Monotone Methods for Equilibrium Selection under Perfect Foresight Dynamics

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Abstract

This paper studies a dynamic adjustment process in a large society of forward-looking agents where payoffs are given by a normal form supermodular game. The stationary states of the dynamics correspond to the Nash equilibria of the stage game. It is shown that if the stage game has a monotone potential maximizer, then the corresponding stationary state is uniquely linearly absorbing and globally accessible for any small degree of friction. Among binary supermodular games, a simple example of a unanimity game with three players is provided where there are multiple globally accessible states for a small friction.

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KEYWORDS: equilibrium selection; perfect foresight dynamics; supermodular game; strategic complementarity; stochastic dominance; potential; monotone potential.

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1 Introduction

Supermodular games capture the key concept of strategic complementarity in various economic phenomena. Examples include oligopolistic competition, adoption of new technologies, bank runs, currency crises, and economic development. Strategic complementarity plays an important role in particular in Keynesian macroeconomics (Cooper (1999)). From a theoretical viewpoint, those games have appealing properties due to their monotone structure (Topkis (1979), Milgrom and Roberts (1990), Vives (1990), and Athey (2001)).

A salient feature of supermodular games is that there typically exist multiple Nash equilibria due to strategic complementarities, which raises the question as to which equilibrium is likely to be played. To address the problem of equilibrium selection, game theory has so far proposed two major strands of approaches besides the classic one by Harsanyi and Selten (1988). One is to consider the stability of Nash equilibria in the context of evolutionary dynamics (Kandori, Mailath, and Rob (1993), Young (1993), Kandori and Rob (1995) for stochastic models; Hofbauer (1999) for a deterministic model); the other is to embed the original game in a static incomplete information game and examine the robustness of equilibrium outcomes to a small amount of uncertainty (Carlsson and van Damme (1993), Frankel, Morris, and Pauzner (2003), Morris and Shin (2003); Kajii and Morris (1997), Morris and Ui (2005)).

Our approach in the present paper is to examine the stability of Nash equilibria in supermodular games under perfect foresight dynamics, first introduced by Matsui and Matsuyama (1995) for 2×2 games.[1] We consider a dynamics adjustment process in a large society where agents make irreversible decisions (e.g., career or sector choices as considered in Matsuyama (1991)) and instantaneous payoffs are given by a normal form supermodular game. In contrast to most evolutionary models where agents are myopic and boundedly rational, our model has forward-looking, rational agents. Combined with a dynamic environment with frictions in action revisions, this gives rise to the possibility that self-fulfilling beliefs destabilize strict Nash equilibria, which allows us to discriminate among Nash equilibria. Indeed, Matsui and Matsuyama (1995) demonstrate that in 2×2 coordination games, the society can escape from the risk-dominated equilibrium to the risk-dominant equilibrium but not vice versa, provided that the friction is sufficiently small. The purpose of this paper is to derive sufficient conditions for the stability of Nash equilibria for broader classes of supermodular games, thereby providing a link between ours and other approaches. In particular, we show that for games with monotone potentials (Morris and

[1] This class of dynamics is also studied by Matsuyama (1991) but with nonlinear payoff functions in the context of development economics. See also Matsuyama (1992) and Kaneda (1995).

Ui (2005)), our condition coincides with that for the robustness to incomplete information (Kajii and Morris (1997)). On the other hand, there are also disagreements: e.g., in unanimity games with more than two players, the equilibrium selection criterion based on Nash product (Harsanyi and Selten (1988)) is not supported.

We employ the following framework. The society consists of N large populations of infinitesimal agents, who are repeatedly and randomly matched to play an N-player normal form game. There are frictions: each agent must make a commitment to a particular action for a random time interval. Opportunities to revise actions follow Poisson processes which are independent across agents. The dynamics thus exhibits inertia in that the action distribution in the society changes continuously. Unlike in standard evolutionary games, each agent forms his belief about the future path of the action distribution and, when given a revision opportunity, takes an action to maximize his expected discounted payoff. A *perfect foresight path* is defined to be a feasible path of action distribution along which every revising agent takes a best response to the future course of play. While the stationary states of this dynamics corresponds to the Nash equilibria of the stage game, there may also exist a perfect foresight path that escapes from a strict Nash equilibrium when the degree of friction, defined as the discounted average duration of the commitment, is sufficiently small. We say that a Nash equilibrium a^* is *linearly absorbing* if the feasible path converging linearly to a^* is the unique perfect foresight path whenever the initial state is close enough to a^*; a^* is *globally accessible* if for any initial state, there exists a perfect foresight path converging to a^*.[2] If a Nash equilibrium is both linearly absorbing and globally accessible, then self-fulfilling expectations cannot destabilize this equilibrium, whereas from any other equilibrium, expectations may lead the society to this equilibrium; that is to say, it is the unique equilibrium that is robust to the possibility of self-fulfilling prophecies.

Several equilibrium selection results based on the perfect foresight dynamics have been obtained so far. Matsui and Matsuyama (1995) demonstrate that in 2×2 coordination games, a strict Nash equilibrium is linearly absorbing and globally accessible for any small degree of friction if and only if it is the risk-dominant equilibrium. Beyond 2×2 games, Oyama (2002) appeals to the notion of p-dominance to identify (in a single population setting) a class of games where one can explicitly characterize the set of perfect foresight paths relevant for stability considerations, showing that a p-dominant equilibrium with $p < 1/2$ is selected.[3] Hofbauer and Sorger (2002) and Kojima (2003) obtain related results based on other generalizations of

[2] Since there may exist multiple perfect foresight paths for a given initial state, it is possible that a state is globally accessible but not linearly absorbing. Indeed, we provide an example where there exist multiple globally accessible states when the friction is small; by definition, none of them is linearly absorbing.

[3] Tercieux (2004) considers set-valued stability concepts and obtains a similar result.

the risk-dominance concept in a multiple population setting.[4] Hofbauer and Sorger (1999, 2002) establish the selection of the unique potential maximizer for potential games, both in a single population setting and in a multi-population setting.[5] Their results rely on the relationship between the perfect foresight paths and the solutions to an associated optimal control problem as well as the Hamiltonian structure that the dynamics has when the stage game is a potential game.

In this paper, we consider supermodular games and games that have some monotonic relationship with supermodular games, by employing methods of analysis based on monotonicity and comparison. An underlying observation is that a perfect foresight path is characterized as a fixed point of the best response correspondence defined on the set of feasible paths. We observe that if the stage game is supermodular, this correspondence is monotone with respect to the partial order over feasible paths induced by the stochastic dominance order. We then compare the perfect foresight paths of two different stage games that are comparable in terms of best responses and show the following analogue to the comparison theorem from the theory of monotone dynamical systems (Smith (1995)):[6] if at least one of the two games is supermodular, then the order of best responses between the games is preserved in the perfect foresight dynamics. This fact allows us to transfer stability properties from one game to the other.

We apply our monotone methods to the class of games with monotone potentials introduced by Morris and Ui (2005), who show that a monotone potential maximizer (MP-maximizer) is robust to incomplete information (Kajii and Morris (1997)).[7] A normal form game is said to have a *monotone potential* if it is comparable (in terms of best responses) to a potential game, and the action profile that maximizes the potential is said to be an *MP-maximizer*. Monotone potential games include both potential games and, interestingly, games with a **p**-dominant equilibrium with $\sum_i p_i < 1$. By invoking the potential game results due to Hofbauer and Sorger (2002),

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[4]Kim (1996) establishes a similar result for binary games with many identical players.

[5]To be precise, they show that a unique potential maximizer a^* is *absorbing* (and globally accessible for small friction): i.e., any perfect foresight path, which may or may not be unique, from a neighborhood of a^* must converge to a^*. It is not known whether a potential maximizer is linearly absorbing. In supermodular games, as we show, absorption and linear absorption are equivalent.

[6]Hofbauer and Sandholm (2002) show that when the underlying game is supermodular, the perturbed best response dynamics forms a monotone dynamical system. The perfect foresight dynamics, on the other hand, cannot be considered as a dynamical system due to the multiplicity of perfect foresight paths.

[7]More generally, Morris and Ui (2005) show that a generalized potential maximizer is robust to incomplete information. A monotone potential induces a generalized potential in the case considered here. Frankel, Morris, and Pauzner (2003) show that under certain conditions, a local potential maximizer (LP-maximizer) is selected in global games with strategic complementarities. In games with marginal diminishing returns, an LP-maximizer is an MP-maximizer.

our main result shows that if the stage game or the monotone potential is supermodular, then an MP-maximizer is globally accessible for any small degree of friction and (generically) linearly absorbing for any degree of friction. Our result thus unifies and extends the previous results using potential maximization and p-dominance, as done by Morris and Ui (2005) for the robustness of equilibria to incomplete information.

We then study the class of binary supermodular games, for which we obtain complete characterizations for linear absorption and for global accessibility. These characterizations are applied to three subclasses. First, for unanimity games, we show that our selection criterion is not in agreement with that in terms of Nash product.[8] In fact, the perfect foresight dynamics fails to select a single Nash equilibrium for some unanimity games. A nondegenerate example (Example 5.2.1 in Subsection 5.2) demonstrates that the two strict Nash equilibria are mutually accessible, actually globally accessible, for a small friction. Second, for games with linear incentives (Selten (1995)), we find a connection to the concept of spatial dominance due to Hofbauer (1999). It is shown that if a strict Nash equilibrium is globally accessible under the perfect foresight dynamics with a small friction, then it is spatially dominant. This implies in particular that for (generic) games with linear incentives, a globally accessible equilibrium is unique if it exists.[9] Third, we introduce the class of games with invariant diagonal, in which all players receive the same payoffs when they all play the same mixed strategy. For this class of games, we obtain the generic existence of a linearly absorbing and globally accessible equilibrium for a small friction.

The concept of perfect foresight path requires that agents optimize against their beliefs about the future path of the action distribution and that those beliefs coincide with the actual path. Relaxing the latter requirement, Matsui and Oyama (2002) introduce the model of rationalizable foresight dynamics, where while the rationality of the agents as well as the structure of the society is common knowledge, beliefs about the future path are not necessarily coordinated among the agents. It is instead assumed that the agents form their beliefs in a rationalizable manner: in particular, they may misforecast the future. A *rationalizable foresight path* is a feasible path along which every revising agent optimizes against another rationalizable foresight path. We show that in supermodular games, a linearly absorbing and globally accessible state is the unique state from which no rationalizable foresight path escapes. That is, our stability results for supermodular games also hold under the less demanding assumption of rationalizable foresight.

[8] Hofbauer (1999) shows that in unanimity games, the Nash equilibrium with the higher Nash product is selected in his spatio-temporal model.

[9] Takahashi (2004) considers many-action supermodular games with linear incentives and shows that, for generic payoffs, there always exists a unique Nash equilibrium that is linearly absorbing and globally accessible for zero degree of friction, and it is also spatially dominant.

The paper is organized as follows. Section 2 introduces the perfect foresight dynamics for general finite N-player games and provides a characterization of perfect foresight paths as the fixed points of the best response correspondence defined on the set of feasible paths. Section 3 studies monotone properties of the perfect foresight dynamics and proves our comparison theorem. It also examines the relationship between the stability concepts under perfect foresight and those under rationalizable foresight. Section 4 considers games with monotone potentials and establishes the selection of MP-maximizer. Section 5 gives complete characterizations for the stability of strict Nash equilibria in binary supermodular games. Detailed analyses are conducted for unanimity games, games with linear incentives, and games with invariant diagonal. Section 6 concludes.

2 Perfect Foresight Dynamics

2.1 Stage Game

Let $G = (I, (A_i)_{i\in I}, (u_i)_{i\in I})$ be a normal form game with $N \geq 2$ players, where $I = \{1, 2, \ldots, N\}$ is the set of players, $A_i = \{0, 1, \ldots, n_i\}$ the finite set of actions for player $i \in I$, and $u_i \colon \prod_{i\in I} A_i \to \mathbb{R}$ the payoff function for player i. We denote $\prod_{i\in I} A_i$ by A and $\prod_{j\neq i} A_j$ by A_{-i}.

Denote by $\mathbb{R}_+$ the set of all nonnegative real numbers and by $\mathbb{R}_{++}$ the set of all positive real numbers. The set of mixed strategies for player i is denoted by

$$\Delta(A_i) = \Big\{x_i = (x_{i0}, x_{i1}, \ldots, x_{in_i}) \in \mathbb{R}_+^{n_i+1} \,\Big|\, \textstyle\sum_{h\in A_i} x_{ih} = 1\Big\},$$

which is identified with the n_i-dimensional simplex. We sometimes identify each action in A_i with the element of $\Delta(A_i)$ that assigns one to the corresponding coordinate. The polyhedron $\prod_{i\in I} \Delta(A_i)$ is a subset of the n-dimensional real space endowed with the sup norm $|\cdot|$, where $n = \sum_{i\in I}(n_i + 1)$. For $x \in \prod_i \Delta(A_i)$ and $\varepsilon > 0$, $B_\varepsilon(x)$ denotes the ε-neighborhood of x relative to $\prod_i \Delta(A_i)$, i.e., $B_\varepsilon(x) = \{y \in \prod_i \Delta(A_i) \mid |y - x| < \varepsilon\}$.

Payoff functions $u_i(h, \cdot)$ are extended to $\prod_{j\neq i} \Delta(A_j)$, and $u_i(\cdot)$ to $\prod_{j\in I} \Delta(A_j)$, i.e.,

$$u_i(h, x_{-i}) = \sum_{a_{-i}\in A_{-i}} \Big(\textstyle\prod_{j\neq i} x_{ja_j}\Big) u_i(h, a_{-i})$$

for $x_{-i} \in \prod_{j\neq i} \Delta(A_j)$, and

$$u_i(x) = \sum_{h\in A_i} x_{ih} u_i(h, x_{-i})$$

for $x \in \prod_{j\in I} \Delta(A_j)$. Let $br^i(x_{-i})$ be the set of best responses to $x_{-i} \in$

$\prod_{j\neq i}\Delta(A_j)$ in pure strategies, i.e.,

$$\begin{aligned} br^i(x_{-i}) &= \arg\max_{h\in A_i} u_i(h, x_{-i}) \\ &= \{h \in A_i \mid u_i(h, x_{-i}) \geq u_i(k, x_{-i}) \text{ for all } k \in A_i\}. \end{aligned}$$

An element $x^* \in \prod_i \Delta(A_i)$ is a Nash equilibrium if for all $i \in I$ and all $h \in A_i$,

$$x^*_{ih} > 0 \Rightarrow h \in br^i(x^*_{-i}),$$

and x^* is a strict Nash equilibrium if for all $i \in I$ and all $h \in A_i$,

$$x^*_{ih} > 0 \Rightarrow \{h\} = br^i(x^*_{-i}).$$

Let $\Delta(A_{-i})$ be the set of probability distributions on A_{-i}. We sometimes extend $u_i(h, \cdot)$ to $\Delta(A_{-i})$. For $\pi_i \in \Delta(A_{-i})$, we write $u_i(h, \pi_i) = \sum_{a_{-i}\in A_{-i}} \pi_i(a_{-i})u_i(h, a_{-i})$ and $br^i(\pi_i) = \arg\max_{h\in A_i} u_i(h, \pi_i)$.

2.2 Perfect Foresight Paths

Given an N-player normal form game, which will be called the stage game, we consider the following dynamic societal game. Society consists of N large populations of infinitesimal agents, one for each role in the stage game. In each population, agents are identical and anonymous. At each point in time, one agent is selected randomly from each population and matched to form an N-tuple and play the stage game. Agents cannot switch actions at every point in time. Instead, every agent must make a commitment to a particular action for a random time interval. Time instants at which each agent can switch actions follow a Poisson process with the arrival rate $\lambda > 0$. The processes are independent across agents. We choose without loss of generality the unit of time in such a way that $\lambda = 1$.[10]

The action distribution in population $i \in I$ at time $t \in \mathbb{R}_+$ is denoted by

$$\phi_i(t) = (\phi_{i0}(t), \phi_{i1}(t), \ldots, \phi_{in_i}(t)) \in \Delta(A_i),$$

where $\phi_{ih}(t)$ is the fraction of agents who are committing to action $h \in A_i$ at time t. Let $\phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_N(t)) \in \prod_i \Delta(A_i)$. Due to the assumption that the switching times follow independent Poisson processes with arrival rate $\lambda = 1$, $\phi_{ih}(\cdot)$ is Lipschitz continuous with Lipschitz constant 1, which implies in particular that it is differentiable at almost all $t \geq 0$. Moreover, its speed of adjustment is bounded: $\dot{\phi}_{ih}(t) \geq -\phi_{ih}(t)$, where $\sum_{h\in A_i} \dot{\phi}_{ih}(t) = 0$. We call such a path $\phi(\cdot)$ a feasible path.

[10] We can alternatively assume as follows. Each agent exits from his population according to the Poisson process with parameter λ and is replaced by his successor. Agents make once-and-for-all decisions upon entry, i.e., one cannot change his action once it is chosen.

Definition 2.1. A path $\phi\colon \mathbb{R}_+ \to \prod_i \Delta(A_i)$ is said to be *feasible* if it is Lipschitz continuous, and for all $i \in I$ and almost all $t \geq 0$, there exists $\alpha_i(t) \in \Delta(A_i)$ such that

$$\dot{\phi}_i(t) = \alpha_i(t) - \phi_i(t). \tag{2.1}$$

In Equation (2.1), $\alpha_i(t) \in \Delta(A_i)$ denotes the action distribution of the agents in population i who have a revision opportunity during the short time interval $[t, t+dt)$. In particular, if for some action profile $a = (a_i)_{i\in I} \in A$, $\alpha_i(t) = a_i$ for all $i \in I$ and all $t \geq 0$, then the resulting feasible path, which converges linearly to a, is called a *linear path* to a.

Denote by Φ^i the set of feasible paths for population i, and let $\Phi = \prod_i \Phi^i$ and $\Phi^{-i} = \prod_{j\neq i} \Phi^j$. For $x \in \prod_i \Delta(A_i)$, the set of feasible paths starting from x is denoted by $\Phi_x = \prod_i \Phi^i_x$. For each $x \in \prod_i \Delta(A_i)$, Φ_x is convex and compact in the topology of uniform convergence on compact intervals.[11]

An agent in population i anticipates the future evolution of the action distribution, and, if given the opportunity to switch actions, commits to an action that maximizes his expected discounted payoff. Since the duration of the commitment has an exponential distribution with mean 1, the expected discounted payoff of committing to action $h \in A_i$ at time t with a given anticipated path $\phi \in \Phi$ is represented by

$$\begin{aligned}V_{ih}(\phi)(t) &= (1+\theta)\int_0^\infty \int_t^{t+s} e^{-\theta(z-t)} u_i(h, \phi_{-i}(z))\, dz\, e^{-s} ds \\ &= (1+\theta)\int_t^\infty e^{-(1+\theta)(s-t)} u_i(h, \phi_{-i}(s))\, ds,\end{aligned}$$

where $\theta > 0$ is a common discount rate (relative to $\lambda = 1$). We view the discounted average duration of a commitment, $\theta/\lambda = \theta$, as the *degree of friction.* Note that V is well-defined whenever $\theta > -1$.

Given a feasible path $\phi \in \Phi$, let $BR^i(\phi)(t)$ be the set of best responses in pure strategies to $\phi_{-i} = (\phi_j)_{j\neq i}$ at time t, i.e.,

$$BR^i(\phi)(t) = \operatorname*{arg\,max}_{h\in A_i} V_{ih}(\phi)(t).$$

Note that for each $i \in I$, the correspondence $BR^i\colon \Phi \times \mathbb{R}_+ \to A_i$ is upper semi-continuous since V_i is continuous.

A perfect foresight path is a feasible path along which each agent optimizes against the correctly anticipated future path.

Definition 2.2. A feasible path ϕ is said to be a *perfect foresight path* if for all $i \in I$, all $h \in A_i$, and almost all $t \geq 0$,

$$\dot{\phi}_{ih}(t) > -\phi_{ih}(t) \Rightarrow h \in BR^i(\phi)(t). \tag{2.2}$$

[11] One can instead use the topology induced by the discounted sup norm.

Note that $\dot{\phi}_{ih}(t) > -\phi_{ih}(t)$ (i.e., $\alpha_{ih}(t) > 0$ in (2.1)) implies that action h is taken by some positive fraction of the agents in population i having a revision opportunity during the short time interval $[t, t+dt)$. The definition says that such an action must be a best response to the path ϕ itself.

2.3 Best Response Correspondence

For a given initial state $x \in \prod_i \Delta(A_i)$, a best response path for population i to a feasible path $\phi \in \Phi_x$ is a feasible path $\psi_i \in \Phi_x^i$ along which every agent takes an optimal action against ϕ. This defines the *best response correspondence* $\beta_x^i \colon \Phi_x \to \Phi_x^i$ which maps each feasible path $\phi \in \Phi_x$ to the set of best response paths for population i:

$$\beta_x^i(\phi) = \{\psi_i \in \Phi_x^i \mid \dot{\psi}_{ih}(t) > -\psi_{ih}(t) \Rightarrow h \in BR^i(\phi)(t) \text{ a.e.}\}. \tag{2.3}$$

Let $\beta_x \colon \Phi_x \to \Phi_x$ be defined by $\beta_x(\phi) = \prod_i \beta_x^i(\phi)$. We denote by $\beta \colon \Phi \to \Phi$ the extension of β_x to Φ, i.e., $\beta(\phi) = \beta_{\phi(0)}(\phi)$ for $\phi \in \Phi$.

A perfect foresight path ϕ with $\phi(0) = x$ is a fixed point of $\beta_x \colon \Phi_x \to \Phi_x$, i.e., $\phi \in \beta_x(\phi)$. The existence of perfect foresight paths follows, due to Kakutani's fixed point theorem, from the fact that β_x is a nonempty-, convex-, and compact-valued upper semi-continuous correspondence. This fact can be shown by either of the two characterizations given below.

Remark 2.1. For a given feasible path $\phi \in \Phi_x$, a best response path $\psi \in \beta_x(\phi)$ is a Lipschitz solution to the differential inclusion

$$\dot{\psi}(t) \in F(\phi)(t) - \psi(t) \text{ a.e.}, \qquad \psi(0) = x, \tag{2.4}$$

where $F \colon \Phi \times \mathbb{R}_+ \to \prod_i \Delta(A_i)$ is defined by

$$F_i(\phi)(t) = \{\alpha_i \in \Delta(A_i) \mid \alpha_{ih} > 0 \Rightarrow h \in BR^i(\phi)(t)\}, \tag{2.5}$$

which is the convex hull of $BR^i(\phi)(t)$. Since $F(\phi)(\cdot)$ is convex- and compact-valued, and upper semi-continuous, the existence theorem for differential inclusions (see, e.g., Aubin and Cellina (1984, Theorem 2.1.4)) implies the nonemptiness of the set of solutions, $\beta_x(\phi)$. The convexity of $\beta_x(\phi)$ is obvious. Furthermore, we can show that $\beta_x(\phi)$ is compact and depends upper semi-continuously on ϕ. For these properties of β_x, we only need the upper semi-continuity of BR^i, which is in turn implied by the continuity of V_i.

Lemma 2.1. *β_x is compact-valued and upper semi-continuous.*

Proof. Since the values are contained in the compact set Φ_x, it is sufficient to show that β_x has a closed graph. Let $\{\phi^k\}_{k=1}^\infty$ and $\{\psi^k\}_{k=1}^\infty$ be such that $\psi^k \in \beta_x(\phi^k)$, and assume that $\phi^k \to \phi$ and $\psi^k \to \psi$ as $k \to \infty$. Take any $i \in I$, $h \in A_i$, and $t \geq 0$ such that $\dot{\psi}_{ih}(t) > -\psi_{ih}(t)$. We want to show that $h \in BR^i(\phi)(t)$.

Observe that for any $\varepsilon > 0$, there exists $\bar{k}$ such that for all $k \geq \bar{k}$,

$$\dot{\psi}_{ih}^{k}(t_k) > -\psi_{ih}^{k}(t_k)$$

for some $t_k \in (t-\varepsilon, t+\varepsilon)$. Take a sequence $\{\varepsilon_\ell\}_{\ell=1}^{\infty}$ such that $\varepsilon_\ell > 0$ and $\varepsilon_\ell \to 0$ as $\ell \to \infty$. Then, we can take a subsequence $\{\psi^{k_\ell}\}_{\ell=1}^{\infty}$ of $\{\psi^k\}_{k=1}^{\infty}$ such that $\dot{\psi}_{ih}^{k_\ell}(t_\ell) > -\psi_{ih}^{k_\ell}(t_\ell)$ holds for some $t_\ell \in (t-\varepsilon_\ell, t+\varepsilon_\ell)$. By assumption, $h \in BR^i(\phi^{k_\ell})(t_\ell)$ for all ℓ. Now let $\ell \to \infty$. Since $BR^i(\cdot)(\cdot)$ is upper semi-continuous, we have $h \in BR^i(\phi)(t)$. ▮

Remark 2.2. The correspondence β_x^i is actually the best response correspondence for an associated differential game, as constructed in Hofbauer and Sorger (2002). With the stage game G, the discount rate $\theta > 0$, and an initial state $x \in \prod_i \Delta(A_i)$ given, the associated differential game is an N-player normal form game in which the set of actions for player $i \in I$ is Φ_x^i and the payoff function for player i is given by

$$J_i(\phi) = \int_0^\infty e^{-\theta t} u_i(\phi(t))\, dt. \tag{2.6}$$

As shown by Hofbauer and Sorger (2002), the perfect foresight paths are precisely the Nash equilibria of this game, due to the following fact.

Lemma 2.2. *For a feasible path $\phi \in \Phi_x$,*

$$\beta_x^i(\phi) = \operatorname*{arg\,max}_{\psi_i \in \Phi_x^i} J_i(\psi_i, \phi_{-i}).$$

Proof. Follows from Lemma 3.1 in Hofbauer and Sorger (2002). ▮

The continuity of J_i, the quasi-concavity of $J_i(\cdot, \phi_{-i})$, and the compactness of Φ_x^i therefore imply the desired properties of β_x^i.

2.4 Stability Concepts

The constant path $\bar{\phi}$ given by $\bar{\phi}(t) = x^* \in \prod_i \Delta(A_i)$ for all $t \geq 0$ is a perfect foresight path if and only if x^* is a Nash equilibrium of the stage game. Nevertheless, there may exist another perfect foresight path starting at x^* which converges to a different Nash equilibrium; that is to say, self-fulfilling beliefs may enable the society to escape from a Nash equilibrium. When the degree of friction $\theta > 0$ is sufficiently small, this may happen even from a strict Nash equilibrium. In fact, in 2×2 coordination games, there exists a perfect foresight path from the risk-dominated equilibrium to the risk-dominant equilibrium for small $\theta > 0$, but not vice versa. This motivates the following stability concepts.

Definition 2.3. (a) $x^* \in \prod_i \Delta(A_i)$ is *absorbing* if there exists $\varepsilon > 0$ such that any perfect foresight path from any $x \in B_\varepsilon(x^*)$ converges to x^*.

(b) $a^* \in A$ is *linearly absorbing* if there exists $\varepsilon > 0$ such that for any $x \in B_\varepsilon(a^*)$, the linear path to a^* is a unique perfect foresight path from x.

(c) $x^* \in \prod_i \Delta(A_i)$ is *accessible* from $x \in \prod_i \Delta(A_i)$ if there exists a perfect foresight path from x that converges to x^*. x^* is *globally accessible* if it is accessible from any x.

If x^* is absorbing and the current state is close enough to x^*, then along any (not necessarily unique) perfect foresight path, the behavior pattern of the society converges to x^*. Linear absorption is a stronger concept than absorption:[12] if a^* is linearly absorbing and the current state is close enough to a^*, then the perfect foresight path is unique, along which every agent at revision opportunity takes the action prescribed in a^*. If a (linearly) absorbing state is also globally accessible, then it is the unique (linearly) absorbing state; if a globally accessible state is also absorbing, then it is the unique globally accessible state.

A globally accessible state is not necessarily absorbing, as there are generally multiple perfect foresight paths from a given initial state. We present a (nondegenerate) example in Subsection 5.2 (Example 5.2.1) that has two globally accessible states for small θ; by definition, neither of them is absorbing.

Any absorbing or globally accessible state is a Nash equilibrium of the stage game, which follows from the proposition below.

Proposition 2.3. *If $x^* \in \prod_i \Delta(A_i)$ is the limit of a perfect foresight path, then x^* is a Nash equilibrium.*

Proof. Suppose that x^* is the limit of a perfect foresight path ϕ^*. Let $\bar{\phi}$ be the constant path at x^*, i.e., $\bar{\phi}(t) = x^*$ for all $t \geq 0$. Let ϕ^t be the feasible path defined by $\phi^t(s) = \phi^*(s+t)$ for all $s \geq 0$. Then, $\{\phi^t\}_{t \geq 0}$ converges to $\bar{\phi}$ as $t \to \infty$.

Take any $i \in I$ and any $h \in A_i$ with $x^*_{ih} > 0$. Then, there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \to \infty$ as $k \to \infty$ and $h \in BR^i(\phi^*)(t_k) = BR^i(\phi^{t_k})(0)$ for any k since ϕ^* is a perfect foresight path that converges to x^*. Let $k \to \infty$. By the upper semi-continuity of $BR^i(\cdot)(0)$, we have $h \in BR^i(\bar{\phi})(0) = br^i(x^*_{-i})$. ∎

3 Supermodularity and Monotonicity

Supermodular games are games in which actions are ordered so that each player's marginal payoff to any increase in his action is nondecreasing in

[12] No example is known of a state that is absorbing but not linearly absorbing. We will show that in supermodular games, any absorbing state is a strict Nash equilibrium and is also linearly absorbing.

other players' actions. In this section, we first identify monotone properties of the perfect foresight dynamics for supermodular stage games. In particular, we observe the monotonicity of the best response correspondence β with respect to a partial order on Φ induced by the stochastic dominance relation over mixed strategies. We then prove a comparison theorem for the perfect foresight paths associated with two different stage games that are comparable in terms of best responses. This theorem implies that if at least one of the two games is supermodular, then one game inherits stability properties from the other. Finally, we show that for supermodular games, stability under perfect foresight is equivalent to that under rationalizable foresight (Matsui and Oyama (2002)).

3.1 Supermodular Games

For $x_i, y_i \in \Delta(A_i)$, we write $x_i \precsim y_i$ if y_i stochastically dominates x_i, i.e.,

$$\sum_{k=h}^{n_i} x_{ik} \leq \sum_{k=h}^{n_i} y_{ik}$$

for all $h \in A_i$. For $x, y \in \prod_i \Delta(A_i)$, we write $x \precsim y$ if $x_i \precsim y_i$ for all $i \in I$ and $x_{-i} \precsim y_{-i}$ if $x_j \precsim y_j$ for all $j \neq i$. Moreover, we define $\phi_i \precsim \psi_i$ for $\phi_i, \psi_i \in \Phi^i$ by $\phi_i(t) \precsim \psi_i(t)$ for all $t \geq 0$; $\phi \precsim \psi$ for $\phi, \psi \in \Phi$ by $\phi_i \precsim \psi_i$ for all $i \in I$; and $\phi_{-i} \precsim \psi_{-i}$ for $\phi_{-i}, \psi_{-i} \in \Phi^{-i}$ by $\phi_j \precsim \psi_j$ for all $j \neq i$. Note that if $\phi(0) \precsim \psi(0)$ and $\dot{\phi}(t) + \phi(t) \precsim \dot{\psi}(t) + \psi(t)$ for almost all $t \geq 0$, then $\phi \precsim \psi$.

The game G is said to be *supermodular* if whenever $h < k$, the difference $u_i(k, a_{-i}) - u_i(h, a_{-i})$ is nondecreasing in $a_{-i} \in A_{-i}$, i.e., if $a_{-i} \leq b_{-i}$, then

$$u_i(k, a_{-i}) - u_i(h, a_{-i}) \leq u_i(k, b_{-i}) - u_i(h, b_{-i}).$$

It is well known that this property extends to mixed strategies: if $h < k$ and $x_{-i} \precsim y_{-i}$, then

$$u_i(k, x_{-i}) - u_i(h, x_{-i}) \leq u_i(k, y_{-i}) - u_i(h, y_{-i}).$$

The expected discounted payoff function V_i preserves this property, implying that BR^i is monotone with respect to the partial order on Φ.

Lemma 3.1. *Suppose that the stage game is supermodular. For $\phi, \psi \in \Phi$, if $\phi_{-i} \precsim \psi_{-i}$, then for all $i \in I$ and all $t \geq 0$,*

$$V_{ik}(\phi)(t) - V_{ih}(\phi)(t) \leq V_{ik}(\psi)(t) - V_{ih}(\psi)(t)$$

for $h < k$, and

$$\min BR^i(\phi)(t) \leq \min BR^i(\psi)(t),$$
$$\max BR^i(\phi)(t) \leq \max BR^i(\psi)(t).$$

Proof. Suppose $\phi_{-i} \precsim \psi_{-i}$ and fix any t. If $k > h$, then

$$\begin{aligned} & V_{ik}(\phi)(t) - V_{ih}(\phi)(t) \\ & \quad = (1+\theta) \int_t^\infty e^{-(1+\theta)(s-t)} \big\{ u_i(k, \phi_{-i}(s)) - u_i(h, \phi_{-i}(s)) \big\}\, ds \\ & \quad \leq (1+\theta) \int_t^\infty e^{-(1+\theta)(s-t)} \big\{ u_i(k, \psi_{-i}(s)) - u_i(h, \psi_{-i}(s)) \big\}\, ds \\ & \quad = V_{ik}(\psi)(t) - V_{ih}(\psi)(t). \end{aligned}$$

Next, let $k = \min BR^i(\phi)(t)$. For any $h < k$,

$$V_{ik}(\psi)(t) - V_{ih}(\psi)(t) \geq V_{ik}(\phi)(t) - V_{ih}(\phi)(t) > 0$$

since $h \notin BR^i(\phi)(t)$. Hence, if $\ell \in BR^i(\psi)(t)$, then $\ell \geq k = \min BR^i(\phi)(t)$. We thus have $\min BR^i(\psi)(t) \geq \min BR^i(\phi)(t)$.

The other claim that $\max BR^i(\phi)(t) \leq \max BR^i(\psi)(t)$ can be proved similarly. ▌

The next proposition establishes the monotonicity of the best response correspondence β^i over Φ. For $\phi \in \Phi$, a feasible path $\phi_i^- \in \beta^i(\phi)$ is the smallest element of $\beta^i(\phi)$ if $\phi_i^- \precsim \phi_i'$ for all $\phi_i' \in \beta^i(\phi)$, and $\phi_i^+ \in \beta^i(\phi)$ is the largest element of $\beta^i(\phi)$ if $\phi_i' \precsim \phi_i^+$ for all $\phi_i' \in \beta^i(\phi)$.

Proposition 3.2. *Suppose that the stage game is supermodular. For $\phi \in \Phi$, $\beta^i(\phi)$ has the smallest element $\min \beta^i(\phi)$ and the largest element $\max \beta^i(\phi)$. If $\phi_i(0) \precsim \psi_i(0)$ and $\phi_{-i} \precsim \psi_{-i}$, then*

$$\begin{aligned} \min \beta^i(\phi) &\precsim \min \beta^i(\psi), \\ \max \beta^i(\phi) &\precsim \max \beta^i(\psi). \end{aligned}$$

Proof. Take ϕ and ψ such that $\phi_i(0) = x_i$, $\psi_i(0) = y_i$, $x_i \precsim y_i$, and $\phi_{-i} \precsim \psi_{-i}$. First, we construct $\phi_i^- = \min \beta^i(\phi)$; the construction of $\max \beta^i(\phi)$ is similar. Define

$$\alpha_i(t) = \min BR^i(\phi)(t),$$

where the right hand side is considered as a mixed strategy. Note that α_i is lower semi-continuous, and hence, measurable, since $BR^i(\phi)(\cdot)$ is an upper semi-continuous correspondence. Then, the unique solution ϕ_i^- to

$$\dot{\phi}_i^-(t) = \alpha_i(t) - \phi_i^-(t) \quad \text{a.e.}, \qquad \phi_i^-(0) = x_i$$

is given by

$$\phi_i^-(t) = e^{-t} x_i + \int_0^t e^{s-t} \alpha_i(s)\, ds.$$

By construction, $\phi_i^- \in \beta^i(\phi)$, and $\phi_i^- \precsim \phi_i'$ for all $\phi_i' \in \beta^i(\phi)$, i.e., ϕ_i^- is the smallest element of $\beta^i(\phi)$.

On the other hand, any path $\psi'_i \in \beta^i(\psi)$ is given by

$$\psi'_i(t) = e^{-t} y_i + \int_0^t e^{s-t} \alpha'_i(s)\, ds$$

for some $\alpha'_i \colon \mathbb{R}_+ \to \Delta(A_i)$ such that $\alpha'_i(t) \in F_i(\psi)(t)$ for almost all $t \geq 0$, where $F_i(\psi)$ is defined by (2.5). Since $\phi_{-i} \precsim \psi_{-i}$, it follows from Lemma 3.1 that

$$\min BR^i(\phi)(t) \leq \min BR^i(\psi)(t),$$

and hence, $\alpha_i(t) \precsim \alpha'_i(t)$ for almost all t. Together with the assumption that $x_i \precsim y_i$, this implies that $\phi_i^- \precsim \psi'_i$, thereby completing the proof of $\min \beta^i(\phi) \precsim \min \beta^i(\psi)$. ▌

3.2 Comparison Theorem

Fix the set of players, I, and the set of action profiles, A. Consider two games $G = (I, (A_i)_{i\in I}, (u_i)_{i\in I})$ and $G' = (I, (A_i)_{i\in I}, (v_i)_{i\in I})$ satisfying that for all $i \in I$ and all $\pi_i \in \Delta(A_{-i})$,

$$\min br^i_{v_i}(\pi_i) \leq \min br^i_{u_i}(\pi_i), \tag{3.1}$$

or that for all $i \in I$ and all $\pi_i \in \Delta(A_{-i})$,

$$\max br^i_{v_i}(\pi_i) \leq \max br^i_{u_i}(\pi_i), \tag{3.2}$$

where $br^i_{u_i}(\pi_i)$ and $br^i_{v_i}(\pi_i)$ are the sets of best responses to π_i in games G and G', respectively. In this subsection, we study the relationship between the perfect foresight paths for the stage game G and those for G'. Note that the state space $\prod_i \Delta(A_i)$ is common in both cases. We will show that if G or G' is supermodular, then the perfect foresight dynamics preserves the order of best responses between G and G', and therefore, G inherits stability properties from G'.

To specify the payoff functions, we denote by $BR^i_{u_i}(\phi)(t)$ ($BR^i_{v_i}(\phi)(t)$, resp.) the set of best responses for population i to a feasible path ϕ at time t when the stage game is G (G', resp.). Note that this can be written as

$$BR^i_{u_i}(\phi)(t) = br^i_{u_i}(\pi^t_i(\phi))$$

with a probability distribution $\pi^t_i(\phi) \in \Delta(A_{-i})$ which is given by

$$\pi^t_i(\phi)(a_{-i}) = (1+\theta) \int_t^\infty e^{-(1+\theta)(s-t)} \left(\prod\nolimits_{j\neq i} \phi_{ja_j}(s)\right) ds.$$

Thus, if (3.1) is satisfied, then for any $\phi \in \Phi$ and any $t \geq 0$,

$$\min BR^i_{v_i}(\phi)(t) \leq \min BR^i_{u_i}(\phi)(t), \tag{3.3}$$

while if (3.2) is satisfied, then for any $\phi \in \Phi$ and any $t \geq 0$,

$$\max BR^i_{v_i}(\phi)(t) \leq \max BR^i_{u_i}(\phi)(t). \tag{3.4}$$

The following lemma is a key to our comparison theorem. The proof relies on a fixed point argument together with the monotonicity of BR^i.

Lemma 3.3. *Let $x, y \in \prod_i \Delta(A_i)$ be such that $y \precsim x$.*

(a) *Suppose that G and G' satisfy (3.1) and that G or G' is supermodular. If a feasible path $\phi \in \Phi_x$ satisfies that for all $i \in I$, all $h \in A_i$, and almost all $t \geq 0$,*

$$\dot{\phi}_{ih}(t) > -\phi_{ih}(t) \Rightarrow h \geq \min BR^i_{u_i}(\phi)(t), \tag{3.5}$$

then there exists a perfect foresight path $\psi^* \in \Phi_y$ for G' such that $\psi^* \precsim \phi$.

(b) *Suppose that G and G' satisfy (3.2) and that G or G' is supermodular. If a feasible path $\psi \in \Phi_y$ satisfies that for all $i \in I$, all $h \in A_i$, and almost all $t \geq 0$,*

$$\dot{\psi}_{ih}(t) > -\psi_{ih}(t) \Rightarrow h \leq \max BR^i_{v_i}(\psi)(t), \tag{3.6}$$

then there exists a perfect foresight path $\phi^* \in \Phi_x$ for G such that $\psi \precsim \phi^*$.

Proof. We only show (a). Given $x, y \in \prod_i \Delta(A_i)$ with $y \precsim x$ and $\phi \in \Phi_x$ satisfying (3.5), define the convex and compact subset $\tilde{\Phi}_y \subset \Phi_y$ to be

$$\tilde{\Phi}_y = \{\psi \in \Phi_y \,|\, \psi \precsim \phi\}.$$

Let $\beta_{G'}$ be the best response correspondence for the stage game G'. We define a convex- and compact-valued and upper semi-continuous correspondence $\tilde{\beta}_{G'} \colon \tilde{\Phi}_y \to \tilde{\Phi}_y$ by

$$\tilde{\beta}_{G'}(\psi) = \beta_{G'}(\psi) \cap \tilde{\Phi}_y \qquad (\psi \in \tilde{\Phi}_y).$$

We want to show that $\tilde{\beta}_{G'}(\psi)$ is nonempty for any $\psi \in \tilde{\Phi}_y$. Then, it follows from Kakutani's fixed point theorem that $\tilde{\beta}_{G'}$ has a fixed point $\psi^* \in \tilde{\beta}_{G'}(\psi^*) \subset \tilde{\Phi}_y$, which is a perfect foresight path for G' and satisfies $\psi^* \precsim \phi$.

For $\psi \in \tilde{\Phi}_y$, take any $i \in I$, $h \in A_i$, and $t \geq 0$ such that $\dot{\phi}_{ih}(t) > -\phi_{ih}(t)$. If G is supermodular, then

$$h \geq \min BR^i_{u_i}(\phi)(t) \geq \min BR^i_{u_i}(\psi)(t) \geq \min BR^i_{v_i}(\psi)(t),$$

where the second inequality follows from the supermodularity of G and Lemma 3.1, and the third inequality follows from the assumption of (3.1). If G' is supermodular, then

$$h \geq \min BR^i_{u_i}(\phi)(t) \geq \min BR^i_{v_i}(\phi)(t) \geq \min BR^i_{v_i}(\psi)(t),$$

where the second inequality follows from the assumption of (3.1), and the third inequality follows from the supermodularity of G' and Lemma 3.1. Therefore, in each case, we have

$$h \geq \min BR^i_{v_i}(\psi)(t)$$

for all h such that $\dot{\phi}_{ih}(t) > -\phi_{ih}(t)$.

Now let $\psi' \in \Phi_y$ be given by

$$\dot{\psi'}_i(t) = \min BR^i_{v_i}(\psi)(t) - \psi'_i(t) \ \text{ a.e.}, \qquad \psi'_i(0) = y_i$$

for all $i \in I$. By construction, we have $\psi' \in \beta_{G'}(\psi)$. Since $\psi'(0) = y \precsim x = \phi(0)$ and $\dot{\psi}'(t) + \psi'(t) \precsim \dot{\phi}(t) + \phi(t)$ for almost all t, we also have $\psi' \precsim \phi$. Therefore, we have $\psi' \in \tilde{\beta}_{G'}(\psi)$, which implies the nonemptiness of $\tilde{\beta}_{G'}(\psi)$. ∎

As a corollary, we have the following result, which is an analogue to the comparison theorem from the theory of differential equations (Walter (1970)) or monotone (cooperative) dynamical systems (Smith (1995)) and to the comparative statics theorem (Milgrom and Roberts (1990)).

Theorem 3.4. *Let $x, y \in \prod_i \Delta(A_i)$ be such that $y \precsim x$.*

(a) *Suppose that G and G' satisfy* (3.1) *and that G or G' is supermodular. For any perfect foresight path ϕ^* for G with $\phi^*(0) = x$, there exists a perfect foresight path ψ^* for G' with $\psi^*(0) = y$ such that $\psi^* \precsim \phi^*$.*

(b) *Suppose that G and G' satisfy* (3.2) *and that G or G' is supermodular. For any perfect foresight path ψ^* for G' with $\psi^*(0) = y$, there exists a perfect foresight path ϕ^* for G with $\phi^*(0) = x$ such that $\psi^* \precsim \phi^*$.*

Suppose that G or G' is supermodular. This theorem implies that if G is comparable (in terms of best responses) to G', then G inherits stability properties from G'. First, assume that G and G' satisfy (3.1) and that action profile $\max A = (n_i)_{i \in I}$ is (linearly) absorbing in G'. Take any state $x \in B_\varepsilon(\max A)$ for a sufficiently small $\varepsilon > 0$ and any perfect foresight path ϕ^* for G with $\phi^*(0) = x$. By Theorem 3.4(a), there exists a perfect foresight path ψ^* for G' with $\psi^*(0) = x$ such that $\psi^* \precsim \phi^*$. By the assumption that $\max A$ is (linearly) absorbing in G', ψ^* converges (linearly) to $\max A$, so that ϕ^* also converges (linearly) to $\max A$. This implies that $\max A$ is (linearly) absorbing in G as well.

Second, assume that G and G' satisfy (3.2) and that $\max A$ is globally accessible in G'. Take any state $x \in \prod_i \Delta(A_i)$. By the assumption that $\max A$ is globally accessible in G', there exists a perfect foresight path ψ^* for G' with $\psi^*(0) = x$ that converges to $\max A$. By Theorem 3.4(b), there exists a perfect foresight path ϕ^* for G with $\phi^*(0) = x$ such that $\psi^* \precsim \phi^*$. Since ψ^* converges to $\max A$, ϕ^* also converges to $\max A$. This implies that $\max A$ is globally accessible in G as well.

Note that by reversing the orders of actions, the above arguments can be applied to $\min A$.

A candidate for the game G' is a potential game. Such a case is considered, with some refinement, in Section 4.

Lemma 3.3 with $G' = G$ (i.e., $v_i = u_i$ for all $i \in I$) yields the following corollary. We say that a feasible path ϕ is a *superpath* if

$$\dot{\phi}_{ih}(t) > -\phi_{ih}(t) \Rightarrow h \geq \min BR^i(\phi)(t) \tag{3.7}$$

for all $i \in I$, all $h \in A_i$, and almost all $t \geq 0$; a feasible path ψ is a *subpath* if

$$\dot{\psi}_{ih}(t) > -\psi_{ih}(t) \Rightarrow h \leq \max BR^i(\psi)(t) \tag{3.8}$$

for all $i \in I$, all $h \in A_i$, and almost all $t \geq 0$.

Lemma 3.5. *Suppose that the stage game is supermodular. Let $x, y \in \prod_i \Delta(A_i)$ be such that $y \precsim x$.*

(a) *If there exists a superpath ϕ with $\phi(0) = x$, then there exists a perfect foresight path ψ^* with $\psi^*(0) = y$ such that $\psi^* \precsim \phi$.*

(b) *If there exists a subpath ψ with $\psi(0) = y$, then there exists a perfect foresight path ϕ^* with $\phi^*(0) = x$ such that $\psi \precsim \phi^*$.*

This lemma is used in Section 5 as well as in the following propositions.

Proposition 3.6. *Suppose that the stage game is supermodular. If $x^* \in \prod_i \Delta(A_i)$ is absorbing, then it is a strict Nash equilibrium.*

Proof. In light of Proposition 2.3, it is sufficient to show that any Nash equilibrium that is not a strict Nash equilibrium is not absorbing. Suppose that x^* is a non-strict Nash equilibrium. We show the existence of an escaping path from x^*.

Let a_i' (a_i'', resp.) be the smallest (the largest, resp.) in $br^i(x_{-i}^*)$ for each player i, and let $a' = (a_i')_{i \in I}$ and $a'' = (a_i'')_{i \in I}$, which are considered as mixed strategy profiles. Note that $a' \precsim x^* \precsim a''$ and, by the definition of a non-strict Nash equilibrium, $a' \neq a''$, so that a' or a'' is different from x^*. Let us assume that $a' \neq x^*$.

Now denote by $\bar{\phi}$ the constant path such that $\bar{\phi}(t) = x^*$ for all t. Note that $BR^i(\bar{\phi})(t) = br^i(x_{-i}^*)$, so that $\min BR^i(\bar{\phi})(t) = a_i'$ for all t. Let ϕ be the feasible path starting from x^* and converging linearly to a', i.e.,

$$\phi(t) = e^{-t} x^* + (1 - e^{-t}) a'.$$

This path satisfies $\phi \precsim \bar{\phi}$, $\phi \neq \bar{\phi}$, and $\dot{\phi}_{ih}(t) > -\phi_{ih}(t)$ only for $h = a_i'$. We also have

$$a_i' = \min BR^i(\bar{\phi})(t) \geq \min BR^i(\phi)(t),$$

where the inequality follows from Lemma 3.1. This means that ϕ is a superpath. Therefore, it follows from Lemma 3.5 that there exists a perfect foresight path ψ^* from x^* such that $\psi^* \precsim \phi$, which does not converge to x^*. ▮

The next proposition shows the equivalence of absorption and linear absorption for supermodular games.

Proposition 3.7. *Suppose that the stage game is supermodular. If $a^* \in A$ is absorbing, then it is linearly absorbing.*

Proof. See Appendix. ▮

A globally accessible state need not be a strict Nash equilibrium in general. Even for the class of strict supermodular games, there are degenerate games where a non-strict, pure-strategy Nash equilibrium is globally accessible. In the game given by Figure 1, the non-strict Nash equilibrium $(0, 1)$ is globally accessible for any degree of friction. It is an open problem whether every globally accessible state must be a pure Nash equilibrium in generic supermodular games.

	0	1
0	1, 1	1, 1
1	0, 0	1, 1

Figure 1: Globally accessible, non-strict Nash equilibrium

3.3 Stability under Rationalizable Foresight

The concept of perfect foresight path requires that agents maximize their future discounted payoffs against their beliefs about the future path of the action distribution and that those beliefs coincide with the actual path. Relaxing the latter requirement, Matsui and Oyama (2002) introduce the model of rationalizable foresight dynamics. In this model, while the rationality of agents as well as the structure of the society is common knowledge, beliefs about the future path are not necessarily coordinated among agents. It is instead assumed that agents form their beliefs in a rationalizable manner: in particular, they may misforecast the future. In this subsection, we consider stability under the rationalizable foresight dynamics and show that in supermodular games, an absorbing and globally accessible state under the perfect foresight dynamics is uniquely absorbing under the rationalizable foresight dynamics as well.

Following Matsui and Oyama (2002), we define rationalizable foresight paths as follows. First let Ψ^0 be the set of all feasible paths, Φ. Then for each positive integer k, define Ψ^k to be

$$\begin{aligned}\Psi^k = \Big\{\psi \in \Psi^{k-1} \,\Big|\, \forall\, i \in I,\ \forall\, h \in A_i,\ \text{a.a.}\, t \geq 0 : \big[\dot{\psi}_{ih}(t) > -\psi_{ih}(t) \\ \Rightarrow \exists\, \psi' \in \Psi^{k-1} : \psi'(s) = \psi(s)\ \forall\, s \in [0,t] \text{ and } h \in BR^i(\psi')(t)\big]\Big\}.\end{aligned}$$

Along a path in Ψ^k, an agent with a revision opportunity at time t takes a best response to some path in Ψ^{k-1} while knowing the past history up to time t.[13] Let $\Psi^* = \bigcap_{k=0}^{\infty} \Psi^k$.

Definition 3.1. A path in Ψ^* is a *rationalizable foresight path.*

Our concept of rationalizable foresight path differs from rationalizability in the associated differential game defined in Remark 2.2. The former incorporates the feature of societal games that different agents in a population can have different beliefs and a single agent can have different beliefs at different revision opportunities, while for the latter, each population acts as a single player, who makes his decision only at time zero.

Along every rationalizable foresight path, each agent optimizes against some, possibly different, rationalizable foresight path. We state this without a proof, as it is essentially the same as Proposition 3.3 in Matsui and Oyama (2002).

Proposition 3.8. *A feasible path $\psi \in \Phi$ is contained in Ψ^* if and only if for all $i \in I$, all $h \in A_i$, and almost all $t \geq 0$ such that $\dot{\psi}_{ih}(t) > -\psi_{ih}(t)$, there exists $\psi' \in \Psi^*$ such that $\psi'(s) = \psi(s)$ for all $s \in [0,t]$ and $h \in BR^i(\psi')(t)$.*

As in a one-shot game, we have the following relationship between perfect and rationalizable foresight paths. This is verified by observing that every perfect foresight path is contained in each Ψ^k.

Lemma 3.9. *A perfect foresight path is a rationalizable foresight path.*

We define absorption under rationalizable foresight analogously to that under perfect foresight.[14]

Definition 3.2. $x^* \in \prod_i \Delta(A_i)$ is *absorbing* under *rationalizable foresight* if there exists $\varepsilon > 0$ such that any rationalizable foresight path from any $x \in B_\varepsilon(x^*)$ converges to x^*.

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[13] Since the environment is stationary and $BR^i(\phi)(t)$ depends only on the behavior of ϕ after time t, in the definition of Ψ^k one can equivalently take ψ' as a path in Ψ^{k-1} that only satisfies $\psi'(t) = \psi(t)$.

[14] We can also define global accessibility under rationalizable foresight in a similar manner. Due to Lemma 3.9, it is weaker than that under perfect foresight.

An absorbing state under rationalizable foresight is also absorbing under perfect foresight due to Lemma 3.9, but not vice versa in general. See Examples 3.1 and 4.1 in Matsui and Oyama (2002). For supermodular games, however, we can show the converse.

Theorem 3.10. *Suppose that the stage game is supermodular. Then, $x^* \in \prod_i \Delta(A_i)$ is absorbing under rationalizable foresight if and only if it is absorbing under perfect foresight.*

Therefore, in supermodular games, an absorbing and globally accessible state under perfect foresight is the unique state that is absorbing under rationalizable foresight.

The "if" part of this theorem follows from the lemma below. For $x \in \prod_i \Delta(A_i)$, let $\Psi_x^k = \Psi^k \cap \Phi_x$ and $\Psi_x^* = \bigcap_{k=0}^{\infty} \Psi_x^k$. Note that $\Psi_x^* = \Psi^* \cap \Phi_x$, i.e., Ψ_x^* is the set of rationalizable foresight paths from x.

Lemma 3.11. *Suppose that the stage game is supermodular. Then, Ψ_x^* has the smallest and the largest elements, and these elements are perfect foresight paths.*

Proof. We show that Ψ_x^* has the smallest element and that it is a perfect foresight path. Let ϕ^0 be the smallest feasible path from x (i.e., the linear path from x to $\min A$) and ϕ^k the smallest best response path to ϕ^{k-1}, which is given by

$$\dot{\phi}_i^k(t) = \min BR^i(\phi^{k-1})(t) - \phi_i^k(t) \;\text{ a.e.,} \qquad \phi_i^k(0) = x_i.$$

Then, $\{\phi^k\}_{k=0}^{\infty}$ is an increasing sequence in the compact set Φ_x, so that $\{\phi^k\}_{k=0}^{\infty}$ converges to some $\phi^* \in \Phi_x$. By the upper semi-continuity of β_x, ϕ^* is a perfect foresight path, and hence, an element of Ψ_x^* by Lemma 3.9.

It suffices to show that ϕ^* is a lower bound of Ψ_x^*. Let us show that ϕ^k is a lower bound of Ψ_x^k $(\supset \Psi_x^*)$ for all k. Then, it follows that the limit ϕ^* is also a lower bound of Ψ_x^*.

First, ϕ^0 is a lower bound of Ψ_x^0. Then, suppose that ϕ^{k-1} is a lower bound of Ψ_x^{k-1}. Fix any $\psi \in \Psi_x^k$, and take any i and any t such that ϕ_i^k and ψ_i are differentiable at t. For any h such that $\dot{\psi}_{ih}(t) > -\psi_{ih}(t)$, we have $h \in BR^i(\psi')(t)$ for some $\psi' \in \Psi_x^{k-1}$. Since $\phi^{k-1} \precsim \psi'$ by assumption, it follows from the supermodularity and Lemma 3.1 that $\min BR^i(\phi^{k-1})(t) \leq \min BR^i(\psi')(t) \leq h$. Therefore, we have $\dot{\phi}_i^k(t) + \phi_i^k(t) \precsim \dot{\psi}_i(t) + \psi_i(t)$ for almost all t, which implies that $\phi^k \precsim \psi$. Hence, ϕ^k is a lower bound of Ψ_x^k. ▌

Proof of Theorem 3.10. "If" part: Take any rationalizable foresight path ψ from x sufficiently close to x^*. By Lemma 3.11, there exist perfect foresight paths ϕ and ϕ' from x such that $\phi \precsim \psi \precsim \phi'$. If x^* is absorbing under perfect foresight, then both ϕ and ϕ' converge to x^*, and therefore, ψ also converges to x^*.

"Only if" part: Follows from Lemma 3.9. ▌

Remark 3.1. All the results in this section, as well as Lemma 2.1, hold under more general settings (after appropriate modifications of replacing "$\phi_{-i} \precsim \psi_{-i}$" with "$\phi \precsim \psi$", and br^i with BR^i) where $V_i(\cdot)(\cdot)\colon \Phi \times \mathbb{R}_+ \to \mathbb{R}^{n_i+1}$ is continuous and $V_i(\cdot)(t)\colon \Phi \to \mathbb{R}^{n_i+1}$ is supermodular, i.e., if $\phi \precsim \psi$, then

$$V_{ik}(\phi)(t) - V_{ih}(\phi)(t) \leq V_{ik}(\psi)(t) - V_{ih}(\psi)(t)$$

for $k > h$. Examples of such functions include the expected discounted payoffs induced by the stage game where the payoff to an agent in population i taking action $h \in A_i$ is given by a continuous function $g_{ih}\colon \prod_i \Delta(A_i) \to \mathbb{R}$. Note here that the payoff function for an agent in population i may depend on the action distribution within population i itself and may not be N-linear in $\prod_i \Delta(A_i)$. Such payoff functions can describe random matching models within a single population, considered in Matsui and Matsuyama (1995), Hofbauer and Sorger (1999), and Oyama (2002), as well as models with nonlinear payoffs, considered in Matsuyama (1991, 1992) and Kaneda (1995). In alternative settings, V_i may depend on the past behavior of ϕ.

4 Games with Monotone Potentials

This section applies the monotonicity argument developed in the previous section to games with monotone potentials introduced by Morris and Ui (2005). Suppose that games G and G' satisfy (3.1) or (3.2). Roughly speaking, G has a monotone potential if G' can be chosen as a potential game, and action profile $\max A$ is a monotone potential maximizer of G if it is the unique potential maximizer of G'. For potential games, Hofbauer and Sorger (2002) show that the unique potential maximizer is absorbing and globally accessible for any small degree of friction. Therefore, we can conclude from Theorem 3.4 and the subsequent discussion that if G or G' is supermodular, then $\max A$ is absorbing (if (3.1) is satisfied) and globally accessible (if (3.2) is satisfied) for any small degree of friction in the stage game G.

For the precise definition, which is given in the subsection below, two remarks are in order. First, when G' is a potential game, a condition weaker than both (3.1) and (3.2) is sufficient for the global accessibility result. Morris and Ui's (2005) definition of monotone potential employs this weaker version (Definition 4.1), while (3.1) corresponds to what we call strict monotone potential (Definition 4.2). Second, in order to define the concept for action profiles a^* other than $\max A$ or $\min A$, we need to divide the set of actions for each player i into two parts: the actions below a_i^* and those above a_i^*.

4.1 Monotone Potential Maximizer

Fix an action profile $a^* \in A$. Let $A_i^- = \{h \in A_i \,|\, h \leq a_i^*\}$ and $A_i^+ = \{h \in A_i \,|\, h \geq a_i^*\}$. For a function $f\colon A \to \mathbb{R}$, a probability distribution $\pi_i \in \Delta(A_{-i})$, and a nonempty set of actions $A_i' \subset A_i$, let

$$br_f^i(\pi_i|A_i') = \arg\max_{h \in A_i'} f(h, \pi_i),$$

where $f(h, \pi_i) = \sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) \, f(h, a_{-i})$. We employ the following simplified version of monotone potential.[15]

Definition 4.1. The action profile $a^* \in A$ is a *monotone potential maximizer*, or an *MP-maximizer*, of G if there exists a function $v\colon A \to \mathbb{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ such that for all $i \in I$ and all $\pi_i \in \Delta(A_{-i})$,

$$\min br_v^i(\pi_i|A_i^-) \leq \max br_{u_i}^i(\pi_i|A_i^-), \tag{4.1}$$

and

$$\max br_v^i(\pi_i|A_i^+) \geq \min br_{u_i}^i(\pi_i|A_i^+). \tag{4.2}$$

Such a function v is called a *monotone potential function* for a^*.

In addition, we introduce a slight refinement of MP-maximizer.[16]

Definition 4.2. The action profile $a^* \in A$ is a *strict monotone potential maximizer*, or a *strict MP-maximizer*, of G if there exists a function $v\colon A \to \mathbb{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ such that for all $i \in I$ and all $\pi_i \in \Delta(A_{-i})$,

$$\min br_v^i(\pi_i|A_i^-) \leq \min br_{u_i}^i(\pi_i|A_i^-), \tag{4.3}$$

and

$$\max br_v^i(\pi_i|A_i^+) \geq \max br_{u_i}^i(\pi_i|A_i^+). \tag{4.4}$$

Such a function v is called a *strict monotone potential function* for a^*.

A (strict) MP-maximizer is a (strict) Nash equilibrium. A strict MP-maximizer is always an MP-maximizer, but the converse is not true. In a degenerate game (with at least two action profiles) where payoffs are constant for each player, all the action profiles become MP-maximizers, while none of them is a strict MP-maximizer. For a generic choice of payoffs, an MP-maximizer is a strict MP-maximizer. For supermodular games, a strict MP-maximizer is unique if it exists, due to Theorems 4.1 and 4.2 given below.

[15] In Morris and Ui (2005), a monotone potential function is defined on a given partition of A.

[16] Morris (1999) introduces a version of MP-maximizer, which is stronger than our concept of strict MP-maximizer: if a^* is an MP-maximizer in the sense of Morris (1999), then it is a strict MP-maximizer, but not vice versa in general.

MP-maximizer unifies several existing concepts. A unique weighted potential maximizer is a strict MP-maximizer. A (strict) **p**-dominant equilibrium with $\sum_{i\in I} p_i < 1$ is a (strict) MP-maximizer. For games with diminishing marginal returns, MP-maximizer reduces to local potential maximizer (Morris (1999) and Morris and Ui (2005)). See Subsection 4.3 for details.

4.2 Results

For a function $f\colon A \to \mathbb{R}$, a feasible path ϕ, and a nonempty set of actions $A'_i \subset A_i$, let

$$BR^i_f(\phi|A'_i)(t) = \operatorname*{arg\,max}_{h\in A'_i}\ (1+\theta)\int_t^\infty e^{-(1+\theta)(s-t)} f(h, \phi_{-i}(s))\, ds,$$

where $f(h, x_{-i}) = \sum_{a_{-i}\in A_{-i}} (\prod_{j\neq i} x_{ja_j}) f(h, a_{-i})$ for $x_{-i} \in \prod_{j\neq i}\Delta(A_j)$. Let $G_v = (I, (A_i)_{i\in I}, (v)_{i\in I})$ be the potential game in which all players have the common payoff function v. We have the following two theorems. Their proofs are given in the Appendix.

Theorem 4.1. *Suppose that the stage game G has an MP-maximizer a^* with a monotone potential function v. If G or G_v is supermodular, then there exists $\bar{\theta} > 0$ such that a^* is globally accessible for all $\theta \in (0, \bar{\theta})$.*

Theorem 4.2. *Suppose that the stage game G has a strict MP-maximizer a^* with a strict monotone potential function v. If G or G_v is supermodular, then a^* is linearly absorbing for all $\theta > 0$.*

In particular, a strict MP-maximizer is the unique linearly absorbing (and globally accessible) state for any small degree of friction, if G or G_v is supermodular.

Given an MP-maximizer a^* and a monotone potential v, observe that the restricted games $G^-_v = (I, (A^-_i)_{i\in I}, (v)_{i\in I})$ and $G^+_v = (I, (A^+_i)_{i\in I}, (v)_{i\in I})$ are potential games with the unique potential maximizer a^*. The proofs of Theorems 4.1 and 4.2 utilize this observation to apply results on potential games by Hofbauer and Sorger (2002).

The proofs proceed as follows. Suppose that a^* is an MP-maximizer with a monotone potential function v. Observe (for the case where $a^* = \max A$) that (4.1) is weaker than (3.2). We thus need feasible paths ϕ^- and ϕ^+ such that

$$\begin{aligned}
\dot{\phi}^-_i(t) &= \min BR^i_v(\phi^-|A^-_i)(t) - \phi^-_i(t) \quad \text{a.e.}, \qquad &\phi^-_i(0) = \min A_i,\\
\dot{\phi}^+_i(t) &= \max BR^i_v(\phi^+|A^+_i)(t) - \phi^+_i(t) \quad \text{a.e.}, \qquad &\phi^+_i(0) = \max A_i
\end{aligned}$$

for all $i \in I$, and $\lim_{t\to\infty} \phi^-(t) = \lim_{t\to\infty} \phi^+(t) = a^*$. Notice that ϕ^- (ϕ^+, resp.) is a feasible path on $\prod_i \Delta(A^-_i)$ ($\prod_i \Delta(A^+_i)$, resp.), and actually a perfect foresight path for the stage game G^-_v (G^+_v, resp.).

To obtain these paths, we use the fact that if the stage game is a potential game, then any solution to a certain optimal control problem is a perfect foresight path, and when the friction $\theta > 0$ is sufficiently small, it converges to the potential maximizer a^*. Fix such a small θ. We show that a minimal (maximal, resp.) solution to the optimal control problem associated with G_v^- (G_v^+, resp.) satisfies the above conditions.

Then, an argument similar to that in the proof of Lemma 3.3 allows us to show that if G or G_v is supermodular, then for any $x \in \prod_i \Delta(A_i)$, there exists a perfect foresight path ϕ^* with $\phi^*(0) = x$ such that $\phi^- \precsim \phi^* \precsim \phi^+$. Since ϕ^- and ϕ^+ converge to a^*, ϕ^* also converges to a^*. This implies that a^* is globally accessible for a small friction.

Next, suppose that a^* is a strict MP-maximizer with a strict monotone potential function v. Take any perfect foresight path ϕ^* starting from a state sufficiently close to a^*. As in the proof of Lemma 3.3, we can show that if G or G_v is supermodular, then there exist feasible paths ϕ^- and ϕ^+ starting from states sufficiently close to a^* such that $\phi^- \precsim \phi^* \precsim \phi^+$ and that ϕ^- and ϕ^+ are perfect foresight paths for the restricted games G_v^- and G_v^+, respectively. Since a^*, the potential maximizer of G_v^- and G_v^+, is absorbing in G_v^- and G_v^+, ϕ^- and ϕ^+ converge to a^*, and therefore, ϕ^* also converges to a^*. In the case where G is supermodular, this implies that a^* is linearly absorbing in G, due to Proposition 3.7. In the case where G_v is supermodular, a^* is linearly absorbing in G_v^- and G_v^+, so that ϕ^- and ϕ^+ converge linearly to a^*. Therefore, ϕ^* also converges linearly to a^*, implying the linear absorption of a^* in G.

4.3 Examples

This subsection provides special cases of games with monotone potentials. For games with no monotone potential, see Examples 5.2.1 and 5.4.1.

4.3.1 p-Dominance

Let $\mathbf{p} = (p_1, \ldots, p_N) \in [0, 1)^N$. The notion of $\mathbf{p}$-dominance (Kajii and Morris (1997)) is a many-player, many-action generalization of risk-dominance.

Definition 4.3. (a) An action profile $a^* \in A$ is a $\mathbf{p}$-*dominant equilibrium* of G if for all $i \in I$, $a_i^* \in br^i(\pi_i)$ holds for all $\pi_i \in \Delta(A_{-i})$ with $\pi_i(a_{-i}^*) \geq p_i$.

(b) An action profile a^* is a *strict* $\mathbf{p}$-*dominant equilibrium* of G if for all $i \in I$, $\{a_i^*\} = br^i(\pi_i)$ holds for all $\pi_i \in \Delta(A_{-i})$ with $\pi_i(a_{-i}^*) > p_i$.

A $\mathbf{p}$-dominant equilibrium with low enough $\mathbf{p}$ is an MP-maximizer with a monotone potential function that is supermodular (with appropriate reordering of actions).

Lemma 4.3. *If a^* is a (strict)* $\mathbf{p}$*-dominant equilibrium with $\sum_{i\in I} p_i < 1$, then a^* is a (strict) MP-maximizer with the (strict) monotone potential v given by*

$$v(a) = \begin{cases} 1 - \sum_{i\in I} p_i & \text{if } a = a^*, \\ -\sum_{i\in C(a)} p_i & \text{otherwise,} \end{cases}$$

where $C(a) = \{i \in I \mid a_i = a_i^*\}$.

Proof. See Appendix. ∎

By relabeling actions so that $a_i^* = \max A_i$ for all $i \in I$, we can have v supermodular. Therefore, we have the following result as a corollary to Theorems 4.1 and 4.2, which generalizes a result for symmetric two-player games by Oyama (2002, Theorem 3).

Corollary 4.4. (a) *A* $\mathbf{p}$*-dominant equilibrium with $\sum_{i\in I} p_i < 1$ is globally accessible for any small degree of friction.*

(b) *A strict* $\mathbf{p}$*-dominant equilibrium with $\sum_{i\in I} p_i < 1$ is linearly absorbing for any degree of friction.*

In particular, a strict $\mathbf{p}$-dominant equilibrium with $\sum_{i\in I} p_i < 1$ is the unique linearly absorbing (and globally accessible) state for any small degree of friction.

Remark 4.1. Hofbauer and Sorger (2002) consider the following concept of 1/2-dominance and show that for games with linear incentives, it implies linear absorption and global accessibility for small frictions. An action profile $a^* \in A$ is said to be 1/2-dominant if for all $i \in I$, $\{a_i^*\} = br^i(x_{-i})$ holds for all $x_{-i} \in \prod_{j\neq i} \Delta(A_j)$ such that $x_{ja_j^*} \geq 1/2$ for all $j \neq i$. For two-player games, 1/2-dominance is equivalent to strict $\mathbf{p}$-dominance with $p_i < 1/2$ for any $i \in I$, so that Corollary 4.4 covers their result. For games with more than two players, there is no obvious relation. Note the difference between π_i and x_{-i} in the definitions.

4.3.2 Local Potential Maximizer

We consider a simplified version of local potential maximizer introduced by Morris and Ui (2005) as well as its refinement.

Definition 4.4. (a) An action profile $a^* \in A$ is a *local potential maximizer*, or an *LP-maximizer*, of G if there exists a function $v\colon A \to \mathbb{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ such that for all $i \in I$, there exists a function $\mu_i\colon A_i \to \mathbb{R}_+$ such that if $h < a_i^*$, then for all $a_{-i} \in A_{-i}$,

$$\mu_i(h)\big(v(h+1, a_{-i}) - v(h, a_{-i})\big) \leq u_i(h+1, a_{-i}) - u_i(h, a_{-i}),$$

and if $h > a_i^*$, then for all $a_{-i} \in A_{-i}$,

$$\mu_i(h)\big(v(h-1, a_{-i}) - v(h, a_{-i})\big) \leq u_i(h-1, a_{-i}) - u_i(h, a_{-i}).$$

Such a function v is called a *local potential function* for a^*.

(b) An action profile a^* is a *strict local potential maximizer*, or a *strict LP-maximizer*, of G if there exists a function $v\colon A \to \mathbb{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ such that for all $i \in I$, there exists a function $\mu_i\colon A_i \to \mathbb{R}_{++}$ such that if $h < a_i^*$, then for all $a_{-i} \in A_{-i}$,

$$\mu_i(h)\big(v(h+1, a_{-i}) - v(h, a_{-i})\big) \leq u_i(h+1, a_{-i}) - u_i(h, a_{-i}),$$

and if $h > a_i^*$, then for all $a_{-i} \in A_{-i}$,

$$\mu_i(h)\big(v(h-1, a_{-i}) - v(h, a_{-i})\big) \leq u_i(h-1, a_{-i}) - u_i(h, a_{-i}).$$

Such a function v is called a *strict local potential function* for a^*.

An LP-maximizer is a strict LP-maximizer if one can take strictly positive numbers for the weights μ_i.[17]

The game G is said to have *diminishing marginal returns* if for all $i \in I$, all $h \neq 0, n_i$, and all $a_{-i} \in A_{-i}$,

$$u_i(h, a_{-i}) - u_i(h-1, a_{-i}) \geq u_i(h+1, a_{-i}) - u_i(h, a_{-i}).$$

In games with diminishing marginal returns, the MP-maximizer conditions reduce to the LP-maximizer conditions.

Lemma 4.5. *If the game G has a (strict) LP-maximizer a^* with a (strict) local potential function v and if G or G_v has diminishing marginal returns, then a^* is a (strict) MP-maximizer with the same function v.*

Proof. See Appendix. ∎

We have the following result as a corollary to Theorems 4.1 and 4.2.

Corollary 4.6. (a) *Suppose that the stage game G has an LP-maximizer a^* with a local potential function v. If G or G_v has diminishing marginal returns and if G or G_v is supermodular, then a^* is globally accessible for any small degree of friction.*

(b) *Suppose that the stage game G has a strict LP-maximizer a^* with a strict local potential function v. If G or G_v has diminishing marginal returns and if G or G_v is supermodular, then a^* is linearly absorbing for any degree of friction.*

In particular, a strict LP-maximizer is the unique linearly absorbing (and globally accessible) state for any small degree of friction, if G or G_v has diminishing marginal returns and G or G_v is supermodular.

[17] Morris (1999) and Frankel, Morris, and Pauzner (2003) give a slightly different definition of LP-maximizer, which is weaker than strict LP-maximizer.

4.3.3 Symmetric 3×3 Supermodular Games

Consider symmetric 3×3 games with three strict Nash equilibria, where $I = \{1, 2\}$, $A_1 = A_2 = \{0, 1, 2\}$, $u_1(h, k) = u_2(k, h)$ for all $h, k \in \{0, 1, 2\}$, and $u_1(h, h) > u_1(k, h)$ for all $k \neq h$. Assume strict supermodularity, i.e., $u_1(h, k) - u_1(h', k) > u_1(h, k') - u_1(h', k')$ if $h > h'$ and $k > k'$. We show that this class of games generically have a strict MP-maximizer.[18]

For $h, k \in \{0, 1, 2\}$, let $u_1(h, k) = w_{hk}$ and

$$\Delta^{hk}_{h'k'} = w_{h'h} + w_{h'k} - w_{k'h} - w_{k'k}.$$

The inequality $\Delta^{hk}_{h'k'} > 0$ means that action h' is better than action k' against the 50-50 mixture of actions h and k. Note that $\Delta^{kh}_{h'k'} = \Delta^{hk}_{h'k'}$ and $\Delta^{hk}_{k'h'} = -\Delta^{hk}_{h'k'}$. Note also that $\Delta^{hk}_{hk} > 0$ if and only if h pairwise risk-dominates k. We have the following complete characterization (for generic games) of the strict MP-maximizer.

Lemma 4.7.
(1) $\Delta^{02}_{01} > 0$ *and* $\Delta^{02}_{02} > 0$.
  $(0, 0)$ *is the strict MP-maximizer.*

(2) $\Delta^{20}_{21} > 0$ *and* $\Delta^{20}_{20} > 0$.
  $(2, 2)$ *is the strict MP-maximizer.*

(3) $\Delta^{02}_{10} > 0$ *and* $\Delta^{02}_{12} > 0$.
  (a) *If* $\Delta^{10}_{10} > 0$ *and* $\Delta^{12}_{12} > 0$*, then* $(1, 1)$ *is the strict MP-maximizer.*
  (b) *If* $\Delta^{01}_{01} > 0$ *and* $\Delta^{12}_{12} > 0$*, then* $(0, 0)$ *is the strict MP-maximizer.*
  (c) *If* $\Delta^{21}_{21} > 0$ *and* $\Delta^{10}_{10} > 0$*, then* $(2, 2)$ *is the strict MP-maximizer.*
  (d) *If* $\Delta^{01}_{01} > 0$ *and* $\Delta^{21}_{21} > 0$ *and*
    (i) *if* $\Delta^{02}_{10}/\Delta^{01}_{01} < \Delta^{02}_{12}/\Delta^{21}_{21}$*, then* $(0, 0)$ *is the strict MP-maximizer.*
    (ii) *if* $\Delta^{02}_{10}/\Delta^{01}_{01} > \Delta^{02}_{12}/\Delta^{21}_{21}$*, then* $(2, 2)$ *is the strict MP-maximizer.*

Proof. See Appendix. ▮

We therefore have the following result as a corollary to Theorems 4.1 and 4.2.

Corollary 4.8. *For a generic symmetric 3×3 supermodular game, there exists a unique linearly absorbing and globally accessible state for any small degree of friction, given by the MP-maximizer in Lemma 4.7.*

This corrects Theorem 4.3 in Hofbauer and Sorger (2002)[19] and resolves their conjecture on the generic existence of an absorbing state.

[18]Morris (1999) establishes the generic existence of LP-maximizers for this class of games. There is an open subset of games, however, that have two strict LP-maximizers. Note that this class of games do not necessarily have diminishing marginal returns. For symmetric 4×4 supermodular games, Morris (1999) presents a (nondegenerate) example with diminishing marginal returns that has no robust equilibrium, and hence, no MP-maximizer.

[19]Their result only considers Case (3) in Lemma 4.7.

4.3.4 Young's Example

Consider the 3×3 game given in Figure 2(a), taken from Young (1993). Oyama (2002) shows by direct computation that $(2, 2)$ is linearly absorbing and globally accessible for a small degree of friction. In fact, $(2, 2)$ is a strict MP-maximizer with a strict monotone potential function that is supermodular (Figure 2(b)), while the original game is not supermodular (for any ordering of actions). Therefore, our results, Theorems 4.1 and 4.2, also apply to this game.

Note that $(1, 1)$ is stochastically stable (Young (1993)), while it is neither absorbing nor globally accessible when the friction is small.

	0	1	2
0	6, 6	0, 5	0, 0
1	5, 0	7, 7	5, 5
2	0, 0	5, 5	8, 8

(a) Original game

	0	1	2
0	6	5	0
1	5	7	5
2	0	5	8

(b) Monotone potential function

Figure 2: Young's example

5 Binary Supermodular Games

In this section, we restrict our attention to supermodular games with two actions for each player, where $A_i = \{0, 1\}$ for all $i \in I$. Note that in this case, the stochastic dominance order reduces to the following simple form: $\phi \precsim \psi$ if and only if $\phi_{i1}(t) \leq \psi_{i1}(t)$ for all $i \in I$ and all $t \geq 0$. Denoting $p_j = x_{j1}$, we define the *incentive function* $d_i \colon [0, 1]^N \to \mathbb{R}$ for player i by

$$d_i(p_1, \ldots, p_N) = u_i(1, x_{-i}) - u_i(0, x_{-i}).$$

In the following, we identify $a = (a_i)_{i \in I} \in A$ with $\mathbf{p} = (p_1, \ldots, p_N) \in [0, 1]^N$ such that $p_i = 0$ if $a_i = 0$ and $p_i = 1$ if $a_i = 1$. We assume that action profiles $\mathbf{0}$, where all players play 0, and $\mathbf{1}$, where all players play 1, are strict Nash equilibria, i.e.,

$$d_i(\mathbf{0}) < 0 < d_i(\mathbf{1})$$

for all i. We further assume that d_i is nondecreasing in each p_j ($j \neq i$) so that the game is supermodular. In the first subsection, we give complete characterizations for the strict Nash equilibrium **1** to be globally accessible and to be absorbing (or, equivalently, linearly absorbing), respectively. By reversing the orders of actions, the results can be applied to the other Nash equilibrium **0**. The subsequent subsections then consider three subclasses of binary supermodular games.

For a feasible path ϕ, denote

$$\begin{aligned}\Delta V_i^\theta(\phi)(t) &= V_{i1}(\phi)(t) - V_{i0}(\phi)(t) \\ &= (1+\theta)\int_t^\infty e^{-(1+\theta)(s-t)} d_i(\phi(s))\,ds.\end{aligned}$$

We write the superscript θ of ΔV to specify the discount rate. Note that ΔV^θ is well-defined also for $\theta = 0$. Recall from Lemma 3.1 that if $\phi \precsim \psi$, then $\Delta V_i^\theta(\phi)(t) \leq \Delta V_i^\theta(\psi)(t)$ for all $i \in I$ and all $t \geq 0$ due to the supermodularity.

5.1 General Results

For $\mathbf{T} = (T_i)_{i\in I} \in \mathbb{R}_+^N$, let $\phi_\mathbf{T}^\mathrm{u}$ be the feasible path given by

$$(\phi_\mathbf{T}^\mathrm{u})_{i1}(t) = \begin{cases} 0 & \text{if } t < T_i \\ 1 - e^{-(t-T_i)} & \text{if } t \geq T_i, \end{cases} \tag{5.1}$$

which starts at **0** and converges to **1**. Along $\phi_\mathbf{T}^\mathrm{u}$, agents in population $i \in I$ start choosing action 1 at time T_i.

Denote $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$. For $\mathbf{T} = (T_i)_{i\in I} \in \bar{\mathbb{R}}_+^N$, let $\psi_\mathbf{T}^\mathrm{d}$ be the feasible path given by

$$(\psi_\mathbf{T}^\mathrm{d})_{i1}(t) = \begin{cases} 1 & \text{if } t < T_i \\ e^{-(t-T_i)} & \text{if } t \geq T_i \end{cases} \qquad \text{for } i \in S, \tag{5.2}$$

and

$$(\psi_\mathbf{T}^\mathrm{d})_{i1}(t) = 1 \qquad \text{for } i \notin S, \tag{5.3}$$

where $S = \{i \in I \,|\, T_i \neq \infty\}$. Let $\mathbf{0}_S$ be the action profile such that i chooses 0 if $i \in S$ and 1 if $i \notin S$. Along $\psi_\mathbf{T}^\mathrm{d}$, which starts at **1** and converges to $\mathbf{0}_S$, agents in population $i \in S$ start choosing action 0 at time T_i, while those in population $i \notin S$ always play action 1.

First, we provide necessary and sufficient conditions for the state **1** to be globally accessible for a given degree of friction (Proposition 5.1.1) and for any small degree of friction (Proposition 5.1.2), respectively. Each condition is equivalent to the existence of a subpath of the form (5.1).

Proposition 5.1.1. *Let $\theta > 0$ be given. The strict Nash equilibrium* **1** *is globally accessible for θ if and only if there exists $\mathbf{T} = (T_i)_{i\in I} \in \mathbb{R}_+^N$ such that for all $i \in I$,*

$$\Delta V_i^\theta(\phi_\mathbf{T}^\mathrm{u})(T_i) \geq 0.$$

Proof. See Appendix. ∎

Proposition 5.1.2. *There exists $\bar{\theta} > 0$ such that the strict Nash equilibrium* **1** *is globally accessible for all $\theta \in (0, \bar{\theta})$ if and only if there exists $\mathbf{T} = (T_i)_{i \in I} \in \mathbb{R}_+^N$ such that for all $i \in I$,*

$$\Delta V_i^0(\phi_{\mathbf{T}}^{\mathrm{u}})(T_i) > 0.$$

Proof. See Appendix. ∎

Next, we provide necessary and sufficient conditions for the state **1** to be absorbing for a given degree of friction (Proposition 5.1.3) and for any degree of friction (Proposition 5.1.4), respectively. Each condition is equivalent to the nonexistence of a superpath of the form (5.2)–(5.3) with $\mathbf{0}_S$ being a Nash equilibrium of the stage game.

Proposition 5.1.3. *Let $\theta > 0$ be given. The strict Nash equilibrium* **1** *is absorbing for θ if and only if for any $\mathbf{T} = (T_i)_{i \in I} \in \bar{\mathbb{R}}_+^N$ such that $S = \{i \in I \,|\, T_i \neq \infty\}$ is nonempty and $\mathbf{0}_S$ is a Nash equilibrium, there exists $i \in S$ such that*

$$\Delta V_i^\theta(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i) > 0.$$

Proof. See Appendix. ∎

Proposition 5.1.4. *The strict Nash equilibrium* **1** *is absorbing for all $\theta > 0$ if and only if for any $\mathbf{T} = (T_i)_{i \in I} \in \bar{\mathbb{R}}_+^N$ such that $S = \{i \in I \,|\, T_i \neq \infty\}$ is nonempty and $\mathbf{0}_S$ is a Nash equilibrium, there exists $i \in S$ such that*

$$\Delta V_i^0(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i) \geq 0.$$

Proof. See Appendix. ∎

5.2 Unanimity Games

This subsection considers N-player unanimity games. The stage game is given by

$$u_i(a) = \begin{cases} y_i & \text{if } a = \mathbf{0} \\ z_i & \text{if } a = \mathbf{1} \\ 0 & \text{otherwise,} \end{cases} \tag{5.4}$$

where $y_i, z_i > 0$. The incentive function for player i is then given by

$$d_i(p_1, \cdots, p_N) = z_i \prod_{j \neq i} p_j - y_i \prod_{j \neq i}(1 - p_j).$$

Note that this game is supermodular.

For $\mathbf{T} = (T_i)_{i \in I} \in \mathbb{R}_+^N$, let

$$\pi_i(\mathbf{T}) = \int_{T_i}^{\infty} e^{-(t-T_i)} \prod_{j \neq i} \left[0 \vee \left\{ 1 - e^{-(t-T_j)} \right\} \right] dt$$
$$= \int_{\max_j T_j}^{\infty} e^{-(t-T_i)} \prod_{j \neq i} \left\{ 1 - e^{-(t-T_j)} \right\} dt, \tag{5.5}$$

and

$$\rho_i(\mathbf{T}) = \int_{T_i}^{\infty} e^{-(t-T_i)} \prod_{j \neq i} \left\{ 1 \wedge e^{-(t-T_j)} \right\} dt. \tag{5.6}$$

5.2.1 Global Accessibility

For a feasible path $\phi_{\mathbf{T}}^{\mathrm{u}}$ defined by (5.1) with a given $\mathbf{T} = (T_i)_{i \in I} \in \mathbb{R}_+^N$, the discounted payoff difference is given by

$$\Delta V_i^0(\phi_{\mathbf{T}}^{\mathrm{u}})(T_i) = z_i \pi_i(\mathbf{T}) - y_i \rho_i(\mathbf{T}),$$

so that $\Delta V_i^0(\phi_{\mathbf{T}}^{\mathrm{u}})(T_i) > 0$ if and only if $z_i/y_i > \rho_i(\mathbf{T})/\pi_i(\mathbf{T})$.

We immediately have the following from Proposition 5.1.2.

Proposition 5.2.1. *Suppose that the stage game is a unanimity game given by* (5.4)*. Then there exists $\bar{\theta} > 0$ such that $\mathbf{1}$ is globally accessible for all $\theta \in (0, \bar{\theta})$ if and only if there exists $\mathbf{T} \in \mathbb{R}_+^N$ such that for all $i \in I$,*

$$\frac{z_i}{y_i} > \frac{\rho_i(\mathbf{T})}{\pi_i(\mathbf{T})}.$$

Symmetrically, there exists $\bar{\theta} > 0$ such that $\mathbf{0}$ is globally accessible for all $\theta \in (0, \bar{\theta})$ if and only if there exists $\mathbf{T} \in \mathbb{R}_+^N$ such that for all $i \in I$,

$$\frac{y_i}{z_i} > \frac{\rho_i(\mathbf{T})}{\pi_i(\mathbf{T})}.$$

5.2.2 Absorption

For a feasible path $\psi_{\mathbf{T}}^{\mathrm{d}}$ defined by (5.2) with a given $\mathbf{T} = (T_i)_{i \in I} \in \mathbb{R}_+^N$, the discounted payoff difference is given by

$$\Delta V_i^0(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i) = z_i \rho_i(\mathbf{T}) - y_i \pi_i(\mathbf{T}),$$

so that $\Delta V_i^0(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i) \geq 0$ if and only if $z_i/y_i \geq \pi_i(\mathbf{T})/\rho_i(\mathbf{T})$.

We have the following from Proposition 5.1.4. Observe that in this case, S satisfies the condition in Proposition 5.1.4 only if $S = I$.

Proposition 5.2.2. *Suppose that the stage game is a unanimity game given by* (5.4). *Then* **1** *is absorbing for all* $\theta > 0$ *if and only if for any* $\mathbf{T} \in \mathbb{R}_+^N$, *there exists* $i \in I$ *such that*

$$\frac{z_i}{y_i} \geq \frac{\pi_i(\mathbf{T})}{\rho_i(\mathbf{T})}.$$

Symmetrically, **0** *is absorbing for all* $\theta > 0$ *if and only if for any* $\mathbf{T} \in \mathbb{R}_+^N$, *there exists* $i \in I$ *such that*

$$\frac{y_i}{z_i} \geq \frac{\pi_i(\mathbf{T})}{\rho_i(\mathbf{T})}.$$

5.2.3 Two-Player Case

In the case where $N = 2$, there exists $\mathbf{T} \in \mathbb{R}_+^2$ such that

$$\frac{z_1}{y_1} > \frac{\rho_1(\mathbf{T})}{\pi_1(\mathbf{T})}, \qquad \frac{z_2}{y_2} > \frac{\rho_2(\mathbf{T})}{\pi_2(\mathbf{T})}$$

if and only if $z_1 z_2 > y_1 y_2$. Therefore, by Propositions 5.2.1 and 5.2.2, **1** is absorbing and globally accessible for any small degree of friction if and only if **1** has the higher Nash product over **0**. In the two-player case, this is equivalent to that **1** risk-dominates **0**.

5.2.4 Three-Player Case

When $N \geq 3$, the complete characterizations given in Propositions 5.2.1 and 5.2.2 turn out to be rather complex. Here we consider three-player binary games with a symmetry between players 2 and 3. We demonstrate that even for this simple class of games, both Nash equilibria **1** and **0** may be simultaneously globally accessible states when the friction is small.

Specifically, we consider the case where

$$(z_1/y_1, z_2/y_2, z_3/y_3) = (r, s, s). \tag{5.7}$$

We can exploit the symmetry due to the following fact. Note here that if $T_i = T_j$, then $\pi_i(\mathbf{T}) = \pi_j(\mathbf{T})$ and $\rho_i(\mathbf{T}) = \rho_j(\mathbf{T})$.

Lemma 5.2.3. *Suppose that the stage game is given by* (5.4). *Then* **1** *is globally accessible for any small degree of friction if and only if there exists* $\mathbf{T}$ *such that for all* $i \in I$,

$$\frac{z_i}{y_i} > \frac{\rho_i(\mathbf{T})}{\pi_i(\mathbf{T})}, \tag{5.8}$$

and

$$\frac{z_i}{y_i} \geq \frac{z_j}{y_j} \Rightarrow T_i \leq T_j. \tag{5.9}$$

Proof. It suffices to show that if there exists $\mathbf{T}$ that satisfies (5.8), then there exists $\mathbf{T}'$ that satisfies both (5.8) and (5.9).

Take $\mathbf{T}$ that satisfies (5.8) and define $\mathbf{T}'$ by

$$T'_i = \min_{j\colon z_j/y_j \le z_i/y_i} T_j$$

for each i. Note that $T'_i \le T_i$ for any i.

Here we fix any i. By definition, there exists j such that $T'_i = T_j$ and $z_j/y_j \le z_i/y_i$. Take such a j. Note that $\mathbf{T}_{-j} \ge \mathbf{T}'_{-j}$ and $T_j = T'_j$. Since $\mathbf{T}$ satisfies (5.8), π_j is decreasing in $\mathbf{T}_{-j}$, and ρ_j is increasing in $\mathbf{T}_{-j}$, we have

$$\frac{z_j}{y_j} > \frac{\rho_j(\mathbf{T})}{\pi_j(\mathbf{T})} \ge \frac{\rho_j(\mathbf{T}')}{\pi_j(\mathbf{T}')}.$$

On the other hand, $\pi_i(\mathbf{T}') = \pi_j(\mathbf{T}')$ and $\rho_i(\mathbf{T}') = \rho_j(\mathbf{T}')$ since $T'_i = T'_j$. Therefore, it follows from $z_j/y_j \le z_i/y_i$ that

$$\frac{z_i}{y_i} \ge \frac{z_j}{y_j} > \frac{\rho_j(\mathbf{T}')}{\pi_j(\mathbf{T}')} = \frac{\rho_i(\mathbf{T}')}{\pi_i(\mathbf{T}')},$$

which completes the proof. ▮

A direct computation utilizing Lemma 5.2.3 shows that $\mathbf{1}$ is globally accessible for a small friction if and only if there exists $u \ge 1$ such that

$$r < s, \quad r > \frac{1}{3u^2 - 3u + 1}, \quad s > \frac{3u^2 - 1}{3u - 1},$$

or there exists $v \ge 1$ such that

$$r \ge s, \quad r > 3v - 2, \quad s > \frac{2}{3v - 1}.$$

The above condition is equivalent to that

$$r < s \quad \text{and} \quad r > \frac{2}{(s-1)\sqrt{9s^2 - 12s + 12} + 3s^2 - 5s + 4},$$

or

$$r \ge s \quad \text{and} \quad r > \frac{2}{s} - 1.$$

In the game given by (5.7), $\mathbf{1}$ has the higher Nash product over $\mathbf{0}$ if $rs^2 > 1$. A direct comparison between $r > 1/s^2$ and the above expressions gives the following sufficient condition in terms of Nash product.

Proposition 5.2.4. *In the game given by* (5.7)*, the Nash equilibrium with the higher Nash product is globally accessible for any small degree of friction.*

The converse is not true.

Example 5.2.1. Let $y_1 = 6 + c > 0$, $y_2 = y_3 = 1$, and $z_1 = z_2 = z_3 = 2$ (see Figure 3). This game is a modified version of an example in Morris and Ui (2005, Example 1).[20] If $c > 0$, then $\mathbf{0}$ is globally accessible for a small friction, while if $c < 2\sqrt{6}$, then $\mathbf{1}$ is globally accessible for a small friction. Therefore, if $0 < c < 2\sqrt{6}$, the game has two globally accessible states simultaneously when the friction is small. Note that $\mathbf{0}$ ($\mathbf{1}$, resp.) has the higher Nash product if $c > 2$ ($c < 2$, resp.).

On the other hand, one can show that if $c \leq 0$, then $\mathbf{1}$ is absorbing for any degree of friction, while if $c \geq 2\sqrt{6}$, then $\mathbf{0}$ is absorbing for any degree of friction.

	0	1
0	$6+c$, 1, 1	0, 0, 0
1	0, 0, 0	0, 0, 0

0

	0	1
0	0, 0, 0	0, 0, 0
1	0, 0, 0	2, 2, 2

1

Figure 3: Multiple globally accessible states

5.3 Binary Games with Linear Incentives

This subsection considers N-player binary supermodular games with linear incentives (Selten (1995)). This class of games is prominent in the theory of equilibrium selection. It includes incomplete information 2×2 games with finitely many types (see Selten (1995)). Such games also arise as superposition of pairwise matches.

A binary game is said to have linear incentives if the incentive function for player i takes the form (with $p_j = x_{j1}$)

$$d_i(p_1, \ldots, p_N) = \sum_{j=1}^{N} \alpha_{ij} p_j - s_i$$

with $\alpha_{ii} = 0$. If $0 < s_i < \sum_{j=1}^{N} \alpha_{ij}$ for all i, then both $\mathbf{0}$ and $\mathbf{1}$ are strict Nash equilibria. We assume that $\alpha_{ij} \geq 0$ so that the game is supermodular. (The

[20] One can verify that $\mathbf{0}$ is not an MP-maximizer for any c, while $\mathbf{1}$ is an MP-maximizer (and hence, robust to incomplete information) if and only if $c < -2$. In the case where $c \geq -2$, nothing seems to be known about the robustness of equilibria.

special case $\alpha_{ij} = \alpha_{ji}$ leads to a potential game and has been considered in Hofbauer and Sorger (2002).)

We restate the characterization for global accessibility given in Proposition 5.1.2 in the following form.

Lemma 5.3.1. *In a binary supermodular game,* **1** *is globally accessible for any small* $\theta > 0$ *if and only if there exists* $(r_i)_{i\in I} \in \mathbb{R}^N$ *such that*

$$\int_0^\infty e^{-t} d_i(\Psi(r_1 - r_i + t), \dots, \Psi(r_N - r_i + t))\, dt > 0 \tag{5.10}$$

for all $i \in I$*, where* Ψ *is given by*

$$\Psi(z) = \begin{cases} 0 & \text{for } z \leq 0, \\ 1 - e^{-z} & \text{for } z > 0. \end{cases}$$

Proof. Given $(T_i)_{i\in I}$ in Proposition 5.1.2, set, for example, $r_i = -T_i$. ∎

There is a relation to the concept of *spatial dominance* due to Hofbauer (1999). He considers a spatial model with populations of agents each of which is distributed along the real line, where agents move randomly on it and interact locally across populations. This can be modeled mathematically by a system of reaction-diffusion equations for the spatial distributions of actions. The local interaction is assumed to be governed by the myopic best response dynamics introduced by Gilboa and Matsui (1991). Each Nash equilibrium corresponds to a spatially homogeneous stationary action distribution. A Nash equilibrium $\mathbf{p}^* \in [0,1]^N$ is called *spatially dominant* if its basin of attraction contains an open set in the compact-open topology. If initially the population is close to $\mathbf{p}^*$ on a sufficiently large (but finite) interval, then it will converge to $\mathbf{p}^*$ everywhere. This implies that every game has at most one spatially dominant equilibrium. Hence this model provides a way of selecting a unique equilibrium for many important games; e.g., in 2×2 coordination games the risk-dominant equilibrium is spatially dominant. However, many games have no spatially dominant equilibrium at all.

The connection with the perfect foresight dynamics follows from the following fact, which holds for general binary supermodular games.

Lemma 5.3.2 (Hofbauer (1999, Lemma 1))**.** *In a binary supermodular game,* **1** *is spatially dominant if there exists* $(r_i)_{i\in N} \in \mathbb{R}^N$ *such that*

$$d_i(\Phi(r_1 - r_i), \dots, \Phi(r_N - r_i)) > 0 \tag{5.11}$$

for all $i \in I$*, where* Φ *is given by*

$$\Phi(z) = \begin{cases} e^z/2, & \text{for } z \leq 0, \\ 1 - e^{-z}/2 & \text{for } z > 0. \end{cases}$$

We need the following.

Lemma 5.3.3. $\int_0^\infty e^{-t}\Psi(z+t)dt = \Phi(z)$.

Proof. If $z \le 0$,

$$\int_0^\infty e^{-t}\Psi(z+t)dt = \int_{-z}^\infty e^{-t}\{1-e^{-(z+t)}\}dt = e^z/2,$$

and if $z > 0$,

$$\int_0^\infty e^{-t}\Psi(z+t)dt = \int_0^\infty e^{-t}\{1-e^{-(z+t)}\}dt = 1-e^{-z}/2,$$

as claimed. ▮

Lemma 5.3.4. *If d_i is linear, then the two conditions* (5.10) *and* (5.11) *are equivalent.*

Proof. By Lemma 5.3.3 and the linearity of d_i,

$$\begin{aligned}
&\int_0^\infty e^{-t}d_i(\Psi(r_1-r_i+t),\ldots,\Psi(r_N-r_i+t))dt \\
&= d_i\left(\int_0^\infty e^{-t}\Psi(r_1-r_i+t)dt,\ldots,\int_0^\infty e^{-t}\Psi(r_N-r_i+t)dt\right) \\
&= d_i(\Phi(r_1-r_i),\ldots,\Phi(r_N-r_i)),
\end{aligned}$$

which implies the claim. ▮

Combining Lemmas 5.3.1, 5.3.2, and 5.3.4 establishes the following implication.

Proposition 5.3.5. *In a binary supermodular game with linear incentives, if the strict Nash equilibrium* **1** *(or* **0***) is globally accessible for any small degree of friction, then it is spatially dominant.*

Since a game has at most one spatially dominant equilibrium, this proposition implies in particular that in binary supermodular games with linear incentives, **0** and **1** cannot be simultaneously globally accessible (in contrast to the example of unanimity games in Subsection 5.2).[21]

The linearity of the incentive functions d_i is crucial in the proof of Lemma 5.3.4. Indeed, the agreement between the selected equilibrium by spatial dominance and the one by the perfect foresight dynamics fails for nonlinear incentives. One class of examples are unanimity games in Subsection 5.2, for which the equilibrium with the higher Nash product is spatially dominant (see Hofbauer (1999)). Another example will be given in Subsection 5.4.

[21] Takahashi (2004) analyzes supermodular games with linear incentives (not necessarily binary) and shows that, for generic payoffs, there always exists a unique equilibrium that is linearly absorbing and globally accessible under the dynamics with $\theta = 0$, and it is also spatially dominant.

5.4 Binary Games with Invariant Diagonal

This subsection considers N-player binary supermodular games with invariant diagonal. A binary game is said to have an *invariant diagonal* if the incentive functions satisfy

$$d_1(p,\dots,p) = \cdots = d_N(p,\dots,p)$$

for all $p \in [0,1]$. This class of games includes games with "equistable biforms" introduced in Selten (1995). We assume that d_i is nondecreasing in each p_j $(j \neq i)$ so that the game is supermodular.

Denote by $D(p)$ the restriction of any d_i to the diagonal $p = p_1 = \cdots = p_N$. Observe that $D(p)$ is nondecreasing in p. This game has a potential function along the diagonal, which is defined by

$$v(p) = \int_0^p D(q)\,dq. \tag{5.12}$$

Proposition 5.4.1. *Suppose that the stage game is a binary supermodular game with an invariant diagonal. Let v be the potential function along the diagonal given by* (5.12)*. If $v(1) > v(0)$, then*

(a) *there exists $\bar{\theta} > 0$ such that $\mathbf{1}$ is globally accessible for all $\theta \in (0,\bar{\theta})$;*

(b) *$\mathbf{1}$ is absorbing for all $\theta > 0$.*

Proof. (a) Along the linear path ϕ from $\mathbf{0}$ to $\mathbf{1}$, which is given by $\phi_{i1}(t) = 1 - e^{-t}$ for all $i \in I$,

$$\begin{aligned}\Delta V_i^0(\phi)(0) &= \int_0^\infty e^{-s} D(1-e^{-s})\,ds \\ &= \int_0^1 D(p)\,dp = v(1).\end{aligned}$$

Hence, if $v(1) > v(0) = 0$, then $\Delta V_i^0(\phi)(0) > 0$, implying that $\mathbf{1}$ is globally accessible for any small $\theta > 0$ by Proposition 5.1.2.

(b) If $v(1) > v(0) = 0$, then there exists $p < 1$ such that $v(p) > 0$. Take such a p and any perfect foresight path ϕ with $\phi_{i1}(0) \geq p$ for all $i \in I$. Note that $\phi_{i1}(t) \geq pe^{-t}$. Then,

$$\begin{aligned}\Delta V_i^\theta(\phi)(0) &= (1+\theta)\int_0^\infty e^{-(1+\theta)s} d_i((\phi_{i1}(s))_{i\in I})\,ds \\ &\geq (1+\theta)\int_0^\infty e^{-(1+\theta)s} D(pe^{-s})\,ds \\ &\geq \int_0^\infty e^{-s} D(pe^{-s})\,ds \\ &= \frac{1}{p}\int_0^p D(q)\,dq = \frac{v(p)}{p} > 0,\end{aligned}$$

where the first inequality follows from the monotonicity of d_i, and the second inequality follows from the stochastic dominance relation between the distributions on $[0, \infty)$ with the density functions $(1+\theta)e^{-(1+\theta)s}$ and e^{-s}. Hence, we have $\phi_{i1}(t) = 1 - (1 - \phi_{i1}(0))e^{-t}$ for all $t \geq 0$, and therefore, ϕ converges to $\mathbf{1}$, implying that $\mathbf{1}$ is absorbing (independently of $\theta > 0$). ▮

Similarly, if $v(0) > v(1)$, then $\mathbf{0}$ is globally accessible for any small $\theta > 0$ and absorbing for any $\theta > 0$. Therefore, for generic binary supermodular games with invariant diagonal, either $\mathbf{0}$ or $\mathbf{1}$ is a unique absorbing and globally accessible state for any small degree of friction (even though there may be other strict equilibria).

Remark 5.4.1. A state $x^* \in \prod_i \Delta(A_i)$ is *linearly stable* if for any $x \in \prod_i \Delta(A_i)$, the linear path from x to x^* is a perfect foresight path. One can verify that for binary supermodular games with invariant diagonal, if $v(1) > v(0)$, then $\mathbf{1}$ is linearly stable for any small degree of friction $\theta > 0$.

Remark 5.4.2. The above result extends to the class of games with "monotone diagonal". Let $D_i(p) = d_i(p, \ldots, p)$ and $v_i(p) = \int_0^p D_i(q)\,dq$. It can be shown precisely in the same way as in Proposition 5.4.1 that if $v_i(1) > v_i(0)$ for all $i \in I$, then $\mathbf{1}$ is globally accessible for any small $\theta > 0$ and absorbing for any $\theta > 0$.

Example 5.4.1. Consider the following three player game (see Figure 4). If all three players match their actions, then their payoffs are given by $u_i(\mathbf{0}) = a > 0$ and $u_i(\mathbf{1}) = d > 0$. For other action profiles, if i matches $i+1$ with action 0, then i's payoff is $b > 0$; if i matches $i+1$ with action 1, then i's payoff is $c > 0$; otherwise, all players receive payoff 0. Suppose here that $a > b$ and $d > c$. Note that this game is supermodular and has an invariant diagonal.[22] Proposition 5.4.1 implies that if $2a + b > c + 2d$, then $\mathbf{0}$ is absorbing and globally accessible for a small friction, while if $2a+b < c+2d$, then $\mathbf{1}$ is absorbing and globally accessible for a small friction.

The selection criterion based on MP-maximization, on the other hand, yields a limited prediction: One can verify that $\mathbf{0}$ is an MP-maximizer if and only if $a > c + d$, while $\mathbf{1}$ is an MP-maximizer if and only if $a + b < d$. For this game, the notion of u-dominance introduced by Kojima (2003) gives the same condition: $\mathbf{0}$ is u-dominant if and only if $a > c + d$, while $\mathbf{1}$ is u-dominant if and only if $a + b < d$.[23]

Spatial dominance selects a different equilibrium for this game, namely, the equilibrium with the larger best response region on the diagonal, i.e., $\mathbf{0}$ is spatially dominant if and only if $a + b > c + d$, while $\mathbf{1}$ is spatially dominant if and only if $a + b < c + d$.

[22]This game is not a (weighted) potential game, since it has a better reply cycle.
[23]In general, MP-maximization and u-dominance give different conditions.

	0	1
0	a, a, a	0, 0, b
1	0, b, 0	c, 0, 0

0

	0	1
0	b, 0, 0	0, c, 0
1	0, 0, c	d, d, d

1

Figure 4: Game with invariant diagonal

6 Conclusion

In this paper, we have considered a dynamic adjustment process in a large society of forward-looking agents where decisions made by the agents are irreversible as in Matsuyama (1991), but where instantaneous payoffs are given by a normal form game as in Matsui and Matsuyama (1995). The stationary states of the dynamics coincide with the Nash equilibria of the stage game. Different stationary states may have different stability properties when the degree of friction is small, which allows us to discriminate among strict Nash equilibria of the stage game. If a Nash equilibrium a^* is linearly absorbing, then in its neighborhood, a perfect foresight path exists uniquely and converges linearly to a^*; namely, self-fulfilling expectations cannot upset a^*. If in addition, a^* is globally accessible, then from any other equilibrium, self-fulfilling expectations may lead the society to a^*. Our task was thus to derive conditions for a strict Nash equilibrium to be uniquely robust to the possibility of self-fulfilling prophecies for any small degree of friction.

We have focused on supermodular games and generalizations thereof, those games that have a monotone relation, in terms of best responses, with supermodular games, and elucidated the induced monotone structure of the dynamics. We have proved, in particular, the stability of monotone potential maximizer, which is shown to be robust to incomplete information by Morris and Ui (2005). We have also shown that for supermodular games, stability in the perfect foresight dynamics coincides with that under the less demanding assumption of rationalizable foresight. On the other hand, we have found that in certain unanimity games, no Nash equilibrium has the required stability property.

Appendix

A.1 Proof of Proposition 3.7

Suppose that the stage game is supermodular and that $a^* \in A$ is absorbing (recall from Proposition 3.6 that in supermodular games, any absorbing state is a pure-strategy state). We first show that a perfect foresight path from a^* is unique. Denote by $\bar{\phi}$ the constant path at a^*.

Lemma A.1.1. *Suppose that the stage game is supermodular. If a^* is absorbing, then $\bar{\phi}$ is the unique perfect foresight path from a^*.*

Proof. Suppose that a^* is absorbing. Let ϕ^- and ϕ^+ be the smallest and the largest perfect foresight paths from a^*, respectively (these exist, as demonstrated in Lemma 3.11, due to the supermodularity of the stage game). We show that ϕ^- is nonincreasing in the sense that $\phi^-(s) \precsim \phi^-(t)$ if $t \leq s$; a dual argument shows that ϕ^+ is nondecreasing. Then, ϕ^- and ϕ^+ must be constant at a^*; otherwise, ϕ^- or ϕ^+ would not converge to a^*, contradicting the absorption of a^*.

For each $i \in I$, denote by $\underline{a}_i$ the smallest action among those h's such that $\min BR^i(\psi)(t) = h$ for some $t \geq 0$. Note that $\underline{a}_i \leq a_i^*$, since $\phi^- \precsim \bar{\phi}$ and hence $\min BR^i(\phi^-)(t) \leq \min BR^i(\bar{\phi})(t) \leq a_i^*$ for all $t \geq 0$ by Lemma 3.1. Then, define for each $i \in I$ a sequence $T_i^{\underline{a}_i}, \dots, T_i^{a_i^*}$ by

$$T_i^h = \inf\{t \geq 0 \mid \min BR^i(\phi^-)(t) \leq h\}$$

for $h = \underline{a}_i, \dots, a_i^*$. Note that $0 = T_i^{a_i^*} \leq T_i^{a_i^*-1} \leq \cdots \leq T_i^{\underline{a}_i+1} \leq T_i^{\underline{a}_i} < \infty$. Now define $\alpha\colon \mathbb{R}_+ \to \prod_i \Delta(A_i)$ by

$$\alpha_{ih}(t) = 1 \quad \text{if } t \in [T_i^h, T_i^{h-1}),$$

where $T_i^{\underline{a}_i-1} = \infty$, and let ϕ be the feasible path given by

$$\dot{\phi}_i(t) = \alpha_i(t) - \phi_i(t) \ \text{ a.e.}, \qquad \phi_i(0) = a_i^*$$

for all $i \in I$. Observe that ϕ is nonincreasing and that $\phi \precsim \phi^-$.

Let us show that ϕ is a superpath. Take any $i \in I$, $h \in A_i$, and $t \geq 0$ such that $\dot{\phi}_{ih}(t) > -\phi_{ih}(t)$. By the definition of ϕ, $t \in [T_i^h, T_i^{h-1})$. Then,

$$h \geq \min BR^i(\phi^-)(T_i^h) \geq \min BR^i(\phi)(T_i^h) \geq \min BR^i(\phi)(t),$$

where the first inequality follows from the definition of T_i^h, the second from the fact that $\phi \precsim \phi^-$, and the third from the fact that ϕ is nonincreasing. This means that ϕ is a superpath. It therefore follows from Lemma 3.5 that there exists a perfect foresight path ψ^* from a^* such that $\psi^* \precsim \phi$.

On the other hand, ϕ^- is the smallest perfect foresight path from a^*. Therefore, we must have $\phi^- \precsim \psi^*$, so that $\psi^* = \phi = \phi^-$. This concludes that ϕ^- is nonincreasing. ∎

We now show that a^* is linearly absorbing. Note that

$$BR^i(\bar{\phi})(t) = \{a_i^*\} \text{ for all } i \in I \text{ and all } t \geq 0, \tag{A.1}$$

since a^* is a strict Nash equilibrium by Proposition 3.6.

Proof of Proposition 3.7. Suppose that a^* is absorbing. For $\varepsilon \in [0,1]$, let $x_\varepsilon^- = \varepsilon \min A + (1-\varepsilon)a^*$ and $x_\varepsilon^+ = \varepsilon \max A + (1-\varepsilon)a^*$. In order to show the linear absorption of a^*, it is sufficient to prove that there exists $\bar{\varepsilon} > 0$ such that the smallest perfect foresight path from $x_{\bar{\varepsilon}}^-$, ϕ^-, and the largest perfect foresight path from $x_{\bar{\varepsilon}}^+$, ϕ^+, satisfy $BR^i(\phi^-)(t) = BR^i(\phi^+)(t) = \{a_i^*\}$ for all $i \in I$ and $t \geq 0$. Then, for any perfect foresight path ϕ from $B_\varepsilon(a^*)$, which satisfies $\phi^- \precsim \phi \precsim \phi^+$ by Lemma 3.5, we have $BR^i(\phi)(t) = \{a_i^*\}$ for all $i \in I$ and $t \geq 0$, so that ϕ converges linearly to a^*.

Take any sequence $\{\varepsilon^k\}_{k=0}^\infty$ such that $\varepsilon^0 > \varepsilon^1 > \cdots > 0$ and $\lim_{k\to\infty} \varepsilon^k = 0$, and let $\phi^{k,-}$ and $\phi^{k,+}$ be the smallest perfect foresight path from $x_{\varepsilon^k}^-$ and the largest perfect foresight path from $x_{\varepsilon^k}^+$, respectively. Here, we assume that ε^0 is small enough so that both $\phi^{0,-}$ and $\phi^{0,+}$ converge to a^*. We only show that for some k, $\min BR^i(\phi^{k,-})(t) \geq a_i^*$ for all $i \in I$ and all $t \geq 0$; a dual argument shows that for some k', $\max BR^i(\phi^{k',+})(t) \leq a_i^*$ for all $i \in I$ and all $t \geq 0$. Then, setting $\bar{\varepsilon} = \min\{\varepsilon^k, \varepsilon^{k'}\}$ completes the proof. Note that $\phi^{0,-} \precsim \phi^{1,-} \precsim \cdots \precsim \bar{\phi}$ and that $\{\phi^{k,-}\}_{k=0}^\infty$ converges, as $k \to \infty$, to some perfect foresight path from a^*, which must be $\bar{\phi}$ by Lemma A.1.1.

Seeking a contradiction, suppose that for each k, there exists T^k such that $\min BR^{\tilde{\imath}}(\phi^{k,-})(T^k) < a_i^*$ for some $\tilde{\imath} \in I$, where $\tilde{\imath}$ can be taken independently of k due to the finiteness of I. Since a^* is absorbing (and a strict Nash equilibrium), there exists $\bar{T}$ such that $\min BR^{\tilde{\imath}}(\phi^{0,-})(t) = a_{\tilde{\imath}}^*$ for all $t \geq \bar{T}$. Since $\phi^{0,-} \precsim \phi^{k,-}$ $(\precsim \bar{\phi})$, it follows that for all k, $\min BR^{\tilde{\imath}}(\phi^{k,-})(t) = a_{\tilde{\imath}}^*$ for all $t \geq \bar{T}$. Therefore, it must be true that $T^k < \bar{T}$ for all k, so that there exists a convergent subsequence of $\{T^k\}_{k=0}^\infty$ with some limit T^*. By the lower semi-continuity of $\min BR^{\tilde{\imath}}$, we have $\min BR^{\tilde{\imath}}(\bar{\phi})(T^*) < a_{\tilde{\imath}}^*$, which contradicts (A.1). ▮

A.2 Proof of Theorem 4.1

Suppose that a^* is an MP-maximizer with a monotone potential function v. Let $A_i' \subset A_i$ denote a set of actions for player i that contains a_i^*. This set will be taken as $A_i^- = \{h \in A_i \,|\, h \leq a_i^*\}$ or $A_i^+ = \{h \in A_i \,|\, h \geq a_i^*\}$. For the potential game $G_v' = (I, (A_i')_{i\in I}, (v)_{i\in I})$ with the unique potential maximizer $a^* \in A'$, consider the following optimal control problem with a given initial state $z \in \prod_i \Delta(A_i')$:

$$\text{maximize} \quad J(\phi) = \int_0^\infty e^{-\theta t} v(\phi(t))\, dt \tag{A.2a}$$

$$\text{subject to} \quad \phi \in \Phi_z', \tag{A.2b}$$

where Φ'_z is the set of feasible paths defined on $\prod_i \Delta(A'_i)$ with the initial state z. The state z will be taken as $\min A = (0, \dots, 0)$ or $\max A = (n_1, \dots, n_N)$.

Lemma A.2.1. *There exists $\bar{\theta} > 0$ such that for any $\theta \in (0, \bar{\theta})$ and any $z \in \prod_i \Delta(A'_i)$, any optimal solution to the optimal control problem* (A.2) *converges to a^*.*

Proof. Apply Lemma 1 in Hofbauer and Sorger (1999) and Lemmas 4.2 and 4.3 in Hofbauer and Sorger (2002) to the restricted potential game G'_v. ▮

Lemma A.2.2. *Let X be a nonempty compact set endowed with a preorder $\precsim$. Suppose that for all $x \in X$, the set $L_x = \{y \in X \mid y \precsim x\}$ is closed. Then X has a minimal element.*

Proof. Take any totally ordered subset of X, and denote it by X'. Since $\{L_x\}_{x \in X'}$ consists of nonempty closed subsets of a compact set and has the finite intersection property, $L^* = \bigcap_{x \in X'} L_x$ is nonempty. Any element $x^* \in L^*$ is a lower bound of X' in X. Therefore, it follows from Zorn's lemma that X has a minimal element. ▮

Lemma A.2.3. *For any $z \in \prod_i \Delta(A'_i)$, there exist optimal solutions to the optimal control problem* (A.2)*, ϕ^- and ϕ^+, such that*

$$\dot{\phi}_i^-(t) = \min BR_v^i(\phi^- | A'_i)(t) - \phi_i^-(t),$$
$$\dot{\phi}_i^+(t) = \max BR_v^i(\phi^+ | A'_i)(t) - \phi_i^+(t)$$

for all $i \in I$ and almost all $t \geq 0$.

Proof. Fix $z \in \prod_i \Delta(A'_i)$. We only show the existence of ϕ^-; the existence of ϕ^+ is shown similarly. Since the functional J is continuous on Φ'_z, the set of optimizers is a nonempty, closed, and hence compact subset of Φ'_z. Hence a minimal optimal solution (with respect to the order $\phi \precsim \psi$, defined by $\phi(t) \precsim \psi(t)$ for all $t \geq 0$) exists by Lemma A.2.2. Let ϕ^- be such a minimal solution.

Take any $i \in I$ and consider the feasible path ϕ_i given by $\phi_i(0) = z_i$ and

$$\dot{\phi}_i(t) = \min BR_v^i(\phi^- | A'_i)(t) - \phi_i(t)$$

for almost all $t \geq 0$. Since by Lemma 2.2, for almost all $t \geq 0$ there exists $\alpha_i(t)$ in the convex hull of $BR_v^i(\phi^- | A'_i)(t)$ such that

$$\dot{\phi}_i^-(t) = \alpha_i(t) - \phi_i^-(t),$$

we have $\phi_i \precsim \phi_i^-$. On the other hand, since ϕ_i is a best response to ϕ_{-i}^- for the game G'_v by construction, we have

$$J(\phi_i, \phi_{-i}^-) \geq J(\phi^-) = \max_{\psi \in \Phi'_z} J(\psi)$$

by Lemma 2.2, meaning that the path (ϕ_i, ϕ^-_{-i}) is also optimal. Hence, the minimality of ϕ^- implies $\phi^-_i(t) = \phi_i(t)$ for all $t \geq 0$. Therefore, we have

$$\dot{\phi}^-_i(t) = \min BR^i_v(\phi^-|A'_i)(t) - \phi^-_i(t)$$

for almost all $t \geq 0$, as claimed. ∎

Lemma A.2.4. *There exists $\bar{\theta} > 0$ such that the following holds for all $\theta \in (0, \bar{\theta})$: there exists a feasible path ϕ^- such that*

$$\dot{\phi}^-_i(t) = \min BR^i_v(\phi^-|A^-_i)(t) - \phi^-_i(t) \quad a.e., \qquad \phi^-_i(0) = \min A_i$$

for all $i \in I$ and $\lim_{t\to\infty} \phi^-(t) = a^*$; there exists a feasible path ϕ^+ such that

$$\dot{\phi}^+_i(t) = \max BR^i_v(\phi^+|A^+_i)(t) - \phi^+_i(t) \quad a.e., \qquad \phi^+_i(0) = \max A_i$$

for all $i \in I$ and $\lim_{t\to\infty} \phi^+(t) = a^*$.

Proof. Follows from Lemmas A.2.1 and A.2.3. ∎

Proof of Theorem 4.1. Suppose that v is a monotone potential function for a^*. Take ϕ^- and ϕ^+ as in Lemma A.2.4. In what follows, we fix a sufficiently small $\theta > 0$ so that both ϕ^- and ϕ^+ converge to a^*.

Now fix any $x \in \prod_i \Delta(A_i)$. Note that $\phi^- \precsim \phi^+$ and $\phi^-(0) \precsim x \precsim \phi^+(0)$. Consider the best response correspondence β_G for the stage game G. Let $\tilde{\Phi}_x = \{\phi \in \Phi_x \,|\, \phi^- \precsim \phi \precsim \phi^+\}$. We will show, as in the proof of Lemma 3.3, that $\tilde{\beta}_G(\phi) = \beta_G(\phi) \cap \tilde{\Phi}_x$ is nonempty for any $\phi \in \tilde{\Phi}_x$. Then, since $\tilde{\Phi}_x$ is convex and compact, it follows from Kakutani's fixed point theorem that there exists a fixed point $\phi^* \in \tilde{\beta}_G(\phi^*) \subset \tilde{\Phi}_x$, which is a perfect foresight path in G and satisfies $\phi^- \precsim \phi^* \precsim \phi^+$. Since both ϕ^- and ϕ^+ converge to a^*, ϕ^* also converges to a^*.

Take any $\phi \in \tilde{\Phi}_x$. Suppose first that the original game G is supermodular. Then, we have

$$\min BR^i_v(\phi^-|A^-_i)(t) \leq \max BR^i_{u_i}(\phi^-|A^-_i)(t) \leq \max BR^i_{u_i}(\phi|A^-_i)(t),$$

where the first inequality follows from the assumption that v is a monotone potential, and the second inequality follows from the supermodularity of u_i and Lemma 3.1. Similarly, we have

$$\max BR^i_v(\phi^+|A^+_i)(t) \geq \min BR^i_{u_i}(\phi^+|A^+_i)(t) \geq \min BR^i_{u_i}(\phi|A^+_i)(t).$$

Suppose next that the potential game G_v is supermodular. Then, we have

$$\min BR^i_v(\phi^-|A^-_i)(t) \leq \min BR^i_v(\phi|A^-_i)(t) \leq \max BR^i_{u_i}(\phi|A^-_i)(t),$$

where the first inequality follows from the supermodularity of v and Lemma 3.1, and the second inequality follows from the assumption that v is a monotone potential. Similarly, we have

$$\max BR^i_v(\phi^+|A^+_i) \geq \max BR^i_v(\phi|A^+_i)(t) \geq \min BR^i_{u_i}(\phi|A^+_i)(t).$$

Therefore, in each case, we have

$$\max BR^i_{u_i}(\phi|A^-_i)(t) \geq \min BR^i_v(\phi^-|A^-_i)(t),$$
$$\min BR^i_{u_i}(\phi|A^+_i)(t) \leq \max BR^i_v(\phi^+|A^+_i)(t)$$

for all $i \in I$ and all $t \geq 0$, so that there exists $h \in BR^i_{u_i}(\phi)(t)$ such that

$$\min BR^i_v(\phi^-|A^-_i)(t) \leq h \leq \max BR^i_v(\phi^+|A^+_i)(t).$$

Define

$$\tilde{F}_i(\phi)(t) = F_i(\phi)(t) \cap \left[\min BR^i_v(\phi^-|A^-_i)(t), \max BR^i_v(\phi^+|A^+_i)(t)\right],$$

where

$$F_i(\phi)(t) = \{\alpha_i \in \Delta(A_i) \,|\, \alpha_{ih} > 0 \Rightarrow h \in BR^i_{u_i}(\phi)(t)\},$$

and $[\alpha_i, \alpha'_i] = \{\alpha''_i \in \Delta(A_i) | \alpha_i \precsim \alpha''_i \precsim \alpha'_i\}$ denotes the order interval. Then the differential inclusion

$$\dot{\psi}(t) \in \tilde{F}(\phi)(t) - \psi(t), \qquad \psi(0) = x$$

has a solution ψ as in Remark 2.1. Since $\tilde{F}_i(\phi)(t) \subset F_i(\phi)(t)$, we have $\psi \in \beta_G(\phi)$. By the construction of ϕ^-, ϕ^+, and ψ, we have $\phi^- \precsim \psi \precsim \phi^+$. Thus, we have $\psi \in \tilde{\beta}_G(\phi) = \beta_G(\phi) \cap \tilde{\Phi}_x$, implying the nonemptiness of $\tilde{\beta}_G(\phi)$. ▮

A.3 Proof of Theorem 4.2

Suppose that a^* is a strict MP-maximizer with a strict monotone potential function v. For a nonempty set of actions $A'_i \subset A_i$ that contains a^*_i, consider the potential game $G'_v = (I, (A'_i)_{i\in I}, (v)_{i\in I})$.

Lemma A.3.1 (Hofbauer and Sorger (2002))**.** *Suppose that G'_v is a potential game with a unique potential maximizer $a^* \in A'$. Then, a^* is absorbing for all $\theta > 0$.*

Proof of Theorem 4.2. Suppose that v is a strict monotone potential function with the strict MP-maximizer a^*, and let $A^-_i = \{h \in A_i \,|\, h \leq a^*_i\}$ and $A^+_i = \{h \in A_i \,|\, h \geq a^*_i\}$. By Lemma A.3.1, a^* is absorbing in each of the restricted potential games $G^-_v = (I, (A^-_i)_{i\in I}, (v)_{i\in I})$ and $G^+_v = (I, (A^+_i)_{i\in I}, (v)_{i\in I})$. Let

$$x^-_\varepsilon = \varepsilon \min A + (1-\varepsilon)a^*,$$
$$x^+_\varepsilon = \varepsilon \max A + (1-\varepsilon)a^*$$

for $\varepsilon \in [0, 1]$.

Choose a small $\varepsilon > 0$ so that any perfect foresight path for G_v^- from x_ε^- and for G_v^+ from x_ε^+ converges to a^*. Fix any state $x \in \prod_i \Delta(A_i)$ close to a^* satisfying

$$x_\varepsilon^- \precsim x \precsim x_\varepsilon^+,$$

and let ϕ^* be any perfect foresight path from x in the original game G.

In the following, we find perfect foresight paths ϕ^- and ϕ^+ for G_v^- and G_v^+, respectively, such that $\phi^-(0) = x_\varepsilon^-$, $\phi^+(0) = x_\varepsilon^+$, and $\phi^- \precsim \phi^* \precsim \phi^+$. Then, since a^* is absorbing both in G_v^- and in G_v^+, ϕ^- and ϕ^+ converge to a^*, and thus, ϕ^* also converges to a^*. In the case where G is supermodular, this implies that a^* is linearly absorbing in G by Proposition 3.7. In the case where G_v is supermodular, a^* is linearly absorbing in G_v^- and in G_v^+, by Proposition 3.7, so that ϕ^- and ϕ^+ linearly converge to a^*, and therefore, ϕ^* also converges linearly to a^*, implying the linear absorption of a^* in G. We only show the existence of ϕ^-; the existence of ϕ^+ is proved similarly.

Let $\tilde{\Phi}_{x_\varepsilon^-} = \{\phi \in \Phi_{x_\varepsilon^-} \mid \phi \precsim \phi^* \text{ and } \phi(t) \in \prod_i \Delta(A_i^-) \text{ for all } t \geq 0\}$. Consider the best response correspondence $\beta_{G_v^-}$ for the stage game G_v^-. We will show that $\tilde{\beta}_{G_v^-}(\phi) = \beta_{G_v^-}(\phi) \cap \tilde{\Phi}_{x_\varepsilon^-}$ is nonempty for any $\phi \in \tilde{\Phi}_{x_\varepsilon^-}$. Then, since $\tilde{\Phi}_{x_\varepsilon^-}$ is convex and compact, it follows from Kakutani's fixed point theorem that there exists a fixed point $\phi^- \in \tilde{\beta}_{G_v^-}(\phi^-) \subset \tilde{\Phi}_{x_\varepsilon^-}$, as desired.

Take any $\phi \in \tilde{\Phi}_{x_\varepsilon^-}$. If G is supermodular, then

$$\min BR_v^i(\phi|A_i^-)(t) \leq \min BR_{u_i}^i(\phi|A_i^-)(t) \leq \min BR_{u_i}^i(\phi^*|A_i^-)(t),$$

where the first inequality follows from the assumption that v is a strict monotone potential, and the second inequality follows from the supermodularity of u_i and Lemma 3.1.

If G_v is supermodular, then

$$\min BR_v^i(\phi|A_i^-)(t) \leq \min BR_v^i(\phi^*|A_i^-)(t) \leq \min BR_{u_i}^i(\phi^*|A_i^-)(t),$$

where the first inequality follows from the supermodularity of v and Lemma 3.1, and the second inequality follows from the assumption that v is a strict monotone potential.

Therefore, in each case, we have

$$\min BR_v^i(\phi|A_i^-)(t) \leq \min BR_{u_i}^i(\phi^*|A_i^-)(t),$$

so that there exists $h \in BR_v^i(\phi|A_i^-)(t)$ such that

$$h \leq \min BR_{u_i}^i(\phi^*|A_i^-)(t).$$

Then, there exists a best response ψ to ϕ in the game G_v^- such that $\psi(0) = x_\varepsilon^-$ and $\psi \precsim \phi^*$, which can be constructed as in the proof of Proposition 3.2. ▮

A.4 Proofs for Subsection 4.3

Proof of Lemma 4.3. Let v be given as in the lemma. We only show the conditions (4.1) and (4.3) for A_i^-; (4.2) and (4.4) are proved similarly. Fix any $i \in I$ and $\pi_i \in \Delta(A_{-i})$. If $a_i^* = \min A_i$, then (4.1) and (4.3) are satisfied. Then consider the case of $a_i^* > \min A_i$. Observe that $v(h, \pi_i) = \sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) v(h, a_{-i})$ is constant for all $h < a_i^*$, so that $\min br_v^i(\pi_i | A_i^-)$ is either $\min A_i$ or a_i^*. It is sufficient to consider the case where $a_i^* = \min br_v^i(\pi_i | A_i^-)$.

Since

$$
\begin{aligned}
v(a_i^*, \pi_i) - v(\min A_i, \pi_i) &= \pi_i(a_{-i}^*) \cdot (1 - p_i) + \sum_{a_{-i} \neq a_{-i}^*} \pi_i(a_{-i}) \cdot (-p_i) \\
&= \pi_i(a_{-i}^*) - p_i,
\end{aligned}
$$

it follows from $a_i^* = \min br_v^i(\pi_i | A_i^-)$ that $\pi_i(a_{-i}^*) > p_i$.

Therefore, if a^* is a **p**-dominant equilibrium, then $a_i^* \in br_{u_i}^i(\pi_i | A_i^-)$, i.e., $a_i^* = \max br_{u_i}^i(\pi_i | A_i^-)$; if a^* is a strict **p**-dominant equilibrium, then $\{a_i^*\} = br_{u_i}^i(\pi_i | A_i^-)$, i.e., $a_i^* = \min br_{u_i}^i(\pi_i | A_i^-)$. ∎

Proof of Lemma 4.5. (a) Suppose that a^* is an LP-maximizer with a local potential function v. We show that if G or G_v has diminishing marginal returns, then a^* is an MP-maximizer with this function v. Fix any $i \in I$ and $\pi_i \in \Delta(A_{-i})$. We show that $\max br_v^i(\pi_i | A_i^-) \leq \max br_{u_i}^i(\pi_i | A_i^-)$. Let $\overline{a}_i = \max br_v^i(\pi_i | A_i^-)$. It is sufficient to consider the case where $\overline{a}_i > \min A_i$.

Since a^* is an LP-maximizer, for all $h < \overline{a}_i$ there exists $\mu_i(h) \geq 0$ such that

$$
\mu_i(h)\big(v(h+1, a_{-i}) - v(h, a_{-i})\big) \leq u_i(h+1, a_{-i}) - u_i(h, a_{-i})
$$

for all $a_{-i} \in A_{-i}$, so that we have

$$
\mu_i(h)\big(v(h+1, \pi_i) - v(h, \pi_i)\big) \leq u_i(h+1, \pi_i) - u_i(h, \pi_i)
$$

for all $h < \overline{a}_i$. On the other hand, we have

$$
v(\overline{a}_i, \pi_i) - v(\overline{a}_i - 1, \pi_i) \geq 0
$$

by the definition of $\overline{a}_i$.

Suppose first that G has diminishing marginal returns. Then, we have

$$
\begin{aligned}
u_i(h+1, \pi_i) - u_i(h, \pi_i) &\geq u_i(\overline{a}_i, \pi_i) - u_i(\overline{a}_i - 1, \pi_i) \\
&\geq \mu_i(\overline{a}_i - 1)\big(v(\overline{a}_i, \pi_i) - v(\overline{a}_i - 1, \pi_i)\big) \\
&\geq 0
\end{aligned}
$$

for any $h < \overline{a}_i$. Hence, we have

$$u_i(\overline{a}_i, \pi_i) - u_i(h, \pi_i) \geq 0$$

for all $h < \overline{a}_i$, which implies that $\overline{a}_i \leq \max br^i_{u_i}(\pi_i | A_i^-)$.

Suppose next that G_v has diminishing marginal returns. Then, we have

$$\begin{aligned} u_i(h+1, \pi_i) - u_i(h, \pi_i) &\geq \mu_i(h)\big(v(h+1, \pi_i) - v(h, \pi_i)\big) \\ &\geq \mu_i(h)\big(v(\overline{a}_i, \pi_i) - v(\overline{a}_i - 1, \pi_i)\big) \\ &\geq 0 \end{aligned}$$

for any $h < \overline{a}_i$. Hence, we have

$$u_i(\overline{a}_i, \pi_i) - u_i(h, \pi_i) \geq 0$$

for all $h < \overline{a}_i$, which implies that $\overline{a}_i \leq \max br^i_{u_i}(\pi_i | A_i^-)$.

(b) Suppose that a^* is a strict LP-maximizer with a strict local potential function v. We show that if G or G_v has diminishing marginal returns, then a^* is a strict MP-maximizer with the same function v. Fix any $i \in I$ and $\pi_i \in \Delta(A_{-i})$. We show that $\min br^i_v(\pi_i | A_i^-) \leq \min br_{u_i}(\pi_i | A_i^-)$. Let $\underline{a}_i = \min br^i_v(\pi_i | A_i^-)$. It is sufficient to consider the case where $\underline{a}_i > \min A_i$.

Since a^* is a strict LP-maximizer, for all $h < \underline{a}_i$ there exists $\mu_i(h) > 0$ such that

$$\mu_i(h)\big(v(h+1, a_{-i}) - v(h, a_{-i})\big) \leq u_i(h+1, a_{-i}) - u_i(h, a_{-i})$$

for all $a_{-i} \in A_{-i}$, so that we have

$$\mu_i(h)\big(v(h+1, \pi_i) - v(h, \pi_i)\big) \leq u_i(h+1, \pi_i) - u_i(h, \pi_i)$$

for all $h < \underline{a}_i$. On the other hand, we have

$$v(\underline{a}_i, \pi_i) - v(\underline{a}_i - 1, \pi_i) > 0$$

by the definition of $\underline{a}_i$.

Suppose first that G has diminishing marginal returns. Then, we have

$$\begin{aligned} u_i(h+1, \pi_i) - u_i(h, \pi_i) &\geq u_i(\underline{a}_i, \pi_i) - u_i(\underline{a}_i - 1, \pi_i) \\ &\geq \mu_i(\underline{a}_i - 1)\big(v(\underline{a}_i, \pi_i) - v(\underline{a}_i - 1, \pi_i)\big) \\ &> 0 \end{aligned}$$

for any $h < \underline{a}_i$. Hence, we have

$$u_i(\underline{a}_i, \pi_i) - u_i(h, \pi_i) > 0$$

for all $h < \underline{a}_i$, which implies that $\underline{a}_i \leq \min br^i_{u_i}(\pi_i | A_i^-)$.

Suppose next that G_v has diminishing marginal returns. Then, we have

$$\begin{aligned} u_i(h+1,\pi_i) - u_i(h,\pi_i) &\geq \mu_i(h)\bigl(v(h+1,\pi_i) - v(h,\pi_i)\bigr) \\ &\geq \mu_i(h)\bigl(v(\underline{a}_i,\pi_i) - v(\underline{a}_i - 1,\pi_i)\bigr) \\ &> 0 \end{aligned}$$

for any $h < \underline{a}_i$. Hence, we have

$$u_i(\underline{a}_i,\pi_i) - u_i(h,\pi_i) > 0$$

for all $h < \underline{a}_i$, which implies that $\underline{a}_i \leq \min br^i_{u_i}(\pi_i | A_i^-)$. ▮

Proof of Lemma 4.7. Case (1): $(0,0)$ is a strict $\mathbf{p}$-dominant equilibrium with $p_1 = p_2 < 1/2$, so that Lemma 4.3 applies.

Case (2): Symmetric with Case (1).

Case (3–a): $(1,1)$ is a strict $\mathbf{p}$-dominant equilibrium with $p_1 = p_2 < 1/2$.

Case (3–b): A monotone potential function v for $(0,0)$ is the following:

	0	1	2
0	$\varepsilon\Delta^{01}_{01}$	$\varepsilon(w_{01} - w_{11})$	$\varepsilon(w_{02} - w_{12}) + (w_{21} - w_{11})$
1	$\varepsilon(w_{01} - w_{11})$	0	$w_{21} - w_{11}$
2	$\varepsilon(w_{02} - w_{12}) + (w_{21} - w_{11})$	$w_{21} - w_{11}$	0

where $\varepsilon > 0$ is sufficiently small. All entries but $v(0,0)$ are less than or equal to zero. By verifying that

$$\begin{aligned} &v(0,k) - v(1,k) = \varepsilon\bigl(u_1(0,k) - u_1(1,k)\bigr), \\ &v(1,k) - v(2,k) \leq u_1(1,k) - u_1(2,k), \\ &v(0,k) - v(2,k) \leq u_1(1,k) - u_1(2,k) \end{aligned}$$

for all k (let ε be sufficiently small, and use $w_{20} - w_{10} < w_{21} - w_{11}$ and $\Delta^{12}_{12} > 0$), one can show that the conditions in Definition 4.2 (with $a^* = (0,0)$) are satisfied.

Case (3–c): Symmetric with Case (3–b).

Case (3–d–i): A monotone potential function v for $(0,0)$ is the following:

	0	1	2
0	ε	$\varepsilon+\lambda_1(w_{10}-w_{00})$	$\lambda_1(w_{02}-w_{12})$ $+\lambda_2(w_{12}-w_{22})$
1	$\varepsilon+\lambda_1(w_{10}-w_{00})$	$-\lambda_2\Delta^{21}_{21}$	$\lambda_2(w_{12}-w_{22})$
2	$\lambda_1(w_{02}-w_{12})$ $+\lambda_2(w_{12}-w_{22})$	$\lambda_2(w_{12}-w_{22})$	0

where $\varepsilon > 0$ is sufficiently small, and $\lambda_1 > 0$ and $\lambda_2 > 0$ are such that

$$\frac{\Delta^{21}_{21}}{\Delta^{01}_{01}} < \frac{\lambda_1}{\lambda_2} < \frac{\Delta^{02}_{12}}{\Delta^{02}_{10}}.$$

This is the local potential function given in Morris (1999). Verify that

$$\begin{aligned}
&v(0,k) - v(1,k) \le \lambda_1\bigl(u_1(0,k) - u_1(1,k)\bigr),\\
&v(1,k) - v(2,k) \le \lambda_2\bigl(u_1(1,k) - u_1(2,k)\bigr),\\
&v(0,k) - v(2,k) \le (\lambda_2+\lambda_3)\bigl(u_1(1,k) - u_1(2,k)\bigr)
\end{aligned}$$

for all k, where $\lambda_3 > 0$ is such that

$$\frac{w_{22}-w_{12}}{w_{12}-w_{02}} < \frac{\lambda_1}{\lambda_3} < \frac{w_{10}-w_{20}}{w_{12}-w_{02}}.$$

Case (3–d–ii): Symmetric with Case (3–d–i). ∎

A.5 Proofs for Subsection 5.1

We will need the following lemma.

Lemma A.5.1. *For all $i \in I$ and all $t \ge 0$,*

(a) *for any $\mathbf{T} \in \mathbb{R}^N_+$, $\Delta V^\theta_i(\phi^{\mathrm{u}}_{\mathbf{T}})(t)$ is decreasing in $\theta \ge 0$,*

(b) *for any $\mathbf{T} \in \bar{\mathbb{R}}^N_+$ with $S = \{i \in I \mid T_i \ne \infty\}$, $\Delta V^\theta_i(\psi^{\mathrm{d}}_{\mathbf{T}})(t)$ is nondecreasing in $\theta \ge 0$, and is increasing in $\theta \ge 0$ if $d_i(\mathbf{1}) > d_i(\mathbf{0}_S)$.*

This lemma is a consequence of the stochastic dominance relation among distributions on $[t,\infty)$ induced by discount rates: the distribution on $[t,\infty)$ with density function $(1+\theta)e^{-(1+\theta)(s-t)}$ strictly stochastically dominates the one with density function $(1+\theta')e^{-(1+\theta')(s-t)}$ for $0 \le \theta < \theta'$. The statements follow from the facts that $d_i((\phi^{\mathrm{u}}_{\mathbf{T}})_1(s))$ is nondecreasing in $s \ge 0$ and increasing in $s \ge \max_{j\in I} T_j$, and that $d_i((\psi^{\mathrm{d}}_{\mathbf{T}})_1(s))$ is nonincreasing in $s \ge 0$, and decreasing in $s \ge \max_{j\in S} T_j$ if $d_i(\mathbf{1}) > d_i(\mathbf{0}_S)$.

We first prove the global accessibility results.

Proof of Proposition 5.1.1. "If" part: Suppose that there exists $\mathbf{T} = (T_i)_{i \in I}$ such that for all i,

$$\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) \geq 0.$$

Since $\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(t)$ is increasing in t, $\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(t) \geq 0$ holds for all $i \in I$ and all $t \geq T_i$. This implies that $\phi^{\mathrm{u}}_{\mathbf{T}}$ satisfies

$$(\dot{\phi}^{\mathrm{u}}_{\mathbf{T}})_{i1}(t) > -(\phi^{\mathrm{u}}_{\mathbf{T}})_{i1}(t) \Rightarrow 1 = \max BR^i(\phi^{\mathrm{u}}_{\mathbf{T}})(t)$$

for almost all $t \geq 0$, so that $\phi^{\mathrm{u}}_{\mathbf{T}}$ is a subpath. It follows from Lemma 3.5 that for any $x \in \prod_i \Delta(A_i)$, there exists a perfect foresight path ϕ^* from x such that $\phi^{\mathrm{u}}_{\mathbf{T}} \precsim \phi^*$. Since $\phi^{\mathrm{u}}_{\mathbf{T}}$ converges to $\mathbf{1}$, ϕ^* also converges to $\mathbf{1}$. Therefore, $\mathbf{1}$ is globally accessible.

"Only if" part: Suppose that $\mathbf{1}$ is globally accessible, so that there exists a perfect foresight path ϕ such that $\phi(0) = \mathbf{0}$ and $\lim_{t \to \infty} \phi(t) = \mathbf{1}$. Take such a perfect foresight path ϕ and let

$$T_i = \inf\{t \geq 0 \,|\, \dot{\phi}_{i1}(t) > -\phi_{i1}(t)\}$$

for each $i \in I$. Note that $T_i < \infty$ for all $i \in I$.

For $\mathbf{T} = (T_i)_{i \in I}$ defined above, define $\phi^{\mathrm{u}}_{\mathbf{T}}$ as in (5.1). Since $\phi \precsim \phi^{\mathrm{u}}_{\mathbf{T}}$, we have

$$\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) \geq \Delta V_i^\theta(\phi)(T_i) \geq 0$$

due to the supermodularity. ∎

Proof of Proposition 5.1.2. "If" part: Take a $\mathbf{T} = (T_i)_{i \in I} \in \mathbb{R}^N_+$ such that

$$\Delta V_i^0(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) > 0$$

for all $i \in I$. Since $\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i)$ is continuous in θ, there exists $\bar{\theta} > 0$ such that for all $\theta \in (0, \bar{\theta})$,

$$\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) > 0$$

for all $i \in I$, implying that $\mathbf{1}$ is globally accessible for all $\theta \in (0, \bar{\theta})$ by Proposition 5.1.1.

"Only if" part: Suppose that $\mathbf{1}$ is globally accessible for a small $\theta > 0$. Then, by Proposition 5.1.1 there exists $\mathbf{T}$ such that

$$\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) \geq 0$$

for all $i \in I$. Since $\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i)$ is decreasing in θ by Lemma A.5.1, it follows that

$$\Delta V_i^0(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) > \Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) \geq 0$$

for all $i \in I$. ∎

Next we prove the absorption results. For Proposition 5.1.3, we show the following.

Lemma A.5.2. *Let $\theta > 0$ be given. The state $\mathbf{1}$ is absorbing for θ if and only if for any $\mathbf{T} = (T_i)_{i \in I} \in \bar{\mathbb{R}}_+^N$ such that $S = \{i \in I \mid T_i \neq \infty\}$ is nonempty, there exists $i \in S$ such that*

$$\mathit{\Delta} V_i^{\theta}(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i) > 0.$$

Proof. "If" part: Note first that by the uniform continuity of d_i, for each positive integer m, there exists $\varepsilon^m > 0$ such that for any $\mathbf{p} = (p_j)_{j \in I}$, $\mathbf{q} = (q_j)_{j \in I} \in [0,1]^N$ with $p_j \geq q_j - \varepsilon^m$ for all $j \in I$, we have

$$d_i(\mathbf{p}) \geq d_i(\mathbf{q}) - \frac{1}{m}$$

for all $i \in I$. Then, for any feasible paths ϕ and ψ such that $\phi_{j1}(t) \geq \psi_{j1}(t) - \varepsilon^m$ for all $j \in I$ and $t \geq 0$, we have

$$\mathit{\Delta} V_i^{\theta}(\phi)(t) \geq \mathit{\Delta} V_i^{\theta}(\psi)(t) - \frac{1}{m}$$

for all $i \in I$ and $t \geq 0$.

Suppose that $\mathbf{1}$ is not absorbing. Take any positive integer m and the corresponding ε^m given above. There exist $x \in \prod_i \Delta(A_i)$ with $x_{1i} > 1 - \varepsilon^m$ and a perfect foresight path ϕ^m with $\phi^m(0) = x$ that does not converge to $\mathbf{1}$. Take any such perfect foresight path ϕ^m for each m.

Define

$$T_i^m = \inf\{t \geq 0 \mid \dot{\phi}_{i1}^m(t) < 1 - \phi_{i1}^m(t)\},$$

and $S^m = \{i \in I \mid T_i^m \neq \infty\}$. Note that S^m is nonempty as ϕ^m does not converge to $\mathbf{1}$. Since ϕ^m is a perfect foresight path and $\mathit{\Delta} V_i^{\theta}(\phi^m)(t)$ is continuous in t, we must have

$$\mathit{\Delta} V_i^{\theta}(\phi^m)(T_i^m) \leq 0 \tag{A.3}$$

for $i \in S^m$.

Define $\tilde{\mathbf{T}}^m = (\tilde{T}_i^m)_{i \in I}$ by $\tilde{T}_i^m = T_i^m - \min_j T_j^m$. Take feasible paths $\psi_{\mathbf{T}^m}^{\mathrm{d}}$ and $\psi_{\tilde{\mathbf{T}}^m}^{\mathrm{d}}$ as in (5.2) and (5.3).

Observe that

$$\phi_{i1}^m(t) \geq (\psi_{\mathbf{T}^m}^{\mathrm{d}})_{i1}(t) - \varepsilon^m$$

for all $i \in I$ and $t \geq 0$. It follows from the definition of ε^m that

$$\begin{aligned} \mathit{\Delta} V_i^{\theta}(\phi^m)(T_i^m) &\geq \mathit{\Delta} V_i^{\theta}(\psi_{\mathbf{T}^m}^{\mathrm{d}})(T_i^m) - \frac{1}{m} \\ &= \mathit{\Delta} V_i^{\theta}(\psi_{\tilde{\mathbf{T}}^m}^{\mathrm{d}})(\tilde{T}_i^m) - \frac{1}{m}, \end{aligned}$$

so that

$$\mathit{\Delta} V_i^{\theta}(\psi_{\tilde{\mathbf{T}}^m}^{\mathrm{d}})(\tilde{T}_i^m) - \frac{1}{m} \leq 0 \tag{A.4}$$

for any $i \in S^m$ by (A.3).

Now let $m \to \infty$. Since the set of feasible paths Φ is compact, $\{\psi_{\tilde{\mathbf{T}}^m}\}_{m=1}^\infty$ has a convergent subsequence $\{\psi^{\mathrm{d}}_{\tilde{\mathbf{T}}^{m(k)}}\}_{k=1}^\infty$ with a limit, which is written as $\psi^{\mathrm{d}}_{\mathbf{T}}$ for some $\mathbf{T} \in \bar{\mathbb{R}}^N_+$. Note that $\lim_{k\to\infty} \tilde{\mathbf{T}}^{m(k)} = \mathbf{T}$. Since $\min_{i\in I} \tilde{T}^m_i = 0$ for all m, $S = \{i \in I \mid T_i \neq \infty\}$ is nonempty due to the finiteness of I. Moreover, since $\mathit{\Delta} V^\theta_i$ is continuous on $\Phi \times \mathbb{R}_+$, we have

$$\mathit{\Delta} V^\theta_i(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) \leq 0$$

for any $i \in S$ by (A.4).

"Only if" part: Suppose that there exists $\mathbf{T} = (T_i)_{i\in I} \in \bar{\mathbb{R}}^N_+$ such that $S = \{i \in I \mid T_i \neq \infty\}$ is nonempty, and

$$\mathit{\Delta} V^\theta_i(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) \leq 0$$

for any $i \in S$. Since $\mathit{\Delta} V^\theta_i(\psi^{\mathrm{d}}_{\mathbf{T}})(t)$ is decreasing in t, $\mathit{\Delta} V^\theta_i(\psi^{\mathrm{d}}_{\mathbf{T}})(t) \leq 0$ holds for all i and all $t \geq T_i$. This implies that $\psi^{\mathrm{d}}_{\mathbf{T}}$ satisfies

$$(\dot{\psi}^{\mathrm{d}}_{\mathbf{T}})_{i0}(t) > -(\psi^{\mathrm{d}}_{\mathbf{T}})_{i0}(t) \Rightarrow 0 = \min BR^i(\psi^{\mathrm{d}}_{\mathbf{T}})(t)$$

for almost all $t \geq 0$, so that $\psi^{\mathrm{d}}_{\mathbf{T}}$ is a superpath. It follows from Lemma 3.5 that there exists a perfect foresight path ϕ^* from $\mathbf{1}$ such that $\phi^* \precsim \psi^{\mathrm{d}}_{\mathbf{T}}$. Since $\psi^{\mathrm{d}}_{\mathbf{T}}$ is such that $(\psi^{\mathrm{d}}_{\mathbf{T}})_{i1}(t) \to 0$ as $t \to \infty$ for $i \in S$, it follows that ϕ^* does not converge to $\mathbf{1}$. Therefore, $\mathbf{1}$ is not absorbing. ▮

Proof of Proposition 5.1.3. By Lemma A.5.2, we only need to show that if for any $\mathbf{T}$ such that $S = \{i \in I \mid T_i \neq \infty\}$ is nonempty and $\mathbf{0}_S$ is a Nash equilibrium, there exists $i \in S$ such that $\mathit{\Delta} V^\theta_i(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) > 0$, then the same condition holds for any $\mathbf{T}$ such that $\mathbf{0}_S$ is not necessarily a Nash equilibrium. Suppose not, and choose $\mathbf{T}$ and S such that S is maximal among all subsets that violate the condition. Then $\mathit{\Delta} V^\theta_i(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) \leq 0$ for any $i \in S$. Since $\mathbf{0}_S$ is not a Nash equilibrium, (i) there exists $j \in S$ such that $d_j(\mathbf{0}_S) > 0$, or (ii) there exists $j \notin S$ such that $d_j(\mathbf{0}_S) < 0$. In case (i), however, by the supermodularity,

$$d_j(\mathbf{0}_S) \leq \mathit{\Delta} V^\theta_j(\psi^{\mathrm{d}}_{\mathbf{T}})(T_j) \leq 0,$$

which is a contradiction. Therefore, case (ii) holds. Choose such a j.

Define $\mathbf{T}' = (T'_1, \dots, T'_N)$ by $T'_i = T_i$ for $i \neq j$ and T'_j as a sufficiently large but finite number. Then $\psi^{\mathrm{d}}_{\mathbf{T}'} \precsim \psi^{\mathrm{d}}_{\mathbf{T}}$, so that

$$\mathit{\Delta} V^\theta_i(\psi^{\mathrm{d}}_{\mathbf{T}'})(T'_i) \leq \mathit{\Delta} V^\theta_i(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) \leq 0$$

for $i \in S$ by the supermodularity. Moreover, since $\mathit{\Delta} V^\theta_j(\psi^{\mathrm{d}}_{\mathbf{T}'})(T'_j)$ converges to $d_j(\mathbf{0}_S) < 0$ as $T'_j \to \infty$, we have

$$\mathit{\Delta} V^\theta_j(\psi^{\mathrm{d}}_{\mathbf{T}'})(T'_j) < 0.$$

This contradicts the maximality of S. ▮

Proposition 5.1.4 follows immediately from the following.

Lemma A.5.3. *The following conditions are equivalent:*

(a) $\mathbf{1}$ *is absorbing for all* $\theta > 0$*;*

(b) *there exists* $\bar{\theta}$ *such that* $\mathbf{1}$ *is absorbing for all* $\theta \in (0, \bar{\theta})$*;*

(c) *for any* $\mathbf{T} = (T_i)_{i \in I} \in \bar{\mathbb{R}}_+^N$ *such that* $S = \{i \in I \mid T_i \neq \infty\}$ *is nonempty and* $\mathbf{0}_S$ *is a Nash equilibrium of the stage game, there exists* $i \in S$ *such that*

$$\Delta V_i^0(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i) \geq 0.$$

Proof. (a) $\Rightarrow$ (b): Obvious.

(b) $\Rightarrow$ (c): Suppose that there exists $\mathbf{T} = (T_i)_{i \in I} \in \bar{\mathbb{R}}_+^N$ such that $S = \{i \in I \mid T_i \neq \infty\}$ is nonempty, $\mathbf{0}_S$ is a Nash equilibrium, and $\Delta V_i^0(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i) < 0$ for all $i \in S$. Fix such a $\mathbf{T}$. Since $\Delta V_i^\theta(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i)$ is continuous in θ, there exists $\bar{\theta} > 0$ such that for all $\theta \in (0, \bar{\theta})$,

$$\Delta V_i^\theta(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i) < 0$$

for all $i \in S$, implying that $\mathbf{1}$ is not absorbing for any $\theta \in (0, \bar{\theta})$ by Proposition 5.1.3.

(c) $\Rightarrow$ (a): Suppose (c). For each $\mathbf{T} = (T_i)_{i \in I} \in \bar{\mathbb{R}}_+^N$ such that $S = \{i \in I \mid T_i \neq \infty\}$ is nonempty and $\mathbf{0}_S$ is a Nash equilibrium, take $i \in S$ as in (c).

By the monotonicity of d_i, we have $d_i(\mathbf{1}) \geq d_i(\mathbf{0}_S)$. If $d_i(\mathbf{1}) = d_i(\mathbf{0}_S)$, then for any $\theta > 0$,

$$\Delta V_i^\theta(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i) = d_i(\mathbf{1}) > 0$$

by the monotonicity of d_i. If $d_i(\mathbf{1}) > d_i(\mathbf{0}_S)$, then $\Delta V_i^\theta(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i)$ is increasing in θ by Lemma A.5.1, so that for any $\theta > 0$,

$$\Delta V_i^\theta(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i) > \Delta V_i^0(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i) \geq 0.$$

It follows that $\mathbf{1}$ is absorbing for all $\theta > 0$ by Proposition 5.1.3. ▮

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