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Chu, Angus C. and Liao, Chih-Hsing and Liu, Xiangbo and Zhang, Mengbo

University of Liverpool, Chinese Culture University, Renmin University, University of California, Los Angeles

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Angus C. Chu Chih-Hsing Liao Xiangbo Liu Mengbo Zhang

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Abstract

This study explores the effects of inflation on economic growth in a monetary search-and-matching model with productive government expenditure. Our results can be summarized as follows. When labor intensity in the production function is below a threshold value, the economy features a unique balanced growth equilibrium in which inflation reduces economic growth. When labor intensity in the production function is above a threshold value, the economy may feature multiple balanced growth paths. Multiple equilibria (i.e., global indeterminacy) arise when the matching probability in the decentralized market is sufficiently large. In this case, the high-growth equilibrium features a negative effect of inflation on economic growth whereas the low-growth equilibrium features a U-shaped effect of inflation on growth. Furthermore, under a sufficiently large matching probability in the decentralized market, both equilibria are locally determinate, and hence, either equilibrium may emerge in the economy.

Keywords: Economic growth; inflation; money; random matching; indeterminacy

JEL Classification: E30, E40, O42

Chu: anguscccc@gmail.com. University of Liverpool Management School, University of Liverpool, Liverpool, United Kingdom. Liao: chihhsingliao@gmail.com. Department of Economics, Chinese Culture University, Taipei, Taiwan. Liu: xiangbo.liu@gmail.com. Hanqing Advanced Institute of Economics and Finance and International Monetary Institute, Renmin University, Beijing, China. Zhang: mbzhangucla@g.ucla.edu. Department of Economics, University of California, Los Angeles, United States.

1 Introduction

This study explores the effects of inflation on economic growth in a monetary search-and-matching model with equilibrium indeterminacy. We consider a two-sector search-and-matching model from Lagos and Wright (2005) and follow Aruoba *et al.* (2011) and Waller (2011) to incorporate endogenous capital accumulation into the model. The novelty of our study is that we allow for capital externality via productive government spending as in the seminal study by Barro (1990) in order to generate endogenous economic growth. The resulting monetary search-and-matching model with productive government spending features equilibrium indeterminacy that is absent in the Barro model and the Lagos-Wright model.

Our results can be summarized as follows. When labor intensity in the production function is below a threshold value, the economy features a unique and determinate balanced growth equilibrium in which an increase in the money growth rate leads to a lower growth rate of output. Interestingly, when labor intensity in the production function is above a threshold value, the economy either features multiple balanced growth equilibria or exhibits no equilibrium. Multiple equilibria (i.e., global indeterminacy) arise when the matching probability in the decentralized market is above a threshold value. When the matching probability is above this threshold but not too large, the low-growth equilibrium is locally determinate whereas the high-growth equilibrium is locally indeterminate and subject to sunspot fluctuations around it. When the matching probability is sufficiently large, both equilibria are locally determinate. In this case, either equilibrium could emerge in the economy. When multiple equilibria are present, the high-growth equilibrium always features a negative effect of inflation on economic growth whereas the low-growth equilibrium features a U-shaped effect of inflation on growth.

The intuition behind the different effects of inflation on growth can be explained as follows. A higher inflation rate increases the cost of consumption in the decentralized market where consumption requires the use of money as a medium of exchange. Due to this lower demand for consumption goods in the decentralized market, individuals have less incentives to accumulate physical capital, which is a factor input for the production of consumption goods in the decentralized market. As a result, higher inflation reduces capital accumulation and causes a negative effect on economic growth. This negative *capital-accumulation* effect of inflation is standard in the literature. However, with the presence of productive government spending, inflation has an additional positive *labor-market* effect on growth. When inflation reduces the demand for consumption in the decentralized market, it also shifts the demand for consumption to the centralized market, where money is not needed for transaction purposes. This increase in consumption causes the individuals to also want to consume more leisure and reduces their supply of labor in the centralized market. Given that the labor demand curve may become upward sloping in the presence of productive government spending, the shift in labor supply in this case leads to a surprising increase in equilibrium labor input, which in turn increases the levels of output and capital investment. At the low-growth equilibrium, both this positive labor-market effect and the negative capital-accumulation effect are present to generate a non-monotonic effect of inflation on economic growth in the form of a U-shape.

This study relates to the literature on matching models of money and capital; see for example, Shi (1999), Menner (2006), Williamson and Wright (2010), Aruoba *et al.* (2011), Bencivenga and Camera (2011) and Waller (2011). Our study differs from these studies by allowing for *endogenous* economic growth in the long run. Chu *et al.* (2014) also consider the effects of inflation on endogenous economic growth in a matching model of money and capital, but their model does not exhibit equilibrium indeterminacy. Our model features a unique equilibrium with the same comparative static effects of inflation as in Chu *et al.* (2014) under one parameter space but also multiple equilibria with different comparative static effects of inflation under another parameter space. In other words, the analysis in this study nests the analysis in Chu *et al.* (2014) as a special case.

The study also relates to the literature on inflation and economic growth; see for example, Wang and Yip (1992), Gomme (1993), Dotsey and Ireland (1996), Ho *et al.* (2007), Chang *et al.* (2007), Chen *et al.* (2008) and Chu and Cozzi (2014). Some studies, such as Farmer (1997), Itaya and Mino (2003) and Lai and Chin (2010), also explore the effects of inflation on equilibrium indeterminacy.[1] Studies in this literature model money demand using the classical approaches, such as a cash-in-advance constraint, money in utility and transaction costs, without considering search and matching. This study attempts to relate this literature to the literature on matching models of money and capital in order to highlight the implications of random matching on growth and indeterminacy. We find that the matching probability in the decentralized market is a key determinant of the dynamic properties of the multiple equilibria in which monetary policy has different effects on economic growth.

The rest of this study is organized as follows. Section 2 presents the model. Section 3 analyzes the dynamics of the model. Section 4 studies the effects of inflation. The final section concludes.

2 The model

We consider an economy that consists of a unit continuum of identical and infinitely-lived individuals in discrete time. In each period, there are economic activities in two markets: individuals first enter a decentralized market (hereafter DM) and then a centralized market (hereafter CM). Following the literature, we assume that there is no discounting within each period, while the discount factor is $\beta \in (0, 1)$ between any two consecutive periods.

2.1 Individuals' optimization in the CM

In the CM, individuals consume and invest the general goods to maximize their lifetime discounted utility. Their instantaneous utility function is represented by

$$u_t = \theta \ln x_t - \gamma h_t,$$

where x_t is the consumption of general goods, h_t is the supply of labor, and the parameters $\gamma > 0$ and $\theta > 0$ determine respectively the disutility of labor supply and the importance of consumption. Let's denote $W(m_t, k_t)$ and $V(m_t, k_t)$ as the period-t value functions for

[1] See Benhabib and Farmer (1994, 1996) and Schmitt-Grohe and Uribe (1997) for seminal studies on equilibrium indeterminacy. Benhabib and Farmer (1999) provide a survey of this literature.

individuals in the CM and the DM, respectively. For the maximization problem of individuals in the CM, we have

$$W\left(m_t, k_t\right) = \max_{x_t, h_t, m_{t+1}, k_{t+1}} \{\theta \ln x_t - \gamma h_t + \beta V\left(m_{t+1}, k_{t+1}\right)\}, \tag{1}$$

subject to a sequence of budget constraints given by

$$k_{t+1} + \frac{m_{t+1}}{p_t} = \left(1 - \tau_t\right)\left(w_t h_t + r_t k_t\right) + \left(1 - \delta\right) k_t - x_t + z_t + \frac{m_t}{p_t}, \tag{2}$$

where p_t is the price of general goods x_t, w_t is the real wage rate, r_t is the real rental price of capital, $\tau_t \in (0,1)$ denotes the income tax rate, k_t denotes the capital stock owned by an individual, and m_t is the nominal money balance in period t. The parameter $\delta \in (0,1)$ is the depreciation rate of capital. z_t denotes a real lump-sum transfer from the government.

If we use the budget constraint to substitute h_t into equation (1), then standard dynamic optimization leads to the following first-order conditions:

$$\frac{\theta}{x_t} = \frac{\gamma}{\left(1 - \tau_t\right) w_t}, \tag{3}$$

$$\frac{\theta}{x_t} = \beta V_k\left(m_{t+1}, k_{t+1}\right), \tag{4}$$

$$\frac{\theta}{p_t x_t} = \beta V_m\left(m_{t+1}, k_{t+1}\right). \tag{5}$$

Equation (3) represents a horizontal labor supply curve. Furthermore, equations (3) to (5) imply that all individuals enter the DM in the next period with the same holdings of capital and money because x_t is the same across individuals, due to their quasi-linear preference, as shown in (3). Finally, the envelope conditions are given by

$$W_k\left(m_t, k_t\right) = \frac{\theta\left[1 - \delta + \left(1 - \tau_t\right) r_t\right]}{x_t}, \tag{6}$$

$$W_m\left(m_t, k_t\right) = \frac{\theta}{p_t x_t}. \tag{7}$$

2.2 Individuals' optimization in the DM

In the DM, firms do not operate, and a special good is produced and traded privately among individuals. We denote $\sigma \in (0, 0.5)$ as the probability of an agent becoming a buyer. Similarly, with probability σ an agent becomes a seller, and with probability $1 - 2\sigma$ he is a nontrader. Following Lagos and Wright (2005), one buyer meets one seller randomly and anonymously with a matching technology and buyers pay money in trade. Given this matching setup, the value of entering the DM is given by

$$V\left(m_t, k_t\right) = \sigma V^b\left(m_t, k_t\right) + \sigma V^s\left(m_t, k_t\right) + \left(1 - 2\sigma\right) W\left(m_t, k_t\right), \tag{8}$$

where $V^b(m_t, k_t)$ and $V^s(m_t, k_t)$ are the values of being a buyer and a seller, respectively.

To analyze $V^b(.)$ and $V^s(.)$, we consider the following functional forms for the buyers' preference and the sellers' production technology. In the DM, each buyer's utility $\ln q_t^b$ is increasing and concave in the consumption of special goods. Each seller produces special goods q_t^s by combining her capital k_t and effort e_t subject to the following Cobb-Douglas production function:

$$q_t^s = F(k_t, A_t e_t) = k_t^\alpha (A_t e_t)^{1-\alpha}, \tag{9}$$

where the parameter $\alpha \in (0,1)$ determines labor intensity $1-\alpha$ in production and A_t is the level of labor productivity. As in the seminal study by Barro (1990), labor productivity is determined by productive government expenditure; i.e., we assume that $A_t = G_t$. Rewriting equation (9), we can express the utility cost of production in terms of effort as

$$e\left(\frac{q_t^s}{G_t}, \frac{k_t}{G_t}\right) = \left(\frac{q_t^s}{G_t}\right)^{1/(1-\alpha)} \left(\frac{k_t}{G_t}\right)^{-\alpha/(1-\alpha)}. \tag{10}$$

Buyers purchase special goods q_t^b by spending money d_t^b, whereas sellers earn money d_t^s by producing special goods q_t^s. Given these terms of trade, the values of being a buyer and a seller are respectively

$$V^b(m_t, k_t) = \ln q_t^b + W(m_t - d_t^b, k_t), \tag{11}$$

$$V^s(m_t, k_t) = -e\left(\frac{q_t^s}{G_t}, \frac{k_t}{G_t}\right) + W(m_t + d_t^s, k_t). \tag{12}$$

Differentiating (11) and (12) and substituting them into (8), we can obtain the following envelope condition for m_t:

$$\begin{aligned} V_m(m_t, k_t) &= (1-2\sigma)W_m(m_t, k_t) + \sigma\left[\frac{1}{q_t^b}\frac{\partial q_t^b}{\partial m_t} + W_m(m_t - d_t^b, k_t)\left(1 - \frac{\partial d_t^b}{\partial m_t}\right)\right] \\ &\quad + \sigma\left[-e_1\left(\frac{q_t^s}{G_t}, \frac{k_t}{G_t}\right)\frac{1}{G_t}\frac{\partial q_t^s}{\partial m_t} + W_m(m_t + d_t^s, k_t)\left(1 + \frac{\partial d_t^s}{\partial m_t}\right)\right], \end{aligned} \tag{13}$$

where $W_m(m_t, k_t) = W_m(m_t - d_t^b, k_t) = W_m(m_t + d_t^s, k_t) = \theta/(p_t x_t)$ from (7). Similarly, we can obtain the following envelope condition for k_t:

$$\begin{aligned} V_k(m_t, k_t) &= (1-2\sigma)W_k(m_t, k_t) + \sigma\left[\frac{1}{q_t^b}\frac{\partial q_t^b}{\partial k_t} - W_m(m_t - d_t^b, k_t)\frac{\partial d_t^b}{\partial k_t} + W_k(m_t - d_t^b, k_t)\right] \\ &\quad + \sigma\left[-e_1\left(\frac{q_t^s}{G_t}, \frac{k_t}{G_t}\right)\frac{1}{G_t}\frac{\partial q_t^s}{\partial k_t} - e_2\left(\frac{q_t^s}{G_t}, \frac{k_t}{G_t}\right)\frac{1}{G_t} + W_m(m_t + d_t^s, k_t)\frac{\partial d_t^s}{\partial k_t} + W_k(m_t + d_t^s, k_t)\right], \end{aligned} \tag{14}$$

where $W_k(m_t, k_t) = W_k(m_t - d_t^b, k_t) = W_k(m_t + d_t^s, k_t) = \theta\left[(1-\tau_t)r_t + (1-\delta)\right]/x_t$ from (6).

To solve the marginal value of holding money (13) and capital (14), we consider a competitive equilibrium with price taking as in Aruoba *et al.* (2011) and Waller (2011).[2] Under

[2] We cannot consider bargaining in this model because the bargaining condition is incompatible with endogenous growth; see Appendix A in Chu *et al.* (2014) for a detailed discussion.

price taking, once buyers and sellers are matched, they both act as price takers. Given the price $\widetilde{p}_t$ of special goods, buyers choose q_t^b to maximize

$$V^b(m_t, k_t) = \max_{q_t^b}[\ln q_t^b + W(m_t - \widetilde{p}_t q_t^b, k_t)] \tag{15}$$

subject to the budget constraint

$$d_t^b = \widetilde{p}_t q_t^b \leq m_t. \tag{16}$$

In the DM, buyers spend all their money, so that the money constraint implies that

$$q_t^b = m_t / \widetilde{p}_t. \tag{17}$$

As for sellers' maximization problem in the DM, it is given by

$$V^s(m_t, k_t) = \max_{q_t^s}\left[-e\left(\frac{q_t^s}{G_t}, \frac{k_t}{G_t}\right) + W(m_t + \widetilde{p}_t q_t^s, k_t)\right]. \tag{18}$$

Sellers' optimal supplies of special goods can be obtained from the following condition:

$$e_1\left(\frac{q_t^s}{G_t}, \frac{k_t}{G_t}\right)\frac{1}{G_t} = \widetilde{p}_t W_m(m_t + \widetilde{p}_t q_t^s, k_t) \Leftrightarrow \frac{1}{1-\alpha} e\left(\frac{q_t^s}{G_t}, \frac{k_t}{G_t}\right) = \theta \frac{\widetilde{p}_t q_t^s}{p_t x_t}, \tag{19}$$

where the second equality of (19) makes use of (7) and (10).

Using (17) and (19), we can obtain $\partial q_t^b / \partial m_t = 1/\widetilde{p}_t$, $\partial d_t^b / \partial m_t = 1$, and $\partial d_t^s / \partial k_t = \widetilde{p}_t \left(\partial q_t^s / \partial k_t\right)$, whereas the other partial derivatives, $\partial q_t^b / \partial k_t$, $\partial d_t^b / \partial k_t$, $\partial q_t^s / \partial m_t$ and $\partial d_t^s / \partial m_t$, in (13) and (14) are zero. Substituting these conditions, $q_t^b = q_t^s = q_t$ and (19) into (13) and (14), we can derive the following conditions:

$$V_m(m_t, k_t) = \frac{(1-\sigma)\,\theta}{p_t x_t} + \frac{\sigma}{\widetilde{p}_t q_t}, \tag{20}$$

$$V_k(m_t, k_t) = \frac{\theta\left[(1-\tau_t)\, r_t + (1-\delta)\right]}{x_t} - \frac{\sigma}{G_t} e_2\left(\frac{q_t}{G_t}, \frac{k_t}{G_t}\right). \tag{21}$$

The intuition behind these two conditions can be explained as follows. The marginal value of money holding is the expected gain in utility by either consuming more special goods q_t in the DM with probability σ or consuming more general goods x_t in the CM with probability $1-\sigma$. The marginal value of capital holding is the gain in utility by consuming more general goods x_t in the CM with the after-tax net capital income $(1-\tau_t)\, r_t + 1 - \delta$ plus the expected gain in utility by incurring less production effort as a seller in the DM with probability σ.[3]

[3] Recall that $e_2(q_t/G_t, k_t/G_t) < 0$; see equation (10).

2.3 Firms' optimization in the CM

In the CM, there is a large number of identical firms. In each period, each firm produces general goods using capital K_t and labor H_t. The production function is given by

$$Y_{x,t} = K_t^{\alpha} \left(A_t H_t\right)^{1-\alpha}, \tag{22}$$

where labor productivity is determined by productive government spending as before; i.e., $A_t = G_t$. Taking factor prices and the government's expenditure as given, the representative firm chooses H_t and K_t to maximize its profits. Interior solutions of the firm's problem are characterized by the first-order conditions as follows:

$$r_t = \alpha K_t^{\alpha-1} \left(G_t H_t\right)^{1-\alpha}, \tag{23}$$

$$w_t = (1-\alpha) K_t^{\alpha} H_t^{-\alpha} G_t^{1-\alpha}. \tag{24}$$

In equilibrium, $K_t = k_t$ and $H_t = h_t$.

2.4 Government

In this economy, the government plays the following two roles: it implements fiscal and monetary policies. In each period, the government's public expenditure is financed by imposing an income tax on individuals. Therefore, the government's budget constraint is given by

$$G_t = \tau_t Y_{x,t}. \tag{25}$$

The government also issues money at an exogenously given rate at $\mu_t = (m_{t+1} - m_t)/m_t$ to finance a lump-sum transfer that has a real value of $z_t = (m_{t+1} - m_t)/p_t = \mu_t m_t / p_t$. We separate the fiscal and monetary components of the government in order to allow for monetary policy independence. In other words, we do not consider the case in which the government can use the central bank to finance its fiscal spending.[4]

2.5 Equilibrium

The equilibrium is defined as a sequence of allocations $\{G_t, x_t, h_t, Y_{x,t}, q_t, d_t, m_{t+1}, k_{t+1}\}_{t=0}^{\infty}$, a sequence of prices $\{r_t, w_t, p_t, \tilde{p}_t\}_{t=0}^{\infty}$ and a sequence of policies $\{\mu_t, \tau_t, z_t\}_{t=0}^{\infty}$, with the following conditions satisfied in each period.

- In the CM, individuals choose $\{x_t, h_t, m_{t+1}, k_{t+1}\}$ to maximize (1) subject to (2), taking $\{r_t, w_t, p_t\}$ and $\{\mu_t, \tau_t, z_t\}$ as given;
- In the DM, buyers and sellers choose $\{q_t, d_t\}$ to maximize (11) and (12) respectively, taking $\{\tilde{p}_t\}$ as given;

[4] In the case of seigniorage, higher inflation would increase tax revenue for productive government spending, and hence, it would have another positive effect on economic growth.

- Firms in the CM produce $\{Y_{x,t}\}$ competitively to maximize profit taking $\{r_t, w_t\}$ and $\{G_t\}$ as given;
- The real aggregate consumption includes consumption in CM and DM such that

$$c_t = (p_t x_t + \sigma \tilde{p}_t q_t) / p_t;$$

- The real aggregate output includes output in CM and DM such that

$$Y_t = (p_t Y_{x,t} + \sigma \tilde{p}_t q_t) / p_t;$$

- The capital stock accumulates through investment from general goods such that

$$k_{t+1} = Y_{x,t} - x_t - G_t + (1 - \delta) k_t;$$

- The government balances its budget in every period such that $G_t = \tau_t Y_{x,t}$ and $z_t = \mu_t m_t / p_t$.
- All markets clear in every period.

3 Equilibrium indeterminacy

In the rest of the paper, we assume stationary monetary and tax policies, i.e., $\mu_t = \mu$ and $\tau_t = \tau$. It should be noted that the stationary money growth rate has a lower bound, i.e., $\mu \geq \beta - 1$, which is equivalent to a zero lower bound on the nominal interest rate. The dynamical system can be derived as follows. First, we define two transformed variables $\Phi_t \equiv m_t / (p_t x_t)$ and $\Omega_t \equiv x_t / k_t$. Φ_t represents the ratio of real money balance to consumption in the CM, whereas Ω_t represents the consumption-capital ratio in CM. Note that Φ_t and Ω_t are both *jump* variables and they are stationary on a balanced growth path. From equations (5) and (20), we obtain the recursive equation of Φ_t, which is given by

$$\Phi_{t+1} = \frac{1 + \mu}{\beta(1 - \sigma)} \Phi_t - \frac{\sigma}{\theta(1 - \sigma)} \equiv f(\Phi_t). \tag{26}$$

Figure 1 shows that the money-consumption ratio Φ_t jumps immediately to a unique and saddle-point stable steady-state equilibrium Φ.

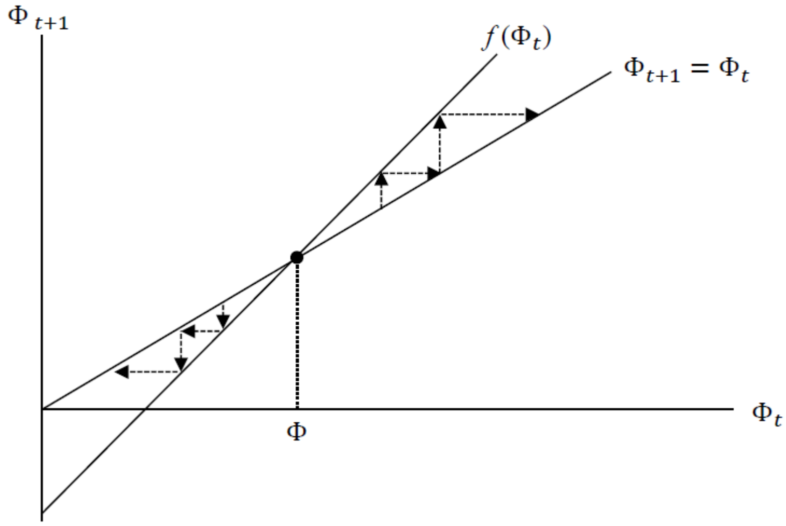

Figure 1: Phase diagram of Φ_t

Manipulating equations (22) and (25) yields $G_t = \tau^{1/\alpha} k_t h_t^{(1-\alpha)/\alpha}$, which is increasing in labor h_t. We then use this condition to rearrange (23) and (24) as

$$r_t = \alpha \left(\tau h_t\right)^{(1-\alpha)/\alpha}, \tag{23a}$$

$$w_t = (1-\alpha)\,\tau^{(1-\alpha)/\alpha} k_t h_t^{(1-2\alpha)/\alpha}. \tag{24a}$$

It is useful to note that (24a) represents the labor demand curve, which is upward sloping if and only if $\alpha < 1/2$ (i.e., labor intensity $1-\alpha > 1/2$). Combining labor demand in (24a) and labor supply in (3), we derive that the following equilibrium relationship between labor h_t and the consumption-capital ratio Ω_t:

$$h_t = \left[\frac{\theta}{\gamma}(1-\tau)(1-\alpha)\,\tau^{(1-\alpha)/\alpha}\right]^{\alpha/(2\alpha-1)} \Omega_t^{\alpha/(1-2\alpha)}, \tag{27}$$

which shows a positive relationship between labor h_t and the consumption-capital ratio Ω_t if and only if $\alpha < 1/2$ (i.e., labor intensity $1-\alpha > 1/2$).

Combining equations (4), (10), (19), (21), (23a) and (27), we obtain the dynamical equation of consumption in the CM:

$$\frac{x_{t+1}}{x_t} = \beta\left[1-\delta+\alpha D\Omega_{t+1}^{\epsilon} + \alpha\sigma\Phi_{t+1}\Omega_{t+1}\right], \tag{28}$$

where we define two composite parameters $\{D, \epsilon\}$ as follows.

$$D \equiv (1-\tau)\,\tau^{(1-\alpha)/\alpha}\left[\frac{\theta}{\gamma}(1-\tau)(1-\alpha)\,\tau^{(1-\alpha)/\alpha}\right]^{(1-\alpha)/(2\alpha-1)} > 0,$$

and $\epsilon \equiv (1-\alpha)/(1-2\alpha)$. For convenience, we plot the value of ϵ against α in Figure 2.

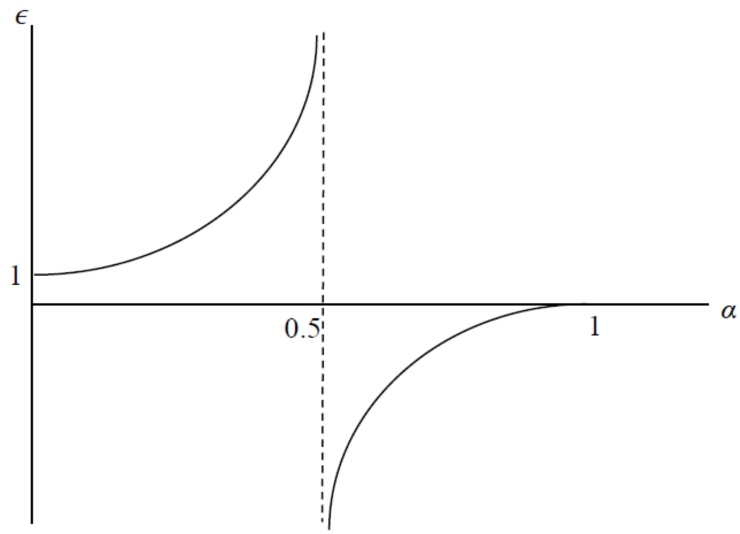

Figure 2: Relationship between α and ϵ

The resource constraint implies the following dynamics of the capital stock k_t:

$$\frac{k_{t+1}}{k_t} = D\Omega_t^{\epsilon} - \Omega_t + 1 - \delta, \tag{29}$$

where we have used (22), (27) and $G_t = \tau^{1/\alpha} k_t h_t^{(1-\alpha)/\alpha}$. Combining equations (28) and (29), we derive the dynamics of $\Omega_t \equiv x_t/k_t$ as follows.

$$\frac{\Omega_{t+1}}{\Omega_t} = \frac{\beta\left[1 - \delta + \alpha D\Omega_{t+1}^{\epsilon} + \alpha\sigma\Phi_{t+1}\Omega_{t+1}\right]}{D\Omega_t^{\epsilon} - \Omega_t + 1 - \delta}, \tag{30}$$

From (26) and (30), the steady-state values of Φ_t and Ω_t, denoted as Φ and Ω, are determined by

$$\Phi = \frac{\sigma\beta}{\theta\left[1 + \mu - (1 - \sigma)\beta\right]}, \tag{31}$$

$$(1 + \alpha\beta\sigma\Phi)\,\Omega = (1 - \alpha\beta)\,D\Omega^{\epsilon} + (1 - \beta)(1 - \delta)\,. \tag{32}$$

We first substitute (31) into (32) and then plot the left-hand side (LHS) and right-hand side (RHS) of (32) in Figure 3.

Figure 3a shows that when $\alpha > 1/2$ (i.e., $\epsilon < 0$), there is a unique steady-state equilibrium value of Ω. In this case, an increase in μ raises the steady-state equilibrium value of Ω. Intuitively, higher inflation increases the cost of consumption in the DM where money is used as a medium of exchange. Due to this lower demand for consumption in the DM, there is less incentives to accumulate physical capital, which is factor input for production in the DM. Furthermore, the lower demand for consumption in the DM shifts the demand for consumption to the CM. Both these effects lead to an increase in the consumption-capital ratio Ω in the CM.

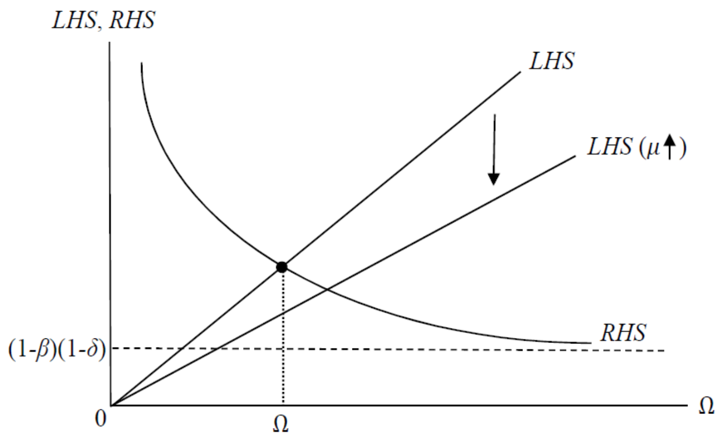

Figure 3a: Unique equilibrium under $\alpha > 1/2$

Figure 3b shows that when $\alpha < 1/2$ (i.e., $\epsilon > 1$) and σ is sufficiently large, there are two steady-state equilibrium values of Ω denoted as $\{\Omega^{low}, \Omega^{high}\}$. In this case, an increase in μ leads to an increase in Ω^{low} but a decrease in Ω^{high}. Given the two equilibria, we have global indeterminacy. The intuition can be understood as follows. Substituting $G_t = \tau^{1/\alpha} k_t h_t^{(1-\alpha)/\alpha}$ into (22) yields $Y_{x,t} = \tau^{(1-\alpha)/\alpha} K_t H_t^{(1-\alpha)/\alpha}$, where $(1-\alpha)/\alpha > 1$ if and only if $\alpha < 1/2$ (i.e., labor intensity $1-\alpha > 1/2$). When $(1-\alpha)/\alpha > 1$, the aggregate production function exhibits increasing returns to scale in labor, which in turn gives rise to an upward-sloping labor demand curve. Together with a horizontal labor supply curve from the quasi-linear preference, global indeterminacy arises. Finally, when $\alpha < 1/2$ (i.e., $\epsilon > 1$) and σ is sufficiently small, there is no equilibrium, and we rule out this parameter space by assumption.

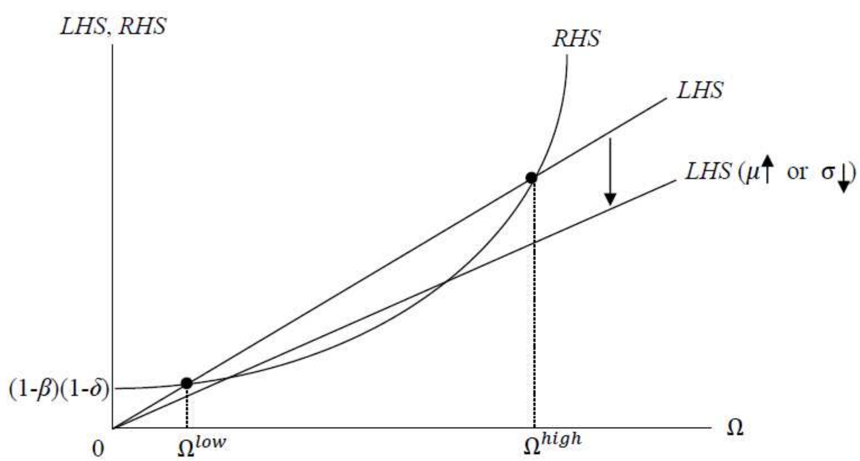

Figure 3b: Multiple equilibria under $\alpha < 1/2$

In (30), the variable Φ_t jumps to its unique steady-state value Φ given in (31). Therefore, equation (30) represents an autonomous one-dimensional dynamical system for Ω_t. Taking a linear approximation around the steady-state equilibrium value Ω and using (32), we derive

$$\Omega_{t+1} = (1-\xi)\Omega + \xi\Omega_t \equiv F(\Omega_t), \tag{33}$$

where $\xi \equiv [(1-\delta) + (1-\epsilon)\, D\Omega^{\epsilon}] \,/\, \{\beta\, [(1-\delta) + \alpha\,(1-\epsilon)\, D\Omega^{\epsilon}]\}$ is the characteristic root of the dynamical system. Figure 4 plots the phase diagram of the *local* dynamics of Ω_t under $\alpha > 1/2$. When $\alpha > 1/2$ (i.e., $\epsilon < 0$), the characteristic root ξ is greater than one. In this case, Figure 4 shows that the unique steady-state equilibrium exhibits saddle-point stability; therefore, Ω_t always jumps to the unique steady state.

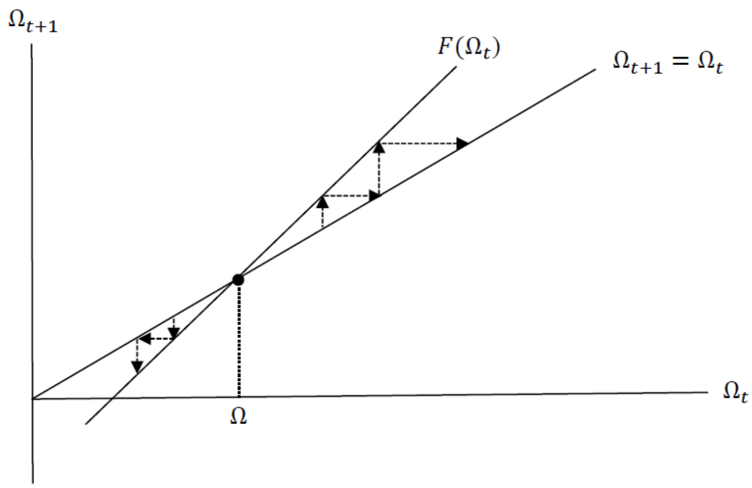

Figure 4: Phase diagram of Ω_t under $\alpha > 1/2$

For the case of $\alpha < 1/2$ (i.e., $\epsilon > 1$), it would be easier to understand the results if we first plot the relationship between the characteristic root ξ and the steady-state equilibrium value Ω. Also, it is useful to recall that $\xi \in (-1, 1)$ implies a dynamically stable (i.e., locally indeterminate) system and that a system is dynamically unstable (i.e., locally determinate) if $\xi < -1$ or $\xi > 1$. Figure 5 shows that the equilibrium Ω^{low} is always dynamically unstable because $\Omega^{low} < \Omega^{*}$ which implies $\xi > 1$, whereas the equilibrium Ω^{high} can be either dynamically unstable (when $\Omega^{high} > \Omega^{**}$ which implies $\xi < -1$ or $\xi > 1$) or dynamically stable (when $\Omega^{high} < \Omega^{**}$ which implies $\xi \in (-1, 1)$).[5]

[5] We will show that $\Omega^{high} > \Omega^{*}$ and also derive Ω^{*} and Ω^{**} in Appendix A.

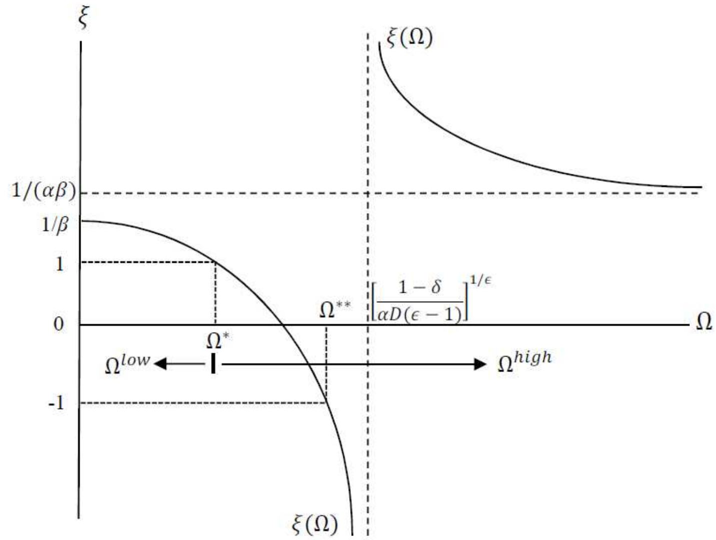

Figure 5: Relationship between ξ and Ω under $\alpha < 1/2$

Recall from Figure 3b that Ω^{high} is increasing in the value of the matching parameter σ. Then, Figure 6a shows that when $\alpha < 1/2$ and σ is not too large,[6] the equilibrium Ω^{high} is locally indeterminate (i.e., dynamically stable) because $\Omega^* < \Omega^{high} < \Omega^{**}$ whereas the equilibrium Ω^{low} is always locally determinate (i.e., dynamically unstable) because $\Omega^{low} < \Omega^*$. When Ω^{low} is unstable and Ω^{high} is stable, Ω_t reaching the unstable equilibrium Ω^{low} is a measure-zero event. In this case, the economy is subject to sunspot fluctuations around the stable equilibrium Ω^{high}.

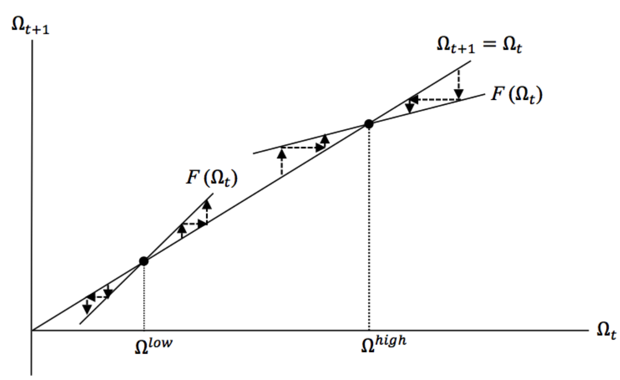

Figure 6a: Phase diagram of Ω_t under
$\alpha < 1/2$ and a small σ

[6]Here we assume that σ is sufficiently large for the presence of equilibria but not excessively large. In the proof of Proposition 1, we explicitly derive these threshold values; see Appendix A.

Figure 6b[7] shows that when $\alpha < 1/2$ and σ is sufficiently large, the two equilibria are both locally determinate (i.e., dynamically unstable) because $\Omega^{high} > \Omega^{**}$ and $\Omega^{low} < \Omega^{*}$. In this case, it is possible for Ω_t to jump to either equilibrium. Therefore, unlike the case with a small σ, we cannot rule out the steady-state equilibrium Ω^{low} under a sufficiently large σ. We summarize these results in Proposition 1.

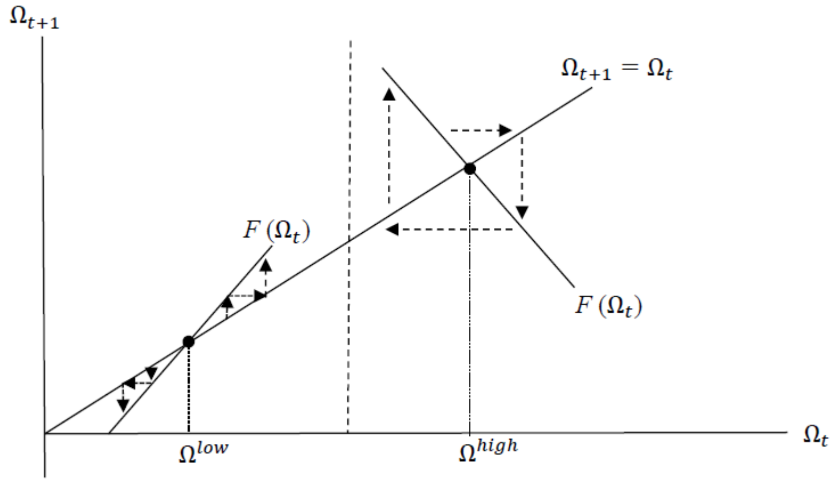

Figure 6b: Phase diagram of Ω_t under $\alpha < 1/2$ and a large σ

Proposition 1 *If $\alpha > 1/2$, then there exists a unique steady-state equilibrium value of Ω_t, which exhibits saddle-point stability. If $\alpha < 1/2$, then there exist two equilibria. One is locally determinate and the other one is locally indeterminate under a sufficiently small σ whereas they are both locally determinate under a sufficiently large σ.*

Proof. See Appendix A. ■

4 Inflation and economic growth

In this section, we examine the relationship between inflation and economic growth. Given that in our analysis we treat the growth rate of money supply m_t as an exogenous policy parameter μ, we first need to discuss the relationship between μ and the endogenous inflation rate π. Along a balanced-growth path, aggregate variables, such as output, consumption, capital and real money balance, grow at the same long-run growth rate g. In other words, the growth rate of m_t/p_t is equal to g, which in turn implies that $(1 + g) = (1 + \mu)/(1 + \pi)$

[7] In this figure, we draw the case in which the characteristic root at the steady-state equilibrium Ω^{high} is $\xi < -1$. One can also draw the case of $\xi > 1$.

because the growth rates of m_t and p_t are respectively μ and π. From the approximation $\ln(1+X) \approx X$, the relationship $(1+\pi) = (1+\mu)/(1+g)$ becomes $\pi = \mu - g(\mu)$, where the long-run growth rate $g(\mu)$ is a function of μ as we will show below. Taking the derivative yields $\partial\pi/\partial\mu = 1 - g'(\mu)$. Therefore, if money growth μ has a negative effect on economic growth g, then it must have a positive effect on inflation π implying also a negative relationship between inflation and economic growth. Even if money growth μ has a positive effect on economic growth g, it would still have a positive effect on inflation π so long as its effect on economic growth is not excessively large (i.e., $g'(\mu) < 1$). In this case, the positive relationship between money growth and economic growth implies also a positive relationship between inflation and economic growth. Therefore, the relationship between money growth and economic growth generally carries over to inflation and economic growth.

Using (29), we obtain the following expression for the long-run growth rate of the economy:

$$g \equiv \frac{k_{t+1}}{k_t} - 1 = D\Omega_t^{\epsilon} - \Omega_t - \delta. \tag{34}$$

In the case of a unique equilibrium (i.e., $\alpha > 1/2$ and $\epsilon < 0$), we have $\partial g/\partial\Omega < 0$. Furthermore, Figure 3a shows that $\partial\Omega/\partial\mu > 0$. Therefore, the overall effect of μ on g is negative. Intuitively, an increase in inflation leads to a higher cost of money holding, which in turn increases the cost of consumption and reduces the level of consumption in the DM. As a result, there are less incentives to accumulate capital for production in the DM, and the lower rate of capital accumulation leads to a lower growth rate. This result is similar to the one in Chu *et al.* (2014). We summarize this result in Proposition 2.

Proposition 2 *If $\alpha > 1/2$, then there exists a unique balanced growth equilibrium in which a higher money growth rate μ reduces economic growth.*

Proof. See Appendix A. ■

In the case of multiple equilibria (i.e., $\alpha < 1/2$ and $\epsilon > 1$), it would be more transparent if we use (28) to express the long-run growth rate of the economy as

$$g \equiv \frac{x_{t+1}}{x_t} - 1 = \beta\left[1 - \delta + \alpha D\Omega^{\epsilon} + \alpha\sigma\Phi\Omega\right] - 1, \tag{35}$$

where Φ is the steady-state ratio of real money balance to consumption in the CM as shown in (31). The ratio of real money balance to consumption in the DM is decreasing in the growth rate of money supply, and this result can be shown as follows:

$$\frac{\partial\Phi}{\partial\mu} = -\frac{\sigma\beta}{\theta\left[(1+\mu) - \beta\left(1-\sigma\right)\right]^2} < 0. \tag{36}$$

Intuitively, a higher money growth rate increases inflation, which in turn raises the cost of money holding. Equation (35) also shows that a larger Ω corresponds to a higher growth rate for a given Φ because ϵ is positive. Therefore, Ω^{high} corresponds to the high-growth equilibrium g^{high} whereas Ω^{low} corresponds to the low-growth equilibrium g^{low}.

Figure 3b shows that Ω^{high} is decreasing in μ. Together with the result that Φ is also decreasing in μ, we find that the high-growth equilibrium growth rate g^{high} is decreasing in the money growth rate μ. Therefore, the effect of inflation on growth in the high-growth equilibrium is the same as in the unique equilibrium. However, the intuition behind these results is different. In the case of the high-growth equilibrium, an increase in inflation reduces the consumption-capital ratio Ω^{high} in the CM, and this counterintuitive result is due to the presence of global indeterminacy. From (34), we see that Ω has a positive effect on g via $D\Omega^{\epsilon}$ (when ϵ is positive) and a negative effect on g via $-\Omega$. The overall relationship between g and Ω in (34) is a U-shaped function[8] as we show in Figure 7.[9] Because Ω^{high} is always on the upward-sloping side of the U-shape, the increase in μ leads to a decrease in both Ω^{high} and g^{high}. In this case, when inflation decreases consumption in the CM, it causes the individuals to also want to consume less leisure and raises their supply of labor in the CM. Given that the labor demand curve is upward sloping due to productive government spending, this shift in labor supply leads to a surprising decrease in equilibrium labor input, which in turn reduces the levels of output and capital investment.

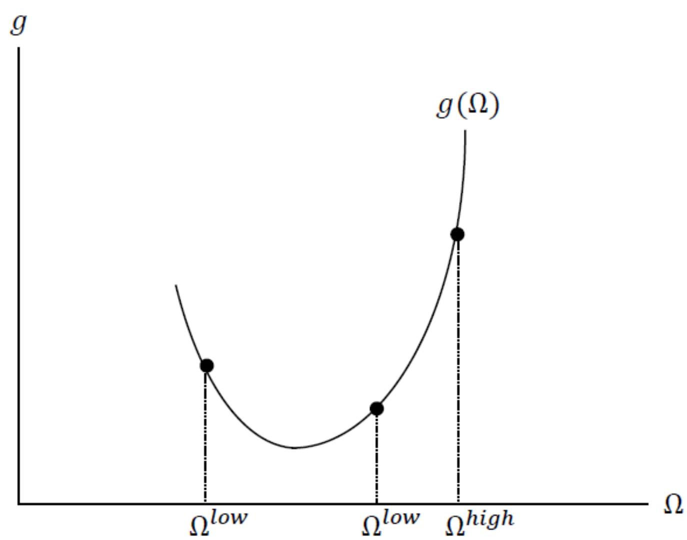

Figure 7: Relationship between g and Ω in (34) when $\epsilon > 1$

As for Ω^{low}, it is increasing in μ as shown in Figure 3b. However, g^{low} can be either increasing or decreasing in μ. Recall from (34) that g is a U-shaped function in Ω when $\epsilon > 1$. Therefore, when Ω^{low} is sufficiently small, the increase in Ω^{low} caused by an increase in μ reduces the growth rate g^{low}. Intuitively, higher inflation reduces both consumption and the incentives to accumulate capital for production in the DM. This lower rate of capital accumulation causes the lower growth rate. This is the standard negative capital-accumulation

[8] Recall that $\epsilon > 1$ when $\alpha < 1/2$.

[9] In Figure 7, the equilibria $\{\Omega^{low}, \Omega^{high}\}$ are determined by the intersection of $g(\Omega)$ in (34) and $g(\Omega)$ in (35), where the latter is a monotonically increasing function in Ω when ϵ is positive. We do not draw (35) in Figure 7 to simplify the diagram.

effect of inflation. In contrast, when Ω^{low} is sufficiently large, the increase in Ω^{low} caused by an increase in μ raises the growth rate g^{low}. Intuitively, when inflation increases consumption in the CM, it causes the individuals to also want to consume more leisure and reduces their supply of labor in the CM. Given that the labor demand curve is upward sloping due to productive government spending, this shift in labor supply leads to a surprising increase in equilibrium labor input, which in turn increases the levels of output and capital investment. This is the novel positive labor-market effect of inflation in the presence of productive government spending. Finally, the overall effect of μ on the low-growth equilibrium growth rate g^{low} follows a U-shaped function.[10] We summarize these results in Proposition 3.

Proposition 3 *If $\alpha < 1/2$, then a higher money growth rate μ has the following effects on economic growth: the high-growth equilibrium g^{high} is decreasing in μ whereas the low-growth equilibrium g^{low} is a U-shaped function in μ.*

Proof. See Appendix A. ■

5 Conclusion

In this study, we have explored the effects of inflation in a monetary search-and-matching model. A novelty of our analysis is that we introduce productive government expenditure into the model in order to generate endogenous economic growth and equilibrium indeterminacy. We find that when labor intensity is below a threshold value, the model features a unique balanced growth equilibrium in which inflation has a negative effect on economic growth as in previous studies. However, when labor intensity is above a threshold value, the model features two balanced growth equilibria, in which the two equilibria display different comparative statics of economic growth with respect to changes in inflation. Specifically, the high-growth equilibrium features a negative effect of inflation on economic growth whereas the low-growth equilibrium features a U-shaped effect of inflation on growth. Furthermore, under a sufficiently large matching probability in the decentralized market, both equilibria are locally determinate. Therefore, either equilibrium may emerge in the economy.

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Appendix A

Proof of Proposition 1. Equation (30) shows that the variable Φ_t jumps to its unique steady state Φ given in (31). We substitute Φ into (30) to obtain the following autonomous one-dimensional dynamical system for Ω_t:

$$\frac{\Omega_{t+1}}{\Omega_t} = \frac{\beta\left[1-\delta+\alpha D\Omega_{t+1}^{\epsilon}+\alpha\sigma\Phi\Omega_{t+1}\right]}{D\Omega_t^{\epsilon}-\Omega_t+1-\delta}. \tag{A1}$$

Taking a linear approximation around the steady-state equilibrium Ω yields

$$\Omega_{t+1} = \Omega + \frac{(1-\delta)+(1-\epsilon)\,D\Omega^{\epsilon}}{\beta\left[(1-\delta)+\alpha\,(1-\epsilon)\,D\Omega^{\epsilon}\right]}\left(\Omega_t-\Omega\right), \tag{A2}$$

where we have used (32). Based on (A2), the characteristic root ξ of the dynamical system can be expressed as

$$\xi \equiv \frac{(1-\delta)+(1-\epsilon)\,D\Omega^{\epsilon}}{\beta\left[(1-\delta)+\alpha\,(1-\epsilon)\,D\Omega^{\epsilon}\right]}. \tag{A3}$$

The local stability properties of the steady state are determined by comparing the number of the stable root with the number of predetermined variables in the dynamical system. In (A2), there is no predetermined variable because Ω_t is a jump variable. As a result, the steady-state equilibrium Ω is locally determinate when the characteristic root is unstable (i.e., $|\xi| > 1$) whereas it is locally indeterminate when the characteristic root is stable (i.e., $|\xi| < 1$). Given these properties, we have the following results. First, if $\alpha > 1/2$ (i.e., $\epsilon < 0$), then the dynamical system exists a unstable root. This result implies that Ω_t displays saddle-point stability and equilibrium uniqueness as shown in Figures 3a and 4.

Second, if $\alpha < 1/2$ (i.e., $\epsilon > 1$), then whether the root is unstable or stable is determined by the steady-state equilibrium value of Ω. The result implies that the multiple equilibria may emerge as shown in Figure 3b. To derive a range for the steady-state equilibrium value of Ω, we first make use of (32) to obtain

$$\frac{\partial LHS}{\partial \Omega} = \frac{\partial RHS}{\partial \Omega} \Rightarrow \Omega^{*} \equiv \left[\frac{(1-\beta)\,(1-\delta)}{(1-\alpha\beta)\,(\epsilon-1)\,D}\right]^{1/\epsilon}, \tag{A4}$$

where Ω^{*} is a threshold value under which $\Omega^{low} < \Omega^{*}$ and $\Omega^{high} > \Omega^{*}$ as shown in Figure 8.

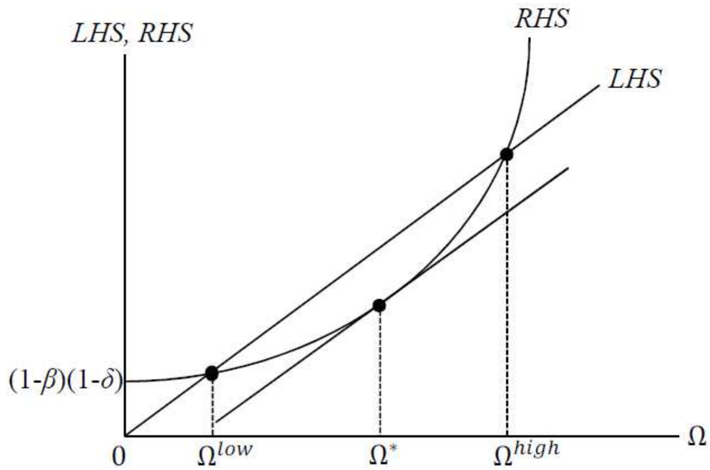

Figure 8

A steady-state equilibrium Ω is dynamically stable if $\xi \in (-1, 1)$. One can manipulate (A3) to show that $\xi \in (-1, 1)$ is equivalent to

$$\Omega^* < \Omega < \Omega^{**}, \tag{A5}$$

where $\Omega^{**} \equiv \{[(1+\beta)(1-\delta)] / [(1+\alpha\beta)(\epsilon-1)D]\}^{1/\epsilon}$. Therefore, a steady-state equilibrium Ω is locally indeterminate if $\Omega \in (\Omega^*, \Omega^{**})$ whereas it is locally determinate if $\Omega < \Omega^*$ or $\Omega > \Omega^{**}$. We can now conclude that Ω^{low} is locally determinate because $\Omega^{low} < \Omega^*$. However, Ω^{high} can be either locally indeterminate when $\Omega^* < \Omega^{high} < \Omega^{**}$ or it can be locally determinate when $\Omega^{high} > \Omega^{**}$.

Next, we examine how the matching probability σ affects the steady-state equilibrium values of $\{\Phi, \Omega\}$, which in turn affect the dynamical properties of Ω_t. Differentiating (31) and (32) with respect to σ yields

$$\frac{\partial \Phi}{\partial \sigma} = \frac{\beta(1+\mu-\beta)}{\theta[(1+\mu)-\beta(1-\sigma)]^2} > 0. \tag{A6}$$

$$\frac{\partial \Omega}{\partial \sigma} = \underbrace{\frac{\alpha\beta\Omega^2}{(1-\alpha\beta)(\epsilon-1)D\Omega^\epsilon-(1-\beta)(1-\delta)}}_{?}\underbrace{\left(\Phi+\sigma\frac{\partial\Phi}{\partial\sigma}\right)}_{+}. \tag{A7}$$

Equation (A6) indicates that increasing σ has a positive effect on Φ. Equation (A7) shows that increasing σ has an ambiguous effect on Ω. Specifically, if and only if $\Omega > \Omega^*$, then Ω is increasing in σ. The result implies that an increase in σ may cause Ω^{high} to change from being locally indeterminate (i.e., $\Omega^{high} < \Omega^{**}$) to being locally determinate (i.e., $\Omega^{high} > \Omega^{**}$). Finally, it can be shown that when σ is sufficiently large (small), we must obtain $\Omega^{high} > \Omega^{**}$ ($\Omega^{high} < \Omega^{**}$). To prove this statement, we make use of (32) to obtain

$$(1+\alpha\beta\sigma\Phi) = (1-\alpha\beta)D\Omega^{\epsilon-1} + \frac{(1-\beta)(1-\delta)}{\Omega}. \tag{A8}$$

The right-hand side (RHS) of (A8) is increasing in Ω, and this result can be shown as follows:

$$\frac{\partial RHS}{\partial \Omega} = \frac{1}{\Omega^2}\underbrace{[(1-\alpha\beta)(\epsilon-1)D\Omega^{\epsilon} - (1-\beta)(1-\delta)]}_{+} > 0. \tag{A9}$$

Given $\Omega^{high} < \Omega^{**}$, we substitute Ω^{**} into the RHS of (A8) to obtain $(1+\alpha\beta\sigma\Phi) < (RHS)_{\Omega=\Omega^{**}}$. Based on $\partial\Phi/\partial\sigma > 0$, there exists a point σ so that $(1+\alpha\beta\sigma\Phi) = (RHS)_{\Omega=\Omega^{**}}$ at $\sigma = \sigma^{**}$, where

$$\sigma^{**} \equiv \frac{1}{2\alpha\beta^2}\left\{\theta\beta(\Theta-1) + \sqrt[2]{[\theta\beta(\Theta-1)]^2 + 4\alpha\theta\beta^2(\Theta-1)\,[1+\mu-\beta]}\right\} > 0, \tag{A10}$$

$$\Theta \equiv (1-\alpha\beta)D(\Omega^{**})^{\epsilon-1} + \frac{(1-\beta)(1-\delta)}{\Omega^{**}} > 1. \tag{A11}$$

By analogous inference, we substitute Ω^{*} into (A8) to derive

$$\sigma^{*} \equiv \frac{1}{2\alpha\beta^2}\left\{\theta\beta(\Psi-1) + \sqrt[2]{[\theta\beta(\Psi-1)]^2 + 4\alpha\theta\beta^2(\Psi-1)\,[1+\mu-\beta]}\right\} > 0, \tag{A12}$$

$$\Psi \equiv (1-\alpha\beta)D(\Omega^{*})^{\epsilon-1} + \frac{(1-\beta)(1-\delta)}{\Omega^{*}} > 1. \tag{A13}$$

As a result, if σ is sufficiently large (i.e., $\sigma > \sigma^{**}$), then Ω^{high} changes from being locally indeterminate to being locally determinate. On the contrary, if σ is sufficiently small (i.e., $\sigma \in (\sigma^{*}, \sigma^{**})$), then Ω^{high} exists and is locally indeterminate. ■

Proof of Proposition 2. Differentiating (32) with respect to μ and using (36) yield

$$\frac{\partial\Omega}{\partial\mu} = \underbrace{\frac{\alpha\beta\sigma\Omega^2}{(1-\alpha\beta)(\epsilon-1)D\Omega^{\epsilon} - (1-\beta)(1-\delta)}}_{?} \times \underbrace{\frac{\partial\Phi}{\partial\mu}}_{-}. \tag{A14}$$

Given $\alpha > 1/2$ and $\epsilon < 0$, we have the following results. First, there is a unique steady-state equilibrium value of Ω which is increasing in μ as reported in Figure 3a. Second, based on (34), the growth rate is monotonically decreasing in the consumption-capital ratio in CM (i.e., $\partial g/\partial\Omega = \epsilon D\Omega^{\epsilon-1} - 1 < 0$). We make use of these results and take the differentials of (34) with respect to μ to obtain

$$\frac{\partial g}{\partial\mu} = \underbrace{\frac{\partial g}{\partial\Omega}}_{-} \times \underbrace{\frac{\partial\Omega}{\partial\mu}}_{+} < 0. \tag{A15}$$

Equation (A15) shows that if $\alpha > 1/2$, there exists a unique balanced-growth equilibrium in which an increase in μ reduces g. ■

Proof of Proposition 3. Given $\alpha < 1/2$ and $\epsilon > 1$, (A14) shows that an increase in μ leads to a decrease in Ω when $\Omega > \Omega^{*}$ whereas it leads to an increase in Ω when $\Omega < \Omega^{*}$.

In other words, when $\alpha < 1/2$ and $\epsilon > 1$, a higher μ increases Ω^{low} and decreases Ω^{high} as shown in Figure 3b. We take the differentials of (35) with respect to μ to obtain

$$\frac{\partial g}{\partial \mu} = \underbrace{\left[\alpha\sigma\beta\Omega + \frac{\alpha^2\beta^2\sigma\Omega^2\left(\epsilon D\Omega^{\epsilon-1} + \sigma\Phi\right)}{\left(1-\alpha\beta\right)\left(\epsilon-1\right)D\Omega^{\epsilon} - \left(1-\beta\right)\left(1-\delta\right)}\right]}_{?} \times \underbrace{\frac{\partial \Phi}{\partial \mu}}_{-}, \tag{A16}$$

where we have used (A14). Equation (A16) shows that when $\Omega > \Omega^*$, g is decreasing in μ. In other words, the high-growth equilibrium g^{high} is always decreasing in μ.

As for the case of $\Omega < \Omega^*$, we substitute (32) into (A16) to derive

$$\frac{\partial g}{\partial \mu} = \underbrace{\frac{\alpha\beta\sigma\Omega^2\left(\epsilon D\Omega^{\epsilon-1} - 1\right)}{\left(1-\alpha\beta\right)\left(\epsilon-1\right)D\Omega^{\epsilon} - \left(1-\beta\right)\left(1-\delta\right)}}_{?} \times \underbrace{\frac{\partial \Phi}{\partial \mu}}_{-}. \tag{A17}$$

Equation (A17) shows that when $\Omega < \Omega^*$, a higher μ has an ambiguous effect on g. Specifically, if $\Omega > \overline{\Omega} \equiv \left[1/\epsilon D\right]^{1/(\epsilon-1)}$, then an increase in μ leads to an increase in g. Moreover, given $\Omega < \Omega^*$, the right-hand side (RHS) of (A8) is decreasing in Ω, and this result can be shown as follows:

$$\frac{\partial RHS}{\partial \Omega} = \frac{1}{\Omega^2}\underbrace{\left[(1-\alpha\beta)(\epsilon-1)D\Omega^{\epsilon} - (1-\beta)(1-\delta)\right]}_{-} < 0. \tag{A18}$$

Substituting $\overline{\Omega}$ into the RHS of (A8) to obtain $(1+\alpha\beta\sigma\Phi) < (RHS)_{\Omega=\overline{\Omega}}$. Given $\partial\Phi/\partial\mu < 0$, there exists a point $\overline{\mu}$ so that $(1+\alpha\beta\sigma\Phi) = (RHS)_{\Omega=\overline{\Omega}}$ at $\mu = \overline{\mu}$, where

$$\overline{\mu} \equiv \frac{\alpha\beta^2\sigma^2}{\theta\left[\left(1-\alpha\beta\right)/\epsilon + (1-\beta)(1-\delta)(\epsilon D)^{1/(\epsilon-1)} - 1\right]} + \beta(1-\sigma) - 1. \tag{A19}$$

As a result, if $\mu > \overline{\mu}$, then we have a positive effect of inflation in the low-growth equilibrium g^{low}. On the contrary, if $\mu < \overline{\mu}$, then we have a negative effect of inflation in the low-growth equilibrium g^{low}. Therefore, the overall effect of μ on g^{low} follows a U-shaped function. ■