Good Approximation of Exponential Utility Function for Optimal Futures Hedging

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Abstract: To get optimal production and hedging decision with normal random variables, Lien (2008) compares the exponential utility function with its second order approximation. In this paper, we first extend the theory further by comparing the exponential utility function with a n-order approximation for any integer n. We then propose an approach with illustration how to get the least n one could choose to get a good approximation.

Keywords: Exponential utility, optimal production, hedging, approximation

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1 Introduction

Using polynomials to approximate the expected utility function is one of the important issues, see, for example, Feldstein (1969), Samuelson (1970), Levy and Markowitz (1979), Pulley (1981), Kroll, Levy, and Markowitz (1984), and Hlawitschka (1994). To obtain the optimal production and hedging decision with normal random variables, Lien (2008) compares the exponential utility function with its second order approximation. In this paper, we extend the theory further by comparing the exponential utility function with a n-order approximation for any integer n. We then propose any approach with illustration how to get the least n one could choose to get a good approximation.

2 The Model

Suppose at time 0, a producer intends to produce q units that are planed to be sold at time 1. The production cost is $c(q)$. Supposing that there is no production risk, we assume that the price, $\tilde{p}$, at time 1 is a random variable following a normal distribution such that $\tilde{p} \sim N(\mu_p, \sigma_p^2)$. In addition, we assume that there is a corresponding futures contract that matures at time 1 with price b at time 0. We also assume the producer wants to hedge against the risk that that price of his/her produced goods may drop so that he/she sells h unit of products under the futures contract and he will deliver the h unit of products against the futures contract at time 1. Let $\tilde{\pi}$ be the profit at time 1, we have

$$\tilde{\pi} = \tilde{p}(q-h) + bh - c(q) \ . \tag{2.1}$$

We further assume that the hedger has an exponential utility function u such that

$$u(\tilde{\pi}) = -\exp(-k\tilde{\pi}) \quad \text{for} \quad k > 0 \ . \tag{2.2}$$

where k is the Arrow-Pratt risk aversion coefficient. Using this modeling setting, one could show that

$$E[u(\tilde{\pi})] = -\exp(-k\mu_\pi)\exp\left[\frac{1}{2}k^2(q-h)^2\sigma_p^2\right] \ , \tag{2.3}$$

where $\mu_\pi = \mu_p(q-h) + bh - c(q)$.

From the literature, such as..., it is known that the firm's optimal production decision q^* depends neither on the risk attitude of the firm nor on the distribution of the underlying price uncertainty. This is the result from the notable separation theorem. The firm's optimal production decision q^* is determined by solving $b = c'(q^*)$. When $b = \mu_p$, the optimal futures position will be equal to the optimal production decision q^*, that's, the firm should completely eliminate its price risk exposure by adopting a full-hedge. To explore the effect of any order approximation of exponential utility function, we follow Lien (2008) and allow $b \neq \mu_p$. We first discuss the second-order approximation in the next section.

3 Second-Order Approximation

Following Tsiang (1972), Gilbert et al. (2006), and Pulley (1981), Lien (2008) considers the following second-order approximation:

$$u_2^a(\tilde{\pi}) = u(\mu_\pi) + u^{(1)}(\mu_\pi)(\tilde{\pi} - \mu_\pi) + \frac{1}{2}u^{(2)}(\mu_\pi)(\tilde{\pi} - \mu_\pi)^2 \ , \qquad (3.1)$$

where $u^{(i)}$ is the i^{th} derivative of the utility function u. Then, one could show that:

$$E\big[u_2^a(\tilde{\pi})\big] = -\exp(-k\mu_\pi)\left[1 + \frac{1}{2}k^2(q-h)^2\sigma_p^2\right] \ . \qquad (3.2)$$

Let (q, h_2) and (q, h_0) be the optimal production levels and futures positions that maximize $E\big[u_2^a(\tilde{\pi})\big]$ and $E\big[u(\tilde{\pi})\big]$ in (3.2) and (2.3), respectively. Lien (2008) shows that if $b > \mu_p$, $q < h_0 < h_2$ and if $b < \mu_p$, $q > h_0 > h_2$. In other words, the deviation between the optimal production level and the optimal futures position under the second-order approximation is always smaller than that under the exponential utility framework.

4 $2n$-Order Approximation

We first extend Lien (2008)'s results to fourth-order approximation and replace the utility function $u_2^a(\tilde{\pi})$ in (3.1) by the following fourth-order approximation:

$$\begin{aligned} u_4^a(\tilde{\pi}) &= u(\mu_\pi) + u'(\mu_\pi)(\tilde{\pi} - \mu_\pi) + \frac{1}{2}u''(\mu_\pi)(\tilde{\pi} - \mu_\pi)^2 \\ &\quad + \frac{1}{3!}u'''(\mu_\pi)(\tilde{\pi} - \mu_\pi)^3 + \frac{1}{4!}u'''(\mu_\pi)(\tilde{\pi} - \mu_\pi)^4. \end{aligned}$$

Then, it can be shown that:

$$E\big[u_4^a(\tilde{\pi})\big] = -\exp(-k\mu_\pi)\left[1 + \frac{1}{2}k^2(q-h)^2\sigma_p^2 + \frac{1}{4!}k^4(q-h)^4K_p\right] ,$$

where $K_p = E(\tilde{\pi} - \mu_p)^4$. For normal distribution, we have $K_p = 3\sigma_p^4$. Thus, we can get

$$E\big[u_4^a(\tilde{\pi})\big] = -\exp(-k\mu_\pi)\left[1 + \frac{1}{2}k^2(q-h)^2\sigma_p^2 + \frac{1}{8}k^4(q-h)^4\sigma_p^4\right] .$$

Let (q, h_4) be the optimal production level and futures position that maximizes $E\big[u_4^a(\tilde{\pi})\big]$. Its first-order condition is:

$$(b-\mu_p)\left[1 + \frac{1}{2}k^2(q-h)^2\sigma_p^2 + \frac{1}{8}k^4(q-h)^4\sigma_p^4\right] + k(q-h)\sigma_p^2 + \frac{1}{2}k^3(q-h)^3\sigma_p^4 = 0.$$

For h_2 that maximizes $E\big[u_2^a(\tilde{\pi})\big]$, we have the following equation:

$$(b-\mu_p)\left[1 + \frac{1}{2}k^2(q-h)^2\sigma_p^2\right] + k(q-h)\sigma_p^2 = 0.$$

From this equation, we can get

$$b - \mu_p = \frac{-k(q-h_2)\sigma_p^2}{1+\frac{1}{2}k^2(q-h_2)^2\sigma_p^2} \ . \tag{4.1}$$

Define $M(h) = (b-\mu_p)\left[1+\frac{1}{2}k^2(q-h)^2\sigma_p^2 + \frac{1}{8}k^4(q-h)^4\sigma_p^4\right] + k(q-h)\sigma_p^2 + \frac{1}{2}k^3(q-h)^3\sigma_p^4$ and incorporate equation (4.1) into the formula of $M(h)$, we get:

$$\begin{aligned} M(h_2) &= (b-\mu_p)\frac{1}{8}k^4(q-h_2)^4\sigma_p^4 + \frac{1}{2}k^3(q-h_2)^3\sigma_p^4 \\ &= \frac{1}{2}k^3(q-h_2)^3\sigma_p^4\left[\frac{k(b-\mu_p)(q-h_2)}{4}+1\right] \\ &= \frac{1}{2}k^3(q-h_2)^3\sigma_p^4 \times \frac{4+k^2(q-h_2)^2\sigma_p^2}{4+2k^2(q-h_2)^2\sigma_p^2}. \end{aligned}$$

Thus, we have $sign\big[M(h_2)\big] = sign\big(q-h_2\big)$. Furthermore, from equation (4.1), we obtain the result that when $b > \mu_p$, $q < h_2$ which, in turn, implies that $M(h_2) < 0$. On the other hand, by definition, we know that $M(h_4) = 0$ and we obtain the following proposition:

Proposition 4.1. *In the above-mentioned model-setting, we have*

a. if $b > \mu_p$, then $h_2 > h_4$, and

b. if $b < \mu_p$, then $h_2 < h_4$.

We now ready to develop the theory for the general situation with $n \geq 2$ for any integer n. Consider the following $2n-$order approximation of the exponential utility function u in (2.2):

$$u_{2n}^a(\tilde{\pi}) = u(\mu_\pi) + u'(\mu_\pi)(\tilde{\pi}-\mu_\pi) + \frac{1}{2}u''(\mu_\pi)(\tilde{\pi}-\mu_\pi)^2 + \cdots$$

$$+\frac{1}{(2n)!}u^{(2n)}(\mu_\pi)(\tilde{\pi}-\mu_\pi)^{2n} \ . \tag{4.2}$$

Take expectation, we get:

$$E\big[u^a_{2n}(\tilde{\pi})\big] = -\exp(-k\mu_\pi)\left[1+\frac{1}{2}k^2(q-h)^2\sigma_p^2+\cdots+\frac{1}{(2n)!}k^{2n}(q-h)^{2n}M_{2n}\right] ,$$

where $M_{2n}=E(\tilde{p}-\mu_p)^{2n}$. Under the assumption of normal distribution, we obtain $M_{2n}=(2n-1)!!\sigma_p^{2n}$. Substituting this into the above equation, we obtain:

$$E\big[u^a_{2n}(\tilde{\pi})\big] = -\exp(-k\mu_\pi)\left[1+\frac{1}{2}k^2(q-h)^2\sigma_p^2+\cdots+\frac{1}{(2n)!!}k^{2n}(q-h)^{2n}\sigma_p^{2n}\right].$$

Let (q,h_{2n}) be the optimal production level and futures position that maximizes $E\big[u^a_{2n}(\tilde{\pi})\big]$. The corresponding first-order condition is:

$$\begin{aligned} V(h) &= (b-\mu_p)\left[1+\frac{1}{2}k^2(q-h)^2\sigma_p^2+\cdots+\frac{1}{(2n)!!}k^{2n}(q-h)^{2n}\sigma_p^{2n}\right] \\ &\quad +k(q-h)\sigma_p^2+\cdots+\frac{1}{(2n-2)!!}k^{2n-1}(q-h)^{2n-1}\sigma_p^{2n}=0. \end{aligned}$$

For h_{2n-2}, the following equation holds:

$$\begin{aligned} &(b-\mu_p)\left[1+\frac{1}{2}k^2(q-h_{2n-2})^2\sigma_p^2+\cdots+\frac{1}{(2n-2)!!}k^{2n-2}(q-h_{2n-2})^{2n-2}\sigma_p^{2n-2}\right] \\ &\qquad +k(q-h_{2n-2})\sigma_p^2+\cdots+\frac{1}{(2n-4)!!}k^{2n-3}(q-h_{2n-2})^{2n-3}\sigma_p^{2n-2}=0. \end{aligned}$$

From this equation, we can get

$$b-\mu_p=-\frac{k(q-h_{2n-2})\sigma_p^2+\cdots+\frac{1}{(2n-4)!!}k^{2n-3}(q-h_{2n-2})^{2n-3}\sigma_p^{2n-2}}{1+\frac{1}{2}k^2(q-h_{2n-2})^2\sigma_p^2+\cdots+\frac{1}{(2n-2)!!}k^{2n-2}(q-h_{2n-2})^{2n-2}\sigma_p^{2n-2}}$$

Plugging this equation in the formula of $V(h)$, we get:

$$\begin{aligned}
V(h_{2n-2}) &= (b-\mu_p)\frac{1}{(2n)!!}k^{2n}(q-h_{2n-2})^{2n}\sigma_p^{2n} + \frac{1}{(2n-2)!!}k^{2n-1}(q-h_{2n-2})^{2n-1}\sigma_p^{2n} \\
&= \frac{1}{(2n-2)!!}k^{2n-1}(q-h_{2n-2})^{2n-1}\sigma_p^{2n}\left[\frac{k(b-\mu_p)(q-h_{2n-2})}{2n}+1\right] \\
&= \frac{1}{(2n-2)!!}k^{2n-1}(q-h_{2n-2})^{2n-1}\sigma_p^{2n} \times \\
& \quad \frac{2n+(n-1)k^2(q-h_{2n-2})^2\sigma_p^2+\cdots+\frac{2}{(2n-2)!!}k^{2n-2}(q-h_{2n-2})^{2n-2}\sigma_p^{2n-2}}{2n+nk^2(q-h_{2n-2})^2\sigma_p^2+\cdots+\frac{2n}{(2n-2)!!}k^{2n-2}(q-h_{2n-2})^{2n-2}\sigma_p^{2n-2}}.
\end{aligned}$$

Thus, we have $sign\big[V(h_{2n-2})\big] = sign\big(q-h_{2n-2}\big)$. Furthermore, from equation (4.3), we obtain the result that when $b>\mu_p$, $q<h_{2n-2}$, which leads to $V(h_{2n-2})<0$. By definition, $V(h_{2n})=0$, and thus, we can conclude that $h_{2n-2}>h_{2n}$ when $b>\mu_p$. Similarly, it can be shown that when $b<\mu_p$, we can have $h_{2n-2}<h_{2n}$. We summarize the results in the following proposition:

Proposition 4.2. *In the above-mentioned model-setting, we have*

a. if $b>\mu_p$, then $h_2>h_4>\cdots>h_{2n}$, and

b. if $b<\mu_p$, then $h_2<h_4<\cdots<h_{2n}$.

5 $2n$-Order Approximation and the True Value

We turn to compare the $2n$-order approximation with the true value. To do so, we first compare with the true utility function.

$$E[u(\tilde{\pi})] = -\exp(-k\mu_\pi)\exp[\frac{1}{2}k^2(q-h)^2\sigma_p^2].$$

let (q, h_0) be the optimal production level and futures position that maximizes $E[u(\tilde{\pi})]$. In this case, the objective function can be simplified to $\mu_\pi - (1/2)k\sigma_\pi^2$ and the first-order condition is

$$(b - \mu_p) + k(q - h)\sigma_p^2 = 0.$$

Note that we can rewrite $V(h)$ as follows:

$$\begin{aligned} V(h) &= (b-\mu_p)\left[1 + \frac{1}{2}k^2(q-h)^2\sigma_p^2 + \cdots + \frac{1}{(2n)!!}k^{2n}(q-h)^{2n}\sigma_p^{2n}\right] \\ &\quad + k(q-h)\sigma_p^2\left[1 + \cdots + \frac{1}{(2n-2)!!}k^{2n-2}(q-h)^{2n-2}\sigma_p^{2n-2}\right] \\ &= \left[(b-\mu_p) + k(q-h)\sigma_p^2\right]\left[1 + \cdots + \frac{1}{(2n-2)!!}k^{2n-2}(q-h)^{2n-2}\sigma_p^{2n-2}\right] \\ &\quad + (b-\mu_p)\frac{1}{(2n)!!}k^{2n}(q-h)^{2n}\sigma_p^{2n}. \end{aligned}$$

As a result, we can have

$$V(h_0) = (b-\mu_p)\frac{1}{(2n)!!}k^{2n}(q-h_0)^{2n}\sigma_p^{2n}.$$

Consequently, sign$V(h_0)$=sign$(b - \mu_p)$. This implies that when $b > \mu_p$, $V(h_0) > 0$. By definition, $V(h_{2n}) = 0$, and thus, we can conclude that $h_0 < h_{2n}$ when $b > \mu_p$. Similarly, it can be shown that when $b < \mu_p$, we have $h_0 > h_{2n}$. We summarize the results in the following proposition:

Proposition 5.1. *In the above-mentioned model-setting, we have*

a. if $b > \mu_p$, then $h_2 > h_4 > \cdots > h_{2n} > h_0$, and

b. if $b < \mu_p$, then $h_2 < h_4 < \cdots < h_{2n} < h_0$.

6 Good Approximation

We now propose an approach to find the least n one could choose to get a good approximation. To do so, we first consider the situation in which $n \to \infty$. Since it is well known that $(2n)!! = 2^n n!$, we can rewrite the $2n-$order approximation to be:

$$E\big[u_{2n}^a(\tilde{\pi})\big] = -\exp(-k\mu_\pi)\left[1 + \frac{1}{2}k^2(q-h)^2\sigma_p^2 + \cdots + \frac{1}{n!2^n}k^{2n}(q-h)^{2n}\sigma_p^{2n}\right].$$

Take limit to both sides of the above equation, we get

$$\begin{aligned}
\lim_{n\to\infty} E\big[u_{2n}^a(\tilde{\pi})\big] &= -\exp(-k\mu_\pi)\lim_{n\to\infty}\Bigg[1 + \frac{1}{2}k^2(q-h)^2\sigma_p^2 \\
&\quad + \cdots + \frac{1}{n!2^n}k^{2n}(q-h)^{2n}\sigma_p^{2n}\Bigg] \\
&= -\exp(-k\mu_\pi)\exp\left[\frac{1}{2}k^2(q-h)^2\sigma_p^2\right] = E\big[u(\tilde{\pi})\big].
\end{aligned}$$

Thus, we can conclude that $h_{2n} \to h_0$. Together with the Cauchy convergence principle, we summarize all the above results in the following theorem:

Theorem 6.1. *Let $\tilde{\pi}$ defined in (2.1) be the profit at time 1 and q be the optimal production level and suppose that h_0 and h_{2n} be the optimal futures position that maximizes $E[u(\tilde{\pi})]$ and $E\big[u_{2n}^a(\tilde{\pi})\big]$ in which u and u_{2n}^a are defined in (2.2) and (4.2), respectively. Under the assumption stated in Section 2, for any integer n, we have*

a. if $b > \mu_p$, then $h_2 > h_4 > \cdots > h_{2n} > h_0$, and

b. if $b < \mu_p$, then $h_2 < h_4 < \cdots < h_{2n} < h_0$.

c. $h_{2n} \to h_0$ *for any* $n \to \infty$*, and*

d. for any $\alpha > 0$*, there exists* N *such that for all* $n > N$*,* $|h_{2n} - h_{2(n-1)}| < \alpha$*.*

Thus, to get a good approximation for $E[u(\tilde{\pi})]$, one may apply part (d) of Theorem 6.1, and decide the level of tolerance, $\alpha > 0$, and compute h_{2n} and $h_{2(n-1)}$ and thereafter get $|h_{2n} - h_{2(n-1)}|$ and choose n if one finds that $|h_{2n} - h_{2(n-1)}| < \alpha$.

7 Illustration

Now we present an example to illustrate our Theorem 6.1. Consider $\tilde{p} \sim N(1,1)$, $u(\tilde{\pi}) = -\exp(-\tilde{\pi})$. That's, we take $\mu_p = \sigma_p = k = 1$. Thus $b = c'(q)$ and $h_0 = q - (k\sigma_p^2)^{-1}(\mu_p - b) = q + b - 1$. While for h_2, it's the solution to the following equation:

$$(b - \mu_p)\left[1 + \frac{1}{2}k^2(q-h)^2\sigma_p^2\right] + k(q-h)\sigma_p^2 = 0.$$

The above equation can also be rewritten as follows:

$$(b-1)\left[1 + \frac{1}{2}(q-h)^2\right] + (q-h) = 0.$$

Solving the above quadratic equation, we can get the solution

$$q - h_2 = \frac{-1 \pm \sqrt{1 - 2(b-1)^2}}{b-1}.$$

Now let $b = 1.5 > 1 = \mu_p$, then

$$q - h_2 = -2 \pm \sqrt{2}.$$

Notice that the second order condition asks that

$$-(b-1)(q-h_2) - 1 < 0.$$

Thus we can conclude that

$$q - h_2 = -2 + \sqrt{2}.$$

Thus $h_2 - h_0 = 1.5 - \sqrt{2} > 0.05$.

Now assume that $b = 0.5 < 1 = \mu_p$, then

$$q - h_2 = 2 \pm \sqrt{2}.$$

According to the second order condition, we can finally obtain that

$$q - h_2 = 2 - \sqrt{2}.$$

Thus $h_0 - h_2 = 1.5 - \sqrt{2} > 0.05$. In both cases, $|h_0 - h_2| > 0.05$.

8 Concluding Remarks

The findings in our paper draw several inference. First, it is generally known that normal distribution coupled with exponential expected utility produces a mean-variance approach. We also know that, a quadratic approximation also leads to a mean-variance approach. In this paper, we find that the result of the exponential expected utility as shown in Section 2 is different from that of the quadratic approximation as shown in Section 3. Thus, the findings in our paper imply that the mean-variance approach generated from using normal distribution coupled with exponential expected utility is different from that generated from using normal distribution coupled with a quadratic utility.

Lastly, Hlawitschka (1994) argues that the usefulness of Taylor series approximations is strictly an empirical issue unrelated to the convergence properties of the infinite series, and, most importantly, that even for a convergent series adding more terms does not necessarily improve the quality of the approximation. We note that our finding suggests the argument from Hlawitschka (1994) may not be correct because in our case adding more terms does improve the quality of the approximation and actually when the number of terms increases, the approximation converges to the true value.

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