A note on reserve price commitments in the Vickrey auction

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Abstract

This note provides a simple explanation why sellers rarely set optimal reserve prices in one-shot auctions. In a standard sealed-bid second-price auction, bidders with private values do not bid truthfully if the seller cannot commit to her announced reserve price. Consequently, expected revenue may be lower than without the announcement of a reserve price.

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1. Introduction

It is well known that the seller in a sealed-bid second-price (or Vickrey) auction maximizes her *ex ante* revenue by setting a positive reserve price (Myerson, 1981; Krishna, 2002). Doing so, she compensates the risk of not selling with an increased price in case the reserve lies between the highest and second highest bids. For bidders, truthful bidding remains an undominated strategy: Blume and Heidhues (2004) characterize all – symmetric and asymmetric – equilibria of the Vickrey auction and show that truthful bidding is a unique equilibrium if the seller imposes a strictly positive reserve price.

In real auctions, however, reserve prices seem unpopular. For example, governments rarely imposed significant reserve prices in recent spectrum licence auctions. Klemperer (2002, p.175) mentions the Swiss UMTS auction and some other "disasters" – auctions with disappointing revenues – and argues they could have been prevented had sellers imposed significantly high reserve prices. In the Swiss UMTS auction, the responsible authority initially withdrew the auction after a large number of bidders had dropped out on short notice. Following complaints from various parties, it had to carry out the auction with unchanged rules. Consequently, the general impression was that the auction finally conducted *had* to end with a sale. As a result, all bids exactly met the (rather low) reserve price and the auction was perceived as a failure (see Wolfstetter, 2003, for more details). Similar situations take place in *eBay* auctions of perishable goods, such as concert tickets, travel tickets or vouchers with limited validity:[1] the seller has to sell them before a certain date and bidders may exploit that fact in their bidding strategies.

In this note, we show that such problems can explain why sellers refrain from setting a reserve price in a one-shot Vickrey auction. In particular, we consider the case of a non-credible reserve price: because of the one-shot nature of the auction, the seller may be tempted *ex post* to accept the highest bid even if it is lower than the previously announced reserve price. As a result, bidders do not bid truthfully. A symmetric equilibrium is to bid the reserve if the valuation exceeds it by a small amount, and to bid the valuation otherwise. With such a non-credible reserve price, the seller's expected revenue may drop even below revenue in the case of Vickrey auction without a reserve price. The auction becomes inefficient.

To our knowledge, the case of a non-credible reserve price in the one-shot Vickrey auction has not been studied before. Instead, an *ex post* sale below the optimal reserve price has been modeled as resale and as negotiations. McAfee and Vincent (1995) introduce the possibility of costless resale in later auctions and show that bidders with valuations just above the reserve price wait to the next round, in which the reserve price is lower. Grant et al. (2002) and Menezes and Ryan (2005) study reserve price commitments for the case when sellers, after an unsuccessful auction round, have the option to re-auction or to negotiate and bidders are short-lived.

[1] Bajari and Hortaçsu (2003) demonstrate that the equilibrium in an *eBay* auction is formally equivalent to the standard equilibrium in a Vickrey auction.

2. Truthful bidding is not an equilibrium strategy

We normalize the value of the object to the seller to 0 and assume that there are n identical bidders with valuations which are their private information. Valuations are independently drawn from the distribution F with density on the normalized support $[0, 1]$.

Suppose the seller chooses a sealed-bid second-price auction and sets a positive reserve price r. However, if the reserve price is not reached, she still offers the object to the highest bidder for a price equal to the second highest bid.[2] If bidders are aware of this lack of commitment, those with valuations below r will also place bids, since they also have a chance to win. The price is then

$$P(n,r) = \begin{cases} b_{2:n} & \text{if} \quad b_{2:n} > r \\ r & \text{if} \quad b_{2:n} \leq r < b_{1:n} \\ b_{2:n} & \text{if} \quad b_{1:n} \leq r, \end{cases} \tag{1}$$

where $b_{i:n}$ is the i-th highest bid.

Truthful bidding, i.e. $b(x) = x$, is not a Nash equilibrium in this auction, since a profitable deviation exists. Consider a bidder with a valuation a little above r. If he bids truthfully, as everybody else, his expected payoff conditional on his valuation is

$$E\pi(x, b = x) = xG(x) - rG(r) - \int_r^x ydG(y), \tag{2}$$

where $G(y)$ is the cumulative distribution function of the highest of $n-1$ valuations, so that $G = F^{n-1}$. However, if he bids r and wins, he pays only the second highest bid and gains

$$E\pi(x, b = r) = xG(r) - \int_0^r ydG(y). \tag{3}$$

The deviation is profitable when $E\pi(x, b = r) - E\pi(x, b = x) > 0$, which can be simplified (cf. appendix A.1) to

$$\int_0^r (r - y)dG(y) > \int_r^x (x - y)dG(y). \tag{4}$$

This condition is always fulfilled for x close enough to r. For example, if F is uniform and $n = 2$, inequality (4) reduces to $x < 2r$, so all bidders with valuations from the set $(r, 2r)$ have an incentive to deviate from truthful bidding.

3. Equilibrium bidding

We propose the following symmetric Nash equilibrium bidding strategy. Bidders with valuations higher than r and lower than a threshold x_1 bid r and all other bidders bid

[2] We focus exclusively on reserve price commitment. An alternative analysis of seller commitment to auction format (for example a switch to a first-price auction in case the reserve price is not met) is outside the scope of this note.

truthfully (x_1 may also equal 1). An example of this pooling strategy is depicted in figure 1. Bidders with valuations below r are not affected and bid truthfully. Consider a bidder

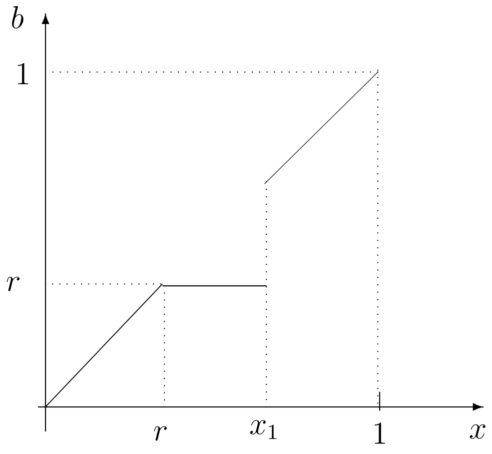

Figure 1: Equilibrium bid

with valuation $x \in (r, x_1)$. He bids r and wins in two cases. If all other bidders have valuations lower than r, the price of the auction equals the second highest valuation. That happens with probability $G(r)$. In the second case, some other bidders also bid r and he wins the tie.[3] That happens with probability

$$R(r, x_1, n) = \sum_{i=1}^{n-1} \binom{n-1}{i} \frac{1}{i+1} (F(x_1) - F(r))^i F(r)^{(n-1)-i}. \tag{5}$$

The price is then r. Hence, the expected profit of our bidder with valuation $x \in (r, x_1)$ is

$$E\pi(x, b = r) = x(G(r) + R(r, x_1, n)) - \int_0^r y dG(y) - rR(r, x_1, n), \tag{6}$$

Using this strategy, the bidder trades off the risk of losing in the tie for the chance of paying a price lower than r. A deviation to a bid higher than x or lower than r is obviously not profitable. A unilateral deviation to a bid $\tilde{b} \in (r, x)$ yields

$$E\pi(x, b = \tilde{b}) = G(x_1)(x - r), \tag{7}$$

because in case of winning the price is r for sure and he wins when all other bidders have valuations lower than x_1. When $E\pi(x, b = r) - E\pi(x, b = \tilde{b}) > 0$, such deviation is not profitable, which using definitions (6) and (7) reduces to

$$(x - r)(G(x_1) - R(r, x_1, n)) < xG(r) - \int_0^r y dG(y). \tag{8}$$

If $x = r$, the condition is trivially satisfied. Both sides of this inequality are linear in x, positively sloped and the left-hand side is steeper, since $G(r) < G(x_1) - R(r, x_1, n)$. They cut in only one point and according to its definition, this point should be x_1. Therefore,

[3] We assume that each bidder in the tie has an equal chance of winning.

we find x_1 by solving

$$(x_1 - r)(G(x_1) - R(r, x_1, n)) = \int_0^r (x_1 - y)dG(y). \tag{9}$$

In case of the uniform distribution and two bidders we have $x_1 = 2r$. For $n = 3$, x_1 decreases to $\frac{3}{2}r$ and when n grows to infinity, x_1 converges to r (cf. appendix A.2).

If $x_1 \geq 1$, all bidders with valuations $x > r$ bid the reserve price, since deviations are not profitable. If $x_1 < 1$, bidders with valuations $x > x_1$ bid truthfully. They expect

$$E\pi(x, b = x) = xG(x) - rG(x_1) - \int_{x_1}^x ydG(y). \tag{10}$$

Any deviation to a bid from the set $[x_1, 1]$ is weakly dominated by the truthful bid. A deviation to (r, x_1) does not decrease the price, but increases the probability of losing, so it cannot be profitable. Finally, a deviation to r is not profitable if (10) is greater than (6). Rearranging terms (cf. appendix A.3), we get

$$\int_0^r (x - y)dG(y) - \int_{x_1}^x (x - y)dG(y) < (x - r)(G(x_1) - R(r, x_1, n)). \tag{11}$$

This condition is fulfilled, since (8) implies that for $x > x_1$ the first integral on the left-hand side alone is lower than the right-hand side of the inequality, and because the second integral is positive.

Our last remark is on the seller's expected revenue. Only in one occasion, the sale price in our case is higher than in the standard Vickrey auction with $r = 0$: when the highest valuation ($Y_{1:n}$) is greater than x_1 and the second-highest ($Y_{2:n}$) lower than r. If $Y_{2:n} \in (r, x_1)$, the sale price is lower. In all other cases, it is the same. Therefore, the difference in expected revenue depends on how much weight distribution F puts on the set (r, x_1) relative to other valuations and how big this set is, which is determined by F and n. If the event $Y_{2:n} \in (r, x_1)$ is probable enough, expected revenue is lower than in the standard Vickrey auction. For example, this is the case when F is uniform and $n = 2$. Since the uniform distribution puts relatively little weight on middle values, more common distributions such as the Gaussian are more prone to this effect.

Summing up, this note provides an explanation why sellers do not have incentives to impose significant reserve prices in settings representable by a one-shot Vickrey auction. The seller's ability to contact the highest bidder after an unsuccessful auction can destroy her commitment and hence the value of a reserve price. As a final caveat, note that the equilibrium proposed here does not carry over to the English auction in a straightforward manner. We therefore believe that reserve price commitments in auctions deserve further study.

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A.1 Derivation of (4)

Using definitions (2) and (3), we rewrite

$$E\pi(x, b = r) - E\pi(x, b = x) > 0 \tag{A1}$$

as

$$xG(r) - \int_0^r ydG(y) - xG(x) + rG(r) + \int_r^x ydG(y) > 0. \tag{A2}$$

Since G is a probability distribution, it holds that

$$G(z) = \int_0^z dG(y), \tag{A3}$$

and substituting this in (A2) we have

$$x\int_0^r dG(y) - \int_0^r ydG(y) - x\int_0^x dG(y) + r\int_0^r dG(y) + \int_r^x ydG(y) > 0. \tag{A4}$$

Simplifying the above we get inequality (4).

A.2 Calculations of x_1 for the uniform distribution

If F is uniform, $R(r, x_1, n)$, which enters definition (6), becomes

$$\frac{r^{n-1}(nx_1 - r(n + (\frac{x_1}{r})^n - 1))}{n(r - x_1)}.$$

For $n = 3$, $R(r, x_1, n)$ reduces to

$$\frac{1}{3}(x_1 - r)(2r - x_1)$$

and equation (9) reduces to

$$\frac{x_1^3}{3} = x_1^2(x_1 - r),$$

which gives $x_1 = \frac{3}{2}r$.

A.3 Derivation of (11)

Using equations (10) and (6), the condition becomes

$$x(G(r) + R(r, x_1, n)) - \int_0^r ydG(y) - rR(r, x_1, n) < xG(x) - rG(x_1) - \int_{x_1}^x ydG(y). \tag{A5}$$

Using $\int_{x_1}^x xdG(y) = x(G(x) - G(x_1))$ and rearranging, we get

$$\int_0^r (x - y)dG(y) + (x - r)R(r, x_1, n) < (x - r)G(x_1) + \int_{x_1}^x (x - y)dG(y), \tag{A6}$$

which after further rearrangement reads as inequality (11).