Heterogeneous EIS and Wealth Distribution in a Neoclassical Growth Model

Nakamoto, Yasuhiro

Faculty of Informatics, Kansai University

August 2015

Heterogeneous EIS and Wealth Distribution in a Neoclassical Growth Model*

Yasuhiro Nakamoto†

Abstract

We introduce the heterogeneities of EIS (elasticities of intertemporal substitution) into the Ramsey version of macrodynamic model with a finite number of agents. The assumption that the degrees of EIS differ among agents means that our economy has various growth rate of private consumption. Then, our contributions are as follows. First, we analytically characterize the steady-state levels of individual capital. Second, we analytically examine the role of heterogeneous EIS for the wealth inequality. Finally, we give numerical examples to see the complicated dynamic motion and the steady-state characterization of wealth inequality.

Keywords: Heterogeneous agents, Elasticity of intertemporal substitution, Convergence speed, Wealth distribution

JEL Classification Code: C00, E13, E21,

---

*We are grateful to Giacomo Corneo, Akira Maeda, Kazuo Mino, Shintaro Nakagawa, Takumi Naito, Ryoji Ohdoi and Masayuki Sato for their help and comments. We have benefited from presentations at the 2012 Japanese Economic Association Spring Meeting in Hokkaido University, PET 2015 at Luxembourg and Macroeconomics workshop in Kyoto University. For financial support, we acknowledge Alexander von Humboldt Foundation fellowship and Grant-in-Aid for Scientific Research (A) Number 23241019 from the Ministry of Education, Culture, Sports, Science and Technology. The author acknowledges financial support from a research project at Grant-in-Aid for Scientific Research, Japan Society for the Promotion of Science(JSPS) and the Alexander von Humboldt fellowship.

†Faculty of Informatics, Kansai University, Reizenji-cho 2-1-1, Takatuki-shi, Osaka-fu, 569-1052, Japan, e-mail: nakamoto@kansai-u.ac.jp

1 Introduction

The relationship between wealth and a consumer's attitude toward risk–as indicated by the degree of risk aversion–is central to many fields of economics. Earliest studies (e.g., Arrow 1970, and Friend and Blume 1975) have focused on examining whether or how the behavior of risk attitudes varies as wealth varies.[1] In recent years, the experimental studies (e.g., Holt and Laury (2002)), which attempt to measure directly the degrees of risk aversion of individuals, examine the measured degrees of risk aversion and wealth. Furthermore, along with the development in the behavioral economics, it has been well-known that the degrees of risk aversion of individuals differ each other (See Holt and Laury 2002, Tanaka et al. 2010). Since the degrees of risk aversion shape the elasticities of intertemporal substitution (EIS), the heterogeneities of risk aversion mean that the intertemporal consumption-saving decisions among individuals differ each other. Then, the important issue which naturally arises is to examine the role of heterogeneous EIS for wealth inequality in the framework of macroeconomic dynamics.

So far, most dynamic general equilibrium macroeconomic theory has treated all agents as identical (the representative-agent paradigm). The simple model yields many predictions, but the assumption that an economy is inhabited by a single type of consumers highly simplifies some general characterization of key elements. The foregoing literature that has addressed wealth inequality also relies heavily on the assumption that the preferences of individuals are identical and homothetic: see Chatterjee (1994), Sorger (2000), Maliar and Maliar (2001), Álvarez-Peláez and Díaz (2005), García-Peñalosa and Turnovsky (2006, 2007, 2008) and Atolia et al (2012). The simplicity of preferences yields the constancy of relative consumption between individuals over time, which implies that the intertemporal choices in saving-consumption are simply given, so that the model can be reduced to the representative-agent model. In that regard, the introduction of heterogeneous EIS yields the lack of simplicity and tractability. In particular, the appearance of a continuum of steady state makes the analytical analysis difficult.[2]

The purpose of this paper is to extend the Ramsey version of dynamic models with

[1] For instance, one important consequence is that individuals facing high exogenous labor income risk, which is normally uninsurable, will be more risk averse. See Pratt and Zeckhauser (1987), Kimball (1993) and Eekhoudt et al. (1996).

[2] As for the preferences, assuming that the rate of time preference is endogenous, or that the publicly provided goods are incorporated into the utility function, it has been known that the dynamic models do not face a continuum of steady state, and then the analysis of wealth inequality is given by the steady-state characterization. For instance, see Epstein (1987a), Epstein (1987b) and Caselli and Ventura (2000).

a finite number of agents by assuming that they have different degrees of EIS (risk aversion), and examines the role of heterogeneous EIS for wealth inequality. In particular, assuming that EIS is heterogeneous among agents deviates from the readily predicted behavior of consumption in the homogeneous EIS economy in the following two respects. Firstly, when the utility function is identical, the initial jump of private consumption depends on only the initial level of capital stock, which means that an initially wealthier household chooses a higher level of initial consumption. On the other hand, under the heterogeneous EIS, since the initial jump of private consumption is dependent of not only the initial level of capital stock but also the degrees of EIS, an initial rich does not necessarily choose a high level of initial consumption when he has the propensity to make the investment willingly. Secondly, the ratio of relative consumption between agents changes over time under the heterogeneous EIS, while it is constant over time under the homogeneous EIS. Because of these differences, we will see various pattern of wealth inequality. Then, we analytically give a solution for the following issues. What is the difference of wealth inequality under the homogeneous and heterogeneous EIS? What elements compose of wealth inequality under the heterogeneous EIS, deviated from that under the homogeneous EIS? How does the dispersion of wealth affect the speeds of convergence? Turning into numerical simulation, we seek further to explore dynamic motion and steady-state characterization of wealth inequality, especially, we examine how wealth inequality is formed by the pattern of dispersed degrees of EIS, how wealth inequality moves over time, and how much the dispersion of wealth impacts the speeds of convergence. Finally, we examine whether the heterogeneous EIS can be the key element of forming the wealth inequality.

Our paper is related to the recent literature in macrodynamic models. Sorger (2002) and García-Peñalosa and Turnovsky (2006, 2008) comprehensively study the wealth and income distribution in one-sector growth models with elastic labor supply. For example, García-Peñalosa and Turnovsky (2008) confirm that the introduction of elastic labor supply yields a drastic change between wealth and income distributions. They make a substantial extension and obtain the outstanding results, while the assumption that agents have identical and homothetic preferences simplifies the analysis of wealth distribution. Therefore, the level of wealth distribution monotonically moves along time although the production side is not complicated.[3] In the current paper, the introduction of heterogeneous EIS yields the complicated motion of wealth inequality.

To examine the role of heterogeneous risk aversion for wealth inequality, Coen-Pirani

[3] García-Peñalosa and Turnovsky (2009) examine if the elasticity of substitution in production has the qualitative effects on the wealth inequality under the homogeneous EIS economy.

(2004, 2005) construct an endowment economy populated by two types of agents with different degrees of risk aversion. Our motivation is alike to his motivation in the sense that both papers examine the role of the heterogeneous preferences for the wealth inequality. Alternatively, our paper is different from Coen-Pirani (2004, 2005) as follows. First, making use of the preferences in Epstein and Zin (1989), Coen-Pirani (2004, 2005) assume that the degrees of risk aversion are different but EIS is homogeneous, so that the intertemporal decisions in consumption-saving are simplified, while our papers assume the heterogeneities of EIS. Second, the evolution in Coen-Pirani (2004, 2005) is characterized by the growth rate of perishable dividend with two-state Markov process in a discrete time model, while our model uses the Ramsey version of macrodynamic model. Third, Coen-Pirani (2004, 2005) simply assume two types of agents, while we adopt a more general setting in the sense that the economy has $N-$types of agents.

Turnovsky (2002) examines the role of EIS for the speeds of convergence. Because he adopts the representative-agent model, the speeds of convergence are determined by macroeconomic variables alone, irrespective of the dispersion of wealth. Alternatively, since we extend Turnovsky (2002) by introducing a finite number of agents with the different degrees of EIS, the dispersion of wealth influences the speeds of convergence.[4]

The paper is organized as follows. Section 2 describes our economy and derives the macroeconomic equilibrium. Section 3 examines the speeds of convergence under the heterogeneous EIS. Section 4 analytically characterizes the distributions of wealth and obtains the main findings of wealth distribution. Section 5 shows numerical examples. Section 6 concludes, while the technical details in characterizing the steady-state levels of individual capital are given in Appendix.

2 Model

We describe our model of a closed economy where time is taken in continuous intervals. The stock of capital is the only net asset held by agents. There are many agents indexed by $i = 1, 2, .., N$, where we assume that the initial holdings of capital stock are different each other. We assume that the population in the whole economy is constant over time.

The representative firm produces a single good according to a constant-returns-to-scale technology expressed by $Y = F(K, L)$ where the production function satisfies neoclassical properties. Here, Y, K and L denote the total output, capital and employment

[4]Turnovsky (2002) is more general in the point that he examines the role of not only EIS but also the elasticities of intratemporal substitution.

of labor, respectively.[5] The wage rate, W, and the return to capital, R, are determined by the marginal products of each production factor:

$$W(K, L) = F_L(K, L), \quad R(K, L) = F_K(K, L). \tag{1}$$

Denoting the levels of capital stock and private consumption, and an amount of labor by an agent i as K_i, C_i and L_i, we assume that she faces a flow budget constraint, such that

$$\dot{K}_i = (R - \delta)K_i + WL_i - C_i \quad i = 1, 2, .., N, \tag{2}$$

where δ is a constant rate of depreciation and the initial level of capital holding K_i^0 is exogenously given. Assuming that the commodity market is competitive, summing (2) among all agents yields the output market in equilibrium:

$$Y = \dot{K} + \delta K + C, \tag{3}$$

where $C = \sum_{i=1}^{N} C_i$ denotes the level of aggregate consumption.

The full-employment conditions are given in:

$$K = \sum_{i=1}^{N} K_i, \quad L = \sum_{i=1}^{N} L_i. \tag{4}$$

2.1 Set up

To keep with our focus on the role of heterogeneous EIS for the wealth inequality, we simplify the consumer's consumption-leisure choice by providing the assumption that each consumer inelastically supplies an identical labor service such that $L_i = \bar{L}(i = 1, 2, ..., N)$. As a result, the cross-sectional differences in income are caused by the differences in capital stock alone, meaning that we can exclude a complicated difference of dynamic behavior between wealth inequality and income one.[6]

The evaluation of life-time utility depends on the consumption profiles alone as follows:

$$U^i = \int_0^{+\infty} e^{-\rho t} \frac{C_i^{1-\beta_i}}{1-\beta_i} dt, \ \rho > 0, \quad \beta_i > 0, \quad i = 1, 2, .., N. \tag{5}$$

We assume that the rate of time preference among agents, ρ, is identical among agents, and that the preference parameter, β_i $(i = 1, ..., N)$, is not necessarily identical each other.

---

[5] We omit time variable t as long as it does not invite confusion.

[6] Turnovsky and García-Peñalosa (2008) show that the introduction of endogenous labor supply itself may lead to opposite movement between income and wealth inequalities in response to a structural change.

Each agent maximizes U^i subject to (2), a constant level of labor supply and the initial holding of capital, K_i^0. Letting the implicit price of capital K_i be q_i, the optimization conditions include

$$C_i^{-\beta_i} = q_i, \quad i = 1, 2, .., N, \tag{6a}$$

$$\frac{\dot{q}_i}{q_i} = \rho + \delta - R, \quad i = 1, 2, .., N, \tag{6b}$$

together with the transversality condition, $\lim_{t\to\infty} e^{-\rho t} q_i K_i = 0$.

Since the right-hand side of $(6b)$ is the same among agents, it holds that $\dot{q}_1/q_1 = \dot{q}_2/q_2 = ... = \dot{q}_N/q_N$ for all $t \geq 0$. Defining $\Omega_{ij} \equiv q_i/q_j$ $(i, j = 1, 2, ..., N)$ where Ω_{ij} is a positive constant parameter, from $(6a)$ we can show that

$$\Omega_{ij} = \frac{C_i^{-\beta_i}}{C_j^{-\beta_j}}, \quad i, j = 1, 2..., N, \quad i \neq j. \tag{7}$$

Because Ω_{ij} $(i, j = 1, 2, ..., N)$ $(i \neq j)$ are undetermined, our model needs to specify trajectory starting from a specific set of initial conditions unlike the representative-agent model.

From (1), $(6a)$ and $(6b)$, we derive the well-known Euler equation:

$$\frac{\dot{C}_i}{C_i} = \frac{R - \delta - \rho}{\beta_i}, \quad i = 1, 2, ..., N. \tag{8}$$

From (8), we pay attention to the following two points. First, agents have different degrees of EIS $1/\beta_i$ $(i = 1, ..., N)$. Therefore, the individual consumption growth (8) cannot be aggregated. Second, noting that $\beta_i > 0$ $(i = 1, 2, ..., N)$, which is the standard assumption that the marginal utility of private consumption decreases as the level of private consumption increases, we show that all agents have an identical sign of consumption growth. In other words, when $K^0 < K^*$ so that $R > \delta + \rho$ over time, $\dot{C}_i/C_i$ $(i = 1, 2, ..., N)$ has a positive sign over time, meaning that the capital stocks held by each agent move in the same direction along time. That is, if $K^0 < K^*$, it holds that $K_i^0 < K_i^*$ for all agents. When $K^0 > K^*$, the reversal relationship can be applied.

Using (1) and (2) under the inelastic labor supply, we obtain:

$$\dot{K}_i = (R(K) - \delta)K_i + W(K)\bar{L} - C_i, \quad i = 1, 2, ..., N, \tag{9a}$$

and the dynamic equation of aggregate capital is given by:

$$\dot{K} = (R(K) - \delta)K + W(K) - C, \tag{9b}$$

where the aggregate labor supply is fixed by $1 = N\bar{L}$.

Finally, in a tractable form, the unknown constant parameter Ω_{ij} is expressed by:

$$\Omega_{ij} = \frac{\beta_i C_i^{-\beta_i - 1}(K_i^0 - K_i^*)}{\beta_j C_j^{-\beta_j - 1}(K_j^0 - K_j^*)}. \tag{10}$$

We give the detail derivation in Appendix A where the utility function is given by a general form.

2.2 The steady state

From (1) and (8), the steady-state level of aggregate capital, K^*, is determined by the modified Golden-Rule condition:

$$R(K^*) = \rho + \delta, \tag{11a}$$

where from (4) we can show that

$$K^* = \sum_{i=1}^{N} K_i^*. \tag{11b}$$

Furthermore, summing up $\dot{K}_i = 0$ among all agents, and using ($11a$) and ($11b$) where $N\bar{L} = 1$, we can determine the steady-state level of aggregate consumption:

$$C^* = \rho K^* + W(K^*). \tag{11c}$$

As a result, the steady-state levels of aggregate capital as well as aggregate consumption are uniquely determined as in the representative-agent model.

Next, we investigate the determination of the steady-state levels of individual capital and consumption. From (7) and (10), we can use the following equations in the steady state:

$$\frac{C_i^*}{C_j^*} = \frac{\beta_i(K_i^0 - K_i^*)}{\beta_j(K_j^0 - K_j^*)}, \quad i, j = 1, 2, ..., N, \quad i \neq j. \tag{12a}$$

Without the loss of generality, we assume that the agent 1 is a base agent:

$$\frac{C_i^*}{C_1^*} = \frac{\beta_i(K_i^0 - K_i^*)}{\beta_1(K_1^0 - K_1^*)}, \quad i = 2, ..., N. \tag{12b}$$

Besides, from $\dot{K}_i = 0$ the steady-state levels of private consumption are given by:

$$C_i^* = \rho K_i^* + W(K^*)\bar{L}, \quad i = 1, 2, ..., N. \tag{13}$$

Under the uniquely determined level of aggregate capital stock at the steady state, using $2N-$equations composed of ($11b$), ($12b$) and (13) yields the following.

Proposition 1 *The steady-state equilibrium is uniquely determined given K_i^0 and β_i ($i = 1, 2, ..., N$).*

Proof. Substituting (13) into ($12b$), we can obtain the following:

$$K_i^* = K_i^*(K_1^*), \quad \frac{\partial K_i^*}{\partial K_1^*} = \frac{(R^*K_1^0 + W^*\bar{L})(W^*\bar{L} + R^*K_i^*)(K_i^0 - K_i^*)}{(R^*K_i^0 + W^*\bar{L})(W^*\bar{L} + R^*K_1^*)(K_1^0 - K_1^*)}(> 0). \tag{14}$$

Since $K_1^0 < (>) K_1^*$ and $K_i^0 < (>) K_i^*$ under $K^0 < (>) K^*$, we can see that $\partial K_i^* / \partial K_1^*$ always has a positive sign.

Next, substituting (14) into (11b), we can obtain the following:

$$K^*(\rho) = K_1^* + \sum_{i=2}^{N} K_i^*(K_1^*). \tag{15}$$

Note that the steady-state level of aggregate capital is determined by (11a) so that the value of left-side hand is fixed.

Because the right-hand side monotonically increases with K_1^*, there is a level of K_1^* that satisfies the equation (15). Finally, from (14) the steady-state levels of individual capital stock held by all agents are determined. ■

At the steady state, we analytically derive the relationship of individual capital stocks between agents i and j as follows:[7]

Proposition 2 *Assume that the initial levels of capital stock held by agents i and j are the same. When $K^0 < K^*$, the heterogeneity $\beta_i > \beta_j$ leads to $K_j^* > K_i^* > K_i^0 = K_j^0$ and $C_j^* > C_i^*$. Alternatively, assuming that $K^0 > K^*$, $\beta_i > \beta_j$ leads to $K_j^* < K_i^* < K_i^0 = K_j^0$ and $C_j^* < C_i^*$.*

Proof. See Appendix B. ■

Proposition 2 shows that a higher degree of EIS (i.e., a lower value of β_i or β_j) leads to a greater level of individual capital stock in the long run under a growing economy such that $K^* > K^0$. The intuitive explanation is as follows. Suppose that $K^0 < K^*$ so that $R > \delta + \rho$ over time, and furthermore that $\beta_i > \beta_j$. Since $1/\beta_j > 1/\beta_i$, the positive growth rate of consumption for the agent j is greater along time. As a result, since the investment by the agent j is higher than that by the agent i, in the steady-state equilibrium it holds that the agent j is richer in wealth, that is, $K_i^* < K_j^*$. If $K^0 > K^*$, the above relationship is reversed.

3 Convergence speed and the dispersion of wealth

To focus on the heterogeneities of EIS, it is helpful to clarify the difference of convergence speeds between the heterogeneous and the homogeneous EIS. When EIS is homogeneous (i.e., $\beta = \beta_i$ for all i in (5)), we can show the Euler equation $\dot{C}/C = (R(K) - \delta - \rho)/\beta$;

[7] In addition, under the assumption that $\beta_i = \beta_j$, we can see that the initially more wealthy agent has a greater amount of capital stock in the steady state relative to the initially less wealthy one. That is, when $K_i^0 > (<) K_j^0$ under $\beta_i = \beta_j$ ($i, j = 1, 2, ..., N$ and $i \neq j$), it holds that $K_i^* > (<) K_j^*$ so that $C_i^* > (<) C_j^*$.

instead, the capital accumulation equation is the same with (9b). Then, the speed of convergence λ_{Rep} under the homogeneous EIS is given by:

$$2\lambda_{\text{Rep}} = \rho - \sqrt{\rho^2 - \frac{4R_K(K^*)C^*}{\beta}}(< 0). \tag{16a}$$

Because of the homogeneity of EIS, the levels of private consumption can be summed up, thereby being able to confirm that the speed of convergence in this case is identical to that in the representative-agent model. That is, since the speed of convergence is determined by the aggregate variable alone, the dispersion of wealth does not have any impacts on the speeds of convergence. Hence, we obtain the well-known fact that $\sigma-$convergence does not influence $\beta-$convergence (See Barro and Sala-i-Martin, 2004).

Next, the convergence speed λ under the heterogeneous EIS is given by:[8]

$$2\lambda = \rho - \sqrt{\rho^2 - 4R_K(K^*)\sum_{i=1}^{N}\left(\frac{C_i^*}{\beta_i}\right)}(< 0). \tag{16b}$$

Since the steady-state levels of private consumption cannot be summed up, it can be seen that the dispersion of wealth at the steady-state equilibrium affects the speed of convergence. More concretely, making use of the individual capital stock relative to the average capital, $k_i \equiv \frac{K_i}{K/N}$, and defining the difference between the level of individual capital and the average level as $\sigma_{i,k} \equiv k_i - 1$, the sum of the reciprocal of absolute risk aversion can be modified by:

$$\sum_{i=1}^{N}\left(\frac{C_i^*}{\beta_i}\right) = \underbrace{C^*\sum_{i=1}^{N}\left(\frac{1}{\beta_i}\right)\frac{1}{N}}_{(\#1)} + \underbrace{\frac{K^*R(K^*)}{N}\sum_{i=1}^{N}\frac{\sigma_{i,k}^*}{\beta_i}}_{(\#2)}(> 0). \tag{17}$$

Note that $\sigma_{i,k}$ has a positive sign (a negative sign) if the level of wealth held by the agent i is more (less) than the mean level.

The $\sum_{i=1}^{N}\left(\frac{1}{\beta_i}\right)\frac{1}{N}$ in (#1) represents the mean level of EIS. The (#2) emerges from the heterogeneity of EIS. Noting that $\sigma_{i,k}^*$ is the relative position of capital held by the agent i compared with the average level of capital, the second term (#2) may be called as the wealth distribution effect. Considering that (#2) has positive or negative sign, the wealth distribution effect makes the speeds of convergence faster or slower. For instance, if the relatively more wealth people ($\sigma_{i,k}^* > 0$) in the long run have greater degrees of EIS on average, the wealth distribution effect positively impacts on the speed of convergence. That is, since the riches like to save for the investment in the future, which implies that the larger ratio of entire wealth is invested, the growth of the whole

[8] See Appendix C for the derivation.

economy is more accelerated. As a result, the economy converges towards the steady-state equilibrium at a faster rate. In that case, the larger the wealth inequality, the faster the speed of convergence. Alternatively, if the opposite case is applied, the expansion of wealth inequality negatively influences the speed of convergence. The results can be summarized as follows.

Proposition 3 *In the long-run unequal economy that the relative-wealth rich agents have greater (smaller) degrees of EIS on average, the expansion of wealth inequality makes the speed of convergence increased (decreased).*

4 Wealth distribution

4.1 The dynamics of the relative wealth

Using (9a) and (9b) as in García-Peñalosa and Turnovsky (2006, 2008, 2009), the dynamic motion of relative wealth is:

$$\dot{k}_i = \frac{W(K)(1-k_i)}{K} + \frac{C(k_i - c_i)}{K}, \quad i = 1, 2, ..., N, \tag{18}$$

where $c_i = \frac{C_i}{C/N}$. Importantly, this dynamic equation clearly gives the difference between the homogeneous and heterogeneous EIS. García-Peñalosa and Turnovsky (2006, 2008, 2009) assume that the utility function is identical and homothetic among the agents. In their setting individual consumption that moves in the same direction changes at the same rate, so that the level of relative consumption of each agent, c_i, stays constant over time. However, considering that the degrees of EIS differ among agents, the levels of relative consumption, c_i, change during the transitional process.

To clarify the role of heterogeneous EIS, we define the following:

$$\theta_i \equiv \frac{c_i/\beta_i}{\sum_{i=1}^{N}(c_i/\beta_i)(1/N)} (> 0). \tag{19}$$

Noting that $\beta_i = \beta$ for all i, it holds $\theta_i = c_i$. In other words, under the homogeneous EIS, the ratio of relative weighted EIS, θ_i, is identical to the ratio of relative consumption, c_i. Moreover, since the ratio of relative consumption, c_i, does not move during the transition, θ_i does not change during the transition as well. That is, it holds that $\theta_i = c_i (= c_i^*)$ along time. Turning to the heterogeneous EIS, the equations do not hold because these values respectively change through time.

Linearizing (18) at the steady state, we can show the following:[9]

$$k_i = k_i^* + \frac{\sigma_{i,k}^* Z_i^* (K^* - K^0)}{\rho - \lambda} e^{\lambda t}, \tag{20}$$

[9]See Appendix D.

Each Z_i^* in the homogeneous and heterogeneous EIS is given by:

$$\text{Heterogeneous EIS:} \quad Z_i^* = \frac{-W_K(K^*) + (\rho - \lambda + R_K(K^*)K^* + W_K(K^*))\left(1 - \frac{\sigma_{i,\theta}^*}{\sigma_{i,k}^*}\right)}{K^*}. \tag{21a}$$

$$\text{Homogeneous EIS:} \quad Z_i^* = Z^* = \frac{W(K^*)\rho\left(1 - \frac{W_K(K^*)K^*}{W(K^*)} + \frac{R_K(K^*)K^*}{\rho} - \frac{\lambda}{\rho}\right)}{C^*K^*}. \tag{21b}$$

where $\sigma_{i,\theta} = \theta_i - 1$. Notice that when EIS is homogeneous, all agents face an identical value of Z^* given in $(21b)$, while if agents have different degrees of EIS, they have each value of Z_i^* in $(21a)$.

Moreover, to simplify our discussion, in what follows we assume that

$$1 \geq \frac{W_K(K^*)K^*}{W(K^*)} - \frac{R_K(K^*)K^*}{\rho}, \tag{22}$$

so that Z^* always has a positive sign in $(21b)$. For instance, when the production function is Cobb-Douglas type $R(K) = \alpha A K^{\alpha-1}$ and $W(K) = (1-\alpha)AK^{\alpha}$ and the rate of depreciation is zero, then the condition (22) is satisfied.[10]

4.2 Catching-up

Using (20), we can show the difference of wealth held by two agents i and j, defined by $\Delta k_{ij} = k_i - k_j$:

$$\text{Heterogeneous EIS:} \quad \Delta k_{ij} = \Delta k_{ij}^0 - \frac{(K^* - K^0)(\sigma_{i,k}^* Z_i^* - \sigma_{j,k}^* Z_j^*)(1 - e^{\lambda t})}{\rho - \lambda}. \tag{23a}$$

If EIS is homogeneous, the equation $(23a)$ can be reduced to:

$$\text{Homogeneous EIS:} \quad \Delta k_{ij} = \Delta k_{ij}^0 - \frac{(K^* - K^0)Z^*\Delta k_{ij}^*(1 - e^{\lambda t})}{\rho - \lambda}. \tag{23b}$$

Then, in a growing economy $K^* > K^0$, substituting $\Delta k_{ij} = 0$ into $(23b)$ yields $\text{sign}(\Delta k_{ij}^0) = \text{sign}(\Delta k_{ij}^*)$, meaning that $k_i^0 > k_j^0$ and $k_i^* > k_j^*$. Considering that the relative consumption is constant during the transition, we can conclude that the catching-up does not arise over time.

Under the heterogeneous EIS, since the relative consumption changes during the transition, the catching-up may arise such that $\Delta k_{ij} = k_i - k_j = 0$. Now, suppose that an agent i is initially relative-wealth richer than an agent j. That is, $\Delta k_{ij}^0 = k_i^0 - k_j^0 > 0$. Defining by T the time that the initially less wealthy household j will catch up with the initially more wealthy household i, which means that $\Delta k_{ij} = k_i - k_j = 0$, we lead to the following:

$$T = \frac{1}{\lambda}\log\left[1 - \frac{\Delta k_{ij}^0(\rho - \lambda)}{(K^* - K^0)(\sigma_{i,k}^* Z_i^* - \sigma_{j,k}^* Z_j^*)}\right]. \tag{24}$$

[10] Making use of the elasticity of substitution between labor and capital in the production function, García-Peñalosa and Turnovsky (2009) mention the possibility of changing the sign of Z^* in $(21b)$.

If T has a positive sign, it holds $\Delta k_{ij} = 0$ at time T. Then, the results can be summarized as follows.

Proposition 4 *Suppose that $K^* > K^0$ and $\Delta k_{ij}^0 > 0$ under (22). Then, the catching-up does not arise in the case of homogeneous EIS. Alternatively, in the case of the heterogeneous EIS, the initially wealth poor j will catch up in wealth if the following condition is satisfied:*

$$0 < \frac{\Delta k_{ij}^0 (\rho - \lambda)}{(K^* - K^0)(\sigma_{i,k}^* Z_i^* - \sigma_{j,k}^* Z_j^*)} < 1. \tag{25}$$

Otherwise, the catching-up does not arise in the heterogeneous EIS.

Proof. Making use of (24), we can derive (25). ■

Since the intuitive explanation in the case of homogeneous EIS is given in the above, we now focus on the case in which the degrees of EIS between agents i and j differ where $K^* > K^0$. The necessary condition of arising the catching-up, $\sigma_{i,k}^* Z_i^* > \sigma_{j,k}^* Z_j^*$, is given by:

$$(0 >)\Delta k_{ij}^* > \left(1 + \frac{W_K(K^*)}{\rho - \lambda + R_K(K^*)K^*}\right)(\theta_i^* - \theta_j^*). \tag{26}$$

where $1 + \frac{W_K(K^*)}{\rho - \lambda + R_K(K^*)K^*} > 0$ under the condition (22). When an initially less wealth agent j catches up with the agent i in wealth, the sign of Δk_{ij}^* is negative. Therefore, at least it is needed to hold that $\theta_j^* > \theta_i^*$ to see the catching-up, which implies that since the value of EIS by the agent j is greater than that by the agent i, the initially less wealthy agent j likes to save more than the agent i.

4.3 The dynamics of the wealth inequality

We now turn to the dynamics of wealth inequality in the entire economy. The relative wealth (20) can be firstly rewritten as:

$$\sigma_{i,k} = \sigma_{i,k}^* \left(1 + \frac{Z_i^*(K^* - K^0)e^{\lambda t}}{\rho - \lambda}\right). \tag{27}$$

Before we derive the wealth inequality, it is useful to differentiate (27) with respect to time:

$$\frac{\partial \sigma_{i,k}}{\partial t} = \frac{\lambda \sigma_{i,k}^* Z_i^*(K^* - K^0)e^{\lambda t}}{\rho - \lambda}. \tag{28}$$

Using (28), we can summarize the dynamic motion of relative wealth as follows.

Proposition 5 *We assume that the economy is growing over time ($K^* > K^0$) under (22). (i) In the case of homogeneous EIS, it holds that $sign\left(\frac{\partial \sigma_{i,k}}{\partial t}\right) = -sign(\sigma_{i,k}^*)$. (ii) In the case of heterogeneous EIS, it holds that $sign\left(\frac{\partial \sigma_{i,k}}{\partial t}\right) = -sign(Z_i^* \sigma_{i,k}^*)$.*

Proposition 5 is helpful to understand the dynamic behavior of wealth inequality. At first, from Proposition 5(i), we can guess that the wealth inequality decreases over time under the homogeneous EIS. For instance, let us consider that an agent i is relatively wealth-rich in the long run (i.e., $\sigma^*_{i,k} > 0$). From Proposition 5(i), the level of wealth held by the agent i relative to the mean level decreases over time $\partial\sigma_{i,k}/\partial t < 0$, which means that the initially unequal distribution shrinks over time. From Proposition 4, since the catching-up does not arise, the difference of wealth between the mean and the agent i persists in the long run, $\sigma^*_{i,k} > 0$. As a result, we conclude that $\sigma^0_{i,k} > \sigma^*_{i,k} > 0$. Similarly, if $\sigma^*_{i,k} < 0$, it holds that $\partial\sigma_{i,k}/\partial t > 0$, thereby showing that the unequal distribution shrinks over time $\sigma^0_{i,k} < \sigma^*_{i,k} < 0$. Considering that the levels of wealth held by all agents are approaching to the mean level over time, we can see that the wealth distribution in the homogeneous EIS monotonically shrinks during the transitional process.

This movement is intuitively reasonable. Consider that an agent i at the initial period is relatively wealth-rich, that is, $\sigma^0_{i,k} > 0$. Due to the difference of the initial capital stock, the initial jump of private consumption by the agent i is larger than that by the average agent, which means that at next period, $\sigma^0_{i,k} > \sigma_{i,k} > 0$. After that, since the relative consumption is constant along time, the relationship that $\sigma^0_{i,k} > \sigma_{i,k} > 0$ continues until the steady-state equilibrium.

These results mean that if we define the index of wealth inequality at time t as follows:

$$S_{kk} \equiv \sum_{i=1}^{N} \sigma^2_{i,k} (> 0), \tag{29}$$

then the difference of the initial jump of private consumption leads to less unequal economy in the long run. In other words, we find that $S^0_{kk} > S^*_{kk} > 0$ and $\partial S_{kk}/\partial t < 0$.

In the case of heterogeneous EIS, the movement of wealth distribution is more complicated. This is because there are two different movements compared with the case in which EIS is homogeneous. First, unlike the homogeneous EIS, the divergence may expand in the heterogeneous EIS. To confirm this movement, suppose that an agent i is initially relative-wealth rich, $\sigma^0_{i,k} > 0$. It holds that $\partial\sigma_{i,k}/\partial t > 0$ when the following inequality is satisfied:

$$1 - \frac{\sigma^*_{i,\theta}}{\sigma^*_{i,k}} < \frac{W_K(K^*)}{\rho - \lambda + R_K(K^*)K^* + W_K(K^*)}. \tag{30}$$

where $\sigma^*_{i,k}$ has a positive sign. If $\sigma^*_{i,\theta} > \sigma^*_{i,k}$, then the inequality (30) is satisfied where the right-hand side of (30) has a positive sign. That is, when the initially relative-wealth rich i has a strong degree of EIS such that $\sigma^*_{i,\theta} > \sigma^*_{i,k}$, the difference between his wealth level and the average one expands over time. Likewise, it is possible that the initially relative-wealth poor i further falls down the relative position in wealth, such that $0 > \sigma^0_{i,k} > \sigma^*_{i,k}$.

Second, we consider the case in which the wealth dispersion shrinks. This case can be seen in the case of homogeneous EIS as well. However, noting that the catching-up can arise in the heterogeneous EIS, the heterogeneity of EIS gives an interesting case. Suppose that an agent i is relative-wealth rich at the beginning of economy. In the homogeneous EIS, the level of relative wealth decreases over time; however, the reversal between the agent i's wealth level and the average one does not arise. That is, it always holds that $\sigma^0_{i,k} > \sigma^*_{i,k} > 0$. When EIS is heterogeneous, it is possible that the agent i holds wealth less than the average level in the long run, $\sigma^0_{i,k} > 0 > \sigma^*_{i,k}$. If the following inequality is satisfied, we can see that $\partial\sigma_{k,t}/\partial t < 0$, such that $\sigma^0_{i,k} > 0 > \sigma^*_{i,k}$:

$$1 - \frac{\sigma^*_{i,\theta}}{\sigma^*_{i,k}} < \frac{W_K(K^*)}{\rho - \lambda + R_K(K^*)K^* + W_K(K^*)}. \tag{31}$$

where $\sigma^*_{i,k}$ has a negative sign. The condition (31) implies that when the initially relative-wealth rich i holds a low degree of EIS, he does not like to save during the transition, and his wealth may relatively fall over time. Concretely, at time T in (24), the reversal between the agent i's wealth level and the average one arises, and since then, the initially relative-wealth rich agent i holds less wealth compared with the average level until the steady state.[11]

Based on the above, we can presume the pattern of movement of wealth distribution to some extent. However, it is not evident which elements compose the wealth inequality in the entire economy. We now derive the wealth inequality S_{kk}. For the purpose of comparison, we can firstly consider the case in which EIS is homogeneous. By raising both sides of the equation (27) to the second power, and summing up it among the agents, we can obtain the following:

$$\text{Homogeneous EIS:} \quad S_{kk} = \frac{\left(1 + \frac{Z^*(K^* - K^0)e^{\lambda t}}{\rho - \lambda}\right)^2}{\left(1 + \frac{Z^*(K^* - K^0)}{\rho - \lambda}\right)^2} S^0_{kk}, \tag{32}$$

where S^0_{kk} is the initial level of wealth inequality. Substituting $t = \infty$ into (32), and assuming a growing economy $K^0 < K^*$, as predicted, we find that the steady state level of wealth inequality is smaller than its initial level $S^0_{kk} > S^*_{kk} > 0$, and that the level of wealth inequality decreases over time $\frac{\partial S_{kk}}{\partial t} < 0$.

[11]Instead of $\sigma^0_{i,k} > 0 > \sigma^*_{i,k}$, we can see $\partial\sigma_{k,t}/\partial t < 0$, such that $\sigma^0_{i,k} > \sigma^*_{i,k} > 0.$, if the following inequality is satisfied:

$$1 - \frac{\sigma^*_{i,\theta}}{\sigma^*_{i,k}} > \frac{W_K(K^*)}{\rho - \lambda + R_K(K^*)K^* + W_K(K^*)}$$

Next, we can derive the wealth inequality in the case of the heterogeneous EIS:[12]

$$S_{kk} = \underbrace{\Gamma_1 S_{kk}^0}_{(\#3)} \underbrace{- \left(\frac{K^* - K^0}{K^*}\right)^2 \left(\Gamma_1 - e^{2\lambda t}\right) S_{\theta\theta}^*}_{(\#4)} \underbrace{+\Gamma_2 S_{k\theta}^*}_{(\#5)}, \tag{33}$$

where $S_{k\theta}^* = \sum_{i=1}^N \sigma_{i,k}^* \sigma_{i,\theta}^*$ and $S_{\theta\theta}^* = \sum_{i=1}^N (\sigma_{i,\theta}^*)^2 (> 0)$. We notice that $S_{\theta\theta}^*$ always has a positive sign, while $S_{k\theta}^*$ has either a positive sign or a negative one. The Γ_1 and Γ_2 are:

$$(1 >)\Gamma_1 = \left(\frac{1 + \frac{(K^* - K^0)e^{\lambda t}}{K^*}\left(1 + \frac{R_K(K^*)K^*}{\rho - \lambda}\right)}{1 + \frac{K^* - K^0}{K^*}\left(1 + \frac{R_K(K^*)K^*}{\rho - \lambda}\right)}\right)^2 (> 0), \tag{34a}$$

$$\Gamma_2 = \frac{2(K^* - K^0)\left(1 + \frac{R_K K^* + W_K}{\rho - \lambda}\right)}{K^*\left(1 + \frac{K^* - K^0}{K^*}\left(1 + \frac{R_K K^*}{\rho - \lambda}\right)\right)^2}$$
$$\times \left\{(1 - e^{\lambda t})\left[1 + \left(1 + \frac{R_K K^*}{\rho - \lambda}\right)^2 \left(\frac{K^* - K^0}{K^*}\right)^2 e^{\lambda t}\right] + (1 - e^{2\lambda t})\left(\frac{K^* - K^0}{K^*}\right)\left(1 + \frac{R_K K^*}{\rho - \lambda}\right)\right\} (> 0). \tag{34b}$$

where we assume that $K^* > K^0$. We must note that Γ_1 and Γ_2 move over time, not fixed.

Then, we can reach the following conclusions about the long-run wealth distribution.

Proposition 6 *We assume that the economy is growing over time ($K^* > K^0$) where we assume* (22)*. In the heterogeneous EIS, the long-run wealth inequality is more than the initial level of inequality if the following inequality is satisfied:*

$$S_{k\theta}^* > \frac{1 - \Gamma_1^*}{\Gamma_2^*} S_{kk}^0 + \left(\frac{K^* - K^0}{K^*}\right) \frac{\Gamma_1^*}{\Gamma_2^*} S_{\theta\theta}^*, \tag{35}$$

while if the opposite inequality is satisfied, we can see that the long-run wealth inequality is less than the initial level. The Γ_1^* and Γ_2^* represent the steady-state values of (34*a*) *and* (34*b*)*:*

$$(1 >)\Gamma_1^* = \frac{1}{\left(1 + \frac{K^* - K^0}{K^*}\left(1 + \frac{R_K(K^*)K^*}{\rho - \lambda}\right)\right)^2} (> 0), \quad \Gamma_2^* = \frac{2(K^* - K^0)\left(1 + \frac{R_K(K^*)K^* + W_K(K^*)}{\rho - \lambda}\right)}{K^*\left(1 + \frac{K^* - K^0}{K^*}\left(1 + \frac{R_K(K^*)K^*}{\rho - \lambda}\right)\right)} (> 0).$$

Proof. Substituting $t = \infty$ into (33), we can see the following:

$$S_{kk}^* - S_{kk}^0 = (\Gamma_1^* - 1)S_{kk}^0 - \left(\frac{K^* - K^0}{K^*}\right)^2 \Gamma_1^* S_{\theta\theta}^* + \Gamma_2^* S_{k\theta}^*, \tag{36}$$

thereby being able to obtain (35). ■

The wealth inequality in (33) consists of three factors. First, the wealth inequality at time t is affected by the initially distributed wealth holding S_{kk}^0 in (#3). Noting

[12] See Appendix E.

that Γ_1 has a positive sign, the larger the dispersion of wealth distribution at the initial period, the larger the level of wealth inequality over time as seen in (33). However, since $0 < \Gamma_1^* < 1$, from (35) it tends to hold that $S^0 > S_{kk}^*$ as the initial level of wealth inequality, S^0, is larger.

If $S_{\theta\theta}^* = S_{k\theta}^* = 0$, (33) can be reduced to (32), which means that the heterogeneity of EIS newly yields the effects given in (#4) and (#5). The (#4) shows the long-run dispersion of heterogeneous EIS. In particular, the coefficient $\Gamma_1 - e^{2\lambda t}$ has a positive sign as follows:

$$\Gamma_1 - e^{2\lambda t} = \frac{(1-e^{2\lambda t}) + \frac{2(K^*-K^0)}{K^*}\left(1+\frac{R_K(K^*)K^*}{\rho-\lambda}\right)e^{\lambda t}\left(1-e^{\lambda t}\right)}{\left(1+\frac{K^*-K^0}{K^*}\left(1+\frac{R_K(K^*)K^*}{\rho-\lambda}\right)\right)^2}(>0), \tag{37}$$

where we assume that $K^* > K^0$. That is, in a growing economy the long-run dispersion in EIS has the negative impacts on the wealth inequality regardless of any spatial arrangement of dispersed degrees of EIS.

Our interests are turned to (#5) in (33). Because Γ_2 has a positive sign, the larger value of $S_{k\theta}$ produces a larger level of long-run wealth inequality S_{kk}^* as seen in Proposition 6. Intuitively, if a lot of long-run riches have large degrees of EIS, which means that the positive value of $S_{k\theta}^*$ is large, the long-run wealth inequality increases.

Finally, we mention the income inequality. Owing to the fixed labor supply, the dynamic movement of income inequality is characterized by only the wealth inequality:[13] That is, because $Y_i = (R(K_i) - \delta)K_i + W(K)\bar{L}$, we can rewrite it as

$$y_i = \left(\frac{R(K)K}{Y} - \delta\frac{K}{Y}\right)k_i + (1-\alpha)\frac{\bar{L}}{1/N}, \tag{38}$$

where $y_i = Y_i/(Y/N)$. Therefore the income inequality is given by:

$$S_{yy} = \left(\alpha - \frac{\delta K}{Y}\right)^2 S_{kk} + \left(\frac{\delta K}{Y}\right)^2, \tag{39}$$

concluding that the larger the level of wealth inequality S_{kk}, the larger the level of income inequality.

5 Numerical examples

To obtain further insights into the dynamic behavior and the steady-state characterization of wealth inequality, we turn to simulate our economy. To do so, we adopt the Cobb-Douglas type of production function $Y = AK^\alpha$ where $0 < \alpha < 1$ and $A > 0$, so that the

[13] García-Peñalosa and Turnovsky (2006, 2008) construct a more general set-up in the sense that the labor supply is endogenous.

inequality (22) is satisfied. That is, in the homogeneous EIS economy such that $\beta_i = \beta$ for all agents i, the level of wealth inequality monotonically decreases over time.

The parameter values we use are mostly conventional:

$$\text{Production parameters: } \alpha = 0.35, \quad A = 0.5, \quad \delta = 0.02$$
$$\text{Taste parameters: } \rho = 0.04, \quad \beta = 2.5$$
$$\text{The number of agents: } N = 201$$
$$\text{Aggregate capital stock: } K^* = 5.1905, \ K^0 = 0.75K^*$$

The rate of time preference of 4% and the parameter of risk aversion, 2.5 are generally used in this field. When we set at $\delta = 0.02$, the interest rate at the steady state, $R(K^*)$ is 6% in the benchmark economy. The choice of $\alpha = 0.35$ implies that 35% of output accrues to private capital and the rest of inelastic labor supply. The level of aggregate capital stock at the initial economy is 75% level of aggregate capital stock at the steady state, that is, we assume a growing economy that $K^* > K^0 (= 0.75K^*)$. We furthermore assume that the number of agents is 201, which means that one person has the average level of capital stock, one-hundred persons are relatively wealth-rich and the rest one-hundred persons are relatively wealth-poor.

Since the agents in this model have the heterogeneities of EIS and the different levels of initial capital, a complicated setting of two heterogeneities leads to the confusion so that it is hard to obtain the intuitive explanation on the dynamic motion of wealth inequality. Therefore, for simplicity, we assume that the initial levels of individual capital stock are uniformly distributed, and furthermore that the initial wealth inequality, S^0_{kk} has an identical value throughout our simulation. In detail, the agents are placed in order based on the initial holding of capital as follows:

$$\frac{K_1^0}{K_{101}^0} = 0.9, \ \frac{K_2^0}{K_{101}^0} = 0.901, \ ..., \frac{K_{100}^0}{K_{101}^0} = 0.999, \ \frac{K_{102}^0}{K_{101}^0} = 1.001, \ \frac{K_{103}^0}{K_{101}^0} = 1.002, ..., \ \frac{K_{201}^0}{K_{101}^0} = 1.1,$$

which means that when the 101-th agent is set as the average agent in the sense that he holds the initial level of capital stock defined by $K^0_{101} = \frac{K^0}{201}$ and $\sigma^0_{101} = 0$, the initial levels of capital holdings held by each agent are uniformly given. In this case, we give $S^0_{kk} = 0.6767$ in all examples where first one-hundred agents are initially poorer in wealth relative to the 101-th agent and the rest one-hundred agents are initially richer.

5.1 EIS, the speed of convergence and wealth distribution

Homogeneous EIS

Table 1 reports the homogeneous EIS economy in which all agents have an identical value of β. In particular, the value $\beta = 2.5$ is widely used in this field. The long-run wealth inequality, S^*_{kk}, takes 0.6143, showing that the long-run wealth inequality is lower than the initial level of inequality $S^0_{kk}(= 0.6767) > S^*_{kk}$ as analytically seen in the previous section, and that the wealth inequality decrease by around 10 % in this homogeneous EIS economy. Since the speed of convergence is 0.0325, it takes around 40 years to complete 25 percent of transitional process when the initial level of aggregate capital is 75 percent of steady-state level of aggregate capital, $K^0 = 0.75K^*$.

Heterogeneous EIS and wealth inequality

In order to investigate the role of heterogeneous EIS for the wealth inequality, we vary β over a large range. In the baseline set-up, to understand the effects of heterogeneous EIS on the wealth inequality more easily, we suppose that the values of EIS are divided into β_{poor} and β_{rich} according to the initial levels of capital holdings. In other words, we assume that the initially less wealthy agents (1st–100-th agents) and the 101-th average agent have β_{poor}, and the rest agents (102-th–201-th agents) have β_{rich}. The separation of risk aversion by wealth is supported in the empirical papers. For instance, Friend and Blume (1975), which is an influential study in this field, examine the relationship between risk aversion and wealth endowment, showing that individuals invest a larger proportion of their wealth in risky assets as wealth increases. That is, individuals are risk lovers as own wealth increases. Recently, Guiso and Paiella (2008) find that the consumer's endowment negatively affects the degree of his risk aversion, which is consistent with the finding in Friend and Blume (1975).[14]

When the values of β_{poor} and β_{rich} respectively change, Table 2 presents four cases. Case 1 assumes that the initially relative-wealth rich (poor) has a larger (smaller) degree of EIS. In the rest three cases, we assume that the initially relative-wealth rich (poor) has a smaller (larger) degree of EIS. Figure 1–4 show the dynamic behavior of wealth inequality in (33), S_{kk}, and the elements (#3), (#4) and (#5) in (33), and the steady-state dispersion of wealth. Concretely, each figure (a) gives the dynamic motion of wealth inequality and the below (b) indicates those of (#3), (#4) and (#5) where a solid curve shows the dynamic motion of (#3), a fine curve (#4) and a thick curve (#5). The figures (c) give the long-run dispersion of wealth.

First, as for the case in which the initially relative-wealth rich has a greater degree of EIS, moving horizontally along the line of Case 1 in Table 2, we can see that $S^*_{kk} = 2.2057$, finding that the degree of wealth inequality in the steady state is larger than three times of initial level. Noting that the speed of convergence is 0.0394, such a more unequal

[14]Considering that wealthier people can make a larger ratio of invest in education, the existing studies make use of the investment in education as a substitute variable for the investment in the risky asset. For instance, see Outreville (2013) with respect to a survey of literature in this field.

economy approaches to the steady-state equilibrium at a faster speed compared with the benchmark homogeneous-EIS economy in Table 1. Based on $K^0 = 0.75K^*$, it takes about 35 years to complete 25 percent of transitional process. In addition, the correlation $S_{k\theta}$ has a positive sign, which means that the agents with the greater degrees of EIS are relatively wealth-richer in the long run. That is, Case 1 expresses the economy in which the initially relative-wealth riches keep the relatively superior position in wealth during the transition. The dispersion of EIS in the steady state, $S^*_{\theta\theta}$, is around 15. As seen later in Table 2, the larger the difference of EIS, the larger the positive value of $S^*_{\theta\theta}$. Looking at the last row in Table 2, the wealth distribution effect, given by (#2) in (17), is 0.0028, which accounts for about 7 percent in the speed of convergence.

Figure 1(a) shows that the level of wealth inequality concavely increases over time. Concretely, Figure 1(a) gives around $S_{kk} = 1.35$ at Time= 20, seeing that the level of wealth inequality increases by a double of initial level of wealth inequality around 6 years, and after that, it reaches a triple level over the rest 29 years. As for each element, the impact of initial wealth inequality, given in (#3), is positive but merely decreases over time. The (#4) together with the dispersion in EIS has negative impact and decreases over time. The impact of (#5) with the correlation $S^*_{k\theta}$ is positive and increases across time. As for (#4) and (#5), notice that there is no impacts at the initial period because $\Gamma^0_1 = 1$ in ($34a$) and $\Gamma^0_2 = 0$ in ($34b$). Turning into Figure 1(c), we can see that the distribution of wealth is separated in two parts. That is, since the initially more wealth people who belong to the upper area of uniform distribution at the initial period save more wealth than the initial poor during the transitional process, the upper area of uniform distribution at the initial period parts from the lower area.

Next, let us consider the cases in which the initially relative-wealth rich has a lower degree of EIS. Moving across the horizontal line of Case 2 in Table 2, we firstly find that the level of wealth inequality decreases in the long run by around 50 percent, and that the level of wealth inequality monotonically decreases in Figure 2(a). Comparing this finding with the homogeneous EIS economy in Table 1, the long-run level of wealth inequality in Case 2 is lower than the homogeneous-EIS economy, showing that the levels of capital held by the initial poor people are closer to that held by the initial riches in the long run; alternatively, we confirmed that about thirty initial poor do not catch up with any initial riches. The reason can be predicted considering that the difference of EIS between the rich and poor group is not large enough. Moreover, this prediction is supported in Figure 2(b), showing that (#4) and (#5) in (33), related to the heterogeneities of EIS, move flat on x-axis = 0, and therefore, they have the insignificant impacts on the wealth inequality over time. One further point we see is that the correlation $S^*_{k\theta}$ has a negative sign in Case

2, meaning that the agents with the greater degrees of EIS are not relatively wealth-rich in the steady state. In other words, although the initially less wealth people have the propensity to save more than the initial riches, they remain relatively wealth-poor in the long run on average. Finally, from Figure 2(c) it seems that the dispersion of wealth in the steady state is similar with the normal distribution, which is composed as follows. Noting that a lot of of initial poor people cannot catch up with a lot of initial riches, the lower (upper) area composes the initial poor (rich) people. In the middle of wealth distribution, there is a mixture of initial riches and poor.

In Case 3, we treat the case that the difference of EIS between the initial poor and rich is large, $\beta_{poor} = 1.5$ and $\beta_{rich} = 2.5$, showing that beyond our prediction, the steady-state level of wealth inequality is larger than that in the case 2. Then, we can predict that a lot of initially relative-wealth poor catch up with the initial riches, and furthermore, the poor holds a lot of wealth in the long run, which can be supported by the positive sign of $S^*_{k\theta}$. We can see from Figure 3(a) that until Time= 30 the level of wealth inequality decreases and reaches a bottom level of around $S_{kk} = 0.36$, and after then, the level of wealth inequality mildly increases until the steady state. Furthermore, unlike Figure 2(b), the large difference of EIS derives the effects of (#4) and (#5) in (33) on wealth inequality along time in Figure 3(b). Figure 3(c) give the distribution of wealth that the rich group in the long run consists of a lot of initial poor, and the poor group consists of the initially relative-wealth riches.

Turning to Case 4, we suppose that the difference of EIS is further large, $\beta_{poor} = 1.2$ and $\beta_{rich} = 2.5$. Interestingly, we can find that the steady-state level of wealth inequality increases by about fifteen percent $S^*_{kk}(= 0.7757) > S^0_{kk}$. Then, Figure 4(a) gives the U-shaped wealth inequality. In detail, the level of wealth inequality sharply decreases until around Time= 17, and thereafter, the level of wealth inequality dramatically increases and is larger than the initial level in the long run. Comparing Figure 3(b) with Figure 4(b), we can see that the dynamic movement of effect (#3) in Figure 4(b) is similar with that in Figure 3(b); however, the impacts of (#4) and (#5) are larger in Case 4 where we have to notice that the scale of vertical axis in Figure 4(b) is the same with the double of that in Figure 3(b). For example, Figure 4(b) shows that the positive impact of the correlation in (#5) is larger than that of (#3) after about Time= 17, and finally, it is larger than about two times of (#3), thereby being able to see that the correlation impact in (#5) derives a dramatic increase in wealth inequality after Time=17. Alternatively, Figure 3(b) does not give such a large effect. Next, looking at Figure 4(c), we can confirm that the distribution of wealth is separated by two parts like Figure 1(c). However, unlike Figure 1(c), the persons who belong to the rich (poor) group in the steady state are initially relative-wealth poor (rich). As for the wealth distribution effect on the speeds of convergence, we can see the larger impact in Case 4 than in Case 3.

Finally, Table 3 gives the catching-up time T in (24), such as $\Delta_{ij} = 0$. In particular, we deal with two agents $i = 91$ and $i = 101$ where the 101-th agent holds the average level of capital stock at the initial period $K^0_{101} = K^0/201$ and the initial level of capital stock held by the 91-th agent is $K^0_{91} = 0.99K^0_{101}$. In Case 3 and 4, these two persons overtake all people who belong to the rich group in the initial period. Table 3 reveals the following three points. First, looking at the horizontal lines, we can see that the larger the initial level of capital stock held by initially relative-wealth riches, the larger the catching-up time T. For instance, the time T that $\Delta_{101,121} = 0$ is larger than the double of time T that $\Delta_{101,111} = 0$ where $K^0_{111} = 1.01K^0_{101}$ and $K^0_{121} = 1.02K^0_{101}$. We furthermore find that the catching-up time T increases in a convex way as the initial levels of capital stock held by the initial riches become large. Second, as predicted easily, the 101-th agent can catch up with each initially relative-wealth rich more rapidly than the 91-th agent because the initial level of capital stock held by the 101-th agent is larger than that held by 91-th agent. Third, comparing Case 3 with Case 4, the greater degree of EIS by the initially relative-wealth poor makes the catching-up time shorter.

5.2 Pareto distribution and multiple types of EIS

Since Pareto (1897), the study of income distribution has a long history. Recently, it has been well-known that Pareto (power) law is only applicable to the upper tail of income distribution, while as the income approaches lower ranges, the distribution of income gradually deviates from Pareto law as firstly clarified in Gibrat (1931). For instance, according to Montroll and Shelesinger (1983) and Souma (2001), the log normal distribution fit the lower tail of observed distribution of US for 1935–1936 and Japan annual income over the 44 years 1955–1998; and alternatively, Clementi and Gallegati (2005) observe that the lower tail follows a two-parameter lognormal distribution in Itary for 1977–2002.[15] Looking at each Figure 1–4 (c) again, we can conclude that the distribution of wealth in the long run does not correspond to the wealth inequality observed in realistic economies.

The purpose of this subsection is to examine whether the heterogeneities of EIS can

[15]Kalecki (1945) argues that the log-normal distribution is not stationary, because its width increases over time. Drăgulescu and Yakovenko (2001) find that the middle portion of the income distribution has an exponential form for the UK during the period 1994–1999 and for the US in 1998. See other influential papers in this feield: Champernowne (1953), Mandelbrot (1960), and Reed (2003).

be the key element of forming realistic wealth (income) inequality when the aggregated wealth is uniformly distributed at the initial economy.[16] In addition to achieve this purpose, the important points we have to care about are the following three points.

[**1**] Risk lovers (the agents with higher degree of EIS) are relatively wealth-riches in the long run.

[**2**] The dynamic behavior of wealth held by the median agent is inverse-U shaped along time.

[**3**] The level of wealth held by the median agent is lower than its average level in the long run.

The point [1] follows the finding in Friend and Blume (1975) and Guiso and Paiella (2008) that a greater proportion of wealth is held in the form of investment as the level of wealth is increased. As for [2] and [3], Wolff (1998) confirms the dynamic motion of wealth held by the median consumer in U.S. economy. Concretely, making use of the survey of consumer finances conducted by the Federal Reserve Board in U.S., he sees an inverse U-shaped motion of median wealth for a U.S. household, that is, the median wealth increased by 7 percent between 1983 and 1989, however it fell by 17 percent from 1989 to 1995. In particular, it is observed that the median level of wealth is lower than the average level.

We now assume the following EIS.

$$\beta_i = 1.8\ (i = 1-3),\ 1.87\ (i = 4-10),\ 2\ (i = 11-30),\ 2.5\ (i = 31-100),$$
$$\beta_i = 3\ (i = 101-140),\ 3.4\ (i = 141-160),\ 3.9\ (i = 161-180),\ 4.4\ (i = 181-201), \tag{40}$$

where each agent holds the initial levels of capital stock in the last subsection. This set-up has the following three characteristics. First, a majority of population (35 percent of population) have a conventional parameter value $\beta = 2.5$. Second, we assume the larger the initial levels of capital stock, the smaller the value of EIS ($1/\beta$), which means that the initially less wealth people like to save during the transition but the initially relative-wealth riches dislike saving. Finally, the average value of risk aversion is given by $\beta = 2.95$, meaning that the average value of β is larger than its majority. In other words, we assume that a small portion of initially relative-wealth poor has greater degrees of EIS and a large portion of initial riches has smaller degrees of EIS, to explain two tail of low incomes and high incomes observed in realistic wealth inequality.

[16] Since the labor supply is inelastic in our model, the shape of dispersion of wealth is identical to that of income.

Making use of this setting, we numerically obtain the following steady-state values:

$$S^*_{kk} = 0.35, \quad S^*_{k\theta} = 0.9584, \quad S^*_{\theta\theta} = 12.4307, \quad \lambda = 0.0303,$$
$$\text{The wealth distribution effect (\#2) in (17)} = 0.0005.$$

In this economy, the steady-state level of wealth inequality is lower than the 52 percent of initial level. Importantly, the distribution of wealth in Figure 5(c) is alike to the form of realistic wealth inequality in the point that the lower tail of wealth follows a log-normal distribution and the upper tail forms the Pareto law.

As for the dynamic motion of wealth inequality, Figure 5(a) shows that the level of wealth inequality decreases until Time= 40, and after then, it mildly increases until the steady state. In that regard, the dynamic movement of wealth inequality in this case is similar with that in Case 3. Similarly, the dynamic motion of (#3), (#4) and (#5) in Figure 3(b) and 5(b) are similar. Figure 5(d) gives the relationship between the levels of long-run wealth and EIS, showing that the larger the degrees of EIS, the larger the long-run levels of wealth. Because the risk lovers are riches in wealth, Figure 5(d) satisfies the point [1] observed in Friend and Blume (1975) and Guiso and Paiella (2008).

To study the dynamic characterization of wealth inequality further, we turn our interests into the dispersion of wealth given in Figure 5(e) and 5(f). Concretely, Figure 5(e) shows the dispersion of wealth at Time= 20 (the blue color) and Time= 50 (the red color), and Figure 5(f) shows the relationship between wealth and EIS at Time= 20 (the circle with the blue color) and Time= 50 (the sign + with the red color). Then, we can find how the uniformly distributed wealth at the initial economy forms a realistic wealth inequality in the long run. Looking at the blue color in Figure 5(e) and 5(f), Figure 5(e) shows that the blue bar graph has two peaks, and Figure 5(f) confirms who shape the two peaks. The first peak around 0.0255 consists of the agents $i = 1 - 30$ and some agents whose degree of EIS $1/\beta$, is 0.4 (i.e., $\beta = 2.5$). That is, when the initially less wealth agents who have the greater degrees of EIS catch up with the majority in wealth, this peak is shaped. On the other hand, we confirm that the other peak consists of a part of agents $i = 31 - 180$, implying that when the initially relative-wealth riches who have the lower degrees of EIS are caught by the majority, the other peak can be shaped. When the time further passes from Time= 20 to 50, the wealth distribution given by red color in Figure 5(e) is alike to the normal distribution, which means that two peaks at Time= 20 disappear, and only one peak can be seen. Then, it can be found from Figure 5(f) that the initial poor with the greater degrees of EIS do not belong to this peak but they are in the upper tail of wealth, and that the peak is composed of a part of majority with $\beta = 2.5$ and the initial riches. Looking back to Figure 5(c) and (d) again, it can be seen that a part of initially relative-wealth riches with the lowest degree of EIS $\beta = 4.4$ forms the lower tail of wealth, and that the initial poor people own a lot of wealth further, which shapes the upper tail of wealth.

Figure 5(g) observes the dynamic movement of personal wealth held by the 1st (top 25th percentile), 2nd (median) and 3rd quantile (bottom 25th percentile) of agents. Firstly, because the average level of $\sigma^*_{i,k}$ is zero, from (20) we find that the average level of wealth is given by the straight dotted line. At the initial economy, the median agent holds the average level of wealth. Secondly, looking at the dot-dash curve in Figure 5(g), the level of wealth held by the median agent increases until about Time= 35, and hereafter, it decreases until the steady state, showing that the dynamic movement of wealth held by the median agent is U-shaped along time. In addition, we can see that the level of wealth held by the median agent is lower than the average level since Time= 45, meaning that more than half of agents own less wealth than the average level in the long run. These characterizations of wealth held by the median agent satisfy [2] and [3] observed in the U.S. economy. Thirdly, the level of wealth held by 3rd quantile of agent, which is given by the solid curve, increases until around Time= 45, and hereafter, it decreases until the steady state. This movement is qualitatively alike to that by the median agent. Alternatively, the level of wealth held by 1rd quantile of agent moves in the opposite way.

Finally, we can conclude that the heterogeneities of EIS play the important role for forming the wealth inequality in the sense that the characteristics of wealth inequality in realistic economies are observed in our numerical simulation. Concretely, Figure 5(c) gives a similar dispersion of wealth in realistic economy, Figure 5(d) imitates the empirical finding [1] and Figure 5(g) imitates the empirical findings [2] and [3]. Because of the heterogeneities of EIS, we can obtain the qualitative characterization of wealth inequality observed in realistic economies, while we cannot imitate the quantitative characteristics of wealth inequality. For instance, dividing by five wealth groups all agents, our case shows that the Gini coefficient in five quantile groups is 0.0078 in the long run, and it is very low compared with those observed in realistic economies. Furthermore, as for [3], Wolff (1998) finds that mean wealth is higher than four times of median wealth in U.S. economy, while our simulation does not give such a large difference.

When the qualitative characteristics are still satisfied, the key element of imitating the quantitative characteristics such as increasing the Gini coefficient may be to increase the number of agents in the economy. For instance, let us change the number of agents from 201 to 401 where the initial levels of capital holdings and the heterogeneities of EIS are given by:[17]

$$\beta_i = 1.8\ (i=1-6),\ 1.87\ (i=7-20),\ 2\ (i=21-60),\ 2.5\ (i=61-200),$$
$$\beta_i = 3\ (i=201-280),\ 3.4\ (i=281-320),\ 3.9\ (i=321-360),\ 4.4\ (i=361-401), \tag{41}$$

$$\frac{K_1^0}{K_{201}^0}=0.9,\ \frac{K_2^0}{K_{201}^0}=0.9005,\ ...,\ \frac{K_{200}^0}{K_{201}^0}=0.9995,\ \frac{K_{202}^0}{K_{201}^0}=1.0005,\ \frac{K_{203}^0}{K_{201}^0}=1.001, ...,\ \frac{K_{401}^0}{K_{201}^0}=1.1,$$

In particular, noting that each proportion of agents for each degree of EIS in (41) is the same with (40) in the previous case, we confirmed that the dynamic motion and the steady-state characterization of wealth inequality do not qualitatively change; however, the Gini coefficient increases from 0.0079 to 0.0097. Similarly, increasing the number of agents from 401 to 2001 yields the high Gini coefficient further, 0.0112, with keeping the qualitative characteristics of wealth distribution. Then, we may predict that the extremely large number of agents leads to the Gini coefficient observed in real economy where the qualitative characteristics are still kept.[18]

6 Conclusion

In this paper, we introduce the heterogeneities of EIS into the Ramsey version of macrodynamic model with a finite number of agents. We obtain two main contributions. First, we analytically characterize the steady-state levels of individual capital, and hence, show the existence of unique steady-state equilibrium given the initial holdings of capital stocks and the different degrees of EIS.

Second, we examine wealth inequality comprehensively. In particular, we confirm that the dynamic motion of wealth inequality under the heterogeneous EIS is characterized by the initially dispersed wealth, the long-run dispersion of EIS and the correlation between EIS and wealth, while the simplification of homogeneous EIS yields only the impact of initially dispersed wealth on wealth inequality. Finally, to obtain further insights into the relationship between EIS and wealth inequality, we conduct the numerical simulation.

[17]When the number of agents further increases, it takes an extreme long time to simulate the dynamic behavior; therefore, we pay attention to the case in which the number of agents is small compared with that in realistic economies.

[18]The changes in the rate of time preference ρ and the productivity A may be useful in the mean that these changes have the quantitative impacts on the wealth inequality, not the qualitative ones. For instance, an increase in the rate of time preference from $\rho = 0.04$ to 0.12 yields $S_{kk}^* = 2.0694$ in Case 1, 0.3028 in Case 2, 0.3020 in Case 3 and 0.5467 in Case 4; a decrease in the productivity from $A = 0.5$ to 0.25 yields $S_{kk}^* = 2.1135$ in Case 1, 0.3138 in Case 2, 0.3378 in Case 3 and 0.6144 in Case 4.

Then, we confirm that the heterogeneities of EIS can be the important elements of shaping realistic distribution of wealth.

References

Álvarez-Peláez, M., and Díaz, A., (2005), "Minimum consumption and transitional dynamics in wealth distribution", *Journal of Monetary Economics* 52, 663–667.

Arrow, K., (1970), *Essays in the theory of risk-bearing.* North Holland.

Atolia, M., Chatterjee, S., and Turnovsky, S., (2012), "Growth and inequality: Dependence on the time path of productivity increases (and other structural changes)", *Journal of Economic Dynamics and Control* 36, 331-348.

Barro, R., and Sala-i-Martin, X., (2004), *Economic growth* (second edition), The MIT Press.

Becker. R., (1980), "On the long-run steady state in a simple dynamic model of equilibrium with heterogeneous households", *Quarterly Journal of Economics* 95, 375–382.

Caselli, F., and Ventura, J., (2000), "A representative consumer theory of distribution", *American Economic Review* 90, 909-926.

Champernowne, D., (1953), "A model of income distribution", *Economic Journal* 63, 318–351.

Clementi, F., and Gallegati, M., (2005), "Power law tails in the Italian personal income distribution", *Physica A: Statistical Mechanics and its Applications* 350, 427–438.

Coen-Pirani, D., (2004), "Effects of differences in risk aversion on the distribution of wealth", *Macroeconomic Dynamics* 8, 612–632.

Coen-Pirani, D., (2005), "Margin requirements and equilibrium asset prices", *Journal of Monetary Economics* 52, 449–475.

Drăgulescu, A., and Yakovenko, V., (2001), "Exponential and power-law probability distributions of wealth and income in the United Kingdom and the United States", *Physica A: Statistical Mechanics and its Applications* 299, 213–221.

Eekhoudt, L., Gollier, C., and Schlesinger, H., (1996), "Changes in background risk and risk taking behavior", *Econometrica* 64, 683–689.

Epstein, L., (1987a), "A simple dynamic general equilibrium model", *Journal of Economic Theory* 41, 68–95.

Esptein, L., (1987b), "The global stability of efficient intertemporal allocations", *Econometrica* 55, 329–355.

Epstein, L., and Zin, S., (1989), "Substitution, risk aversion and the temporal behavior of consumption and asset returns: a theoretical framework", *Econometrica* 57, 937–969.

Friend, I., and Blume, M.E., 1975, The demand for risky assets, *American Economic Review* 65, 900–922.

García-Peñalosa, C. and Turnovsky, S. (2006), "Growth and Income Inequality: A Canonical Model", *Economic Theory* 28, 25-49.
García-Peñalosa, C. and Turnovsky, S. (2007), "Growth, income inequality and fiscal policy: what are the relevant trade-offs?", *Journal of Money, Credit and Banking* 39, 369–394.
García-Peñalosa, C. and Turnovsky, S. (2009), "The dynamics of wealth inequality in a simple Ramsey model: a note on the role of production flexibility", *Macroeconomic Dynamics* 13, 250–262.
Gibrat, (1931), *Les inégalités économiques.* Paris 1931.
Guiso, L., and Paiella, M., (2008), "Risk aversion, wealth, and background risk", *Journal of the European Economic Association* 6, 1109-1150.
Holt and Laury, (2002), "Risk aversion and incentive effects", *American Economic Review* 92, 1644–1655.
Kalecki, M., (1945), "On the Gibrat distribution", *Econometrica* 13, 161–170.
Kimball, M., (1993), "Standard risk aversion", *Econometrica* 61, 589–611.
Li, W., Rychlik, M., Szidarovszky, F., and Chiarella, C., (2003), "On the attractivity of a class of homogeneous dynamic economic systems", *Nonlinear Analysis* 52, 1617–1636.
Mandelbrot, B., (1960), "The Pareto-Lévy law and the distribution of income", *International Economic Review* 1, 79–106.
Montroll, E., and Shlesinger, M. 1983, Maximum entropy formalism, fractals, scaling phenomena, and $1/f$ noise: a tale of tails, *Journal of Statistical Physics* 32, 209–230.
Outreville, F., (2013), "The relationship between relative risk aversion and the level of eduction: a survey and implications for the demand for life insurance", *Journal of Economic Surveys*, 1–15.
Pareto, V., (1987), *Cours d'Economie Politique.* Lausanne and Paris: Rouge and Cie.
Pratt, J., and Zeckhauser, R., (1987), "Proper risk aversion", *Econometrica* 55, 143–154.
Reed, W., (2003), "The Pareto law of incomes–an explanation and an extension", *Physica A: Statistical Mechanics and its Applications* 319, 469–486.
Sarte, G., (1997), "Progressive taxation and income inequality in dynamic competitive equilibrium", *Journal of Public Economics* 66, 145–171.
Sorger, G., (2000), "Income and wealth distribution in a simple model of growth", *Economic Theory* 16, 23–42.
Souma, W., (2001), "Universal structure of the personal income distribution", *Fractals* 9, 463–470.
Tanaka, T., Camerer, C., and Nguyen, Q., 2010, "Risk and time preferences: Linking experimental and household survey data from Vietnam", *American Economic Review* 15,

page 557–571.

Turnovsky, S., (2002), "Intertemporal and Intratemporal Substitution, and the Speed of Convergence in the Neoclassical Growth Model", *Journal of Economic Dynamics and Control* 26, 1765–1785.

Turnovsky, S., and García-Peñalosa, C. (2008),"Distributional dynamics in a neoclassical growth model: The role of elastic labor supply", *Journal of Economic Dynamics and Control* 32, 1399-1431.

Wolff, E., (1998), "Recent trends in the size distribution of household wealth", *Journal of Economic Perspective* 12, 131-150.

Table 1: Homogeneous EIS $\beta = 2.5$

	K^*	S^*_{kk}	Λ
Benchmark economy:	5.1905	0.6143	0.0325

Table 2: Heterogeneous EIS, wealth inequality and convergence speed

”	β_{poor}	β_{rich}	K^*	S^*_{kk}	$S^*_{k\theta}$	$S^*_{\theta\theta}$	λ	(#2) in (17)
Case 1:	2.5	1.5	5.1905	2.2057	5.4552	15.1658	0.0394	0.0028
Case 2:	2	2.5	5.1905	0.328	-0.2094	2.3613	0.0353	-0.0012
Case 3:	1.5	2.5	5.1905	0.4183	1.1964	13.1106	0.0396	-0.0004
Case 4:	1.2	2.5	5.1905	0.7757	3.4641	26.2927	0.0437	0.0017

Table 3: The catching-up time in Case 3 and 4

				The catching-up time T in (24)									
		β_{poor}	β_{rich}	$i = 111$	$i = 121$	$i = 131$	$i = 141$	$i = 151$	$i = 161$	$i = 171$	$i = 181$	$i = 191$	$i = 201$
Case 3:	$i = 91$	1.5	2.5	4.5684	7.165	10.0302	13.2297	16.8567	21.0498	26.0282	32.1693	40.2114	51.9405
	$i = 101$	1.5	2.5	2.1817	4.5456	7.1276	9.9749	13.1519	16.7501	20.9044	25.828	31.8852	39.7826
Case 4:	$i = 91$	1.2	2.5	2.9233	4.5177	6.217	8.0373	9.9984	12.1254	14.4509	17.0178	19.8851	23.136
	$i = 101$	1.2	2.5	1.4153	2.9113	4.4984	6.1896	8.0005	9.9507	12.0649	14.3749	16.923	19.7667

Figure 1

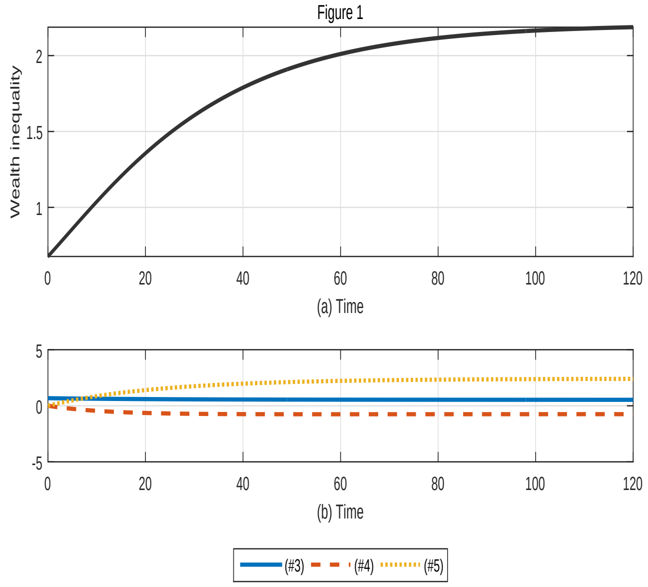


Figure 1(c)

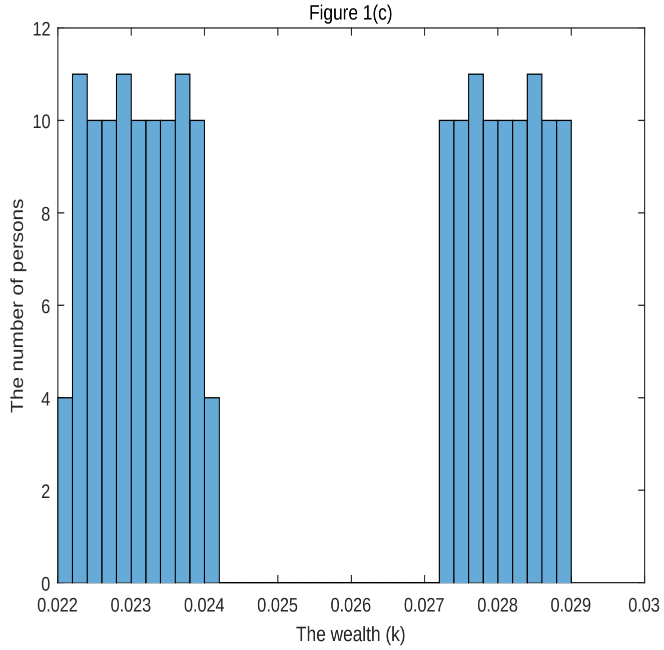


Figure 2

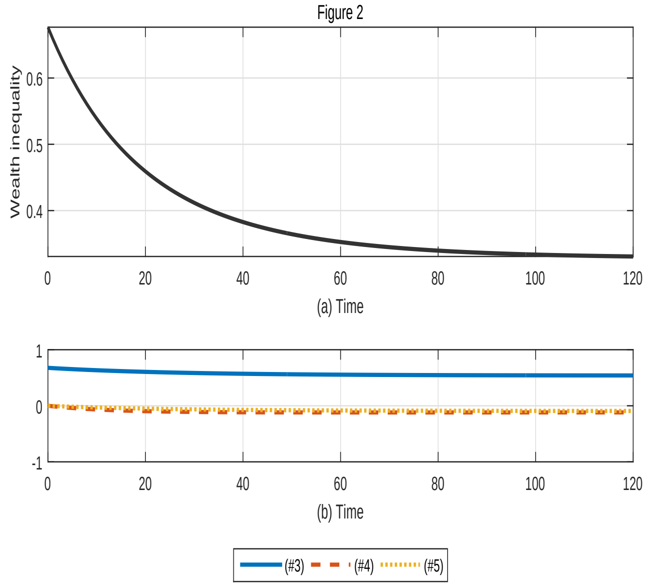
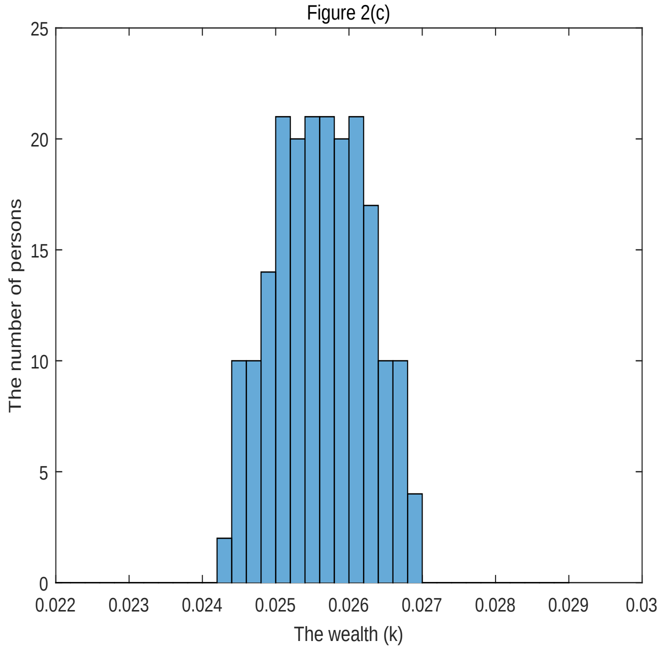


Figure 2(c)



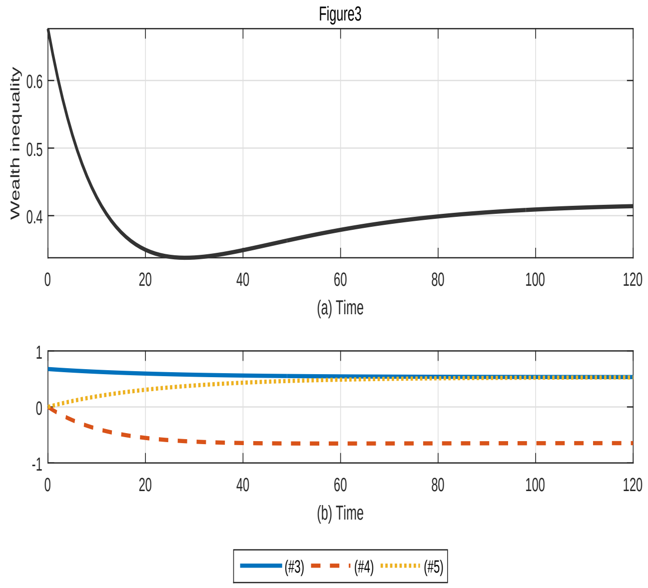


Figure3
Wealth inequality
0.6
0.5
0.4
0
20
40
60
80
100
120
(a) Time
1
0
-1
(b) Time
(#3)
(#4)
(#5)

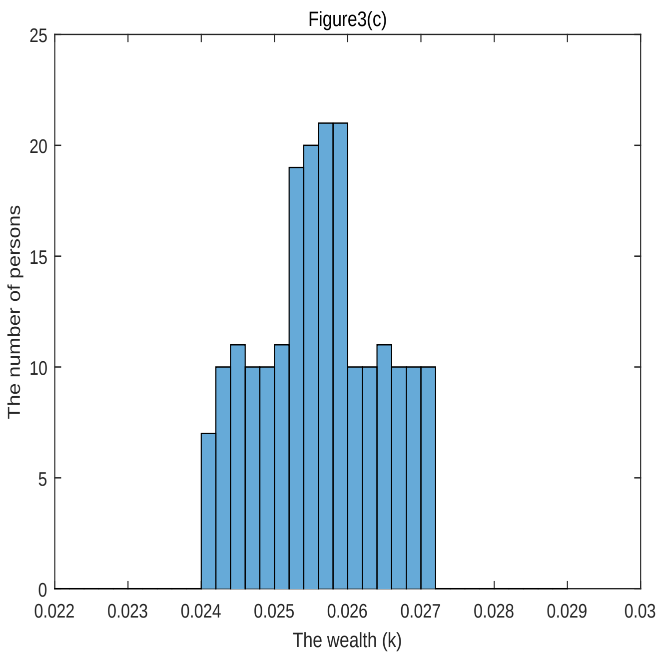


Figure3(c)
The number of persons
25
20
15
10
5
0
0.022
0.023
0.024
0.025
0.026
0.027
0.028
0.029
0.03
The wealth (k)

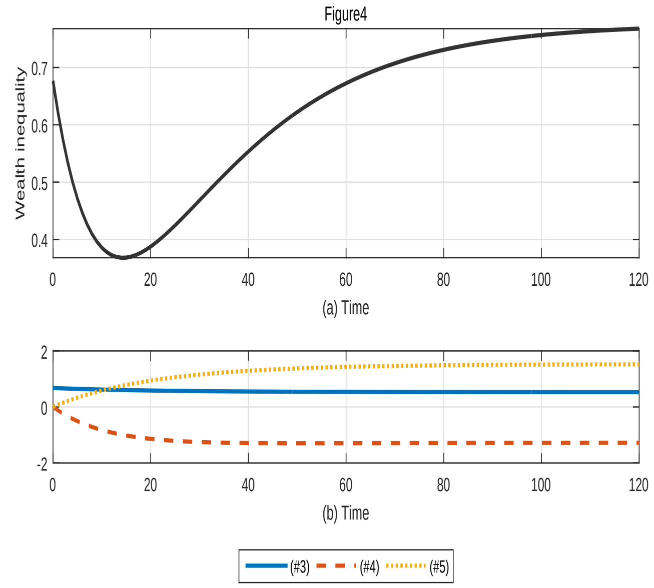


Figure4
Wealth inequality
0.4
0.5
0.6
0.7
0
20
40
60
80
100
120
(a) Time
2
0
-2
0
20
40
60
80
100
120
(b) Time
(#3)
(#4)
(#5)

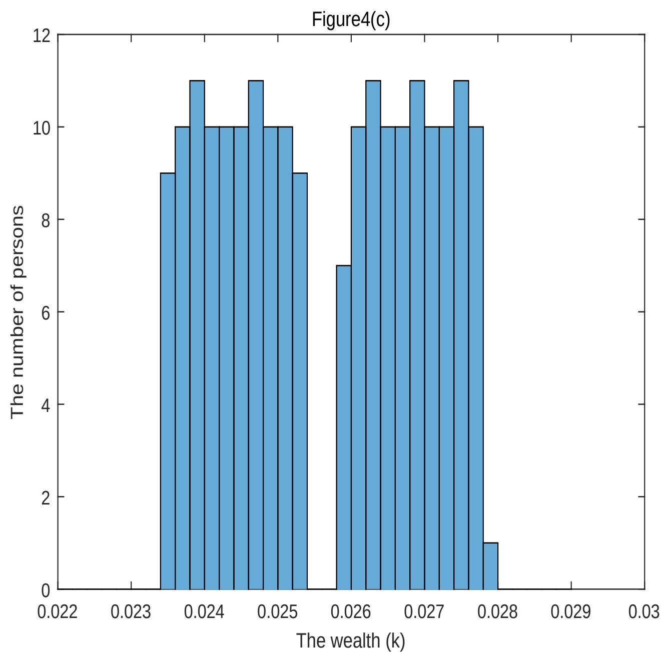


Figure4(c)
The number of persons
0
2
4
6
8
10
12
0.022
0.023
0.024
0.025
0.026
0.027
0.028
0.029
0.03
The wealth (k)

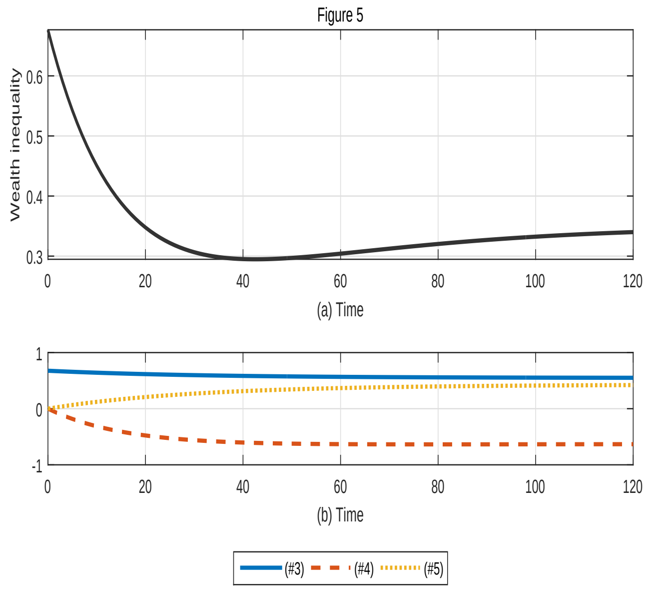


Figure 5
Wealth inequality
0.3
0.4
0.5
0.6
0 20 40 60 80 100 120
(a) Time
1
0
-1
0 20 40 60 80 100 120
(b) Time
(#3) (#4) (#5)

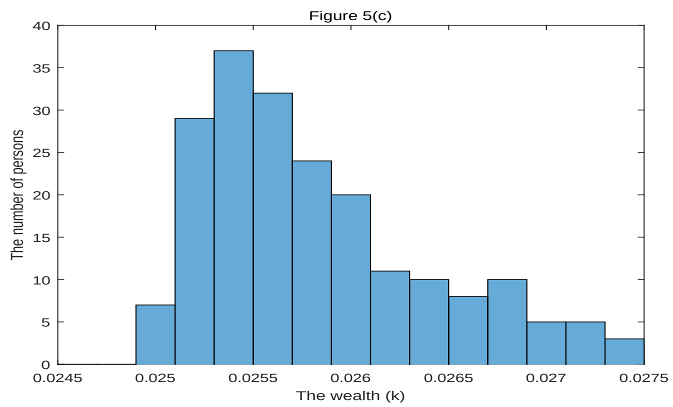


Figure 5(c)
The number of persons
0 5 10 15 20 25 30 35 40
0.0245 0.025 0.0255 0.026 0.0265 0.027 0.0275
The wealth (k)

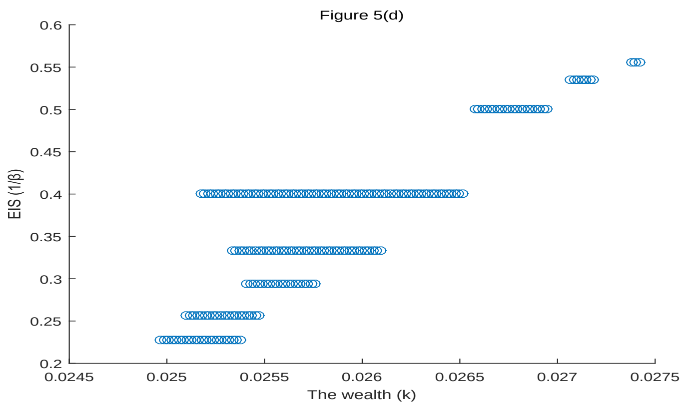


Figure 5(d)
EIS (1/β)
0.2 0.25 0.3 0.35 0.4 0.45 0.5 0.55 0.6
0.0245 0.025 0.0255 0.026 0.0265 0.027 0.0275
The wealth (k)

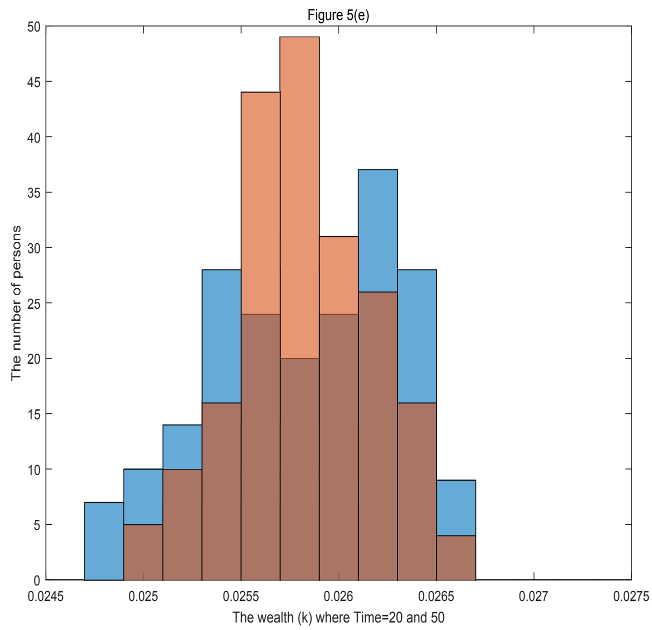


Figure 5(e)
The number of persons
The wealth (k) where Time=20 and 50

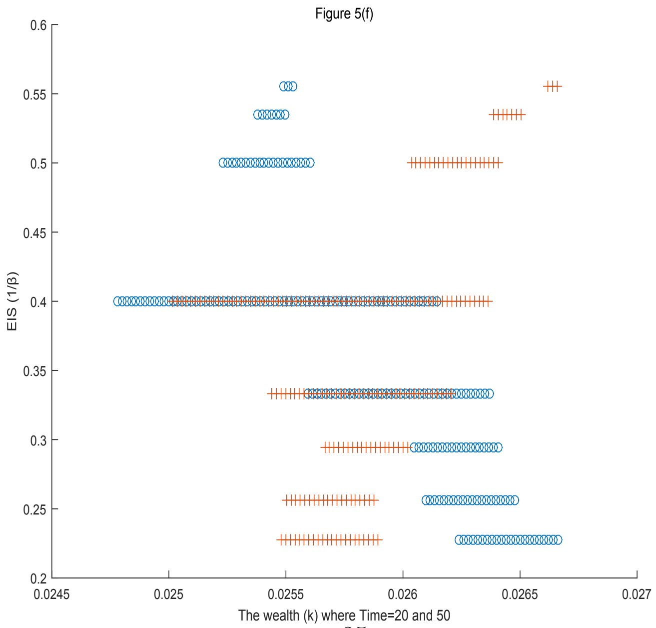
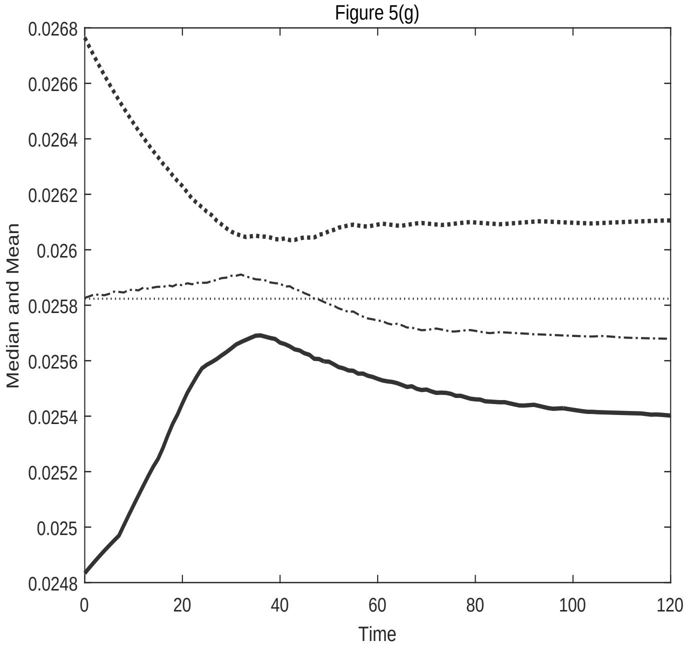


Figure 5(f)
EIS (1/β)
The wealth (k) where Time=20 and 50

Figure 5(g)


0.0268
0.0266
0.0264
0.0262
0.026
0.0258
0.0256
0.0254
0.0252
0.025
0.0248
Median and Mean
0
20
40
60
80
100
120
Time