Networks, Frictions, and Price Dispersion

Donna, Javier and Schenone, Pablo and Veramendi, Gregory

The Ohio State University, Arizona State University, Arizona State University

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Javier Donna — The Ohio State University
Pablo Schenone — Arizona State University
Gregory Veramendi — Arizona State University

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Abstract

This paper models frictions in buyer-seller markets using networks, where buyers are linked with a subset of sellers and sellers are linked with a subset of buyers. Sparser networks are associated with higher search frictions. We use the model to characterize pairwise stable allocations and their supporting prices. Our approach allows for network effects, where a buyer who is not linked to a seller affects the price obtained by that seller. Network effects generate the central finding of our paper: even relatively sparse networks lead to price distributions and allocations that are close to the perfectly competitive outcome where the law of one price holds. We then investigate the role of network effects in a dynamic setting by studying wages in the context of an on-the-job search model. We find two novel predictions relative to the search literature. Lowering frictions (so that workers receive job offers at a higher rate) leads to: (1) *lower* worker mobility and *lower* expected wage growth and (2) *lower* expected wages in markets with high unemployment. We argue that our framework is suited to the analysis of a wide range of real-world markets, such as the labor market and buyer-seller trading platforms like eBay or Amazon.

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Javier Donna
Department of Economics
The Ohio State University
1945 N High St
417 Arps Hall
Columbus, OH 43210
Phone: 614-688-0364
Email: donna.1@osu.edu

Pablo Schenone
Department of Economics
W.P. Carey School of Business
Arizona State University
501 E. Orange Street
Tempe, AZ 85287
Phone: 480-965-5596
Email: pablo.schenone@asu.edu

Gregory Veramendi
Department of Economics
W.P. Carey School of Business
Arizona State University
501 E. Orange Street
Tempe, AZ 85287
Phone: 480-965-0894
Email: gregory.veramendi@asu.edu

1 Introduction

Price dispersion is observed in many markets even after accounting for observable characteristics. Examples include labor markets, where similar workers are paid different wages (Mortensen 2005); buyer-seller trading platforms such as eBay or Amazon, where identical goods are sold by the same seller at different prices (Einav, Kuchler, Levin, and Sundaresan 2015); and markets for automobiles, where identical automobiles are sold at different prices by the same dealer (Morton, Zettelmeyer, and Silva-Risso 2001). A common characteristic of these markets is that buyers interact with multiple, but not all, sellers. In the case of labor markets, firms interview multiple applicants. Bidders on eBay bid in multiple auctions for an identical product. Consumers visit multiple automobile dealerships before making a purchase. A natural question arises: How does the fact that buyers interact with multiple sellers affect price dispersion?

The central insight of this paper is that "indirect competition" plays an important role in determining price dispersion. For example, consider the case of two eBay auctions where there is only one common bidder participating in both auctions. The bidders in one auction *indirectly compete* with the bidders in the other auction because the two auctions are connected through the common bidder.[1] Allowing for indirect competition results in what we call "network effects": an interdependence in the prices between these two auctions caused by indirect competition. Even if many sellers are not directly competing for the same buyer, networks effects can equalize prices across them. How buyers and sellers are connected (linked) determines the extent of the network effects and motivates the use of networks for our analysis. We study prices and allocations in a network using pairwise stability as our matchmaking criterion since this is the weakest criterion for matchmaking that is consistent with Pareto efficiency. To the best of our knowledge, there has been no attempt to study price dispersion in networks using pairwise stability as the matchmaking criterion.[2] See Section 2 for a detailed literature review.

In this paper, we use networks to model buyer-seller markets where a buyer can obtain a good from the seller only if the two are linked. Perfect competition assumes that all buyers are linked to all sellers in the network, leading to the Walrasian outcome. Frictions are present in the market whenever there is at least one seller that is not linked to every buyer. Hence, the level of frictions in the network is determined by the total number of links or sparsity of the network. In this setting, we characterize pairwise stable allocations restricted to the network and the set of prices that sustain them. While we characterize allocations and prices, computing these for a large network is not tractable. Thus, we develop a computationally tractable algorithm that finds the upper and lower bounds of the set of prices that sustain any pairwise stable match.

[1] See section 3 for an example with buyers and sellers.

[2] The only other paper that we are aware that uses pairwise stability as the matchmaking criterion in a network with frictions is Elliott (2015).

A central finding of our paper is that even relatively sparse networks lead to price distributions and allocations that are close to the perfectly competitive outcome.[3] For example, in a network with 10,000 sellers, over 99% of the sellers are paid the same price when less than 0.1% of the possible links are active. The prediction that even sparse networks lead to price profiles that are close to the perfectly competitive outcome is a consequence of network effects. As the number of links increases, indirect competition among buyers rapidly becomes more likely. This indirect competition pushes towards price equalization in markets with a homogeneous good. Indirect competition makes the market look "as if" it was perfectly competitive, hence the result.

We then investigate the role of indirect competition in a dynamic setting by studying wages in the context of an on-the-job search model.[4] In a standard on-the-job search model, employed workers can search for better job offers, while firms cannot search to replace a worker with another one at lower wage. This creates an asymmetry where employed workers always benefit from lower frictions through higher wage growth (workers receive offers at a higher rate) and higher wages in the overall economy (Pissarides, 1990). An implication is that new search technologies (*e.g.* the internet) promise wage gains to workers. In addition, labor market policies designed to reduce frictions (*e.g.* job search assistance programs[5]) are intended to not only help workers find jobs faster, but also help them attain higher wages. Our model predicts that workers do not always benefit from reducing frictions due to network effects. The benefit varies depending on whether the markets are tight (fewer workers than open vacancies at firms) or loose (more workers than vacancies). In loose markets, lowering frictions improves the outside options of firms. Although we include the same asymmetry as in on-the-job search models, firms receive multiple links and are more likely to have a second link with an unemployed worker. This drives expected wages down. In turn, employed workers are more likely to compete with unemployed workers, limiting wage growth and job-to-job mobility. In tight markets, workers benefit from reduced frictions as in a standard search model. Firms compete more strongly for workers, driving wages up. Workers have better outside options and receive a higher initial wage out of unemployment, but this leaves little room for wage growth. Likewise, since workers start off with a higher wage, they have a lower probability of moving to a job that pays more. Hence, even in tight markets, lowering frictions diminishes wage growth and job-to-job mobility. Although the asymmetry

[3] Our finding that pricing behavior in networks with frictions closely resembles the perfectly competitive outcome is consistent with the pricing behavior observed in the laboratory experiments in Charness, Corominas-Bosch, and Fréchette (2007) and Gale and Kariv (2009). See Judd and Kearns (2008) for a survey of experiments in networked markets.

[4] Search models can be mapped into the corresponding network using the same firms and workers where each worker receives a link from a firm when they receive a job offer from that firm in the search model. Many search models are in continuous time. Hence, decreasing frictions in these models increases the offer arrival rates to workers. Yet, since these models are in continuous time, at any instant they only receive one offer.

[5] Gautier, Muller, van der Klaauw, Rosholm, and Svarer (2014) estimate the equilibrium effects of a job search assistance program in Denmark.

between workers and firms is still present in our model, network effects mitigate its impact in loose markets. Hence, our model makes new predictions about the effect of technologies and policies that affect the level of frictions in the economy.

Our paper contributes to the literature on networks. The networks literature typically proposes a concrete buyer-seller game to be played within these networks and identifies conditions under which the equilibria of these games are Pareto efficient.[6] In contrast, we use pairwise stability as the matchmaking criterion and focus on how the distribution of prices depends on the level of frictions in the market. Using pairwise stability as our matchmaking criterion implies that the equilibrium of *any* game that is consistent with Pareto efficiency will be included in our set of matches.

In summary, we make four main contributions: (1) we characterize the set of prices that sustain any pairwise stable matching in an unrestricted network; (2) we develop an algorithm for finding a stable matching and its supporting prices that is tractable for large networks; (3) we study price dispersion in arbitrary sparse networks; and (4) we study wage dispersion and wage growth in an application to labor markets.

Outline of the Paper

The theoretical analysis begins by studying arbitrary buyer-seller networks. These networks are exogenously formed by linking buyers and sellers, but no restrictions are placed on how many sellers a buyer can be linked with. Sellers offer one unit of an indivisible good. Buyers have single unit demand and also differ in their valuation of the good. To keep the model simple, we focus on the homogeneous goods case, where all sellers have the same valuation for the good. Buyers' utility is their valuation less the price if they obtain the good and zero otherwise. Sellers' utility is the price they are payed if they sell the good and their valuation otherwise.

In the model, we assume matches form according to pairwise stability restricted to the network. That is, a buyer obtains a good from a seller at a price p if four conditions hold: first, the buyer and the seller are connected in the network; second, there is no other seller linked to this buyer that is willing to sell at a price lower than p; third, there is no other buyer linked to the seller that is willing to pay a price higher than p; finally, the price p lies between the seller's valuation and the buyer's valuation. This is the weakest criterion for matchmaking that is consistent with Pareto efficiency.

The main theorem characterizes the set of all pairwise stable matchings in an arbitrary network and the set of prices that sustain them. To do that we decompose the original network into a network of fully connected subnetworks. With this, we identify two components that jointly determine the prices that sustain pairwise stable matchings: a pure competitive

[6] See for example Kranton and Minehart (2001), Corominas-Bosch (2004), Manea (2011), Gautier and Holzner (2013), and Elliott (2014). Elliott (2015) uses pairwise stability as the matchmaking criterion in a network with frictions, but does not investigate price dispersion.

component and an outside option component. Fully connected subnetworks are competitive markets with a unique price. The links between these subnetworks introduce an outside option component since a buyer from one subnetwork might choose to participate in another. The final prices that sustain a stable matching can be thought of as the result of these two effects. The theorem is useful because it allows us to rationalize any pairwise stable matching as an outcome of a decentralized market. In particular, we show that the outcome of a mechanism that generates any pairwise stable matching is equivalent to the outcome of a set of independent second-price auctions. We show how one can construct a set of independent auctions for any pairwise stable matching.

We use our characterization of the prices that support pairwise stable matchings to design an algorithm which we use to simulate large markets. For any given pairwise stable matching, calculating the set of prices that sustain it is simple whenever the network is small (*e.g.* 2 sellers and 3 buyers). However, since the applications of interest (the labor market, internet auctions) involve large economies, calculating the set of sustaining prices is intractable. Hence, we design an algorithm that outputs a matching (*i.e* a complete specification of buyer-seller matches) that is pairwise stable, and the prices that sustain it.

The algorithm we use to simulate the model is a deferred acceptance algorithm that works in two stages. In the first stage, one side of the market (*e.g.* sellers) hold "auctions" and the other side (*e.g.* buyers) sequentially "bid" in their linked auctions. Once no bidder wants to make any new bids, the algorithm ends. A corollary of the main theorem is that the outcome of the algorithm is indeed a stable matching. The second stage of the algorithm calculates two sets of prices: (1) the minimum prices that just price out the unmatched agents and support the stable matching; and (2) the maximum prices that give all matched agents nonnegative utility and also support the stable matching. In this way, the second stage finds the lower and upper bounds of the set of prices that support the pairwise-stable matching from the first stage.

We simulate a range of networks to obtain predictions about price distributions. We start with a set of heterogeneous buyers and homogeneous sellers. We parameterize the level of frictions by choosing the number of links per buyer in the network. This determines the total number of links in the network. The simulation then randomly draws links between sellers and buyers. After the network is realized, we run the algorithm and generate a price profile. Given that our algorithm can be applied to arbitrary network structures and it is computationally tractable for both small and large markets, our methodology is applicable to wide range of empirical settings (*e.g.* labor markets, online buyer-seller platforms, automobile markets, *etc.*).

We adapt the buyer-seller model to the labor market to explore questions about wage dispersion and growth. In this case, workers are sellers and firms are buyers of their services. To study wage growth, we extend our model to accommodate multiple periods. In this extension each period has three stages. In the first stage, J new firms enter the market and links are formed with the employed and unemployed workers. The parameters J and the number of links determine the market tightness (ratio of J to unemployed workers, denoted by θ) and the level of frictions. Firms that are employing a worker from a previous period do not receive any new links but retain the link to their employee. In the second stage, firm-worker matches are formed given the new network as in the basic buyer-seller model. Applying the buyer-seller model implies that workers accept the vacancy that pays the highest wage and hence, do not consider other aspects of the match, such as future wage growth (see section 4.4 for more on this point). Finally, at the end of the period, some matches are randomly destroyed. The firms that are unmatched at the end of a period (either because the match was destroyed or they could not form a match in the first place) exit the market. We interpret "firms" as time sensitive vacancies so that, if by the end of a period, a vacancy is not filled, it disappears from the job market. When the next period starts, J new firms enter.

The rest of the paper is organized as follows. In the next Section we describe the related literature and highlight how our paper contributes to the current body of work. In Section 3, we present two motivating examples. In Section 4 we describe the model. Section 5 presents the deferred acceptance algorithm. In Section 6 we describe the results of the simulation. Finally, in Section 7, we discuss how our results can be interpreted in the context of eBay auctions and labor markets, and how they can be used to develop a framework for robust econometric analysis. All proofs are in the appendix.

2 Contributions and Related Literature

Our paper contributes to the literatures on network theory, search and matching, and competing auctions. Most matching market models do not consider frictions nor dynamics in their analyses. While search models feature both dynamics and frictions, they severely restrict the competition between workers and firms. Competitive auction models are either frictionless or restrict the competition between auctions.

In network theory, the closest related papers to ours are Kranton and Minehart (2001), Corominas-Bosch (2004), Manea (2011), Gautier and Holzner (2013), Elliott (2014), and Elliott (2015).[7] These papers analyze static models where multiple buyers negotiate with multiple sellers.[8] Kranton and Minehart (2001) study efficiency when using a public ascending auction to clear the market. They do this in two steps: first, for each exogenously given

[7]There is a vast literature in economics and sociology that studies information transmission in social networks (for example, see Myers and Shultz 1951; Rees 1966; Montgomery 1991; Calvó-Armengol and Jackson 2004, 2007; and the references there). Pairwise stability has been used to study network formation (*e.g.* Jackson and Wolinsky 1996). We do not study information transmission nor network formation. In contrast, we use pairwise stability as the matchmaking criterion in a network with frictions. We characterize pairwise stable allocations and the set of prices that sustain them in networked markets. We then use this characterization to study price dispersion in these networks. See Jackson (2008) for a detailed review of the literature on social and economic networks.

[8]Fainmesser (2012) analyzes repeated games in networks. Galeotti (2010) investigates the role of communication on consumer's search and firms' pricing behavior in networked markets.

network, they study conditions under which the equilibrium outcome is efficient; second, they study endogenous network formation and prove that the endogenously formed networks satisfy their conditions for efficiency. Corominas-Bosch (2004) examines the equilibrium payoffs of an alternating offers game to answer two questions: what is the set of networks that supports a specific allocation (similar to what we call the "Walrasian outcome"), and what are the networks that only support this allocation. Manea (2011) investigates bilateral bargaining in networks. Gautier and Holzner (2013) study efficient allocations in arbitrary bipartite graphs by studying the set of maximal matchings. Elliott (2014) studies the efficiency of the entry decisions of firms and workers in a labor market model.[9] Elliott (2015) extends the Kranton and Mineheart model to consider different levels of bargaining power, different cost shares, negotiated investments, and ex-ante heterogeneous gains from trade.

While the use of networks to model frictions in an economy is common, both the question we seek to answer, and the way we answer it, is novel. The literature typically proposes a concrete game to be played within these networks and focuses under which conditions (if any) the equilibria of these games are Pareto efficient. For example, Kranton and Minehart (2001) and Corominas-Bosch (2004) assume buyers and sellers play in simultaneous auctions restricted by the network (a buyer can bid on a seller's auction only if the two are linked) and study whether the outcome of these auctions is Pareto efficient. However, to an econometrician, the concrete mechanism through which the goods are allocated is rarely observable. Hence, it is useful to impose a minimal set of restrictions on this allocation mechanism. Remaining agnostic with respect to the details of the game allows the researcher to weaken the behavioral assumptions that would be specific to the game otherwise imposed; for example, whether buyers are submitting simultaneous (or sequential) bids to sellers, whether sellers are proposing simultaneous (or sequential) prices to the buyers, whether they are alternating bids and price propositions, *etc.* The weakest criterion for matchmaking that is consistent with Pareto efficiency and agnostic regarding the game details is pairwise stability. To the best of our knowledge, for arbitrary network structures the literature lacks a characterization of pairwise stable matchings and their supporting prices. In this paper we characterize the set of prices that sustains each pairwise stable match. Moreover, our focus is not on efficiency (already embedded in the pairwise stability criterion) but, rather, on the role that network effects play in determining the prices that support stable matchings.

[9] Elliott (2014) shows that, for his particular labor market game, there exists a perfect Bayes Nash equilibrium (PBNE) where the payoffs in a sparse network are the same as the payoffs in the perfectly competitive outcome. In contrast, our results are game-free: the only constraint we impose on allocations is that they be pairwise stable. So we remain agnostic about the mechanism (game) that generates such allocations. By implication, we are also free of equilibrium assumptions (Nash Equilibrium, PBNE, rationalizability, *etc.*). The results are complementary since our model does not nest his and vice-versa. Workers in his model choose which firms to apply to, while workers in our model receive links exogenously. Our assumption is similar to the assumption made in the random search literature, where workers and firms meet randomly; Elliott's assumption is similar to the assumption made in the directed search literature, where workers choose which firms to meet (see Rogerson, Shimer, and Wright 2005 for more on modeling decisions in search models). We have workers and firms meet exogenously since our goal is not to develop a theory of labor market participation, but a parsimonious model of wage dispersion.

Our paper contributes to the literature on the matching role of markets (*e.g.* Gale and Shapley 1962; Shapley and Scarf 1974; Crawford and Knower 1981; Kelso and Crawford 1982; Ausubel and Milgrom 2002; and Hatfield and Kojima 2010).[10] We follow the matching literature by developing a deferred acceptance algorithm that picks specific stable matchings. The algorithm has two stages. The first stage outputs an allocation and is motivated by the wage adjusting process in Crawford and Knower (1981) and Kelso and Crawford (1982). This allocation has the property that there exist prices for which it is pairwise stable. The second stage outputs two prices: the pointwise minimum price at which the stage 1 allocation is stable, and the pointwise maximum price at which the stage 1 allocation is stable.[11]

There is an extensive literature in industrial organization that uses models of search to rationalize price dispersion observed in real world markets.[12] Some of these models include fixed sample and sequential search (*e.g.* Stigler 1961; Rothschild 1973; Reinganum 1979; MacMinn 1980; Burdett and Judd 1983; Carlson and McAfee 1983; Stahl 1989; Janssen and Moraga-González 2004; Janssen, Moraga-Gonzalez, and Wildenbeest 2005; Lester 2011) and clearinghouse models (*e.g.* Salop and Stiglitz 1977; Rosenthal 1980; Varian 1980; Baye and Morgan 2001; Baye, Morgan, and Scholten 2004). These models typically do not use networks to analyze price dispersion. In contrast, it is critical to our analysis to incorporate network effects.

We also contribute to search models with on-the-job search. In wage posting and competitive search models (*e.g.* Burdett and Mortensen 1998 and Moen 1997), there is ex-ante competition among firms for workers. Firms must post and commit to wages before coming into contact with workers. Models with Bertrand competition between firms (*e.g.* Postel-Vinay and Robin 2002) have ex-post competition between firms by allowing a firm to make counter-offers to their employees when they come into contact with a rival firm. In contrast to this literature, we study prices and allocations in a network where we allow firms to negotiate ex-post with more than one worker at a time.

Our model also relates to the literature on directed search with multiple applications (*e.g.* Kranton and Minehart 2001; Albrecht, Gautier, and Vroman 2006; Kircher 2009; Galenianos and Kircher 2009; Walthoff 2012; Albrecht, Gautier, and Vroman 2014). In these models, firms also post wages and workers can apply to more than one vacancy. Since firms post wages, workers are not able to negotiate wages ex-post between different firms. A central question analyzed in this literature is whether the level of entry is efficient. Although an important question, the efficiency of the entry decision is not the focus in our paper. In Albrecht, Gautier, and Vroman (2014), workers post selling mechanisms and firms choose one worker to interact with. They allow for wage competition between firms but not between workers. In contrast to this literature, we allow firms to interact with multiple workers.

[10] Roth (2008) discusses recent progress in the study of deferred acceptance algorithms. See Roth and Sotomayor (1990) for a comprehensive survey of the two-sided matching literature.

[11] See section 5 for details.

[12] See Baye, Morgan, and Scholten (2006) for a detailed survey.

We also contribute to the competitive auctions literature where buyers are typically allowed to bid in only one auction (*e.g.* Wolinsky 1988; McAfee 1993; Peters and Severinov 1997, 2006; Julien, Kennes, and King 2000). By allowing buyers to be linked to many sellers, our model generates competition among sellers typically absent in most competitive auctions models.

There is also a growing literature that uses networks to study trading in financial settings such as over-the-counter (OTC) markets (*e.g.* Gofman 2011; Malamud and Rostek 2013; Babus and Kondor 2013; and Alvarez and Barlevy 2014). They use concrete games to investigate OTC markets where dealers trade with other dealers. In contrast, we study markets where the set of sellers and buyers belong to two disjoint sets: sellers can only trade with buyers while buyers can only trade with sellers (*i.e.* bipartite networks as defined in Section 4).

In the computer science literature, Kakade, Kearns, and Ortiz (2004) study trade using an Arrow-Debreu economy (without firms) where consumers trade goods with other consumers. Kakade, Kearns, Ortiz, Pemantle, and Suri (2004) use a concrete game to study the interaction between the statistical structure of the underlying network and the variation in prices at equilibrium.[13]

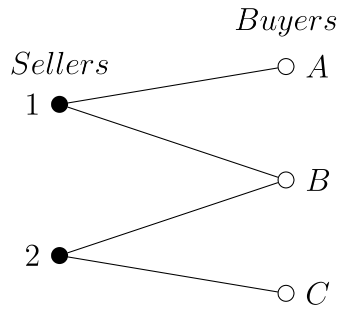
3 Two Motivating Examples

The following two examples illustrate indirect competition and the usefulness of our theoretical tool for characterizing pairwise stable matches in networks. In both examples, we assume that sellers are selling identical goods.

We use Example 1 to get intuition about indirect competition and network effects. Indirect competition is a feature of the structure of the network, whereby buyers that are not connected to the same seller have to compete with each other. Likewise, indirect competition between sellers occurs when sellers that are not connected to the same buyer have to compete. Network effects are defined as the effect of indirect competition on prices. The simplest example that includes these features involves two sellers and three buyers.

[13] Kakade, Kearns, Ortiz, Pemantle, and Suri (2004) analyze networked markets where the numbers of buyers and sellers are equal. They show that, for their particular game, there is no equilibrium price dispersion when the following conditions hold: (1) the number of buyers and sellers go to infinity, (2) the links are formed uniformly at random, and (3) the probability of forming a link is high enough. In their model there is limiting price dispersion (as the number of buyers and sellers go to infinity) when the network is formed via preferential attachment. In contrast, the only constraint we impose on allocations is that they be pairwise stable. So our results are game-free as emphasized in footnote 9. In addition, in our simulations we study price dispersion in bipartite networks varying arbitrarily the number of buyers, the number of sellers, and the number of links per seller or buyer.

Example 1. Assume that buyers A, B and C are ordered in their valuations ($\nu_A > \nu_B > \nu_C > 0$) and sellers 1 and 2 have the same valuation (normalized to 0). Consider the following network:

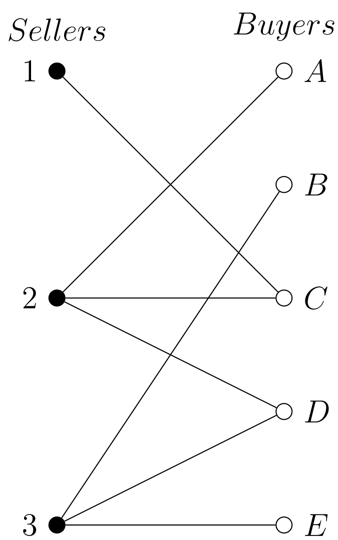


The buyer-preferred stable match is for buyer A to pay ν_C to seller 1 and buyer B to pay ν_C to seller 2. Buyer B cannot pay less than ν_C because buyer C will poach seller 2. Likewise, buyer A cannot pay less than ν_C because buyer B will poach seller 1. In this example, buyer A is indirectly competing with buyer C. Network effects force buyer A to pay ν_C even though buyer C is not linked to seller 1. If buyer C dropped out of the market, then both buyer A and buyer B paying zero is the buyer-preferred pairwise stable match.

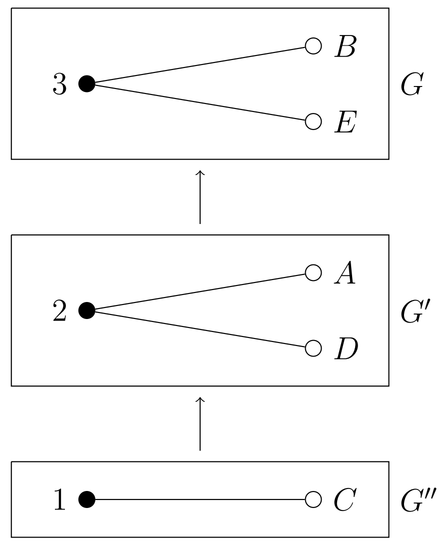
Example 2 demonstrates how we use *abstractions* to highlight the importance of indirect competition and characterize pairwise stable matches. An abstraction in fully connected networks is a decomposition of a network into fully connected subnetworks that satisfy the following properties: (1) each node in the abstraction is a subnetwork of the original network, (2) each link in the original network is either a link within a subnetwork in the abstraction or a link that connects two distinct nodes in the abstraction. This construction uses that fully connected subnetworks are competitive markets with a unique price. The following example demonstrates one possible abstraction of a network.

Example 2. Consider a market with three sellers and five buyers with the following network and an associated abstraction in fully connected networks. Assume that the five buyers are ordered in their valuations ($\nu_A > \nu_B > \nu_C > \nu_D > \nu_E > b$) and the three sellers have the same valuation (*i.e.* $b(1) = b(2) = b(3) = b$).

Network



An Abstraction in Fully Connected Subnetworks

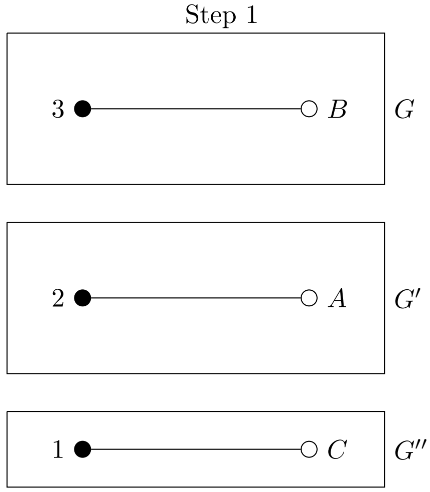


Even though many prices sustain it, there is a unique pairwise stable match: Buyer B buys from seller 3, buyer A buys from seller 2, and buyer C buys from seller 1.

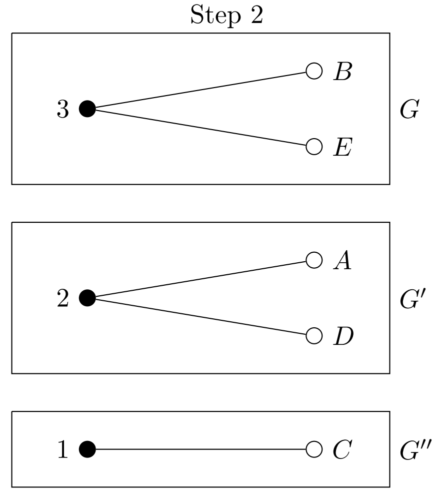
Abstractions are useful to highlight how indirect competition affects price formation.[14] We

[14] One way to construct an abstraction is to follow three steps: (1) Form a subnetwork around each stable match, (2) add the unmatched buyers (sellers) to one of the subnetworks that contains a seller (buyer) to which they are linked, and (3) form a directed link between subnetworks if there is a buyer in one subnetwork that is connected to a seller in another subnetwork. The direction of the link will point from the subnetwork that contains the buyer to the subnetwork that contains the seller. Although there is not a unique assignment in step 2, any assignment will characterize the same set of pairwise stable matches and their supporting prices.

Step 1



Step 2



proceed with five observations: (1) Consider subnetwork G' as an independent subnetwork. In this case, a stable match will be seller 2 selling to buyer A at a price that is greater than buyer D's willingness to pay. (2) Since buyer D is linked to seller 3, buyer D's willingness to pay to seller 2 is less than or equal to the price that prevails in subnetwork G. (3) Thus, A is matched with seller 2 at a price greater than or equal to the price in subnetwork G, even though A is not directly linked to seller 3. (4) We call this an abstraction, because the identity of the buyer in subnetwork G' linked to the seller in subnetwork G is irrelevant. (5) Finally, the direction of the link between subnetwork G and subnetwork G' describes the relationship of the prices that prevails in each subnetwork, namely $p(G') < p(G)$.

Through the use of abstractions, our main theorem helps us understand pairwise stable matchings and their supporting prices. We use abstractions to decompose these prices into a competitive component and an outside-option component. We can do this because abstractions allow us to ignore information about the network that is irrelevant for calculating the prices that sustain a pairwise stable match. Using abstractions and our main theorem, we characterize the set of prices that sustain any given stable matching. Finally, abstractions and our main theorem allows us to prove that our algorithm picks a pairwise stable matching. We build the theory in the next section and discuss these points in detail in subsection 4.3.

4 The Model

4.1 Buyer-Seller Model

We consider buyers and sellers that wish to match pairwise. Sellers differ in their valuation and offer a homogeneous good. For simplicity, we assume that sellers have no idiosyncratic preferences over the buyer they sell to. Buyers differ in their valuation and have single unit demand. A buyer with valuation ν that buys from a seller at price p has utility $\nu - p$ and 0 otherwise. The seller's utility is the price, p, if they sell the good, and their valuation, b, if they do not. Single unit demand implies we focus on pairwise matching.

Matching takes place in exogenous buyer-seller networks. Each buyer is linked with a subset of sellers.That a buyer is not (necessarily) linked to all possible sellers captures search frictions in the environment.

The formal model we use to capture these interactions is a graph-theoretic model. A graph is a set of nodes connected by links (or edges). We say the graph is undirected if the direction of the link does not matter. We say that the graph is bipartite if the set of nodes can be partitioned into two sets such that no two nodes in the same set are connected to each other. In our framework, buyers and sellers constitute a bipartite undirected graph: first, the set of nodes is partitioned into buyers and sellers; second, a buyer is linked to a seller if and only if that seller is linked to that buyer; and third, no buyer (respectively seller) is connected to another buyer (respectively seller). We formalize this in the next definition.

Definition (graph). *Given a set V of* nodes *and a set $E \subset V^2$ of* edges *we say (V, E) is a* graph. *Moreover,*

- *We say a graph (V, E) is trivial if $E = \emptyset$ and V is a singleton.*
- *We say a graph (V, E) is finite if E is finite.*
- *We say the graph is* undirected *when, for each $v, v' \in V$, $(v, v') \in E$ if and only if $(v', v) \in E$.*
- *We say (V, E) is* a bipartite graph *if there exists two disjoint sets, $V_1, V_2 \subset V$, such that $V = V_1 \cup V_2$ and $(v, v') \in E$ only if $v \in V_i \Rightarrow v' \in V_j$, for $i \neq j$. We write these graphs explicitly as (V_1, V_2, E).*
- *We say a bipartite graph (V_1, V_2, E) is* fully connected *if for each $v_1 \in V_1$, $(v_1, v_2) \in E$ for each $v_2 \in V_2$.*

Since graphs tell us which buyers are connected to which sellers, but they do not tell us the valuation of buyers nor the valuation of the sellers, we extend the definition of the graph to the definition of a network. Intuitively, a network is a graph where each node is given a numerical value. This value is interpreted as the "valuation" of the buyer or seller. We define price functions for networks as functions that map every possible buyer-seller edge into a real number. This real number is interpreted as the price that would prevail if the buyer was to buy the good from the seller. The price function is individually rational if, for each buyer-seller edge, it specifies a price that lies between the seller's valuation and the buyer's valuation. For the rest of the paper, even if not explicitly mentioned, $\mathcal{J}$ denotes the set of buyers, and j indexes buyers. Similarly, $\mathcal{I}$ denotes the set of sellers, and i indexes sellers.

Definition (networks and prices). *Let $(\mathcal{J}, \mathcal{I}, E)$ be an undirected bipartite graph, and $M \subset E$ be any subset of edges. Let $\nu : \mathcal{J} \to \mathbb{R}$, $b : \mathcal{I} \to \mathbb{R}$, and $p_M : M \to \mathbb{R}$ be functions such that $p_M((j, i)) = p_M((i, j))$ for each $(j, i) \in M$. We say $N = (\mathcal{J}, \mathcal{I}, E; \nu, b)$ is a* buyer-seller network. *We say the function p_M is a* price function . *We say p_M is* individually rational *(IR) if for each $(j, i) \in M$, $p_M(j, i) \in [b(i), \nu(j)]$.*

Given a network $(\mathcal{J}, \mathcal{I}, E; \nu, b)$, a matching M is any subset $M \subset E$ such that three properties hold: first, each buyer is matched to at most one seller (recall that buyers have unit demand); second, each seller is matched to at most one buyer; and finally, if a seller is matched to a buyer then the buyer is matched to the seller. That is, if $(j, i), (j, i') \in M$, then $i = i'$; if $(j, i), (j', i) \in M$, then $j = j'$; and $(j, i) \in M$ if and only if $(i, j) \in M$. We then say an edge $(j, i) \in M$ is a match or, alternatively, that i and j are matched. Finally, given a matching M, we define $i^* : \mathcal{J} \to \mathcal{I} \cup \{\emptyset\}$ as the function that maps each buyer to the seller with whom it is matched, or to the symbol $\emptyset$ if the buyer is unmatched. Likewise, $j^* : \mathcal{I} \to \mathcal{J} \cup \{\emptyset\}$ is the function that maps each seller to the buyer with whom it is matched, or to the symbol $\emptyset$ if the seller is unmatched. Finally, given a matching M and a price function p_M, the function v summarizes the virtual price each agent (buyer or seller) pays or is getting payed, without having to explicitly distinguish if the they are matched or not. We call these functions v the *payment functions*. In symbols: for each j and i,

$$v(j) = \begin{cases} \nu(j) & \text{if} \quad i^*(j) = \emptyset \\ p_M(i^*(j), j) & \text{if} \quad i^*(j) \neq \emptyset, \end{cases}$$

and

$$v(i) = \begin{cases} b(i) & \text{if} \quad j^*(i) = \emptyset \\ p_M(i, j^*(i)) & \text{if} \quad j^*(i) \neq \emptyset. \end{cases}$$

Next, we define pairwise stability of a matching M with respect to a price function p_M. Pairwise stability means that the edges in M are priced such that individual rationality holds, and there are no mutually benefit matches by agents that are linked but are not matched (*i.e.* agents linked by an edge $e \in E \setminus M$). In other words, any extension of p_M to all edges cannot yield Pareto improvements over the match M executed at prices p_M. Note that pairwise stability only requires that an agent is able to observe the prices of his linked counterparts, but not who they are linked to.

Definition (block). *Let M be a matching and $p_M : M \to \mathbb{R}$. We say an edge $(i, j) \in E \setminus M$* blocks *(M, p_M) if $v(i) < v(j)$.*

Definition (pairwise stability). *Given a non-trivial network $(\mathcal{J}, \mathcal{I}, E; \nu, b)$ and a matching $M \subset E$, we say M is* pairwise stable *at prices p_M if the following hold:*

- No blocking: *no edge $(i, j) \in E \setminus M$ blocks (M, p_M).*
- Individual rationality: *$p_M(i, j) \in [b(i), \nu(j)]$ for all $(i, j) \in M$.*

In fully connected networks it is simple to characterize stable matchings. Indeed, if M is stable with respect to a price function p, then all prices must be the same. To see this, assume i is matched to j, i' is matched to j', and let p be any individually rational extension of p_M to E. Then, $p(j, i) \leq p(j, i') \leq p(j', i') \leq p(j', i) \leq p(j, i)$, where all these terms are well defined because the network is fully connected. As a corollary, all stable matchings can be characterized by whether there are more buyers than sellers or vice versa. Intuitively, stable matchings are those matchings which are maximal and can be sustained by individually rational prices that price out the side of the market (sellers or buyers) that is in excess. In this regard, the matchings and prices we obtain from pairwise stability in fully connected graphs are those that would prevail if this was a perfectly competitive economy. We summarize this in the following remark.

Remark 1. *Let $(\mathcal{J}, \mathcal{I}, E; \nu, b)$ be a fully connected network, with $J = \#\mathcal{J}$, $I = \#\mathcal{I}$. Assume that $\bar{b} = \max\{b(i) : i \in \mathcal{I}\} \leq \min\{\nu(j) : j \in \mathcal{J}\} = \underline{\nu}$. Let $M \subset E$ be a matching.*

- *If $I > J$, M is stable if, and only if,*
  - *All buyers are matched: For each $j \in \mathcal{J}$ there is $i \in \mathcal{I}$ such that $(j,i) \in M$.*
  - *Only lowest valuation sellers are matched: If $i \in \mathcal{I}$ is such that $\#\{i' : b(i) > b(i')\} \geq J$ then there is no $j \in \mathcal{J}$ such that $(j,i) \in M$.*
  - *Seller valuations determine matching prices: For each $(j,i) \in E$, $p(j,i) = p$ where $p \in [\max\{b(i) : (\exists j \in \mathcal{J})$ such that $(j,i) \in M\}, \{\min\{b(i) : (\nexists j \in \mathcal{J})$ such that $(j,i) \in M\}]$.*
- *If $I = J$, M is stable if, and only if,*
  - *All buyers are matched: For each $j \in \mathcal{J}$ there is $i \in \mathcal{I}$ such that $(j,i) \in M$.*
  - *All sellers are matched: For each $i \in \mathcal{I}$ there is $j \in \mathcal{J}$ such that $(j,i) \in M$.*
  - *Sellers sell at an intermediate price: For each $(j,i) \in E$, $p(j,i) = p$ where $p \in [\bar{b}, \underline{\nu}]$.*
- *If $I < J$, M is stable if, and only if,*
  - *Only highest valuation buyers are matched: For each $j \in \mathcal{J}$ if $\{j' : \nu(j') > \nu(j)\} \geq I$ then there is no $i \in \mathcal{I}$ such that $(j,i) \in M$.*
  - *All sellers are matched: For each $i \in \mathcal{I}$ there is $j \in \mathcal{J}$ such that $(j,i) \in M$.*
  - *Buyer valuations determine matching prices: For each $(j,i) \in E$, $p(j,i) = p$ where $p \in [\max\{\nu(j) : (\nexists i \in \mathcal{I})$ such that $(j,i) \in M\}, \min\{\nu(j) : (\exists i \in \mathcal{I})$ such that $(j,i) \in M\}]$.*

4.2 A Theorem

To characterize the set of pairwise stable matchings in a network and identify the set of prices that can sustain them, it is convenient to abstract away from certain links in the original graph and retain only the links that are essential for characterizing these matchings. This yields economic insight into how these prices are determined. Our next definition, the abstraction of a graph, identifies links of any given network that are essential for determining prices that sustain pairwise stable matchings. Slightly abusing notation (see remark 2 below) an abstraction of a graph is a directed graph with nodes and edges as follows: each node is a subgraph of the original graph, and each edge in the original graph is either (i) an edge within a subgraph in the abstraction or (ii) connecting two distinct nodes in the abstraction. We now present a formal definition for the abstraction of a graph, an extension of that definition for networks, and an example.

Definition (graph abstraction). *Let $(\mathcal{J}, \mathcal{I}, E)$ be a buyer-seller graph. We say a directed graph $(\mathcal{G}, E^*)$ is an* abstraction of $(\mathcal{J}, \mathcal{I}, E)$ *if the following hold:*

- *Each $G \in \mathcal{G}$ is a graph $(\mathcal{J}_G, \mathcal{I}_G, E_G)$ such that $\mathcal{J}_G \subset \mathcal{J}$, $\mathcal{I}_G \subset \mathcal{I}$,* $E_G = \{e : e \in E \text{ and } e \in (\mathcal{J}_G \times \mathcal{I}_G) \cup (\mathcal{I}_G \times \mathcal{J}_G)\}$,
- $\{\mathcal{J}_G : G \in \mathcal{G}\}$ *and* $\{\mathcal{I}_G : G \in \mathcal{G}\}$ *are a partition of* $\mathcal{J}$ *and* $\mathcal{I}$ *respectively,*
- $(G, G') \in E^*$ *if and only if there exists* $j \in \mathcal{J}_G, i \in \mathcal{I}_{G'}$ *such that* $(j, i) \in E$.

Moreover, we say that an abstraction is an abstraction in fully connected graphs if each $G \in \mathcal{G}$ is a fully connected graph.

Remark 2. *Generally, nodes in graphs are the smallest object in the graph. As such, a more standard definition of abstraction would let the vertices $G \in \mathcal{G}$ be arbitrary objects in an arbitrary set, and would include a bijective mapping between $\mathcal{G}$ and the relevant subgraphs of $(\mathcal{J}, \mathcal{I}, E)$. However, this would imply adding notation that makes the model unnecessarily cumbersome.*

Remark 3. *Since one-to-one graphs and trivial graphs are fully connected, an abstraction in fully connected graphs always exists.*

Remark 4. *Given a graph $(\mathcal{J}, \mathcal{I}, E)$, abstractions in fully connected graphs will not necessarily be unique. See example 2 above.*

Definition (network abstraction). *Let $(\mathcal{J}, \mathcal{I}, E; \nu, b)$ be a buyer-seller network. We say $(\mathcal{G}, E^*; p)$ is an* abstraction of $(\mathcal{J}, \mathcal{I}, E; \nu, b)$ *if the following hold:*

- $(\mathcal{G}, E^*)$ *is an abstraction of* $(\mathcal{J}, \mathcal{I}, E)$,
- *If $G \in \mathcal{G}$ satisfies $\mathcal{J}_G = \{j\}$, $\mathcal{I}_G = \emptyset$, then $p(G) = \nu(j)$; if $G \in \mathcal{G}$ satisfies $\mathcal{J}_G = \emptyset$, $\mathcal{I}_G = \{i\}$, then $p(G) = b(i)$; otherwise, $p(G) \in [\min\{b(i) : i \in \mathcal{I}_G\}, \max\{\nu(j) : j \in \mathcal{J}_G\}]$.*

Since our objective is to use abstractions to characterize the stable matchings in a graph, our next definition specifies when a matching is stable with respect to an abstraction of that graph. Intuitively, a matching $M \subset E$ is stable with respect to an abstraction when three conditions hold. First, the abstraction does not break M: for each buyer-seller match in M, that pair belongs to the same subgraph in the abstraction. Second, recall that the only price functions that can sustain a stable matching in a fully connected network are those where all edges are priced equally. Therefore, each subgraph G in the abstraction is assigned a number, $p(G)$, that plays the role of this uniform price. The third condition is that buyers are sorted into the fully connected subnetwork whose price is lower than the price of any other subnetwork they have access to.

Definition (stability abstraction). *Let $(\mathcal{J}, \mathcal{I}, E; \nu, b)$ be a buyer-seller network and $(\mathcal{G}, E^*; p^*)$ be an abstraction of it in fully connected graphs. We say that M is stable with respect to the abstraction if three conditions hold:*

- *$\mathcal{G}$ does not break M: for each $e \in M$, $e \in E_G$ for some $G \in \mathcal{G}$.*
- *Prices $p(\cdot)$ induce pairwise stability:*
  - *For each non-trivial $G \in \mathcal{G}$, M restricted to G is stable at prices $p(j,i) = p^*(G)$ for all $(j,i) \in M \cap E_G$,*
  - *If $G = (\{i\}, \emptyset, \emptyset)$ for some j, then $p^*(G) = b(i)$,*
  - *If $G = (\emptyset, \{j\}, \emptyset)$ for some j, then $p^*(G) = \nu(j)$.*
- *Cheapest sorting: if $(G, G') \in E^*$ then $p(G) \leq p(G')$.*

Remark 5. *The cheapest sorting condition is defined to be consistent with the construction of the edges in E^*. Indeed, we define E^* so that $(G, G') \in E^*$ whenever a buyer in G is linked to a seller in G'. Since buyers search for the cheapest price, then we define cheapest sorting as $p(G) \leq p(G')$. If we took the opposite convention for the edges in E^*, that $(G, G') \in E$ whenever a seller in G is linked to a buyer in G', since sellers search for the highest price, then cheapest sorting would be defined as $p(G) \geq p(G')$.*

With these definitions we can state our main theorem.

Theorem 1. *Let $\mathcal{N} = (\mathcal{J}, \mathcal{I}, E; \nu, b)$ be a network. Let M be a matching. Then the following are equivalent:*

1. *There exists p_M such that M is stable with respect to p_M*
2. *There exists an abstraction in fully connected graphs $A = (\mathcal{G}, E^*; p^*)$ such that M is stable with respect to A.*

The proof of the theorem is in section A in the appendix.

4.3 Theorem Application

In this subsection we illustrate four ways in which abstractions and Theorem 1 are useful to understand pairwise stable matchings and their supporting prices. For this we use Example 2.

First, theorem 1 is useful to decompose the prices that sustain pairwise stable matchings into a competitive component and an outside option component. We proceed in four steps. First, consider nodes G' and G as independent graphs and note that they are fully connected graphs. Second, use Remark 1 to conclude three things: that only buyers B and A should match to sellers, that buyer B should pay at least $\nu(E)$, and that buyer A should pay at least $\nu(D)$. Steps one and two imply that we would observe these matches and supporting prices if these were two independent, perfectly competitive economies. The later two constraints are the competitive component of prices that sustain pairwise stable matchings. Third, note that buyer D is linked to seller 3, as indicated by the edge (G', G) in the abstraction. Thus, any price that sustains a pairwise stable matching must also satisfy that buyer A pays seller

2 no more than what B pays seller 3. This is reflected by the cheapest sorting condition, and is what we call the the outside option component. Steps one through three imply that all prices that support a pairwise matching that is stable *with respect to an abstraction* will have these competitive and outside option components. Fourth, Theorem 1 implies that all such prices make this matching stable with respect to the *original* network. Therefore, all prices that support pairwise stable matchings (in the original network) also have these competitive and outside option components.

A corollary of the above decomposition is that, to calculate the prices that sustain a particular stable matching, not all edges are relevant. Indeed, consider a modified graph where edge $(D, 3)$ is replaced by edge $(A, 3)$. The original matching is still stable in the abstraction at the original prices. Thus, this matching is also stable in the modified graph. The relevant aspect of these graphs that sustains the proposed matching at the proposed prices is that at least one buyer in $\{A, D\}$ is connected to seller 3, but the exact identity of the buyer is irrelevant. Conversely, given an abstraction and a pairwise stable matching in that abstraction, any graph obtained by drawing edges in a manner consistent with the abstraction will support the given matching.

Second, Theorem 1 is also useful to pin down the set of all prices that can sustain any given stable matching. In Example 2, the unique stable matching is $M = \{(C, 1), (A, 2), (B, 3)\}$, but there are many prices that can sustain M. To calculate the full set of prices that sustain M we consider the abstraction in fully connected graphs shown in Example 2: in G, the unique stable match is $\{(B, 3)\}$ at a price $p(B, 3) \in [\nu(E), \nu(B)]$; in G' the unique stable match is $\{(A, 2)\}$ at a price $p(A, 2) \in [\nu(D), \nu(A)]$; and in G'' the unique stable match is $\{(C, 1)\}$ at a price $p(C, 1) \in [b, \nu(D)]$. However, since G'' is connected to G' and G' is connected to G, we must also have $p(C, 1) \leq p(A, 2) \leq p(B, 3)$. Therefore, M can only be sustained at prices that satisfy $p(C, 1) \in [b, \nu(c)]$, $p(A, 2) \in [\max\{p(C, 1), \nu(D)\}, \min\{p(B, 3), \nu(A)\}] = [\max\{p(C, 1), \nu(D)\}, p(B, 3)]$, $p(B, 3) \in [\max\{\nu(E), p(A, 2)\}, \nu(B)] = [p(A, 2), \nu(B)]$.

Third Theorem 1 is also useful to prove that the algorithm (see Section 5) finds: (1) pairwise stable matchings in any given network and (2) the upper and lower bounds of the set of prices that sustain those matchings. In Section 5 we present a description of the algorithm and its properties. In Appendix B we present the formal algorithm and formal proofs.

Finally, Theorem 1 allows us to rationalize pairwise stable matchings as the result of equilibrium bidding strategies in simultaneous auction games. In the example above, we can interpret each node in the abstraction as the following second-price auction. Each seller holds an auction. Buyers can only bid in the sellers' auction that belongs to same node in the abstraction. In the example, buyer A only bids in seller 2's auction, buyer B only bids in seller 3's auction, buyer C only bids in seller 1's auction, and so on. Finally, we assign fictitious values to the buyer with the second highest valuation in each node of the abstraction. This fictitious value is determined by the constraints imposed by the edges in the abstraction. In the example, C has fictitious value $\hat{\nu}(C) = p(C, 1)$; D has fictitious value $\hat{\nu}(D) = p(D, 2) \in [\max\{p(C, 1), \nu(D)\}, p]$; and E has fictitious value $\hat{\nu}(E) = p(B, 3) \in [p(A, 2), \nu(B)]$. Such triplets $(p(C, 1), p(A, 2), p(B, 3))$ make the matching M stable with respect to the abstraction. Alternatively, instead of assigning fictitious values to the buyers with the second highest valuation, we can assign reservation prices to the sellers. Again, these reservation prices are determined by the constraints imposed by the edges in the abstraction. The outcome of the independent second-price auctions with fictitious valuations for the buyers is observationally equivalent to the independent second-price auctions with reservation prices for the sellers. Moreover, the outcome of either of these independent second-price auctions is indistinguishable from the outcome of any other mechanism that generates the same matching at the supporting prices.

4.4 Labor Market Model with On-the-job Search

In this section we adapt the buyer-seller model to the labor market, where workers are sellers and firms are buyers. We assume workers do on-the-job search and that firms have single unit demand, so a firm is equivalent to a vacancy.

At the beginning of period 1, a finite bipartite graph is randomly drawn between workers and firms. We use $(\mathcal{I}_1, \mathcal{J}_1, E_1)$ to denote the time 1 graph, I and J to denote the respective number of workers and firms, and we define market tightness as the ratio of J to unemployed workers, denoted as θ_1. Conditional on the graph we assign productivities to firms, denoted with $\nu(\cdot)$, drawn *i.i.d.* from a continuous distribution F with support in the interval $[\underline{\nu}, \overline{\nu}]$. To keep the model simple we assume that all workers have the same reservation wage. Using the notation of the buyer-seller model, $b(i) = b$ for each $i \in \mathcal{I}_1$.

After the period 1 graph has been realized, we pick a specific pairwise stable matching M_1^0 at wages $w_{M_1^0}$ using the algorithm that we describe in Section 5. Matched firms receive a period utility of $\nu(j) - w_{M_1^0}(j, i^*(j))$, matched workers receive a period utility of $w_{M_1^0}(j^*(i), i)$, unmatched workers receive their reservation wage b, and unmatched firms receive utility 0 and leave. After these utilities are realized, there is an exogenous job destruction shock. This means that each link $m \in M_1^0$ is dissolved with probability $\delta > 0$. Firms whose links are dissolved become unmatched and leave. We denote with M_1 the period 1 matching after the exogenous job destruction.

Now consider the beginning of period $t \geq 2$. Given the matching M_{t-1} of period $t-1$, we add J new firms and no new workers (*i.e.* $\mathcal{I}_t = \mathcal{I}_{t-1}$). We draw the productivities for these new firms from the same distribution F. We randomly draw links between workers and firms that satisfy the following three conditions. First, positive probability is assigned only to graphs with vertices in $\mathcal{I}_t \cup \mathcal{J}_t$, where these denote the set of period t workers and firms, respectively. Second, matches from period $t-1$ are not dissolved (that is, $M_{t-1} \subset E_t$, where E_t is the set of period t edges). Finally, matched firms receive no new applications, but matched workers may apply to new firms because they can do on-the-job search (that is, if

$(j,i) \in M_{t-1}$ then $\{i' : (j,i') \in E_t\} = \{i\}$, but no constraints are placed on $\{j' : (j',i) \in E_t\}$). We denote the corresponding graph with $(\mathcal{I}_t, \mathcal{J}_t, E_t)$. The reservation wage of workers who were not matched in $t-1$ is b; the reservation wage for workers who were matched in $t-1$ is the worker's wage ($w_{M_{t-1}}$). As before, the period utility for matched firms are their profits, the period utility for matched workers are their wages, the period utility of unmatched workers are their reservation wages, and the period utility of unmatched firms are 0 and these leave. Finally, period t utilities are discounted at a rate β^t, with $\beta \in (0,1)$.

We are applying the buyer-seller model within each period, so pairwise stable matchings are independently formed period by period. Determining the matches in this way implies that workers accept the vacancy that pays the highest wage and hence, do not consider other aspects of the match, such as future wage growth. From the firm's perspective this is without loss of generality: if they are unmatched at the end of a period they leave, so their static and dynamic problems coincide. From the workers perspective, however, there is a loss of generality. To see this consider worker i that is matched to firm j at wage w at the end of period t, and assume that in period $t+1$ firm j' will only be linked with i. To rule out the trivial case, assume that $\nu(j') > w$. Then worker i will expect a wage in period $t+1$ that belongs to $[\min\{\nu(j), \nu(j')\}, \max\{\nu(j), \nu(j')\}]$. Hence, worker i is willing to work for a more productive firm in period t even if that firm offers slightly lower salaries than the competitors. To the best of our knowledge, there is no standard solution concept for dynamic matching markets when matching opportunities arrive over time. For a more thorough discussion of the complications that arise in dynamic matching models see, for example, Doval (2014). For this reason, relaxing this assumption is left for future work.

5 A Deferred Acceptance Algorithm

We now present our deferred acceptance algorithm. We describe the algorithm as a first-price auction to give intuition of how the algorithm works. A formal description of the algorithm can be found in Section B in the appendix. We denote the agents on the side of the market that are holding the "auctions" as sellers and the agents on the other side that are "bidding" as bidders. Recall that we are approaching this problem from the matching perspective, so we are not making any statement about the actual economic mechanisms or incentives of the agents that determine prices and matches. Bidders bid in increments of $\frac{\Delta}{2}$. The value of Δ is set so that the productivity of firms lie in a Δ grid. Formally, for all j, $\nu(j) = b + k_j\Delta$ for some integer k_j that is randomly drawn at the start of the algorithm. We describe the algorithm for the case where the sellers hold the auctions. When buyers hold the auctions, the bidding starts at their valuation and prices decrease.

The algorithm has two stages. The first stage outputs an allocation and is motivated by the wage adjusting process in Crawford and Knower (1981) and Kelso and Crawford (1982). (See Section B in the appendix for a detailed comparison about the first stage of our algorithm and the algorithms in Crawford and Knower and Kelso and Crawford.) This allocation has the property that there exist prices for which it is pairwise stable. The second stage outputs two prices: the pointwise minimum price at which the stage 1 allocation is stable, and the pointwise maximum price at which the stage 1 allocation is stable.

Stage 1: The Matching Determination Program

The algorithm starts in round $t = 1$ when none of the sellers has received any bid. All bidders are placed into a queue and arrive sequentially. The entering order of the bidders is determined randomly. The *standing bid* of a seller is the last bid accepted by the seller or b if the seller has not received any bids. The *winning bidder* is the bidder who placed the last standing bid.

This is round t of the matching determination program.

1. Take the first bidder in the queue (for concreteness, call it bidder j). Bidder j selects the seller with the lowest standing bid among the linked sellers. If there is more than one such seller, the bidder selects one of these sellers at random. Call it seller i. If the lowest standing bid is greater than $\nu(j) - \frac{\Delta}{2}$, bidder j does nothing and leaves the queue. Otherwise, bidder j bids the standing bid of seller i plus $\frac{\Delta}{2}$.

2. If bidder j makes a bid, seller i accepts the bid from bidder j. The new standing bid of seller i is now the previous standing bid plus $\frac{\Delta}{2}$. Bidder j leaves the queue. If there was a bidder j' who was the winning bidder (before bidder j bid), bidder j' is placed at the end of the queue.

3. The algorithm continues from step 1 with the next bidder in the queue. The algorithm stops when there are no bidders left in the queue. In this case, each seller is matched to the winning bidder.

We now present the second stage, the price determination program. The key insight of this stage is that, if a seller i is matched to a buyer j, and is also linked to an unmatched buyer j', then the price j pays i must price j' out of the market. That is, $p_M(i,j) \geq \nu(j)$. Moreover, if seller i is matched to buyer j, and seller i is also linked to a buyer j' who is also matched (say, to a seller i') then i must be getting payed at least what i' is getting payed. Otherwise, j' would like to block with i.

Stage 2: The Price Determination Program (I)

The program starts in round $t = 1$ with $M \subset E$ produced from stage 1 as its input.

1. Set the "price" of all unmatched sellers to b.

2. For matched sellers, set the price of seller i for buyer j to the maximum $\nu(j')$ amongst all j' that are linked to i but are not matched.

3. We call these prices $(\rho_i^1)_{i\in\mathcal{I}}$.

This is round $t > 1$ of the price determination algorithm. We take $(\rho_i^{t-1})_{i\in\mathcal{I}}$ as inputs for this round.

1. Set the "price" of all unmatched sellers in round t to b.

2. For matched sellers, set the price of each seller i for buyer j to the maximum price in round $t-1$ of the matched buyers that are linked to i. That is, amongst all matched j' that are linked to i, set ρ_i^t to the maximum $\rho_{i^*(j')}^{t-1}$. Note that one such j' is j itself, so these prices form a non-decreasing sequence.

3. If $\rho_i^t = \rho_i^{t-1}$ for all i, stop the algorithm and output these prices. Otherwise, start step $t+1$.

As formally stated in Proposition 1, the Price Determination Program (I) captures the pointwise minimum price function at which M is stable. A modified version of this program, which we call Price Determination Program (II), generates the pointwise maximum price function at which M is stable. Rather than starting with ρ^1 at a low value, with successive iterations iterations rising it, the modified program starts with ρ^1 at high values and successive iterations lower it. Section B contains the formal algorithm, including both versions of the Price Determination Program.

Proposition 1. *The deferred acceptance algorithm has the following properties:*

1. *It stops after a finite number of rounds.*

2. *It outputs a pairwise stable allocation.*

3. *Price Determination program (I) outputs the pointwise minimum price function at which M is stable.*

4. *Price Determination program (II) outputs the pointwise maximum price function at which M is stable.*

The proof of Proposition 1 is in section B in the appendix.

6 Results

In the next two subsections we document the results from simulations of the buyer-seller and the labor-market models. We use the results from the simulation to obtain predictions about the population distribution of prices and the matching process.

6.1 Buyer-Seller Model

6.1.1 Simulation

We now describe the simulation of the buyer-seller model.

There are three parameters in the buyer-seller model: the number of buyers (J), the number of sellers (I), and the expected number of links per buyer (ELB). Every seller begins with one unit of a good (so the number of goods is I). The market tightness, θ, is the ratio of the number of buyers to the number of sellers, $\theta = \frac{J}{I}$. The market tightness is exogenous. We start the baseline simulation with $I = 10,000$ identical sellers and $J = 10,000 \times \theta$ heterogeneous buyers.[15] We consider markets with $J \in [1000, 50000]$, so $\theta \in [0.1, 5]$. We also consider markets with ELB $\in [1, 10]$.[16] The higher the ELB, the lower the search frictions in the market. The product of the number of buyers and the ELB determines the number of active links in the market. The total number of possible links in the market is $J \times I$. The proportion of active links relative to the total number of possible links in a network is a measure of the sparsity of the network. Given the parameters J, I, and ELB, a network is formed by randomly drawing buyers and sellers to form links. Once the network is constructed, we apply the algorithm from Section 5 to the network. The "bids" in the first stage of the algorithm take place on a grid of possible prices with $2J$ grid points.

Buyers' valuation is normalized to range between 0 and 100 which bounds the minimum and maximum prices between those values. One can interpret the reported prices as if the buyers' valuations are drawn from a uniform distribution, whose support is normalized between 0 and 100. Alternatively, one can interpret the prices or valuations as percentiles of any cumulative distribution function of the buyers' valuations. The second interpretation is possible because the allocations and prices only depend on order statistics and not the actual valuations.

We compare the price distributions to the Walrasian outcome, when each buyer is linked to every seller. The Walrasian outcome price, p_{walras}, is given by:

$$p_{walras} = \begin{cases} 0 & if \quad \theta \leq 1 \\ (1 - \frac{1}{\theta}) \times 100 & if \quad \theta > 1. \end{cases}$$

Recall that the Walrasian outcome has a unique price (see Remark 1). When $\theta \leq 1$, there are more sellers than buyers and so there is always a seller who is indifferent between selling the good at 0 or not selling it at all. In other words, the reservation price of the marginal seller is zero, which is what determines the market price. When $\theta > 1$, there are more buyers than sellers. Only $\frac{1}{\theta}$ of the buyers will buy the good. Hence the valuation of the marginal buyer will be $(1 - \frac{1}{\theta}) \times 100$. This buyer will be indifferent between paying $(1 - \frac{1}{\theta}) \times 100$ and leaving the market, and so the market price will be $(1 - \frac{1}{\theta}) \times 100$.

[15] The results do not change substantially using 1,000 or 100,000 sellers. Results are available upon request.
[16] We obtain similar results by varying expected links per seller (ELS) in the simulations.

6.1.2 Results

Distribution of Prices. Figure 1 displays the distribution of prices for the buyer-preferred match by market tightness (horizontal axis in each panel) and ELB (different panels). Each vertical box corresponds to a simulated market characterized by those parameters. Each panel shows the population distribution of prices for different levels of search frictions in different markets. The top-left panel shows the price distribution for high frictions, where ELB equals 1. The top-right and bottom panels show what happens in markets with lower frictions (when ELB equals 2, 3, and 5, respectively). At low levels of θ there are many sellers for each buyer. So low numbers for θ indicate "loose" seller markets where sellers are at a disadvantage. In addition, each panel displays the Walrasian outcome.

For market tightness less than one, the market looks like a monopsony and nearly all sellers are paid their valuation (recall that, for simplicity, all sellers are identical, so we normalized their valuation to zero). This is because it is unlikely for a seller to receive multiple links. Even if a seller receives two links, it is likely that at least one of the buyers has an outside option of zero. This happens if the buyer is also linked with another seller who has no other links.

On the other hand, as market tightness is increased the market becomes more competitive between buyers and more favorable for sellers. The median price increases as does price dispersion. There are now many buyers linked to each seller and the buyers have worse outside options. Even if a buyer is linked to a second seller, it is likely that the second seller is linked to many other buyers. In markets with lower frictions, competition between buyers increases, thus increasing prices until they reach the Walrasian outcome.

Figure 2 shows that similar results to the ones in Figure 1 are obtained using the seller-preferred match. Figure 2 displays, for each market tightness, the distribution of prices using both the seller- and the buyer-preferred match. (For the buyer-preferred match, each vertical box in Figure 2 is identical to the corresponding vertical box in Figure 1.) When ELB equals 5, the 95th and 5th price percentiles coincide with the Walrasian outcome for both the buyer- and the seller- preferred match. The prices in the buyer-preferred matching represents the lower bound of the set of prices that support each match. Likewise, the prices in the seller-preferred match represents the upper-bound of the set of prices that support each match. Since both the seller-preferred and buyer-preferred price distributions mimic the Walrasian outcome when ELB=5, it must be true that the price distribution in any allocation that supports a pairwise stable match must also mimic the Walrasian outcome.

Price Dispersion and the Walrasian Outcome. Price dispersion decreases when search frictions decrease. There are many buyers linked to each seller, but there are also many sellers linked to each buyer, improving the outside options of both parties. These improved outside options reduces price dispersion (*i.e.* the likelihood that a seller has to take a low price is low, but at the same time the probability that a buyer has to pay a high price is also low). Figure 3 shows the evolution of the price distribution for the buyer-preferred match. In the top panel, the figure displays the difference between the 95th and the 5th price percentiles. In the bottom panel, the figure displays the difference between the 99.5th and 0.5th price percentiles. All sellers are paid the same price at the Walrasian outcome, so both differences equal zero at the Walrasian outcome. We are interested in answering the following two questions: How sparse can the network be while 90% and 99% of sellers are paid the same price? While there is price dispersion when there are fewer than four ELB, the price distribution begins to collapse for more ELB. When there are five ELB, there is nearly no difference between the price at the 95th and 5th percentiles. Likewise, when there are eight ELB, there is almost no difference between the price at the 99.5th and 0.5th percentile. In other words, at least 90% or 99% of the sellers are paid the same price when the number of active links relative to the total number of links is only 5/10,000 or 8/10,000, respectively. The price distribution in the model collapses with less than 0.1% of the possible links in the network.

The Effect of Frictions on Mean Prices. Figure 4 displays the evolution of mean prices (where the expectation is taken relative to the population distribution of prices) over ELB for different market tightness. Mean prices represent the buyers' and sellers' *ex ante* expected prices before the network is drawn. The figure shows how mean prices vary with ELB (*i.e.* frictions) in a given market (holding fixed the market tightness), so mean prices are normalized by the mean price when ELB equals 1. Increasing ELB may increase or decrease mean prices, depending on market tightness. For example, consider markets where there are many sellers for each buyer ($\theta \leq 1$, so that sellers are at a disadvantage). When ELB is low, price dispersion is high, even when there are more sellers than buyers (top-left panel in Figure 1), resulting in relatively high mean prices. As ELB increases, the price distribution collapses to the Walrasian outcome (Figure 3). The Walrasian outcome is zero when there are more sellers than buyers. Thus, when there are more sellers than buyers ($\theta \leq 1$), lowering frictions results in *lower* mean prices as a consequence of network effects. Intuitively, since there are more sellers than buyers, increasing ELB improves the outside option of the buyers who now talk to relatively more sellers, even when sellers expect to talk to more buyers.[17]

Distribution of Matched Buyers. Figure 5 shows the distribution of matched buyers for different markets. In loose markets ($\theta < 1$), the probability of finding a match does not depend on the buyer's valuation. Prices are low and there are many unmatched sellers, so buyers have a roughly equal chance of finding a match. The ELB does not change the distributions of matched buyers in loose markets.

As markets become tighter ($\theta > 1$), competition between buyers becomes more important. In these markets, prices are higher and some buyers are priced out of the market. Buyers

[17] Same results are obtained using expected links per seller (ELS) instead of ELB. When there are more sellers than buyers ($\theta \leq 1$), increasing ELS results in lower mean prices. Results are available upon request.

with high valuations (*e.g.* above the Walrasian price) are more likely to buy goods than buyers with low valuations. When ELB is low, buyers with high valuations may be linked to a seller with another high-valuation buyer. Since they have few links, they are priced out of the market. For markets with higher ELB, buyers have better outside options and the probability that a high-valuation buyer is priced out of the market decreases. When ELB=5, the distribution of matched buyers looks close to the Walrasian outcome, where all buyers with $\nu(j) > p_{walras}$ are matched and all buyers with $\nu(j) < p_{walras}$ are priced out of the market.

Welfare. Results on the welfare in the buyer-seller model are in the online appendix (see Figure A1). We analyze welfare using the labor market model on page 28, where the results are similar to the buyer-seller model.

6.2 Labor Market Model

6.2.1 Simulation

We adapt the buyer-seller model to the labor market to explore questions about wage dispersion and growth. In this case, workers are sellers and firms are buyers of their services. Firms have single unit demand for labor and cannot dismiss their employee. We assume that workers are homogeneous and firms are heterogeneous in their productivity ($\nu(j)$).[18] We normalize the reservation wage of the worker to zero. If a worker and firm match at wage w, the worker's utility is w and the firm's profit is $\nu(j) - w$. Wages and firm productivities are normalized to range between 0 and 100 as in the buyer-seller model. To study wage growth, we extend our model to accommodate multiple periods.

In the first period all workers start unemployed and the simulation is identical to the basic buyer-seller model (see subsection 6.1). At the end of the period, some matches are randomly destroyed at rate $\delta \in (0, 1)$. The firms that are unmatched at the end of a period (either because the match was destroyed or they could not form a match in the first place) exit the market. We interpret "firms" as time sensitive vacancies. So if by the end of a period a vacancy is not filled, it disappears from the job market.

At the beginning of the next period, some fraction of workers are employed by old firms and the rest are unemployed. The same number of firms are created (J) and a new network is drawn between all the workers (employed and unemployed) and the new firms. Firms from previous periods maintain the link to their employed worker and lose all other links from previous periods. New firms are placed into the queue and old firms start off as the highest bidders in their employee's "auction". So from the standpoint of the algorithm, the standing bid (or reservation wage) in an employed worker's auction is the wage from the previous

[18] Worker heterogeneity is important for understanding wage dispersion in the data and it is straightforward to add worker heterogeneity to our model. However, not including worker heterogeneity makes the exposition of our results more clear as the Walrasian outcome has a unique wage.

period. The new firms either "bid" for unemployed workers or try to poach employed workers from old firms by bidding in the auctions of the employed workers. Firms from previous periods can only bid in their employee's auction. When firms arrive at the front of the queue, they bid in their subnetworks according to the algorithm (see Section 5). The bidding process ends when there are no more firms in the queue. Matched firms produce with the hired workers and the workers receive their wage. At the end of the period matches receive a job destruction shock at rate δ.

We consider markets that are in steady state. The market is in steady state when the flows into unemployment equal the flows out of unemployment. We find the steady state by simulating the economy for enough periods until the average unemployment remains stable.[19] Then we record 400 periods.

The labor-market model has five parameters: the number of workers (I), which is constant for all periods, the number of firms (J) that enter in each period, the expected number of links per firm (ELF), the job destruction rate (δ), and the relative probability of receiving a link between employed and unemployed workers (λ). The market tightness, θ, is defined as the ratio of firms to unemployed workers, $\frac{J}{U}$. In contrast to the basic buyer-seller model, θ is now endogenous as it depends on the the number of unemployed workers (U).

Both the number of workers and the job destruction rate are fixed for all simulations. Following Shimer (2012), the monthly employment to unemployment rate is set at 2% (prime age men, Figure 3), which translates to a quarterly $\delta = 0.06$. We use 5,000 workers for the simulation of the dynamic labor market model.[20] A market is a combination of J and ELF.

The relative probability of receiving a link between employed and unemployed workers (λ) is an important determinant of the structure of the network. Most empirical studies find different job offer rates between employed and unemployed workers.[21] This is important for understanding allocations and wage growth, since network effects are substantially diminished when $\theta < 1$ and $\lambda = 1$. To understand why, recall that when $\theta < 1$, there are more unemployed workers (U) than vacancies entering the market (J). Even when the unemployment rate is relatively high (10%-20%), firms have a low probability of linking with an unemployed worker when $\lambda = 1$. So even in markets that appear unfavorable for the workers (high unemployment and low market tightness), it is difficult for the firms to link to unemployed workers when $\lambda = 1$. This implies that in these markets, unemployed workers do not have to compete with each other. If we follow the empirical literature by setting $\lambda < 1$, then firms have a higher probability of linking with an unemployed worker and network effects

[19]We compute the average unemployment over 10 consecutive periods and compare it to the average unemployment 10 periods before. We find the steady state when this difference is less than the tolerance level. The convergence is relatively fast. It takes between 30 to 100 periods (depending on the values of the parameters) to find the steady state. Let U_t be the unemployment in period t in a specific market and let $\bar{U}_t = \frac{1}{10}\sum_{j=t-9}^{t} U_j$ be the average unemployment of the 10 periods ending in t. We find the steady state, t_{SS}, as follows: $\mid \bar{U}_{t_{SS}-10} - \bar{U}_{t_{SS}} \mid < \epsilon$, where ϵ is a tolerance level.

[20]The results do not change much when using 1,000 or 10,000 workers. Results are available upon request.

[21]See Holzer (1987) and Blau and Robins (1990) for an investigation on different search intensities and offer rates for employed and unemployed workers.

again become important even when market tightness is low.

To make the comparison to labor-search models, we use a model of Bertrand competition between two firms as our benchmark. For example, Postel-Vinay and Robin (2002) (henceforth PVR) allows two firms to compete over the wages (*à la* Bertrand) of an employed worker and is closest to our model. The PVR framework in terms of networks is as follows. Given a set of firms, workers, and a matching technology *à la* PVR, construct the corresponding network with the same set of firms and workers, where a worker is linked to a firm if, and only if, the worker and the firm are matched by the matching technology. An important assumption in most search models, including PVR, is that a firm can negotiate with at most one worker and, in PVR, a worker can negotiate with at most two firms. This limits the type of networks that are realized and drives most of the differences between search models and our model.

We also compare our results to a perfectly competitive model (Walrasian outcome) in steady state. Recall that a market is in steady state when the flows into unemployment equal the flows out of unemployment. In the Walrasian outcome, all unemployed workers find jobs when $\theta \geq 1$ (*i.e.* there are more vacancies than unemployed workers). So the number of employed workers in steady state at the end of a period is the total number of workers in the market adjusted by the separation rate, $I \times (1-\delta)$. When $\theta < 1$, the fraction of unemployed workers in steady state, u_{SS}, is given by equalizing the flow of workers out of and into unemployment:

$$\big[u_{ss} + \delta(1-u_{ss})\big]\theta = \delta(1-u_{ss}),$$

where u_{ss} is the unemployment rate before the job separation shock.[22] The left-hand side represents the number of firms that hire workers (θ times the fraction of unemployed at the beginning of a period). The right-hand side represents the number of workers that lose their jobs. Thus, $u_{SS} = \frac{\delta(1-\theta)}{(1-\delta)\theta+\delta}$ when $\theta < 1$. Then, the number of employed workers in the Walrasian outcome, E_{SS}, is given by $E_{SS} = I \times (1-u_{ss})$. Due to the multi-period aspect of the model, the number of firms employing a worker in steady state, E_{SS}, is greater than the number of firms that enter in each period, $E_{SS} > J$.

6.2.2 Results

Distribution of Wages. Following the empirical results in Bagger, Fontaine, Postel-Vinay, and Robin (2014), we set $\lambda = 0.05$. This implies that an unemployed worker is twenty times more likely to receive a link than an employed worker. Setting λ below one concentrates the network between the new vacancies and the unemployed workers. Figure 6 displays the

[22] We use lowercase letters for rates (*e.g.* the unemployment rate is u) and uppercase letters for levels (*e.g.* the number of unemployed workers is U).

distribution of wages in different markets when $\lambda = 0.05$.[23] (This figure is similar to Figure 1 but for the dynamic labor market model.) All workers now compete with unemployed workers and the average wage is close to the Walrasian outcome when ELF equals five.

Worker Mobility and Wage Growth. To analyze worker mobility and wage growth we keep $\lambda = 0.05$. Figure 7 shows the wage profiles of workers after an unemployment spell. In loose markets, workers start out with low wages and wages grow very slowly. In tight markets with low frictions, workers attain a high wage immediately out of unemployment and then their wages grow very little over their career. This leads to the result that as you reduce frictions, workers have lower median wage growth. Figure 8 (top panel) shows the median wage growth after twenty periods of employment. As a benchmark, we also display the wage growth for the Bertrand competition benchmark (see subsubsection 6.2.1). When $\theta < 1$, wage growth is reduced because firms have better outside options. When $\theta > 1$, reducing frictions causes firms to compete strongly for workers. This drives initial wages up leaving little room for wage growth.

The lower panel of Figure 8 displays the worker mobility, which is defined as the fraction of workers that make a job-to-job transition in a period. The intuition for why worker mobility decreases as frictions are lowered is similar to wage growth. When $\theta < 1$, firms are less likely to poach workers from another firm because they have better outside options that likely include an unemployed worker. When $\theta > 1$, workers are less likely to move from one firm to another because when they enter the labor market they immediately find a job that pays a high wage. Our results indicate that empirical researchers should be cautious when using worker mobility to make inference about the level of frictions in a market where network effects are present.

Welfare. To analyze welfare we need a welfare criterion. In the labor market model, the utility function of the firms that hire a worker is the production function minus the wage, $\nu(j) - w(j)$, and the utility function of the workers that are hired is the wage, $w(j)$. We use the utilitarian welfare criterion, Ω, that is sum of these utilities and corresponds to $\Omega = \sum_{j=1}^{\tilde{J}} \nu(j)$, where $j = 1, \ldots, \tilde{J}$ index the firms that hire a worker. The unconstrained first-best allocation is the one that maximizes the welfare in the absence of frictions. It corresponds to the Walrasian outcome.[24]

Following Bagger, Fontaine, Postel-Vinay, and Robin (2014), we use the following cumu-

[23] Figure A2 in the online appendix displays the distribution of wages in different markets when $\lambda = 1$. As discussed in the previous section, $\lambda = 1$ is inconsistent with the empirical literature and suppresses network effects. Comparing figures 6 and A2 shows the effect that including indirect competition has on wage dispersion and average wages.

[24] The same welfare analysis holds for the buyer-seller market using buyers, sellers, prices, and a single period. We obtain similar results. See Figure A1 in the online appendix.

lative distribution function for the productivity of firms:

$$F(\nu) = 1 - exp\Big(- \big[c_1(\nu - c_0)\big]^{c_2}\Big),$$

where $c_0 = 5$, $c_1 = 8$, and $c_2 = 0.7$. Thus, the utilitarian welfare in each market, Ω, represents the aggregate production in that market.

Figure 9 displays the aggregate production (welfare) and average labor productivity for different levels of frictions and for the Walrasian outcome (*i.e.* unconstrained first-best allocation).[25] The top panel in Figure 9 shows the ratio of the aggregate production in our model relative to the aggregate production in the Walrasian outcome.[26] As ELF is increased, the aggregate production in our model is close to the aggregate production in the Walrasian outcome. For example, when $\theta = 0.5$ and ELF=5, the aggregate production is approximately 97.5% of the aggregate production in the Walrasian outcome. So even in sparse networks, markets are close to the unconstrained first-best allocation.

We also investigate the allocation of workers to firms. The bottom panel of Figure 9 displays the average productivity of jobs of employed workers. We find that lowering frictions can improve or worsen the allocation of workers to more productive firms. In loose markets (lower θ), reducing frictions lowers average labor market productivity. The intuition is that when frictions are high, firms are less likely to have an outside option when linked with an employed worker. This leads to competition between firms. When frictions are lower, firms are more likely to have another link to an unemployed worker. Hence, there is less competition between firms. Recall that when there are no frictions and $\theta < 1$, there is no competition between firms since there are more unemployed workers than firms. In tight markets (high θ), lower frictions leads to the allocation of workers to the more productive firms as in standard search models.

7 Concluding Remarks

The defining characteristic of markets with frictions is that similar goods or services are traded at different prices, resulting in price dispersion. In this paper we use networks to characterize pairwise stable allocations and their supporting prices in buyer-seller markets with frictions. The central insight of the paper is that including indirect competition causes markets with frictions to have prices and allocations that look close to the Walrasian outcome. To study prices in large networks, we develop a computationally tractable algorithm that finds the upper and lower bounds of the set of prices that sustain any pairwise stable match. Network effects reverse the relationship between the level of frictions and many economic outcomes.

[25] All specifications display the aggregate production in steady state at the end of the period after the destruction shock occurred. So, $E_{SS} \leq I \times (1-\delta)$ in our model, and $E_{SS} = I \times (1-\delta)$ in the Walrasian outcome. The results do not change substantially if the welfare is calculated before the destruction shock occurs.

[26] Figure A5 in the online appendix shows the total aggregate production.

We find that lowering frictions leads to: 1) lower wage growth, 2) lower worker mobility, and 3) lower expected prices in loose markets ($\theta < 1$).

The main finding of our paper, that sparse networks look as if they were perfectly competitive, might seem inconsistent with the price dispersion observed in eBay, labor markets, automobile markets, *etc*. There are three dimensions in which our model provides a richer understanding of frictions and price dispersion. Consider the case of eBay, where search frictions arise because its search engine is quite sensitive to the buyers' search inquiry and to the sellers' title for the product listing. In addition, buyers most likely do not compare all the listings for a given product at a given time. First, one possible explanation for the price dispersion is that search frictions are relatively high (*i.e.* ELB<3 or buyers participate in less than three auctions on average). But this is unlikely to be the whole answer. Second, the structure of the network is an important factor in the formation of prices. Although our networks are generated by randomly forming links, this is clearly not the case at eBay. All buyers who make the same search inquiry receive the same list of items or products. Price dispersion in eBay may be more about the structure of the network and less about the absolute level of frictions. Third, our model makes clear predictions on the distribution of prices for any network. Given information on the participation of buyers in auctions, the actual network can be constructed. This can be used to decompose the sources of price dispersion into frictions and other factors, such as unobserved heterogeneity.

Econometric methods where identification is based on minimal assumptions provide a robust structural framework for inference improving credibility and robustness of the empirical analysis.[27] In this context, using pairwise stability as our matchmaking criterion can be used to develop an empirical framework in the spirit of credible econometrics.[28] Since pairwise stability is the weakest criterion for matchmaking that is consistent with Pareto efficiency, not specifying the game details allows the econometrician to weaken the behavioral assumptions that would otherwise be imposed by a specific game.

Our model sheds some light on a puzzle about the recent recovery from the Great Recession in the US. Empirical studies show that wage growth is lower compared to previous recoveries at the same unemployment rate (Yellen, 2014). Our model predicts that lower frictions imply lower wage growth. If we assume that search frictions in the labor market have been decreasing over time due to new technologies, our model provides a possible explanation for this puzzle.

When considering empirical applications, there are a number of ways the model could be enriched. For example, an application to eBay might consider endogenous search intensity. An application to the labor market might consider a more realistic production function including heterogeneity of both workers and firms, endogenous search intensity and endogenous entry of firms. The goal of this paper is to construct a parsimonious model that demonstrates

[27]For example, see Manski (2003), Tamer (2010), Pakes, Porter, Ho, and Ishii (2014).

[28]For example, see Fox (2010a), Fox (2010b), Agarwal (2015), and the references there. See Fox (2009) for a survey.

the importance of network effects in price dispersion, wage growth, allocations of goods and workers, *etc.* Enriching the model in other dimensions is an avenue for future research.

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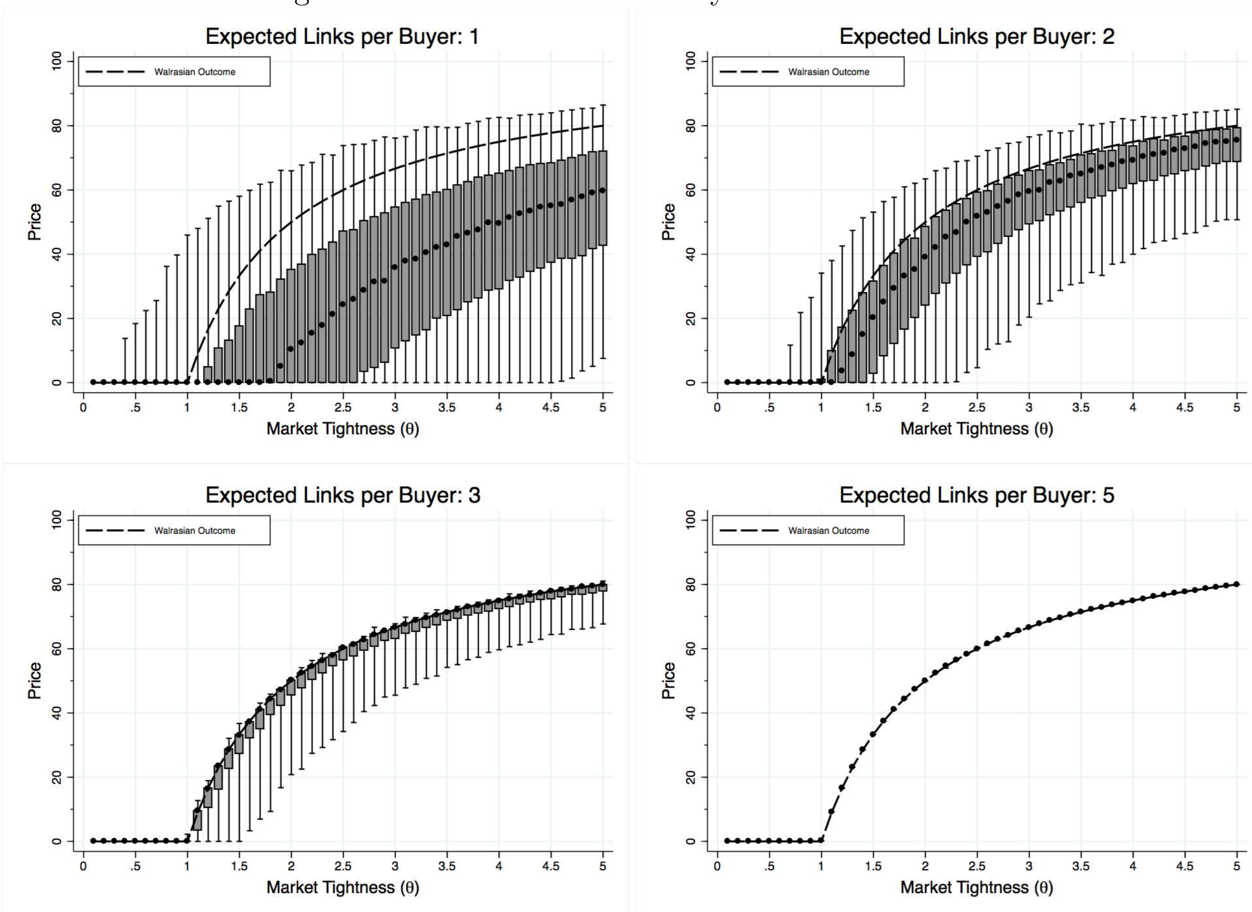
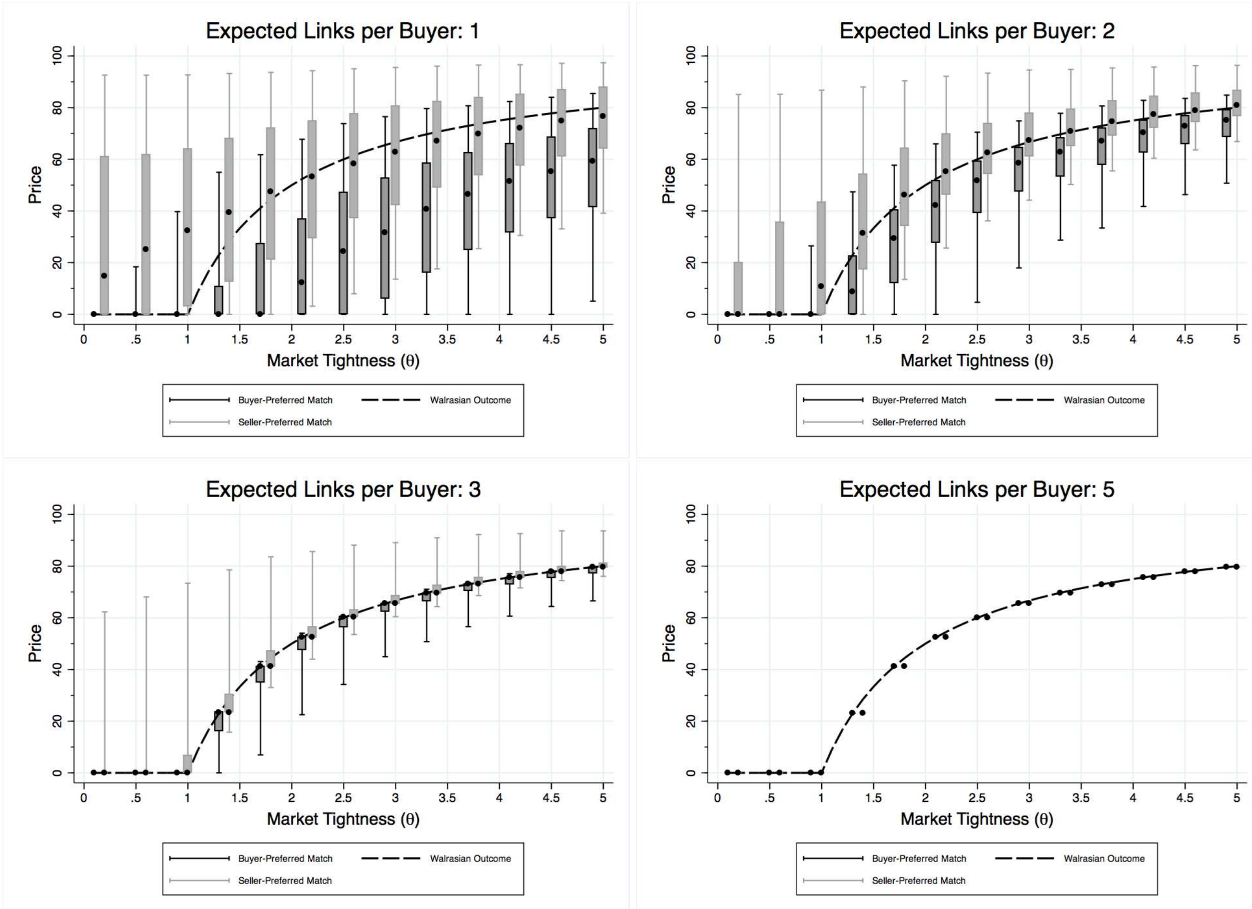
Figure 1: Price Distribution: Buyer-Preferred Match.



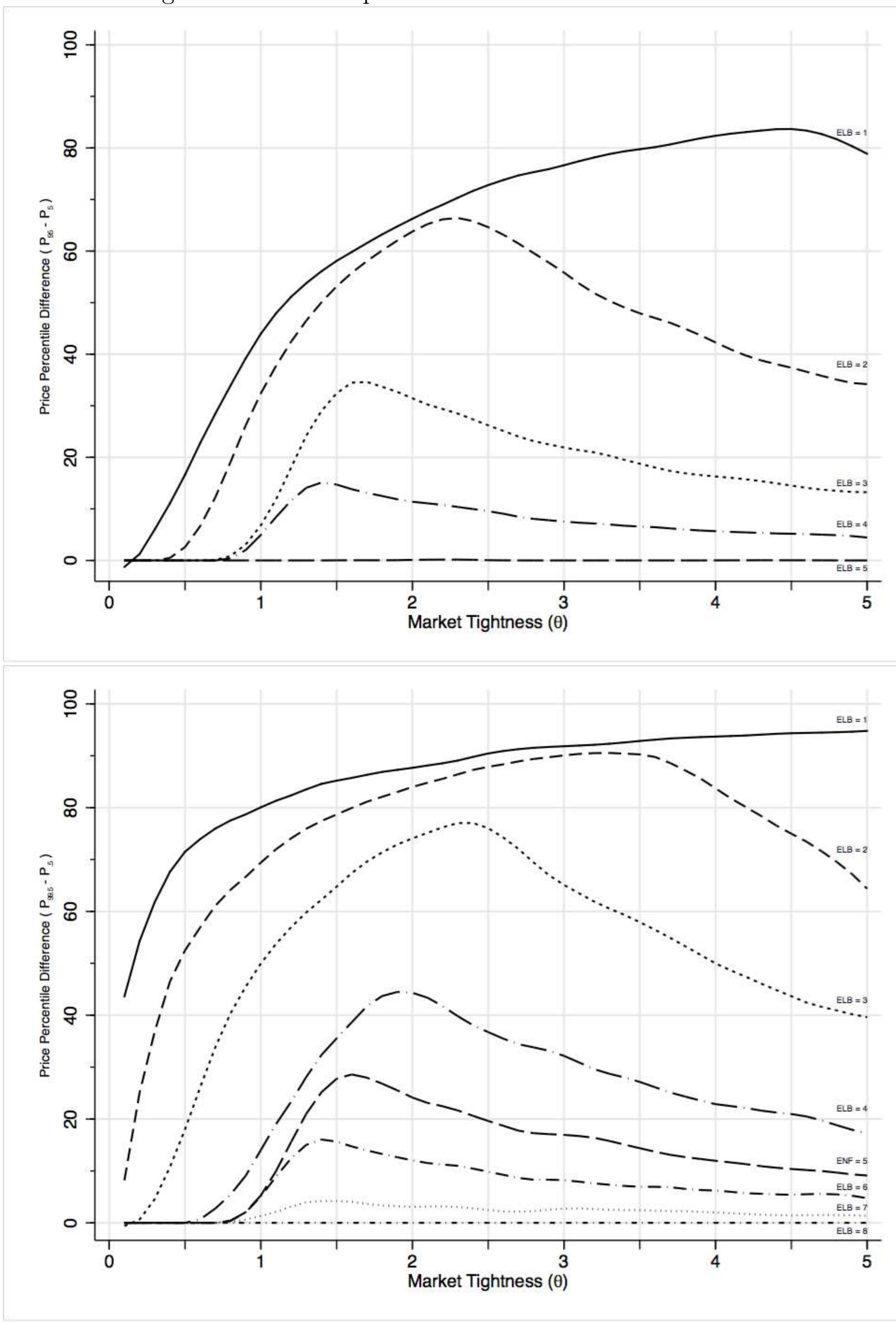
Notes: Starting in the top left, panels 1 to 4 figure display the empirical distribution of prices from the model using the buyer-preferred match disaggregated by: i) Market Tightness (which ranges from 0.1 to 5 in the horizontal axis in each graph) and ii) Expected Links per Buyer (1, 2, 3, and 5). Each vertical box corresponds to a simulated market characterized by those parameters. Each vertical box displays the 95th percentile (upper whisker), 75th percentile (upper hinge), median (black circle marker), 25th percentile (lower hinge), and 5th percentile (lower whisker). Note that buyers' valuation is normalized to range between 0 and 100 which, in turn, bounds the minimum and maximum prices between those values. If the 95th percentile coincides with the 5th percentile, then the figure shows only a dot (which corresponds to the median too). In addition, each panel displays the Walrasian outcome, p_{walras}. We describe how to calculate the Walrasian outcome in subsection 6.1.

Figure 2: Price Distribution: Buyer- vs. Seller-Preferred Matches.



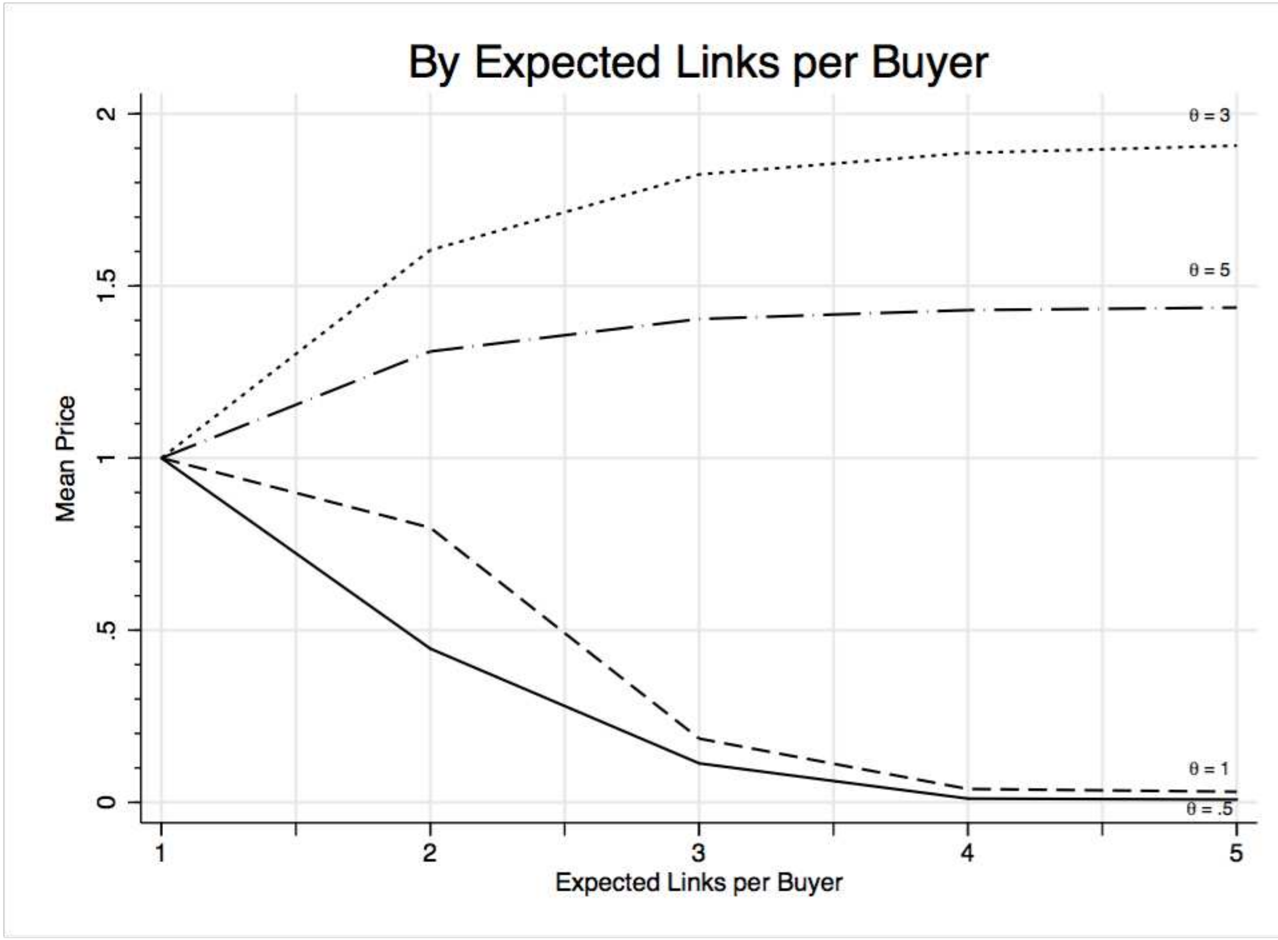
Notes: At each market tightness, panels 1 to 4 display the distribution of prices in the model using the sellers and the buyer-preferred match. For the seller-preferred match, each vertical box in this figure is identical to the corresponding vertical box in Figure 1. In addition, each panel displays the Walrasian outcome, p_{walras}. We describe how to calculate the Walrasian outcome in subsection 6.1. See the notes in Figure 1 for a description of the vertical boxes.

Figure 3: Price Dispersion and the Walrasian Outcome.



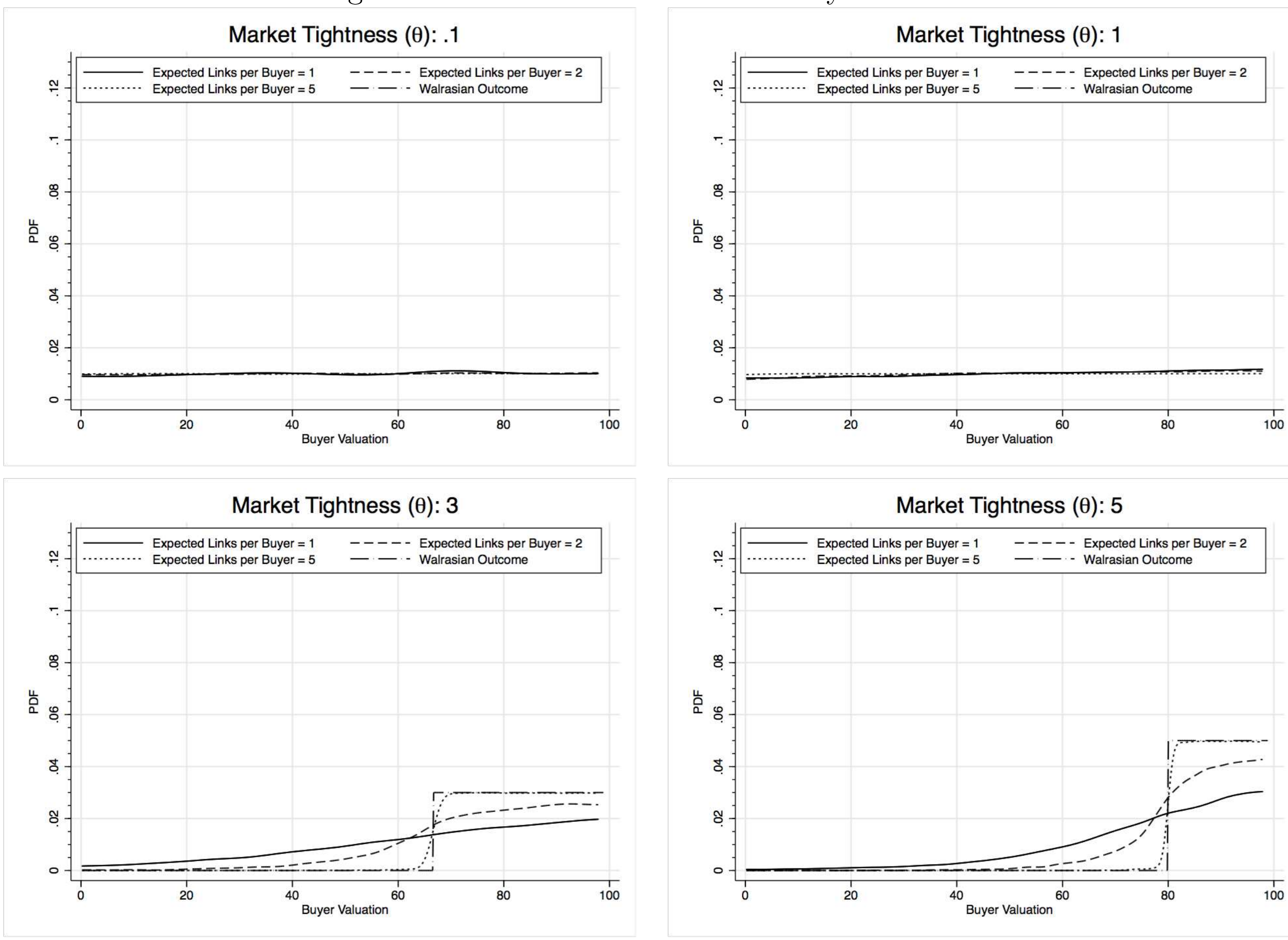
Notes: The top (bottom) figure displays the difference between the 95th (99.5th) price percentile and the 5th (0.5th) price percentile for different market tightness and expected links per buyer using a Nadaraya Watson kernel regression (of the percentile difference on market tightness) with an Epanechnikov kernel with bandwidth selected by cross validation. Price distributions are generated using the buyer-preferred match in a market with 10,000 sellers.

Figure 4: The Effect of Frictions on Mean Prices.



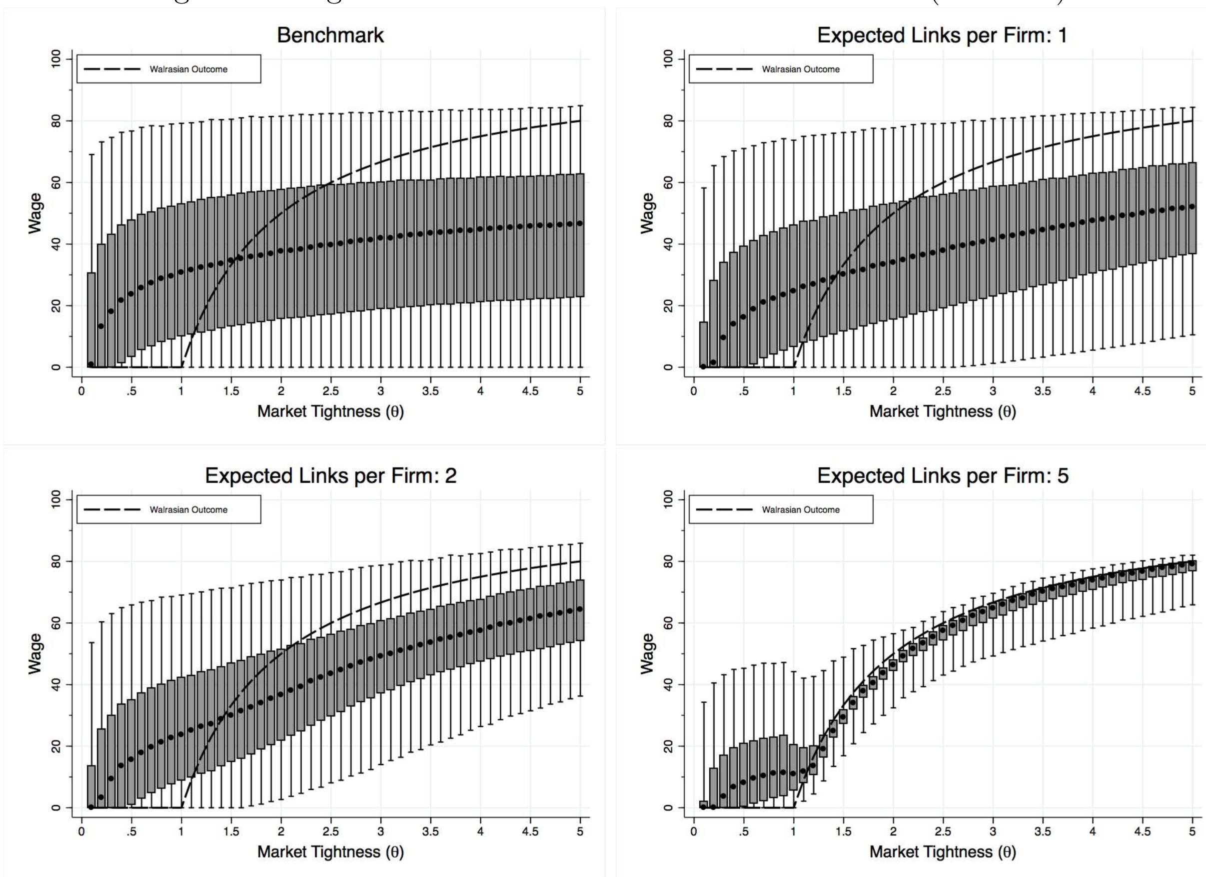
Notes: The figure displays the evolution of mean prices over expected links per buyer for different market tightness. For each market tightness, mean prices are normalized by the mean price when the expected links per buyer is one. So, by construction, mean prices for each market tightness coincide when the expected links per buyer is one.

Figure 5: Distribution of Matched Buyers.



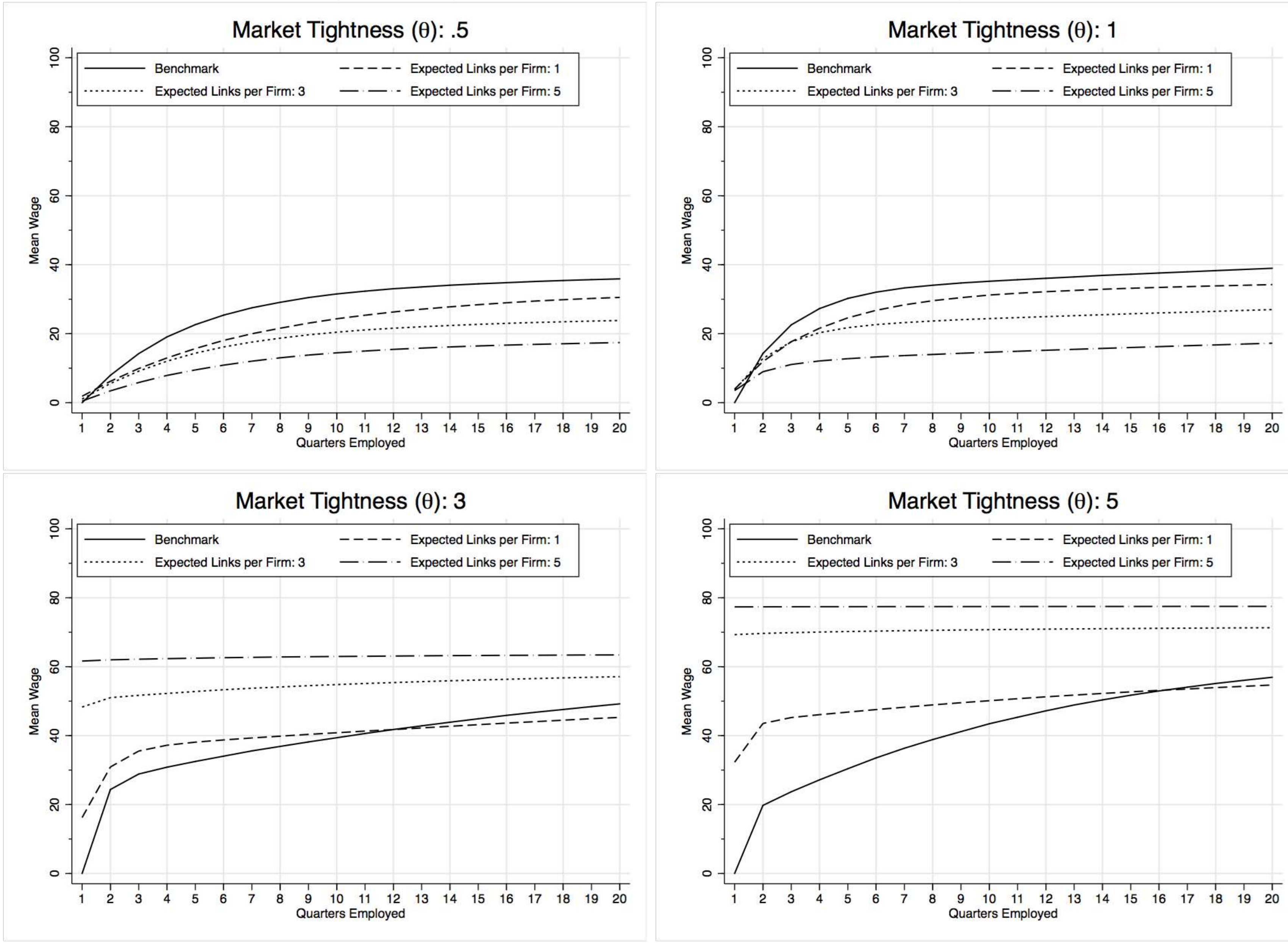
Notes: Each of the four panels in the figure displays the univariate kernel density estimation of the buyers' valuations distribution (buyers who bought a good from a seller) for three markets that differ in the ELB (1, 2, and 5) and for a given market tightness. In addition, each panel displays the distribution of matched buyers in the Walrasian outcome. We describe how to calculate the Walrasian outcome in subsection 6.1. Let ν denote buyers' valuation in the market. We estimate the probability density function in each market, $f(\nu)$, as: $\hat{f}_K(\nu; h) = \frac{1}{N\,h}\sum_{j=1}^{N} K\left(\frac{\nu-\nu(j)}{h}\right)$, where $K(z)$ is a standard univariate gaussian kernel function, h is the bandwidth that we choose by cross validation, and $\nu(j)$, $j = 1, \ldots, N$ are the valuations of the buyers who bought a good in each market. Note that we normalize buyers' valuation to range between 0 and 100 which, in turn, bounds the minimum and maximum prices between those values. Each valuation value between 0 and 100, can be interpreted as the percentile for any distribution of buyers' valuations. Given that the price distribution has its domain bounded we use a renormalization method to deal with the boundaries when estimating the productivity probability density function.

Figure 6: Wage Distribution in the Labor Market Model ($\lambda = 0.05$).



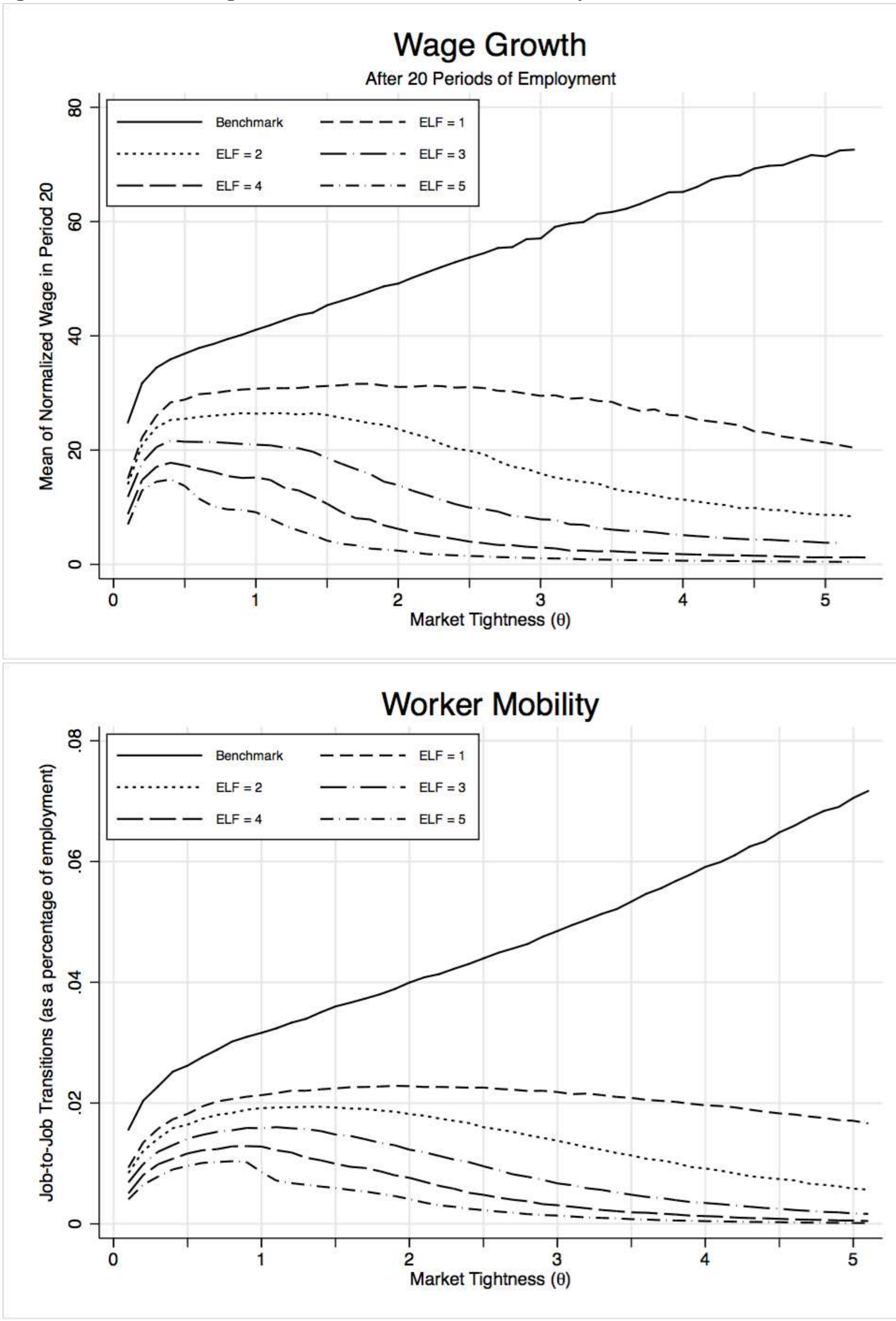
Notes: The figure displays the distribution of wages in the labor market model with $\lambda = 0.05$ (see subsection 6.2.1). All figures display the firm-preferred match. Starting in the top left, panel one shows the empirical distribution of wages for our benchmark, a model with Bertrand competition between at most two firms (see subsection 6.2.1). Panels 2 to 4 figure display the empirical distribution of wages from the model using the firm-preferred match disaggregated by: i) Market Tightness (which ranges from 0.1 to 5 in the horizontal axis in each graph) and ii) Expected Links per firm (1, 2, and 5). Each vertical box corresponds to a simulated market characterized by those parameters. Each vertical box displays the 95th percentile (upper whisker), 75th percentile (upper hinge), median (black circle marker), 25th percentile (lower hinge), and 5th percentile (lower whisker). Note that firms' productivity is normalized to range between 0 and 100 which, in turn, bounds the minimum and maximum wages between those values. If the 95th percentile coincides with the 5th percentile, then the figure shows only a dot (which corresponds to the median too). In addition, each panel displays the Walrasian outcome, w_{walras}. We describe how to calculate the Walrasian outcome in subsection 6.1, where in the case of the labor market $w_{walras} = p_{walras}$.

Figure 7: Wage Profile in the Labor Market Model: Mean Wages.



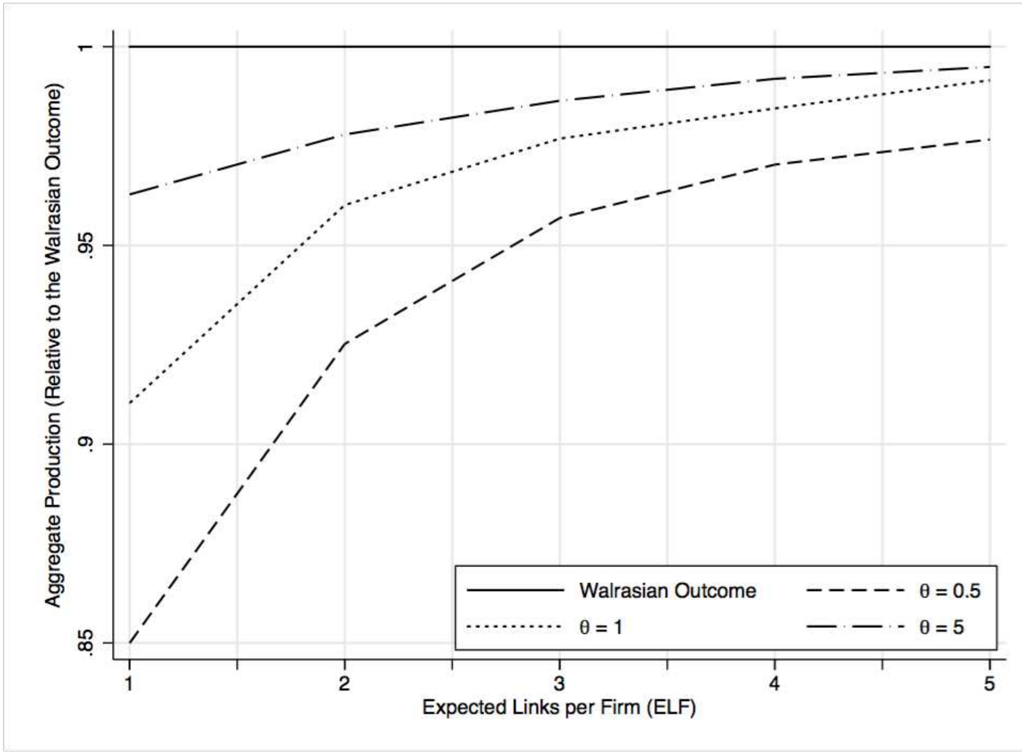
Notes: Each figure displays the wage profile in the labor market model with $\lambda = 0.05$ and the firm-preferred match (see subsection 6.2.1). The horizontal axis shows the number of periods that the worker has been employed. Mean wages are computed by period. The sample is the set of workers that are employed at least 20 consecutive periods in steady state. Each figure shows the wage profile for a given expected number of links per firm (1, 3, and 5) for different market tightness (0.5, 1, 3, and 5). As a benchmark, we also display the wage profile for a model with Bertrand competition between at most two firms (see subsection 6.2.1).

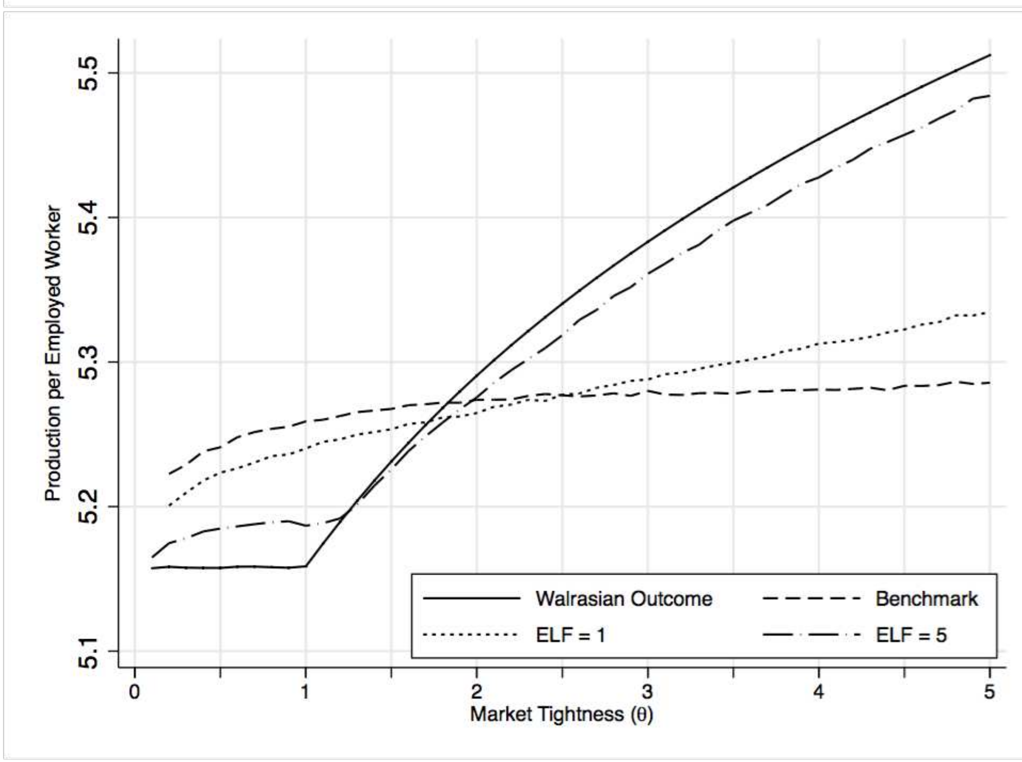
Figure 8: Mean Wage Growth and Worker Mobility in the Labor Market Model.



Notes: The top figure displays wage growth by market tightness and expected number of links per firm. The wage growth is defined as the wage of the worker in period 20 minus the wage of the same worker in period 1. We use the sample of workers who have been employed for at least 20 periods. The bottom figure displays the worker mobility by market tightness and expected number of links per firm. Worker mobility is defined as the probability that an employed worker makes a job-to-job transition in a period. For both figures, we set $\lambda = 0.05$ and use the firm-preferred match (see subsection 6.2.1). As a benchmark, we also display the results for a model restricted to have Bertrand competition between at most two firms (see subsection 6.2.1).

Figure 9: Welfare in the Labor Market Model.





Notes: The top panel displays the aggregate production relative to the Walrasian outcome by ELF and by market tightness. For example, when $\theta = 0.5$ and ELF=5, the aggregate production is approximately 97.5% of the aggregate production in the Walrasian outcome. The aggregate production in each market represents the utilitarian welfare in that market. Figure A5 in the online appendix shows the total aggregate production. The bottom panel displays the average productivity per worker (total production divided by number of employees) for different markets. As a benchmark, we also display the results for a model restricted to have Bertrand competition between at most two firms (see subsection 6.2.1). We set $\lambda = 0.05$ and use the firm-preferred match (see subsection 6.2.1). Following Bagger, Fontaine, Postel-Vinay, and Robin (2014), we use the following cumulative distribution function for the productivity of firms $F(\nu) = 1 - exp\Big(-\big[c_1(\nu - c_0)\big]^{c_2}\Big)$, where $c_0 = 5$, $c_1 = 8$, and $c_2 = 0.7$.

Appendix

A Proof of Theorem 1

In this section we prove Theorem 1.

Theorem 1: Let $\mathcal{N} = (\mathcal{J}, \mathcal{I}, E; \nu, b)$ be a network. Let M be a matching. Then the following are equivalent:

1. There exists p_M such that M is stable with respect to p_M
2. There exists an abstraction in fully connected graphs $A = (\mathcal{G}, E^*; p^*)$ such that M is stable with respect to A.

Proof. 1 implies 2: Let $A = (\mathcal{G}, E^*; p^*)$ be defined as follows: For each $(i,j) \in M$ let $G_{ij} = (\{i\}, \{j\}, \{(i,j),(j,i)\})$. For each $i \in \mathcal{I}$ such that $i^*(j) = \emptyset$ let $G_{i\emptyset} = (\{i\}, \emptyset, \emptyset)$ and for each $j \in \mathcal{J}$ such that $j^*(i) = \emptyset$, let $G_{j\emptyset} = (\emptyset, \{j\}, \emptyset)$. Let $p(G_{ij}) = p_M(i,j)$, $p(G_{i\emptyset}) = b(i)$ and $p(G_{j\emptyset}) = \nu(j)$. Define E^* so that $(G, G') \in E^*$ if and only, in the original graph, the buyer in G is linked with the seller in G'. These ingredients define an abstraction in fully connected graphs. We need only show that M is stable with respect to A. Since subgraphs are either trivial or include a single buyer and a single seller, M restricted to G is vacuously stable. Let $G, G' \in \mathcal{G}$ be such that $(G, G') \in E^*$. Let $j \in \mathcal{J}_G$ and $i \in \mathcal{I}_{G'}$. If $j^*(i) = \emptyset$ then $b(i) \geq v(j)$ because (M, p_M) cannot be blocked. Thus, $p^*(G) = v(j) \leq b(i) = p^*(G')$. If $i^*(j) = \emptyset$ then $\nu(j) \leq v(i)$ because (M, p_M) cannot be blocked. Thus, $p^*(G) = \nu(j) \leq v(i) = p^*(G')$. Otherwise, $p^*(G) = p_M(i^*(j), j)$, $p^*(G') = p_M(i, j^*(i))$. Then, by stability, $p^*(G) \leq p^*(G')$.

2 implies 1: Let A be as in item 2. Note that, $i \in \mathcal{I}_G \Rightarrow v(i) \geq p^*(G)$. In words, if i belongs to a subgraph G, then the virtual price i is receiving must be at least the value of the subgraph (if i is matched, equality hods, if not, $v(i) = b(i)$ and then $b(i) \geq p^*(G)$.) Similarly, $j \in \mathcal{J}_G \Rightarrow v(j) \leq p^*(G)$. For each edge $(i,j) \in M$ let $p_M(i,j) = p^*(G)$ where G is such that $(i,j) \in E_G$. This is well defined because A does not break M and because M restricted to G is stable. We must show that no match can be blocked. Let $(i,j) \in M$ be arbitrarily chosen and G be such that $(i,j) \in E_G$. Because M restricted to G is stable, for all $(i,j) \in E_G \setminus M$, $v(i) \geq v(j)$. Now let i' be such that $(i', j) \in E$ but $i' \notin \mathcal{I}_G$; *i.e.*, i' is linked to j but does not belong to the same subgraph in the abstraction. Let $G' \neq G$ be such that $i' \in \mathcal{I}_{G'}$. Then $(G, G') \in E^*$ so $p^*(G) \leq p^*(G')$ by cheapest sorting. Thus, $v(j) \leq p^*(G) \leq p^*(G') \leq v(i')$ so (i', j) does not block (M, p_M). Since j, i and i' were arbitrarily chose, then no blocks to (M, p_M) exist, so M is stable with respect to p_M. $\square$

B Formal algorithm and proof of Proposition 1

In this appendix we discuss the formal algorithm used in the main paper and prove some of its properties. We now present the basic notation we use in the match determination program. Let $s^t \in \mathbb{R}^{J\times I}$ be a matrix of prices for each buyer-seller pair. Each element, $s^t_{i,j}$, represents the price that buyer j would have to bid for seller i if j were to bid for i in round t. Vector q^t represents the bidding queue in period t: $q^t_n = j \in \mathcal{J}$ represents that in round t, buyer j is the n-th bidder in the queue. The algorithm ends when $l(q) = 0$, where $l(q)$ indicates the length of q. Quantity $D(j)$ indicates j's demand. Quantities with primes will indicate quantities that will carry over to the next round of the algorithm. Finally, for each seller j, we use the following payoff function to model that a buyer can only buy a good from a seller if the two are linked in the network: $u_j : \mathcal{I} \times \mathbb{R}^{I\times J} :\to \bar{\mathbb{R}}$, $u_j(i,s) = \nu(j) - s_{i,j}$ if $(i,j) \in E$ and $u_j(i,s) = -\infty$ otherwise.

Recall some notational conventions: given a matching M, $i^*(\cdot) : \mathcal{J} \to \mathcal{I} \cup \{\emptyset\}$ satisfies $(i^*(j), j) \in M$ for each M-matched j, and $i^*(j) = \emptyset$ if j is M-unmatched. Analogously, $j^*(\cdot) : \mathcal{I} \to \mathcal{J} \cup \{\emptyset\}$ satisfies $(i, j^*(i)) \in M$ for each M-matched i, and $j^*(i) = \emptyset$ if i is M-unmatched. Also, even if not explicitly stated, the network is denoted $\mathcal{N} = (\mathcal{I}, \mathcal{J}, E; b, \nu(\cdot))$, $I = \#\mathcal{I}$, $J = \#\mathcal{J}$.

Match Determination Program.

Input$= (\mathcal{N}, s^0, (u_1, ..., u_J), h^0, q)$ where:

- $s^0 = (s^0_1, ..., s^0_J) \in \mathbb{R}^{J\times I}$, $s^0_j = (b, ..., b) \in \mathbb{R}^I$,
- For each buyer j, and each $t \in \mathbb{N} \cup \{0\}$, $u_j(i, s^t) = \nu(j) - s^t_{i,j}$ if $(i,j) \in E$ and $u_j(i, s^t) = -\infty$ if $(i,j) \notin E$,
- $h^0 = (0, ..., 0) \in \mathbb{R}^{I\times J}$.
- $q^0 \in \mathcal{J}^J$ such that $q^0_n = q^0_m$ iff $m = n$.

Start step $R(1)$:

R(t). Set $h^t = h$, $s^t = s$, $q^t = q$, $j = q_1$.

1. If $\max\{u_j(i,s) : i \in \mathcal{I}\} < 0$ set $s' = s$ and $h' = h$, $q' = (q_2, ..., q_{l(q)})$.
   a. If $l(q') = 0$, stop, set $M = \{(i,j) : h_{i,j} = 1\}$, and *Output*$= M$.
   b. If $l(q') \neq 0$, set $q^{t+1} = q'$, $s^{t+1} = s'$, $h^{t+1} = h'$ and proceed to $R(t+1)$.
2. If $\max\{u_j(i,s) : i \in \mathcal{I}\} \geq 0$ let $D(j) \in \arg\max\{u_j(i,s) : i \in \mathcal{I}\}$.
   a. If $\arg\max\{u_j(i,s) : i \in \mathcal{I}\}$ has more than one element, select $D(j) \in \arg\max_{i\in\mathcal{I}}\{u_j(i,s)\}$ randomly.

3. Set the following parameters:

   a. $s'_{D(j),j} = s_{D(j),j}$; for all $j' \neq j$, $s'_{D(j),j'} = s_{D(j),j} + \frac{\Delta}{2}$; $s'_{i'',j''} = s_{i'',j''}$ elsewhere,

   b. If $h_{D(j),j'} = 0$ for all $j' \neq j$, set $q' = (q_2, ..., q_{l(q)})$; if $h_{D(j),j'} = 1$ for some $j' \neq j$, set $q' = (q_2, ..., q_{l(q)}, j')$,

   c. $h'_{D(j),j} = 1$; for all $j' \neq j$, $h'_{D(j),j'} = 0$; $h'_{i'',j''} = h_{i'',j''}$ elsewhere.

4. If $l(q') = 0$, stop. Set $M = \{(i,j) : h_{i,j} = 1\}$. *Output*$= M$.

   If $l(q) \neq 0$ set $h' = h^{t+1}$ and $s' = s^{t+1}$ and $q' = q^{t+1}$. Then start $R(t+1)$.

Although this algorithm is motivated by Crawford and Knower (1981) and Kelso and Crawford (1982), there are three important differences. The first is that firm productivities increase in increments of Δ whereas bids increase in increments of $\frac{\Delta}{2}$. Since Crawford and Knower (1981) and Kelso and Crawford (1982) work with a discrete core, the algorithm they run produces a stable match when both bid increments and productivities increase by the same amount. However, since we work with a continuous core, it is not true that the matching generated by such an algorithm is stable. One can construct examples where the matching generated by the algorithms in Crawford and Knower (1981) and Kelso and Crawford (1982) (say, M) satisfies that there is no price function p_M such that M is stable with respect to p_M. We provide one example in section B (Figure A6) in the online appendix. The modification we introduce, that bids live in a finer grid than firm productivities, helps us bypass this problem. The second difference with the algorithms in Crawford and Knower (1981) and Kelso and Crawford (1982) is that we only use their program to find the matching, but not the prices that make it stable. The reason is that their algorithm makes prices rise too quickly. While in some networks the price generated by the algorithms in Crawford and Knower (1981) and Kelso and Crawford (1982) is the pointwise minimum price that makes the matching stable, this is not always guaranteed. This is because in our setting we violate the non-indifference assumptions made in Crawford and Knower (1981) and Kelso and Crawford (1982). In order to capture, for each matching, the pointwise maximum and minimum prices at which that matching is stable we run two independent programs. We call these the Price Determination Programs, and we describe them below. The first Price Determination Program (I), outputs the pointwise minimum price function at which a matching is stable. The second Price Determination program (II), outputs the pointwise maximum price function at which a matching is stable. The third difference is that, when a seller i accepts a bid from a buyer j, then any future bid buyer j' submits to i must outbid j's bid. In symbols, if in round t seller i accepts bid $s^t_{i,j}$ from j, then at the end of round t all sellers j' linked to i have their bid price raised to $s^{t+1}_{i,j'} = s^t_{i,j} + \frac{\Delta}{2}$. This modification reduces the run time of the algorithm by a factor of four.

Price Determination Program (I).

Input$= (\mathcal{N}, M)$.

1. For each $i \in \mathcal{I}$ such that $j^*(i) = \emptyset$ set $\rho_i^1 = b$.

2. For each $i \in \mathcal{I}$ such that $j^*(i) \neq \emptyset$ set $\rho_i^1 = \max\{\nu(j) : (i,j) \in E \text{ and } i^*(j) = \emptyset\}$.

3. Set $t = 1$. Start step 4(1).

4(t). Given $(\rho_1^t, ..., \rho_I^t)$:

   a. For each $i \in \mathcal{I}$ such that $j^*(i) = \emptyset$ set $\rho_i^{t+1} = \rho_i^t$.

   b. For each $i \in \mathcal{I}$ such that $j^*(i) \neq \emptyset$, let $j \equiv j^*(i)$. Then, set

$$\rho_i^{t+1} = \max\{\rho_{i'}^t : (\exists j')(i', j') \in M, \, (i, j') \in E\}.$$

   c. If for all $i \in \mathcal{I}$ $\rho_i^{t+1} = \rho_i^t$:

      * For each i such that $j^*(i) \neq \emptyset$ set $p_M(i, j^*(i)) = \rho_i^{t+1}$.
      * *Output*$= (p_M(\cdot))$.

   d. Otherwise, start step $4(t+1)$.

Price Determination program (I) outputs the minimum price at which M can be made stable.

Price Determination Program (II).
Input$= (\mathcal{N}, M)$.

1. For each $i \in \mathcal{I}$ such that $j^*(i) = \emptyset$ set $\rho_i^1 = b$.

2. For each $i \in \mathcal{I}$ such that $j^*(i) \neq \emptyset$ set $\rho_i^1 = \nu(j^*(i))$.

3. Set $t = 1$. Start step 4(1).

4(t). Given $(\rho_1^t, ..., \rho_I^t)$:

   a. For each $i \in \mathcal{I}$ such that $j^*(i) = \emptyset$ set $\rho_i^{t+1} = \rho_i^t$.

   b. For each $i \in \mathcal{I}$ such that $j^*(i) \neq \emptyset$, let $j \equiv j^*(i)$. Then, set

$$\rho_i^{t+1} = \min\{\rho_{i'}^t : (i', j) \in E\}.$$

   c. If for all $i \in \mathcal{I}$ $\rho_i^{t+1} = \rho_i^t$:

      * For each i such that $j^*(i) \neq \emptyset$ set $p_M(i, j^*(i)) = \rho_i^{t+1}$.
      * *Output*$= (p_M(\cdot))$.

   d. Otherwise, start step $4(t+1)$.

Price Determination program (II) outputs the maximum price at which M can be made stable.

In Section 5 we claimed our algorithm has four properties: it ends in finite time, it selects a pairwise stable allocation, and for each allocation it selects the pointwise minimum and maximum prices that sustain it.

Proposition 1: The deferred acceptance algorithm has the following properties:

1. It stops after a finite number of rounds.

2. It outputs a pairwise stable allocation.

3. Price Determination program (I) outputs the pointwise minimum price function at which M is stable.

4. Price Determination program (II) outputs the pointwise maximum price function at which M is stable.

We now prove these items one at a time. In what follows, we use MDP and PDP to abbreviate the Matching Determination Program and the Price Determination Program respectively. Finally, if $(x_i)_{i\in\mathcal{I}}$ is a vector indexed by $\mathcal{I}$ we use the convenient shorthand notation $x_\cdot$ to denote the whole vector, whenever ambiguity is unlikely.

We need two lemmas: the first, shows that, given M produced by the MDP, there exist prices p_M such that M is stable with respect to M. The second shows that the prices generated by the PDP are weakly lower than any p_M such that M is stable with respect to M. To prove these Lemmas, recall that $(\rho_i^t)_{i\in\mathcal{I},t\geq 1}$ from the PDP(I) is defined as follows:

- If $j^*(i) = \emptyset$, $\rho_i^t = b$ for all t.

- If $j^*(i) = j$ for some $j \in \mathcal{J}$, $\rho_i^1 = \max\{\nu(j) : (i,j) \in E, i^*(j) = \emptyset\}$ for each $i \in \mathcal{I}$, and $\rho_i^t = \max\{\rho_{i'}^{t-1} : (\exists j', i') : (j', i') \in M, (j', i) \in E\}$ for all $t \geq 2$.

The following properties imply that there exists a value T such that, for all i and all $t \geq T$, $\rho_i^t = \rho_i^{t+1}$. That is, $(\rho_\cdot^t)_{t\geq 0}$ is eventually constant. We let $\rho_\cdot^\infty \equiv \lim_{t\to\infty} \rho_\cdot^t$.

1. For all i, $\rho_i^t \leq \rho_i^{t+1}$. This follows because $\rho_i^{t-1} \in \{\rho_{i'}^{t-1} : (\exists j') : (j', i') \in M, (j', i) \in E\}$ whenever $j^*(i) = j$ and $\rho_i^t = b$ whenever $j^*(i) = \emptyset$.

2. For all i, $\rho_i^t \leq \max\{\nu(j) : j \in \mathcal{J}\}$.

3. For all i, if $\rho_i^t \neq \rho_i^{t+1}$ then $\rho_i^{t+1} - \rho_i^t \geq \Delta$.

Finally, recall that $\Delta \in \mathbb{R}$ is chosen so that for all $j \in \mathcal{J}$, $\nu(j) = b + k_j\Delta$ for for $k_j \in \mathbb{N} \cup \{0\}$. In particular, $\nu(j) \geq b$ for all j. This normalization only rules out uninteresting cases where a buyer never places a bid and is never matched to a seller.

Lemma 1. *Let M be the matching produced by the MDP. Then, there exists p_M such that M is stable with respect to p_M.*

Proof. For each edge $(i, j) \in M$ define $p_M(i, j) = \rho_i^\infty$ where $(\rho_i^t)_{i \in \mathcal{J}, t \in \mathbb{N} \cup \{\infty\}}$ is as defined in the PDP(I). Also, let T be the last round of the MDP and let $[s_{i,j}^T]_{i \in \mathcal{I}, j \in \mathcal{J}}$ be the matrix of final prices generated by the MDP. We show that M is stable with respect to p_M. Assume first that $(i, j) \in E$ are such that $j^*(i) = i^*(j) = \emptyset$. Then i received no bids, so $s_{i,j}^T = b$. Since the algorithm ended, it must be that $u_j(i, s^T) < 0 \Leftrightarrow \nu(j) < b$, a contradiction. Thus, there does not exist an edge $(i, j) \in E$ such that $j^*(i) = i^*(j) = \emptyset$ so, a fortiori, no such edge $(i, j) \in E$ blocks M. Now let $(i, j) \in M$. Pick $j' \neq j$ such that $(i, j') \in E$. We show $(i, j') \in E$ does not block M. If $i^*(j') = \emptyset$ then $\nu(j') \leq \rho_i^1 \leq \rho_i^\infty = p_M(i, j)$. If $i^*(j') \neq \emptyset$ then $\rho_i^\infty \geq \rho_{i^*(j')}^\infty$ by construction. Thus, $p_M(i, j) \geq p_M(i^*(j'), j')$. Thus, (i, j') does not block M. Pick $i' \neq i$ such that $(i', j) \in E$. We show $(i', j) \in E$ does not block M. If $j^*(i') \neq \emptyset$ then $\rho_{i'}^\infty \geq \rho_i^\infty$ by construction. Thus, $p_M(i', j^*(i')) \geq p_M(i, j)$. Let $j^*(i') = \emptyset$. Then i' never received a bid. Let t be the last time j bids for i. Since bidders bid for the cheapest seller $s_{i,j}^t \leq s_{i',j}^t = b$. By definition of t, $s_{i,j}^t = s_{i,j}^T$ so $s_{i,j}^T = p_M(i, j) = b$. We use this to argue that $\rho_{\hat{i}}^1 = b$ for all matched $\hat{i}$ such that $(i, j^*(\hat{i})) \in E$ (note that i is one such $\hat{i}$). Pick $\hat{i}$ such that $j^*(\hat{i}) \neq \emptyset$ and $(i, j^*(\hat{i})) \in E$. Then, $s_{\hat{i},j^*(\hat{i})}^T \leq s_{i,j^*(\hat{i})}^T$.[29] Since $s_{i,j}^T = b$ and $s_{i,j^*(\hat{i})}^T \leq s_{i,j}^T + \frac{\Delta}{2}$, then $s_{\hat{i},j^*(\hat{i})}^T \leq b + \frac{\Delta}{2}$. If there exists $\tilde{j}$ such that $i^*(\tilde{j}) = \emptyset$ and $(\hat{i}, \tilde{j}) \in E$, then it must be that $\nu(\tilde{j}) = b$. Indeed, if $\nu(\tilde{j}) > b$ then $\nu(\tilde{j}) \geq b + \Delta$ which is a contradiction: since $s_{\hat{i},j^*(\hat{i})}^T \leq b + \frac{\Delta}{2}$ and $i^*(\tilde{j}) = \emptyset$, $u_{\tilde{j}}(\hat{i}, s^T) \geq 0$, which contradicts T being the last round of the MDP. Thus, $\nu(\tilde{j}) = b$. Hence, $\rho_{\hat{i}}^1 = b$. We now conclude the argument in an inductive manner: if $\rho_{\hat{i}}^k = b$ for some k and all $\hat{i}$ that satisfy $j^*(\hat{i}) \neq \emptyset$ and $(i, j^*(\hat{i})) \in E$, then by construction $\rho_i^{k+1} = b$. Thus, $\rho_i^\infty = b = p_M(i, j)$. Thus, (i', j) does not block M. Therefore, M is stable with respect to p_M. □

Lemma 2. *Let M be the matching generated by the MDP. Let p_M be any price function such that M is stable with respect to p_M (which is well defined by our previous lemma) and let v be the associated payment function. Let p_M^* be the price generated by the PDP(I) and v^* the corresponding payment function. Then, $v^* \leq v$.*

Proof. Let M, p_M, v, p_M^* and v^* be as in the statement of the lemma. Then, for all i, $v(i) \geq \rho_i^1$. Indeed, if $j^*(i) = \emptyset$ then $v(i) = b = \rho_i^1$. If $j^*(i) = j$ for some j then, by stability of M with respect to p_M, $v(i) \geq \nu(j')$ for each j' such that $i^*(j') = \emptyset$. Thus, $v(i) \geq \rho_i^1$.

[29] Indeed, let t be the last time $j^*(\hat{i})$ bids for $\hat{i}$. Then, $s_{\hat{i},j^*(\hat{i})}^t = s_{\hat{i},j^*(\hat{i})}^T$, and $s_{\hat{i},j^*(\hat{i})}^t \leq s_{i,j^*(\hat{i})}^t$, where the last inequality holds because buyers always bid for the cheapest sellers. By monotonicity of the matrix of prices, $s_{i,j^*(\hat{i})}^t \leq s_{i,j^*(\hat{i})}^T$. Thus, $s_{\hat{i},j^*(\hat{i})}^T \leq s_{i,j^*(\hat{i})}^T$.

We now show that if $v \geq \rho^t$ for some k, then $v \geq \rho^{t+1}$. Indeed, for all i such that $j^*(i) = \emptyset$, $v(i) = b = \rho_i^t = \rho_i^{t+1}$. For all i such that $j^*(i) = j$, we have the following:

$$\begin{aligned}
\rho_i^{t+1} &= \max\{\rho_{i'}^t : (\exists j', i')(i', j') \in M \text{ and } (i, j') \in E\} \\
&\leq \max\{v(i') : (\exists j', i')(i', j') \in M \text{ and } (i, j') \in E\} \leq v(i),
\end{aligned}$$

where the last inequality follows from stability of M with respect to p_M. Thus, for each t and each i, $\rho_i^t \leq v(i)$. Hence $\rho_\cdot^\infty \equiv v^*(\cdot) \leq v(\cdot)$. □

We now prove items 1 through 4 of Proposition 1.

1. The algorithm ends in finite time.

Proof. By the same arguments as Crawford-Knoer, the matching determination program ends in finite time. Furthermore, let $K \in \mathbb{N}$ satisfy $\max\{\nu(j) : j \in \mathcal{J}\} = b + K\Delta$. Then the price determination program ends in at most $2K$ rounds. □

2. The algorithm outputs a pairwise stable matching.

Lemma 1 already shows that M is stable with respect to p_M when p_M is the price function generated by PDP(I). Lemma 3 below shows that M is also stable with respect to p_M when p_M is the price function generated by PDP(II).

3. Price Determination program (I) outputs the pointwise minimum price function at which M is stable.

Proof. Follows from lemma 2 and that M is stable with respect to p_M, where p_M is the price function generated by the PDP (I). □

4. Price Determination program (II) outputs the pointwise maximum price function at which M is stable.

The proof is analogous to the proof that the PDP(I) outputs the pointwise minimum price function at which M is stable. Reasoning as in Lemma 2, if p_M is such that M is stable at prices p_M, and v is the corresponding payment function, then $v(\cdot) \leq \rho_\cdot^\infty$ (Lemma 4 below). Moreover, M is stable at prices induced by $\rho_\cdot^\infty$ (Lemma 3, below). The result then follows from Lemmas 3 and 4.

Lemma 3. *Let M be the matching generated by the MDP, and let p_M be the prices generated by the PDP(II). M is stable with respect to p_M.*

Proof. Let M be the matching outputted by the matching determination program, and p_M be the prices generated by the price determination program. Assume $(i, j) \in E$, $j^*(i) = i^*(j) = \emptyset$. Since there exists $\hat{p}_M$ such that M is stable with respect to $\hat{p}_M$ then $\nu(j) \leq b$.

Thus (i,j) do not block M at ρ^{∞}. Now consider $(i,j) \in M$. We show no seller and no buyer wishes to block (i,j):

a. *No Buyer blocks:* Let j' be such that $(i,j') \in E$. If $i^*(j') \neq \emptyset$ then, by construction, $\rho^{\infty}_{i^*(j')} \leq \rho^{\infty}_i$, so (i,j') does not block. Assume now that $i^*(j') = \emptyset$. We say a seller i' is indirectly connected to seller j if there exists sequences $(i_1, ..., i_k)$ and $(j_1, ..., j_{k-1})$ such that $(i_1, j) \in E$, $(i_1, j_1) \in E$, $(i_2, j_1) \in E$, ..., $(i_k, j_{k-1}) \in E$, with $i' = i_k$. That is, if a path can be constructed from j to i'. By construction, $\min\{\nu(j^*(i')) :$ i' is indirectly connected to j$\} \leq \rho^{\infty}_i$ where, by convention, $\nu(\emptyset) = b$. Now consider the abstraction used in Theorem 1 item [1]: each matched pair $(\hat{i}, \hat{j}) \in M$ is assigned their own subgraph, and all unmatched buyers/sellers are assigned a trivial subgraph that contains only them. Because there exist prices $\hat{p}_M$ such that M is stable at $\hat{p}_M$, cheapest sorting implies that

$$\begin{aligned}\nu(j') &\leq \hat{p}_M(i,j) \leq \min\{v(i') : i' \text{ is indirectly connected to j}\} \\ &\leq \min\{\nu(j^*(i')) : i' \text{ is indirectly connected to j}\}.\end{aligned}$$

Thus, $\nu(j') \leq \rho^{\infty}_i$ so (i,j') does not block.

b *No Seller blocks:* Let i' be such that $(i',j) \in E$. By construction, $\rho^{\infty}_i \leq \rho^{\infty}_{i'}$. Thus, (i',j) does not block.

□

Lemma 4. *Let M be the matching generated by the MDP. Let p_M be any price function such that M is stable with respect to p_M (which is well defined by our previous lemma) and let v be the associated payment function. Let p^*_M be the price generated by the PDP(II) and v^* the corresponding payment function. Then, $v^* \geq v$.*

Proof. Let M, p_M, v, p^*_M and v^* be as in the statement of the lemma. Then, $v(i) \leq \rho^1_i$ for all i. Indeed, if $j^*(i) = \emptyset$ then $v(i) = b = \rho^1_i$. If $j^*(i) = j$ for some j then, by stability of M with respect to p_M, $v(i) \leq \nu(j) = \rho^1_i$.

We now show that if $v \leq \rho^t$ for some k, then $v \leq \rho^{t+1}$. Indeed, for all i such that $j^*(i) = \emptyset$, $v(i) = b = \rho^t_i = \rho^{t+1}_i$. For all i such that $j^*(i) = j$, we have the following:

$$\begin{aligned}\rho^{t+1}_i &= \min\{\rho^t_{i'} : (i',j) \in E\} \\ &\geq \min\{v(i') : (i',j) \in E\} \geq v(i),\end{aligned}$$

where the last inequality follows from stability of M with respect to p_M. Thus, for each t and eaxh i, $\rho^t_i \geq v(i)$. Hence $\rho^{\infty}_{\cdot} \equiv v^*(\cdot) \geq v(\cdot)$. □