^i_h \leq 1$ for all i, so that an agent's endowment is always in his consumption set. For feasibility of endowment, I require that $e^i_h \geq 0$ for all i, h, and that $\sum_{i\in\mathcal{I}} e^i_h \leq 1$ for all h. Let $e = (e^i)_{i\in\mathcal{I}}$ be the profile of endowments and let $\mathcal{E}$ be the set of endowment profiles. An endowment profile e is *strictly positive* if $e^i_h > 0$ for all i, h.

The set of allocations is $\mathcal{Q} = \{q = (q^1, ..., q^I) : q^i \in Q^i \text{ for each } i\}$. An allocation q is *feasible* if $\sum_{i\in\mathcal{I}} q^i_h \leq 1$ for all h. An allocation q is a *random assignment* if $\sum_{i\in\mathcal{I}} q^i_h = 1$ for all h. A deterministic assignment is a random assignment q in which for each agent i there is a house h such that $q_h^i = 1$ and $q_{h'}^i = 0$ for all $h' \neq h$. Note that at a random assignment q, it must be that $\sum_{h \in \mathcal{H}} q_h^i = 1$ for all i, so each agent's consumption bundle is a proper probability distribution over the houses. A random assignment q can be represented as a $I \times H$ bistochastic matrix where q_h^i is the entry in the i^{th} row and h^{th} column. A deterministic assignment can be represented as a permutation matrix. By the Birkhoff (1946) - von Neumann (1953) theorem, any random assignment can be obtained as a lottery over deterministic assignments. Let $\mathcal{R}$ be the set of all random assignments.

A random assignment problem is completely described by the pair (u, e). A solution to the random assignment is a mapping $\phi : \mathcal{U} \times \mathcal{E} \to \mathcal{R}$.

To facilitate exposition, I assume that *preferences are strict*: for all i, $u_h^i \neq u_{h'}^i$ for any $h \neq h'$ and that *endowments are complete*: $\sum_{i \in \mathcal{I}} e_h^i = 1$ for all h. Unless stated otherwise, these assumptions are being made. They are relaxed in section 4.

2.2. Desirable properties of a solution

Given a random assignment problem (u, e), the following concepts are standard. An allocation is *efficient* if there is no allocation at which all agents are weakly better off and at least one agent is strictly better off. An allocation is *weakly efficient* if there is no allocation at which all agents are strictly better off.

Definition 1. *An allocation q is efficient if there exists no random assignment $\hat{q}$ such that $\hat{q}^i u^i \geq q^i u^i$ for all i, with strict equality for at least some agent i.*

Definition 2. *An allocation q is weakly efficient if there exists no random assignment q such that $\hat{q}^i u^i > q^i u^i$ for all i.*

To make a bidder willing to change to a new assignment, it must be that the new assignment is at least as good for the agent as his endowment. An allocation that has this property is called *individually rational* (IR).

Definition 3. *An allocation q is individually rational if $q^i u^i \geq e^i u^i$ for all i.*

Depending on the application, individual rationality has different interpretations. In the housing market model of Shapley and Scarf (1974), each agent's endowment is a discrete house which he can consume, so individual rationality has the natural interpretation as an agent's property rights. In the lottery-improvement context, though no agent can realize the lottery by himself, he may be entitled to the current lottery that he has. For example, in dorm-room assignment, a student may request that his chances of getting certain rooms be unchanged. In the case of divisible goods, like time-sharing contexts, individual rationality has the interpretation of property rights.

Envy-freeness, the central axiom of fairness, is incompatible with individual rationality. For example, if preferences are strict and identical across all agents, and one agent is endowed with his most preferred house, then any individually rational solution must give this house to this agent, but such a solution is not envy-free. I relax envy-freeness to *justified envy-freeness*, which requires that if agent 1's endowments are weakly more than agent 2's, then agent 1must not strictly prefer agent 2's lottery. This notion of fairness is perhaps the most natural in my setting.

Definition 4. *A random assignment q is justified envy-free if for any two agents i and j, $e^i \geq e^j$ implies that $q^i u^i \geq q^j u^i$.*

2.3. Competitive equilibrium

Given a complete endowment profile e and preference profile u, imagine an exchange economy that is made up of the same set of goods (houses) $\mathcal{H}$ and the same set of agents $\mathcal{I}$ with the same set of preferences u over the same consumption space $(Q^i)_{i\in\mathcal{I}}$ and the same profile of endowments e. In this exchange economy, agent i's endowment is e^i, and given market prices, he can sell his endowment to buy a sub-probability distribution that maximizes his utility. A competitive equilibrium consists of prices and a feasible allocation that is individually optimal for the agents given prices.

For ease of exposition, I will first define the *competitive equilibrium with budgets* in which an agent's wealth consists of his endowment and an exogenous non-negative budget. Let a budget vector $b = (b^i, ..., b_I) \in \mathbb{R}^I_+$ be given. I write $b = 0$ to mean $b^i = 0$ for all i. Similarly, $b > 0$ means $b^i > 0$ for all i. Let the set of price vectors be the H-simplex:

$$P = \{p = (p_1, ..., p_H) \in \mathbb{R}^H_+ : \sum_{h\in\mathcal{H}} p_h = 1\}.$$

Define an agent's choice set as

$$F^i(p, e, b) = \{q^i = (q^i_1, ..., q^i_H) \in \mathbb{R}^H_+ : \sum_{h\in\mathcal{H}} q^i_h \leq 1 \text{ and } q^i p \leq e^i p + b^i\}.$$

An agent's choice set is the set of bundles of goods which do not add up to more than one in total and do not cost more than his wealth. In other words, his choice set is the set of affordable sub-probability distributions. His demand correspondence is

$$\phi^i(p, e, b) = \operatorname*{arg\,max}_{q^i \in F^i(p,e,b)} q^i u^i,$$

that is, he chooses the utility-maximizing bundle(s) from his choice set.

Definition 5. *A competitive equilibrium with budgets b is a price vector p and random assignment $q = (q^i)_{i\in\mathcal{I}}$ such that given p, for each agent i, $q^i \in \phi^i(p, e, b)$.*

An agent's demand correspondence consists of the optimal affordable sub-probability distributions.[2] Though the definition does not require any agent to consume a proper probability distribution, the requirement that the allocation is a random assignment ensures that every agent consumes a proper probability distribution.

A competitive equilibrium in the usual sense (without the extra budgets) is simply a competitive equilibrium with a budget vector that is identically zero.

Definition 6. *A competitive equilibrium is a price vector p and random assignment $q = (q^i)_{i\in\mathcal{I}}$ such that given p, for each agent i, $q^i \in \phi^i(p, e, 0)$.*

[2] Since the desired allocation is a random assignment, I could have imposed the restriction that each agent must consume a proper probability distribution. However, relaxing this restriction by letting each agent consume a sub-probability distribution yields good continuity properties. Hylland and Zeckhauser (1979) imposes the restriction and tackle the continuity problem by always having some zero-priced house in the market, but consequently have to transform the price space to attain convexity and compactness.

Sometimes the existence of competitive equilibrium is not guaranteed, but an *approximate competitive equilibrium* might exist. An $\epsilon-$approximate competitive equilibrium consists of prices and individual demands such that aggregate demand for any house does not exceed aggregate endowments by more than ϵ.

Definition 7. *For $\epsilon > 0$, an ϵ-approximate competitive equilibrium with budgets b is a price vector p and individual demands $q = (q^i)_{i\in\mathcal{I}}, q^i \in \phi^i(p, e, b)$ for all i such that $|\sum_{i\in\mathcal{I}} q^i_h - 1| < \epsilon$ for all h.*

3. Competitive equilibrium

3.1. Existence theorems

I now show the results regarding the existence of competitive equilibrium. I first construct a price-demand correspondence that is guaranteed to have a fixed point by Kakutani's theorem, then argue that the fixed point is a competitive equilibrium.

The aggregate demand correspondence is the sum of individual demands. I abuse notation and write q also to denote aggregate demand.

$$\Phi^D(p, e, b) = \{q : \exists\, (q^i)_{i\in\mathcal{I}} \text{ such that } q^i \in \phi^i(p, e, b) \text{ for each } i \text{ and } \sum_{i\in\mathcal{I}} q^i = q\}.$$

The range of Φ^D is the convex compact set :

$$D = \{q = (q_1, ..., q_H) \in \mathbb{R}^H_+ : \sum_{h\in\mathcal{H}} q_h \leq H\}.$$

Define the price player correspondence $\Phi^P : D \to P$ to be:

$$\Phi^P(q) = \arg\max_{p\in P} p(q - \vec{1}^H),$$

where $\vec{1}^H$ is a vector of length H with all entries being one that represents the aggregate resource of the economy. The term $(q - \vec{1}^H)$ in the definition is called *excess demand*. The price player, given excess demand at the old price vector, sets a new price vector that maximizes the value of excess demand. It is convenient to note that the price player ensures that a house has non-zero price if and only if its excess demand is *maximal* among all the houses. In particular, note that unless excess demand is the same for all houses, the price player will set the price of some house to zero.

For a given endowment profile e and budgets b, define the correspondence $\Phi : P \times D \to P \times D$ by:

$$\Phi(p, q) = (\Phi^P(q), \Phi^D(p, e, b)).$$

The term $(e^i p + b^i)$ is called agent i's *wealth*. Given a budget vector b, the *aggregate wealth* of the economy at any price p is $\sum_{i\in\mathcal{I}} e^i p + b^i = 1 + \sum_{i\in\mathcal{I}} b^i$.

This fixed point approach is quite commonly used in establishing existence of competitive equilibrium. The only points of departure are: (1) the requirement that each consumption bundle must be a sub-probability distribution and (2) an agent's wealth comes from his endowment *and* an exogenous budget.

One condition necessary for Kakutani's fixed point theorem is upper hemi-continuity of demand in prices. The definition of competitive equilibrium requires that each agent consume a proper probability distribution. However, if this is imposed on demand by restricting the consumption space to the hyperplane $\sum_{h\in\mathcal{H}} q^i_h = 1$ then the choice set is not continuous in prices, leading to demand being not upper hemi-continuous in prices.[3] The relaxation of the choice set to include sub-probability distribution eliminates this discontinuity issue and greatly simplifies the analysis.

Lemma 1. *If $e^i p + b^i > 0$ for all p then the correspondence $F^i(p, e, b)$ is continuous in p.*

Proof. See appendix. □

The continuity of the choice set, coupled with the linearity and continuity of vNM utility functions, gives the following result via the theorem of the maximum.

Lemma 2. *If $e^i p + b^i > 0$ for all p, then Φ^i is non-empty, convex-valued, and upper hemi-continuous in prices.*

The aggregate demand correspondence Φ^D is hence upper hemi-continuous. It is also non-empty and convex-valued. The corresponding properties of the price player correspondence are easy to establish. Therefore, Φ is a non-empty, convex-valued upper hemi-continuous correspondence from a non-empty convex compact subset to itself, so by Kakutani's theorem Φ has a fixed point. Denote the fixed point price by $p(e, b)$ and the associated consumption bundles of the agents by $q(e, b) = (q^i(e, b))_{i\in\mathcal{I}}$.

Strictly positive endowment is sufficient to guarantee strictly positive wealth even when the budget vector is identically zero, and hence the existence of a fixed point $(p(e, 0), q(e, 0)$. It can be shown that at this fixed point the market clears, so it is a competitive equilibrium.

Theorem 1. *If endowment is strictly positive, then a competitive equilibrium exists. The corresponding random assignment is efficient, individually rational and justified envy-free.*

Proof. Strictly positive endowment guarantees the existence of fixed point $(p(e, 0), q(e, 0))$. I now argue that this fixed point is market-clearing, i.e., $q(e, 0) = \vec{1}^H$. In this proof, *excess demand* of house h means that $q_h(e, 0) > 1$, and *excess supply* of house h means that $q_h(e, 0) < 1$. Note that because of zero budgets, aggregate wealth is one.

Step 1: There is no excess demand. Suppose in negation that there is excess demand of some houses. Let $\mathcal{K}$ denote set of houses in *maximal* excess demand. The price player must set the prices of houses outside $\mathcal{K}$ to be zero, and the prices of houses in $\mathcal{K}$ to sum to one. Excess demand then implies that aggregate expenditure is strictly greater than one, i.e., $p(e, 0)q(e, 0) > 1$. But this means that aggregate expenditure exceeds aggregate wealth, a contradiction.

Step 2: There is no excess supply. Suppose that there is excess supply. Consider two cases: (1) some prices are zero, or (2) all prices are strictly positive. If there is some zero-priced house, then each agent i's consumption bundle must sum up to one since it is costless to consume this

[3] For example, consider a simple setting where there are two houses A and B and an agent is endowed with $\frac{1}{2}$ of each house. The agent strictly prefers house A to house B. Suppose that the agent can only buy proper probability distributions over A and B. Consider the price sequence $p_A = 1 + \epsilon, p_B = 1 - \epsilon$ with ϵ going to zero. At each small $\epsilon > 0$ the agent cannot buy more than $\frac{1}{2}$ of house A, and demand is $(q_A, q_B) = (\frac{1}{2}, \frac{1}{2})$. However, at the limit when $\epsilon = 0$, the choice set expands to include all combinations, and demand jumps discontinuously to $(q_A, q_B) = (1, 0)$

zero-priced house. So collectively all agents consume a total of I house shares. Since $I = H$, no excess demand then means that there is no excess supply.

If prices are all strictly positive, then excess supply of a house implies equal excess supply for every house (otherwise the price player will set some price to zero). Consequently, aggregate expenditure is less than one, i.e., $p(e, 0)q(e, 0) < 1$, so the total wealth of the economy is not exhausted. Therefore there is at least some agent i who does not use up his wealth. Strict preferences implies that this agent gets his most preferred house in its entirety. But this implies no excess supply of this house, a contradiction.

So at price $p(e, 0)$ there is no excess demand and no excess supply, which means that the market clears exactly. So I agents consume H houses in total at the fixed point. Since each agent's consumption bundle adds up to one or less, it must be the case that each agent's consumption bundle adds up to exactly one, i.e., each agent is consuming a proper probability distribution. $q(e, 0)$ is therefore a random assignment. $(p(e, 0), q(e, 0))$ is a competitive equilibrium with zero budgets, so it must be a competitive equilibrium as well.

Efficiency can be shown via the usual revealed preference argument. Note that strict preferences imply that if $\hat{q}^i u^i \geq q^i u^i$ then it must be that $\hat{q}^i p \geq q^i p$. This is not true if preferences are not strict because an agent may be choosing an optimal bundle that is not necessarily the cheapest.

The fact that agents are optimizing from choice sets given by their endowments ensures justified envy-freeness and individual rationality. □

Theorem 1 can be used to prove a result that is similar in spirit to that of Hylland and Zeckhauser (1979). If none of the agents has any endowments, I can give each of them some strictly positive budget (normalized to sum to one) and let them use their budget to buy house shares from some social auctioneer. A competitive equilibrium with budget is guaranteed to exist. The intuition is that there is no fundamental difference between budget and endowment. Instead of getting budget b^i and no endowment, agent i can be endowed with endowment $e^i = b^i \vec{1}^H$ and no budget. Since prices sum to one, the value of e^i is always exactly b^i. The agents now are endowed with all the houses, and can trade among themselves instead of buying from the social auctioneer. I have essentially transformed the CEEI setting to the exchange economy setting, and theorem 1 applies.

Corollary 1. *If endowment profile e is identically zero, then for any budgets b such that $b^i > 0$ for all i and $\sum_{i \in \mathcal{I}} b^i = 1$ there exists a competitive equilibrium with budgets b.*

Theorem 1 requires the endowment profile to be strictly positive. A endowment profile that is not strictly positive can cause discontinuity in demand at zero prices and so the existence of a fixed point cannot be guaranteed. Additional budget solves this discontinuity problem, but might lead to excess demand. However, there is a close relationship between the budgets and excess demand which is summarized in the following lemma.

Lemma 3. *Given budget $b > 0$ the fixed point $(p(e, b), q(e, b))$ constitutes an ϵ-approximate equilibrium with budgets b where $\epsilon = (H - 1) \sum_{i \in \mathcal{I}} b^i$.*

Proof. Consider the fixed point $(p(e, b), q(e, b))$ given endowment profile e and a budget vector b with $b^i > 0$ for all i. If there is no excess demand then by the same logic in the proof of theorem 1 there is no excess supply, and $q(e, b) - \vec{1}^H = 0$.

Suppose there is excess demand. Let $\mathcal{K}$ be the set of houses with maximal excess demand d, then the price player must set prices of houses in $\mathcal{K}$ to sum to one and all other prices to be zero.

Aggregate expenditure is then $1+d$, which must be bounded above by aggregate wealth $1+\sum_{i\in\mathcal{I}} b^i$, so $d \leq \sum_{i\in\mathcal{I}} b^i$.

There must also be excess supply, otherwise excess demand for every good means I agents are consuming more than I house shares in total which is infeasible. Therefore the price player must set the prices of houses in excess supply to zero. Consequently each agent i's consumption bundle must sum up to one, leading to a total of $I = H$ for all the agents. Consider any house h^*. Since maximal excess demand is d, maximum total amount of shares possible from all the other houses is $(H-1)(1+d)$, so the minimum demand for house h^* is $H-(H-1)(1+d) = 1-(H-1)d$ and the maximum excess supply is therefore $(H-1)d$, which, by the inequality from the preceding paragraph, is bounded above by $(H-1)\sum_{i\in\mathcal{I}} b^i$.

Therefore, maximal excess supply or demand is bounded by $(H-1)\sum_{i\in\mathcal{I}} b^i$. □

The relationship between the approximation error ϵ and the budget vector implies that for any $\epsilon > 0$ I can choose any budget vector $b > 0$ such that $\sum_{i\in\mathcal{I}} b^i < \frac{1}{H-1}\epsilon$ to guarantee the existence of ϵ-approximate equilibrium with budget b. The fixed point $(p(e,b), q(e,b))$ is one such ϵ-approximate equilibrium. If one gives the agents equal budgets, then justified envy-freeness is ensured at the approximate equilibrium allocation. This is summarized by the following result.

Theorem 2. *Let $\epsilon > 0$ be given. Let b be such that $b^i = \frac{1}{I}\frac{1}{H-1}\epsilon$ for all i. An ϵ-approximate competitive equilibrium with budget b exists. The corresponding allocation is efficient, individually rational and justified envy-free.*

A related result is given by Mas-Colell (1992) who show that, under much more general conditions, a competitive with "slacks" (budgets) exists. His result, to be more precise, states that there exists budgets that will sustain a competitive equilibrium. My result, however, states that for *any* budget small enough, there will be an *approximate* competitive equilibrium with correspondingly small approximation error.

3.2. Relationship with competitive equilibrium with indivisibility

So far I have assumed that the houses are infinitely divisible. I now define the competitive equilibrium *with indivisibility* for the setting where the houses are not divisible.

Definition 8. *A* competitive equilibrium with indivisibility *is a price vector p and a deterministic assignment m such that for each agent i, $m^i = \arg\max_{h\in\mathcal{H}, p_h \leq pe^i} u^i_h$.*

An allocation associated with a competitive equilibrium with indivisibility is called a competitive allocation with indivisibility. Roth and Postlewaite (1977) show that if preferences are strict and endowment is discrete, a competitive allocation with indivisibility exists and is unique.

Theorem 3. *(Roth and Postlewaite) If preference profile u is strict and endowment profile e is discrete, then there exists a unique competitive allocation with indivisibility.*

I show that if the endowment profile is discrete, then a competitive equilibrium with divisibility always exists and its allocation is unique and coincides with the unique competitive allocation with indivisibility.

Theorem 4. *If endowment profile e is discrete, then there exists a unique competitive allocation. Moreover, it is the same as the unique competitive allocation with indivisibility.*

Proof. See appendix. □

The result suggests that the uniqueness of competitive allocation with indivisibility is not due to indivisibility but due to the discrete endowment structure and the requirement that each agent's consumption bundle adds up to one.

4. Extensions

4.1. General preferences

The proof of theorem 1 makes it clear that the assumption of strict preferences is only used to argue that market clears at the fixed point.[4] When indifferences are present, one can always perturb the profile of preferences to get strict preferences that are arbitrarily close to the original preferences. The existence of competitive equilibrium at these perturbed preferences and a continuity argument can be used to establish that competitive equilibrium exists even when there are indifferences.[5]

Theorem 5. *Suppose that endowment is strictly positive. A competitive equilibrium exists with general preferences. The corresponding random assignment is weakly efficient, individually rational and justified envy-free.*

Proof. Let endowment be strictly positive. Let u be a profile of preferences with indifferences. Consider the sequence of profile (u^m) converging to u such that u^m contains no indifferences for each m. By theorem 1, for each u^m there exists a competitive equilibrium (p^m, q^m). Since the sequence of competitive equilibria (p^m, q^m) is bounded, there exists a convergent subsequence (p^n, q^n) that converges to, say, (p, q). I argue that (p, q) is a competitive equilibrium for preference profile u.

q^n is a random assignment for each n, so the limit q must also be a random assignment. I need to show that $q^i \in \phi^i(p, e, 0)$ for each i, i.e., q^i is agent i's demand. I suppress agent index i for notational ease. Since $q^n p^n \leq ep^n$, at the limit it is true that $qp \leq ep$, so $q \in F(p, e, 0)$. Consider any other $\hat{q}$ in $F(p, e, 0)$. I must show that $\hat{q}u \leq qu$. By lower hemi-continuity of the choice set $F(p, e, 0)$ in p and the assumption that $\hat{q} \in F(p, e, 0)$, there exists (p^{nk}), a subsequence of (p^n), such that there is a sequence $(\hat{q}^{nk})$, with $\hat{q}^{nk} \in F(p^{nk}, e, 0)$, converging to $\hat{q}$. Since $\hat{q}^{nk} \in F(p^{nk}, e, 0)$, optimality of (q^{nk}) implies that $\hat{q}^{nk}u^{nk} \leq q^{nk}u^{nk}$. Therefore at the limit $\hat{q}u \leq qu$.

For each n, the allocation q^n is efficient at u^n by theorem 1. Suppose that the limiting allocation q is not weakly efficient at preferences u, then q is strictly Pareto-dominated by some $\hat{q}$ at u. Since (q^n) converges to q and (u^n) to u, there is some n such that q^n is also strictly Pareto-dominated by $\hat{q}$ at u^n, contradicting the efficiency of q^n. Therefore q is weakly efficient.

Note that the usual proof approach using revealed preference does not work because of the possibility of satiated agents, i.e., a weakly preferred bundle does not necessarily cost at least as much if the agent is satiated and is not choosing the cheapest bundle.

Individual rationality and justified envy-freeness are established using revealed preference. □

[4] In fact, the only assumption really needed in the argument of the proof is that each agent has a unique house that he values the most.

[5] This result does not contradict a counter example in Hylland and Zeckhauser (1979), which precludes the possibility of having different prices for identical houses.

4.2. Incomplete endowment profile

I have been assuming that the endowment profile is complete, i.e., agents collectively own all the houses. If endowment is not complete, then a competitive equilibrium is not guaranteed to exist. It is easy to see that for the market to clear, total expenditure must be equal to total cost of all the houses, which is 1 (prices are assumed to sum to 1). Therefore any house not completely owned by the agents must have a price of zero in equilibrium. But if all the houses are not completely owned then there must be no equilibrium.

There is, however, a simple solution to incomplete endowment. The goal of the social planner is to distribute the houses in the best way possible, and competitive equilibrium is a means to that end. The social planner can distribute the unowned houses evenly among the agents and use the competitive equilibrium resulting from the modified endowment.

Theorem 6. *Suppose that endowment is strictly positive and incomplete. Distribute the unowned houses evenly among the agents to get a modified endowment. A competitive equilibrium exists at the modified endowment. The corresponding allocation is efficient, individually rational and justified envy-free.*

4.3. Under-capacity

Suppose that there are less houses than agents, i.e., $I - H > 0$. Obviously no random assignment is feasible since there are more agents than houses, so the existing definition of competitive equilibrium automatically implies that a competitive equilibrium does not exist. However, with a slight and natural modification of the definition, much of the results hold true still.

Definition 9. *A competitive equilibrium with under-capacity is price vector p and allocation $q = (q^i)_{i\in\mathcal{I}}$ such that $\sum_{i\in\mathcal{I}} q^i_h = 1$ for all h and $q^i \in \phi^i(p, e, 0)$ for all i.*

A competitive equilibrium is now required to only be market-clearing, not a random assignment. A proof identical to that of theorem 1 can be used to show the following result.

Theorem 7. *If endowment is strictly positive, then a competitive equilibrium with under-capacity exists. The corresponding allocation is efficient, individually rational and justified envy-free.*

Proof. See appendix. □

In a competitive equilibrium with under-capacity, some agents are not consuming a proper probability distribution. These agents can be thought of as consuming a non-zero probability of getting a *null* house, a house that yields no utility to any agent. A competitive equilibrium with under-capacity (p^u, q^u) can also be thought of as a standard competitive equilibrium in which

- $F = I - H$ null houses are added to the set of houses and given prices of zero.
- Non-null houses have the same prices as p^u.
- Agents whose allocation at q^u is not a proper probability distribution add the appropriate amount of null house to their consumption to obtain a proper probability distribution.

4.4. Over-capacity

In the case of over-capacity, there are more houses than agents, i.e., $H - I > 0$. Naturally, no random assignment can be market-clearing. A competitive equilibrium with over-capacity only requires that each agent's allocation is a proper probability distribution.

Definition 10. *A competitive equilibrium with over-capacity is price vector p and allocation $q = (q^i)_{i \in \mathcal{I}}$ such that $\sum_{h \in \mathcal{H}} q^i_h = 1$ for all i and $q^i \in \phi^i(p, e, 0)$ for all i.*

Using a proof similar to the proof of theorem 1, it is easy to show that

Theorem 8. *If endowment is strictly positive, then a competitive equilibrium with over-capacity exists. The corresponding allocation is efficient, individually rational and justified envy-free.*

Proof. See appendix. □

A competitive equilibrium with over-capacity (p^o, q^o) is equivalent to a standard competitive equilibrium in which

- $F = H - I$ fictitious agents are added to the set of agents, each of whom is given a positive budget b_f and is indifferent among all houses.
- Prices are the same as p^o.
- Non-fictitious agents' allocations are the same as q^o.
- Fictitious agents consume the houses not consumed by the non-fictitious agents.

5. Application to the assignment problem

5.1. Mechanisms

The focus of the paper thus far has been on the existence and properties of competitive equilibrium. The random assignment problem is, however, a mechanism design problem: the social planner has to somehow assign house shares among the agents without knowing their preferences. By the revelation principle, the social planner can restrict attention to mechanisms where he first elicits the preferences of the agent and then uses a systematic mapping from preferences and endowment to random assignments. Such a mapping is a *direct revelation mechanism.* Formally, a mechanism is a mapping $\varphi : (\mathcal{U} \times \mathcal{E}) \to 2^{\mathcal{Q}}$ that takes in an assignment problem and returns a random assignment. In general, it is desirable that a mechanism is efficient, individually rational, justified envy-free and strategy-proof.

Definition 11. *A mechanism φ is efficient, individually rational and justified envy-free if, for any assignment problem (u, e), the random assignment $\varphi(u, e)$ is efficient, individually rational and justified envy-free.*

A direct mechanism is *incentive compatible* if truthful reporting of preferences is a dominant strategy for each agent - misreporting preferences never benefits the agent regardless of the reports of the other agents. In the current context, a mechanism is incentive compatible if misreporting can never strictly improve the agent's vNM utility. In the definition below, $\varphi^i(u, e)$ denotes agent i's lottery at the allocation $\varphi(u, e)$.

Definition 12. *A mechanism φ is incentive compatible if for any agent i, $\varphi^i(u^i, u^{-i}, e)u^i \geq \varphi^i(\hat{u}^i, u^{-i}, e)u^i$ for all $\hat{u}^i \in \mathcal{U}^i$.*

5.2. The pseudo-market mechanism

Consider the pseudo-market mechanism, which works as follows. First, the social planner elicits preferences of the agents. Endowment is assumed to be public information. If endowment e is not complete, he distributes the unowned house shares evenly among all agents to get a complete, *modified* endowment profile. If the modified endowment profile is strict, by theorem 1 he can simulate the market for house shares and compute the competitive equilibrium allocation which will be an efficient(weakly efficient if preferences exhibit indifferences), individually rational and justified-envy free random assignment. The results regarding the extensions of the model can be used to adapt the pseudo-market mechanism to accommodate under- and over-capacity. If the modified endowment profile is not strictly positive, by theorem 2 he can give each agent the same small positive budget and compute the ϵ-approximate equilibrium allocation. Note that the approximation error ϵ can be made arbitrarily small by choosing arbitrarily small budgets.

In the case where there are many competitive equilibria, an arbitrary but consistent rule can be used to determine which competitive equilibrium allocation will be used as the output of the pseudo-market mechanism. For example, a simple rule can be: "Choose the allocation that is best for agent 1. If there are many such allocations, choose among those the best for agent 2. If there are many such allocations, choose among those the best for agent 3. And so on."

Since the pseudo-market mechanism outputs a competitive allocation, it is (weakly) efficient, individually rational and justified envy-free. However, it is not strategy-proof. This is to be expected, since Zhou (1990) shows that there is no mechanism that is anonymous, efficient and strategy proof. It is indeed possible for an agent to misreport to manipulate the equilibrium prices and get a better lottery.

It is, however, worth noting that since the pseudo-market mechanism is based on competitive equilibrium, it inherits certain incentive properties of competitive equilibrium. In particular, as the number of agents becomes large through replication, the gain from misreporting goes to zero. The interested reader can refer to Roberts and Postlewaite (1976) for the formal statements and technical details. The intuition is as follows. To manipulate a mechanism based on competitive equilibrium, a bidder must manipulate prices. Because non-equilibrium prices give rise to excess demand, the bidder must, through misreporting of preferences, absorb excess demand to sustain such prices as equilibrium prices. When the economy is replicated infinitely many times, however, this excess demand, no matter how small, becomes too large for the bidder to absorb. Therefore, in the infinitely replicated economy, the only sustainable prices are indeed competitive equilibrium prices. In a sense, prices become more and more exogenous to each bidder. Given these prices and public endowments, the choice set of each bidder is exogenous, and so it is optimal for the bidder to report preferences truthfully.

6. Conclusion

The random assignment problem has attracted attention in recent years. While the notion of *ordinal* efficiency is perhaps the most natural in many contexts (Bogomolnaia and Moulin, 2001; Yilmaz, 2010), in other contexts where utility is cardinal, ex-ante efficiency is a desired feature of a solution. The possibility of fractional endowment (Athanassoglou and Sethuraman, 2010) calls for a solution that is also individually rational and justified envy-free.

In this paper I provide a solution based on the concept of competitive equilibrium in an exchange economy that closely corresponds to the assignment problem with fractional endowment. I first show that, under certain conditions, a competitive equilibrium exists and its allocation can be used as a solution to the random assignment problem that satisfies ex-ante efficiency and individual rationality and justified envy-freeness. I also provide results when the conditions are relaxed, as well as several extensions of the model such as under- and over-capacity and preferences exhibiting indifferences.

The results regarding competitive equilibrium are then used to design a pseudo-market solution for the random assignment problem. Inheriting properties of competitive allocation, the solution is ex-ante efficient, individually rational, justified envy-free but is not incentive compatible.

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A. Appendix

A.1. Proof of lemma 1

I use the sequential characterization to show upper and lower hemi-continuity. For notational convenience I suppress the i-superscript that indexes the agent.

For upper hemi-continuity, suppose that the sequence (p^n) converges to p and the sequence $(q^n), q^n \in F(p^n, e, b)$ for each n, converges to q. I need to show that $q \in F(p, e, b)$. This is straightforward. Since $\sum_{h \in \mathcal{H}} q_h^n \leq 1$ for each n and $(q^n) \to q$, it must be that $\sum_{h \in \mathcal{H}} q_h \leq 1$. Similarly, since $q^n p^n \leq ep^n + b$, it must be that $qp \leq ep + b$.

For lower-hemicontinuity, suppose that (p^n) converges to p and $q \in F(p, e, b)$. I construct a sequence q^n such that $q^n \in F(p^n, e, b)$ for all n and (q^n) converges to q. If $q \in F(p^n, e, b)$ then let $q^n = q$. Otherwise, let $q^n = \frac{ep^n+b}{qp^n} q$. Since $q \notin F(p^n, e, b)$, $ep^n + b < qp^n$ and hence $\frac{ep^n+b}{qp^n} < 1$, so q^n is a valid sub-probability distribution. By construction the cost of the bundle q^n is exactly $ep^n + b$, so q^n is affordable. Therefore $q^n \in F(p^n, e, b)$.

Since $(p^n) \to p$, the ratio $\frac{ep^n+b}{qp^n} < 1$ converges to 1. So I have constructed a sequence (q^n) such that $q^n \in F(p^n, e, b)$ for all n and (q^n) converges to q. □

A.2. Proof of theorem 4

Step 1: A competitive equilibrium exists. By theorem 3, a competitive equilibrium with indivisibility, with price $p = (p_1, ...p_H)$ and a deterministic assignment m, exists. I now construct prices $\hat{p}$ to support it as a competitive equilibrium (allowing for divisibility) from the initial discrete endowment.

Relabel the houses so that $p_1 \leq p_2 \leq ... \leq p_H$. Partition $\mathcal{H}$ into disjoint sets of houses with the same price $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_T$ where $T \leq H$ so that $\mathcal{H}_1$ is the set of houses with the lowest price, $\mathcal{H}_2$ is the set of houses with the second-lowest price, and so on. Let $\hat{p}_1 = 0, \hat{p}_2 = 1$, and for any $2 \leq t \leq T$, construct $\hat{p}_t$ such that

$$\frac{\hat{p}_{t+1}}{\hat{p}_t} > \max_{i,i',i'' \in \mathcal{I}, h, h', h'' \in \mathcal{H}} \frac{u_h^i}{u_{h'}^{i'} - u_{h''}^{i''}}$$

Assign price $\hat{p}_t$ to the houses in $\mathcal{H}_t$. The idea behind this construction is to ensure that the prices are such that agent i can afford exactly the house m^i, and houses more preferred than m^i are too expensive and thus not optimal to consume. I now show that given these prices $\hat{p}$, m^i is agent i's optimal bundle even if he can consume random assignments.

Denote by $UCS^i(m^i) = \{h \in \mathcal{H} : u_h^i > u_{m^i}^i\}$ and $LCS^i(m^i) = \{h \in \mathcal{H} : u_h^i < u_{m^i}^i\}$ the upper and lower contour sets of agent i at house m^i. Consider any house $h \in USC^i(m^i)$. By definition of competitive equilibrium with indivisibility, $p_h > p_{m^i}$, so if m^i is in $\mathcal{H}_t$ and assigned price $\hat{p}_t$ then h is in $\mathcal{H}_{t'}$ where $t' > t$ and assigned price of at least $\hat{p}_{t+1}$. So the price of any house in $UCS^i(m^i)$ is at least $\hat{p}_{t+1}$.

Consider agent i's optimal bundle, given prices. If the bundle adds up to less than one, then the agent must be consuming shares of the house h^* with the maximum "bang for the buck" ratio, i.e., $h^* = \arg\max_{h \in \mathcal{H}} \frac{u_h^i}{\hat{p}_h}$. However, for all $h \in UCS$, $\frac{u_h^i}{\hat{p}_h} \leq \frac{u_h^i}{\hat{p}_{t+1}} < \frac{u_{m^i}^i}{\hat{p}_t}$ by construction. So the agent does not consume any house in UCS. Since he can afford m^i, he must consume house m^i entirely, contradicting the assumption that he is consuming less than one house in total.

Suppose now that the optimal bundle adds up to exactly one, but the agent is consuming from both $UCS^i(m^i)$ and $LCS^i(m^i)$.[6] This is not optimal since the following change is welfare-improving. Reduce consumption from $UCS^i(m^i)$ by small $\epsilon > 0$, increase consumption of m^i by $\frac{\hat{p}_{t+1}}{\hat{p}_t}\epsilon$, reduce consumption of $LCS^i(m^i)$ by $\frac{\hat{p}_{t+1}}{\hat{p}_t}\epsilon$. The resulting bundle is still feasible and affordable. The change in utility can be easily verified to be positive.

So for each agent i, given prices $\hat{p}$, m^i is his optimal bundle. $(\hat{p}, m)$ therefore constitutes a competitive equilibrium (allowing for divisibility).

Step 2: Competitive equilibrium allocation is unique. Consider any competitive allocation resulting from discrete endowment. Consider the set of houses with the lowest price. The agents endowed with these houses, by budget constraint, must consume from these houses exclusively. This implies that these agents exhaust these houses. Moreover, by strict preferences, their optimal bundles will be discrete, i.e., each agent chooses his most preferred house among these lowest-priced houses.

Now consider the houses with the second-lowest price, and the agents who own them. Since the lowest-priced houses have already been exhausted by their initial owners, these agents must not consume from the lowest-priced houses. The same logic as above then implies that they exhaust the second-lowest-priced houses, and that each agent's consumption is discrete. This line of reasoning eventually implies that the competitive equilibrium allocation must be a deterministic assignment. Therefore, any competitive allocation must be a deterministic assignment. But any competitive equilibrium with deterministic allocation is automatically a competitive equilibrium with indivisibility, which is unique. □

A.3. Proof of theorem 7

As mentioned in the main text, a competitive equilibrium with under-capacity (p^u, q^u) can also be thought of as a standard competitive equilibrium with the following modifications.

[6] Other cases are simple. Mixing between m^i and $UCS^i(m^i)$ is not affordable. Mixing between m^i and $LCS^i(m^i)$ is clearly sub-optimal.

- $F = I - H$ null houses are added to the set of houses and given prices of zero.
- Non-null houses have the same prices as p^u.
- Agents whose allocation at q^u is not a proper probability distribution add the appropriate amount of null house to their consumption to obtain a proper probability distribution.

This proof establishes the existence of competitive equilibrium for the economy with the above modifications.

With the null houses added, the fixed point still exists and the argument that there is no excess demand at the fixed point still holds. To argue that there is no excess supply takes more care. Suppose that there is excess supply. If all prices are non-zero, then excess supply implies equal excess supply of all houses (otherwise the price player will set some price to zero). This in turn implies that expenditure is less than one, so at least one agent doesn't exhaust his wealth and must therefore be consuming an entire unit of his most preferred house, contradicting excess supply. So some prices must be zero to sustain excess supply.

If a non-null house has zero price then all agents must be consuming a proper probability distribution since it is costless to buy this non-null house. I agents will then consume I houses, and no excess demand implies no excess supply. Therefore all non-null houses have non-zero prices.

One of the null houses must then have zero price. Other null houses have non-zero prices only if they are demanded more than the zero-price null house (otherwise the price player will set their prices to zero), which can never be true since all null houses are identically worthless to the agents. So all null houses have zero prices.

If the non-null houses have excess supply then, because their prices are all non-zero, they must have equal excess supply. This implies that expenditure on these houses are less than one. Expenditure on zero-priced null houses is of course zero. So total expenditure is less than one, so at least one agent doesn't exhaust his wealth and must therefore be consuming an entire unit of his most preferred house, contradicting excess supply of the non-null houses. □

A.4. Proof of theorem 8

As mentioned in the main text, a competitive equilibrium with over-capacity (p^o, q^o) is equivalent to a standard competitive equilibrium with the following modifications

- $F = H - I$ fictitious agents are added to the set of agents, each of whom is given a positive budget b_f and is indifferent among all houses.
- Prices are the same as p^o.
- Non-fictitious agents' allocations are the same as q^o.
- Fictitious agents consume the houses not consumed by the non-fictitious agents.

This proof establishes the existence of competitive equilibrium for the economy with the above modifications.

I first show that the demand of each fictitious agent is upper-semi continuous. There is probably a faster proof using the assumption that houses are identical to fictitious agents. The following proof follows the method of Hylland and Zeckhauser (1979). I drop the superscript agent index for notational ease.

Suppose $(p^n) \to p$ and $q^n \in \phi(p^n, e, b)$ for each n, and that $(q^n) \to q$. I need to show that $q \in \phi(p, e, b)$. Suppose in negation that $q \notin \phi(p, e, b)$. Since $\phi(p, e, b)$ is non-empty, I can choose $y \in \phi(p, e, b)$. There are two possible cases: either (1) $yu > qu$ or (2) $yu = qu$ and $yp < qp$.

In the first case, I construct a sequence (y^n) converging to y such that $y^n \in F(p^n, e, b)$ for all n. If $y \in F(p^n, e, b)$, let $y^n = y$. Otherwise, let $y^n = \alpha^n y$ where $\alpha^n = \frac{ep^n+b}{yp^n}$. I know that $ep + b > 0$. Also, $\lim_{n\to\infty} yp^n = yp \leq ep + b$, therefore $\lim_{n\to\infty} \alpha^n = 1$. So $(y^n) \to y$.

Since $yu > qu$, there must be some n such that $y^n u > q^n u$, but $y^n \in F(p^n)$, so q^n must not be an optimal choice, a contradiction.

In the second case, since $yp < qp$, there exists some n such that $yp^n < q^n p^n$. So $y \in F(p^n, e, b)$. $q^n \in \phi(p^n, e, b)$ then implies that $q^n u > yu$ (otherwise y would be chosen over q^n, being cheaper and at least as good as q^n). Since $yp < qp \leq ep + b$, there exists some $\lambda \in (0, 1]$ such that $((1-\lambda)y + \lambda q^n)p \leq ep + b$. So the bundle $((1-\lambda)y+\lambda q^n)$ is feasible at p, and gives higher utility than y, so y is not an optimal choice, a contradiction.

Therefore the demand of each fictitious agent is upper-semi continuous. Convexity is obvious. The aggregate demand is therefore also convex-valued and upper-semi continuous. By Kakutani's fixed point theorem, a fixed point of ϕ exists.

At the fixed point, suppose that there is excess demand for some houses. Maximum total demand cannot exceed H, so there must be excess supply for some houses. The price player must set the price of these houses to be zero. The fictitious agent, indifferent among all houses, will then minimize expenditure by consuming these zero-priced houses exclusively, spending zero as a result, so total expenditure come only from the real agents, but then there cannot be excess demand. Therefore there is no excess demand.

If there is a zero-priced house, then each agent consumes a proper probability distribution, and in total H house shares are demanded. No excess demand then implies that market clears exactly.

If all prices are strictly positive, then excess supply implies equal excess supply for every house (otherwise the price player will set some price to zero) and that aggregate expenditure is less than 1, but aggregate wealth is $1+Fb^f$, so at least one real agent does not exhaust his wealth. This implies that this agent consumes his most preferred house in its entirety, and contradicts excess supply. $\square$