A Simple Necessary Condition for Partially Honest Nash Implementation

Doghmi, Ahmed

Univesity of Rabat, University of Caen, National Institute of Statistics and Applied Economics

24 January 2011

A Simple Necessary Condition for Partially Honest Nash Implementation*

Ahmed Doghmi[ab†]

First version: January 2011
October 14, 2015

[a] *University of Rabat, Mohammadia School of Engineering, the QSM Laboratory, Avenue Ibn Sina B.P. 765 Agdal, 10100 Rabat, Morocco.*

[b] *University of Caen Basse-Normandie, Center for Research in Economics and Management (CREM), (CNRS UMR 6211), 19 Rue Claude Bloch 14032 Caen, Cedex, France.*

Abstract. This paper belongs to the recent literature which explores the consequence(s) of allowing some player(s) to be minimally honest in the standard Nash implementation. In this literature, Dutta and Sen(2012) give sufficient conditions for the implementation of social choice correspondences in Nash equilibria provided at least one individual is partially honest, that is Maskin Monotonicity is no longer required. They did not present any necessary condition. Thus, in this paper, we seek to fill the gap by deriving a simple necessary condition called, *Partial-Equivalency.* This condition allows us to lift the silence on the implementability of several social choice correspondences where Dutta and Sen's theorem (2012) does not give any answer. We apply our result to domains of private good economies with single-peaked preferences and we give examples of solutions of the problem of fair division that are not partially honest Nash implementable. We introduce the mild property of *citizen sovereignty* and we prove that Partial-Equivalency is not only necessary, but becomes also sufficient. Also, we extend our result to partially honest environment with incomplete information.

Keywords: Partial honesty; Nash implementation; Partial-Equivalency; Private good economies; Single-peaked preferences.

JEL classification: C72; D71

1 Introduction

Recently, many authors have introduced the concept of honesty in implementation theory. Matsushima (2008a) was the first who studied the effect of a little honesty in the conduct of agents on the implementability of social choice correspondences. He constructed a model in incomplete information and he supposed that the agents have intrinsic preferences for honesty in the sense that they dislike the idea of lying when it does not influence their welfare. In other words, when an agent makes a choice between

*We would like to thank Michele Lombardi for discussions, suggestions and helpful comments. Also, We would like to thank the seminar participants at Maastricht University in Netherland (June 19, 2011) and audiences at the Rabat Academic School workshop in Morocco (Marsh 29-30, 2012). Of course any error is our own responsibility.

†Corresponding author. Tel.:+212 6 17 25 11 98. *E-mail addresses:* doghmi@insea.ac.ma, ahmeddoghmi@hotmail.com (A.Doghmi)

an honest strategy and a dishonest one, such that they each reach outcomes that have the same payoff, the agent plays the honest strategy, defined as the strategy that goes along with the intention of a central planner.

In his model, Matsushima (2008a) considered a social choice function (SCF) which assigns to each possible vNM preference profile a lottery over the basic set of outcomes. He supposed that there is a cost of dishonesty removed from the agents' utility functions. This cost is an increasing function of the proportion of dishonest announcements committed during the game. Matsushima (2008a) proved that if a social choice function is Bayesian-incentive compatible, then it is fully implementable in iterative dominance.

To achieve this implementability, Matsushima (2008a) considered a detail-free mechanism in the sense that the planner does not need to know details of the agents' utility functions or prior belief distributions to design the mechanism. In addition, the mechanism does involve small fines that are to be imposed on players by the planner when certain strategies are played. In his framework, Matsushima (2008a) provided an extraordinary result. However, the fact that the planner would know the incentive compatibility of an implementable social choice function without the knowledge of the details on utility functions and priors that are not needed for the design of the implementing mechanism, is not clear.

To clarify this point, Matsushima (2008b) provided a work in a complete information setting and he gave a similar result for Nash implementation when players suffer a small utility loss from lying. He showed that when there are three or more individuals, every social choice function is implementable in the iterative elimination of strictly dominated strategies, and hence in Nash equilibrium as long as there is aversion to telling lies among the agents. Impressively, if this aversion is absent, the mechanism will have a large multiplicity of Nash equilibria, a multiplicity that disappears the moment even a slight white-lie aversion comes in and we turn to iteratively undominated equilibrium. The mechanism is entirely detail-free without any dependence to the form of the social choice function.

Dutta and Sen (2009,2012) have made a significant contribution that reveals the consequences of partial honesty for implementation theory. They considered a model that differs from that of Matsushima by studying the implementability of social choice correspondences (SCCs), not functions, and assuming Nash equilibrium as the implementing equilibrium notion, rather than iterative undomination. They also considered a model that is purely ordinal, not cardinal like the one of Matsushima. This makes their model very significant to study the implementability of several SCCs and especially those of the voting problems. They showed that when there are at least three individuals, the presence of even a single partially honest individual (whose identity is not known to the planner) can lead to a dramatic increase in the class of Nash implementable SCCs. In particular, all SCCs satisfying no-veto power can be implemented. This result is surprising and stands in stark contrast to the classical results of Maskin (1999) who outlines the fact that Maskin Monotonicity is a necessary condition for an SCC to be Nash implementable and becomes sufficient together with the no-veto power condition. The planner here only needs to know that there is at least a partially honest agent without having any knowledge of its identity. In a domain of strict orders, they also provided necessary and sufficient conditions for implementation in the two-player case when there is exactly one partially honest individual and when both individuals are partially honest.

Dutta and Sen (2009,2012) provided also additional results in a Bayesian setting. They assumed that there exists a particular agent who is partially honest with a strictly positive probability and they showed that when there is at least three players participating in the mechanism, any SCC satisfying no-veto power can be implemented in Bayesian Nash equilibria.

In a Bayesian environment different to that of Dutta and Sen (2012), Korpela (2012) assumed that all individuals are partially honest and he showed that any partially honest Bayesian implementable SCF must satisfy incentive compatibility, and with at least three agents, any SCF satisfying incentive compatibility and no-veto power can be implementable in Bayesian equilibria. Holden et al. (2012) considered full implementation in general environments when agents have an arbitrarily small preference for honesty. They showed that with at least two agents and when the separable punishment condition holds, any SCF can be implemented by a simple mechanism in two rounds of iterated deletion of strictly dominated strategies. Kartik and Tercieux (2012) also studied the problem of implementation when agents are minimally honest, but in using a bit different formalization. They provided a necessary and almost sufficient condition for implementability in a environment where strategies can be costly is given.

Doghmi and Ziad (2013) used weak variant of no-veto power in the many players case into a complete information setting to implement an SCC in Nash equilibria in the presence of at least one partially honest player. They provided economic applications in the domains of single-peaked, single-dipped, and single-plateaued preferences and they showed that any solution of the problem of fair division satisfying unanimity can be implemented in Nash equilibria. Thus, they extended the implementability of SCCs from the monotonic family to the no-monotonic family by a very simple manner.

However, the conditions proposed by Dutta and Sen (2012), and Doghmi and Ziad (2013) are not necessary and so they are silent about the implementability of very demanding no-monotonic correspondences that fail to satisfy no-veto power and its weak versions like strong core correspondence and strong Pareto correspondence in the finite allocation problems, and the voting rules of Borda, Plurality and Anti-plurality in political sciences. This, it remained an open question.

To give an answer in closing this gap, Lombardi and Yoshihara (2013) provided necessary and sufficient conditions for Nash implementation with partially honest agents. These conditions constitute a useful tool to determine whether or not given social correspondences can be implemented with partial honesty. However, they were stated in terms of the existence of some unknown sets as in Moore and Repullo (1990), and hence these properties are not simple. Thus, Lombardi and Yoshihara (2013) presented, in analogical way to that of the Sjöström (1991), algorithms for constructing the unknown sets and testing the (non-)implementability of SCCs. Nevertheless, the conditions of these algorithms are based on the *richness* of preference domain (Lemmas 1-2,4, pp.7-8). This requirement is not checked for the non-unanimous correspondences in some domains that have the property of private preferences like single-peaked and single-plateaued preferences in private good economies.

To simplify the checking process, we give in this paper a new necessary condition relatively easy to check compared to that of Lombardi and Yoshihara (2013) termed, *Partial-Equivalency*, and we show that if a SCC F is partially honest implementable, then F must satisfy the Partial-Equivalency condition. We apply this result to private good economies with single-peaked preferences for detecting the possible non partial honest implementability of several important non-unanimous and non-monotonic SCCs that do not satisfy neither the no-veto power condition nor its weak variants. We use the mild requirement of *citizen sovereignty*, which is trivially checked for many SCCs in private good economies with single-peaked preferences, and we prove that Partial-Equivalency is not only necessary, but becomes also sufficient. Also, we show that our necessary condition remains valid in Bayesian setting. We prove that if an SCC is partial honesty implementable in Bayesian Nash equilibria, then this correspondence must satisfy Partial-Equivalency.

The rest of this paper is organized as follows. In Section 2, we introduce notations and definitions. In Section 3, we state and prove our main result. In Section 4, we give applications in the domains of private good economies with single-peaked preferences. In Section 5, we extend our result to honesty environment with incomplete information. We conclude by remarks.

2 Notations and definitions

Let A be a set of alternatives, and let $N = \{1, ..., n\}$ be a set of individuals, with generic element i. Each individual i is characterized by a preference relation R_i defined over A, which is a complete, transitive, and reflexive relation in some class $\Re_i$ of admissible preference relations. Let $\Re = \Re_1 \times ... \times \Re_n$. Let $\mathscr{D} \subset \Re$ be a domain. An element $R = (R_1, ..., R_n) \in \mathscr{D}$ is a preference profile. The relation R_i indicates the individual's i preference. For $a, b \in A$, the notation aR_ib means that the individual i prefers weakly a to b. The asymmetrical and symmetrical parts of R_i are noted respectively by P_i and $\sim_i$. A social choice correspondence (SCC) F is a multivalued mapping from $\Re$ into $2^A \setminus \{\emptyset\}$, that associates with every R a nonempty subset of A. For all $R_i \in \Re_i$ and all $a \in A$, the lower contour set for agent i at alternative a is noted by: $L(a, R_i) = \{b \in A \mid aR_ib\}$. The strict lower contour set and the indifference lower contour set are noted respectively by $LS(a, R_i) = \{b \in A \mid aP_ib\}$ and $LI(a, R_i) = \{b \in A \mid a \sim_i b\}$.

Now, we model an environment for partial honesty. As Dutta and Sen (2009,2012), we assume that an honest player's preference for honesty is *lexicographic*. Let $S_i = \Re \times C_i$ be the set of strategy profiles for a player i , where C_i denotes the other components of the strategy space (which depend to individual preferences, social states,...). Let $S = S_1 \times ... \times S_n$ be a set of strategy profiles. The elements of S are denoted by $s = (s_1, ..., s_n)$. For each $i \in N$, and $R \in \mathscr{D}$, let $\tau_i(R) = \{R\} \times C_i$ be the set of truthful messages for agent i. We denote by $s_i \in \tau_i(R)$ a truthful strategy as player i is reporting the true preference profile. We extend a player's ordering over A to an ordering over a strategic space S. This, because the players preference between being honest and dishonest depends on strategies that the others played and of the outcomes which they obtained. Let $\succeq_i^R$ be the preference of player i over S in the preference profile R. The asymmetrical and symmetrical parts of R_i are noted respectively by $\succ_i^R$ and $\sim_i^R$. Let Γ be a mechanism (game form) represented by the pair (S, g), where $S_i = \mathscr{D} \times C_i$ and $g : S \to A$ is a payoff function.

Definition 1 *A player i is partially honest whenever for all preference profile $R \in \mathscr{D}$ and for all $(s_i, s_{-i}), (s'_i, s_{-i}) \in S$,*

(i) If $g(s_i, s_{-i}) R_i g(s'_i, s_{-i})$ *and* $s_i \in \tau_i(R)$, $s'_i \notin \tau_i(R)$, *then* $(s_i, s_{-i}) \succ_i^R (s'_i, s_{-i})$.
(ii) In all other cases, $(s_i, s_{-i}) \succeq_i^R (s'_i, s_{-i})$ *iff* $g(s_i, s_{-i}) R_i g(s'_i, s_{-i})$.

A Nash equilibrium of the game $(\Gamma, \succeq^R)$ is a vector of strategies $s \in S$ such that for any i, $g(s) R_i g(b_i, s_{-i})$, for all $b_i \in S_i$, i.e., when the other player chooses s_{-i}, the player i cannot deviate from s_i. Given $N(g, \succeq^R, S)$ the set of Nash equilibria of the game $(\Gamma, \succeq^R)$, a mechanism $\Gamma = (S, g)$ implements a SCC F in Nash equilibria if for all $R \in \mathscr{D}$, $F(R) = g(N(g, \succeq^R, S))$. We say that a SCC F is implementable in Nash equilibria if there is a mechanism which implements it in these equilibria.

To characterize the SCC that can be Nash implemented in partial honest environment, Dutta and Sen (2012) used the following properties.

Assumption A: There exists at least one partially honest individual and this fact is known to the planner. However, the identity of this individual is not known to her.

No-veto power: A SCC F satisfies no-veto power if the following condition holds for each a and each $R \in \mathscr{D}$: if there exists an $i \in N$ such that $L(a, R_j) = A$ for each $j \in N \setminus \{i\}$, then $a \in F(R)$.

Dutta and Sen (2009,2012) proved that under Assumption A, any SCC satisfying no-veto power can be implemented in Nash equilibria. To enlarge the family of monotonic correspondences that are Nash implementable with standard agents to that of no-monotonic correspondences that are Nash implementable with partially honest agents in private good economies with single-peaked, single-dipped, and single-plateaued preferences, Doghmi and Ziad (2013) used the properties of *I-weak no-veto power*[1] and *unanimity* [2] that are weaker variants of no-veto power. They proved that as long as there are at least three agents participating in the mechanism and Assumption A holds, any SCC F satisfying I-weak no-veto power and unanimity can be implemented in Nash equilibria. They exploited these properties as a tool to provide especially their main result in private good economies with single-peaked, single-plateaued, and single-dipped preferences by showing that Unanimity alone suffices to implement any solution of the problem of fair division in the presence of at least one partially honest agent. For the non-unanimous correspondences, neither this result nor that of Dutta and Sen (2012) can inform us on the implementability of these solutions. Lombardi and Yoshihara (2013) gave a full characterization for this topic. They provided necessary and sufficient conditions and algorithms. The conditions are founded on the existence of unknown sets, and hence they are not simple as those of Moore and Repullo (1990). The algorithms for constructing

[1] The I-weak no-veto power property has been initially introduced in Doghmi and Ziad (2012) together with a strong version of monotonicity to characterize standard Nash implementation. To define it, we provide the following notion. *Indifferent options subset*: For any agent'i preference R_i, any alternative $a \in F(R)$, for some $c \in LI(a, R_i)$ with $c \neq a$, we define the indifferent options subset by the subset $I(a, c, R_i) = \{b \in A \setminus \{a, c\}$ s.t. $a \sim_i b \sim_i c\}$. An SCC F satisfies I-weak no-veto power if for each $i \in N$, each $R \in \Re$, each $a \in F(R)$ and some $c \in LI(a, R_i) \setminus \{a\}$, for $R' \in \Re$, $b \in LS(a, R_i) \cup I(a, c, R_i) \subseteq L(b, R'_i)$ and $L(b, R'_j) = A$ for all $j \in N \setminus \{i\}$, then $b \in F(R')$. Other weak variants of the no-veto power condition called *weak no-veto power* and *strict weak no-veto power* have been provided by Doghmi and Ziad (2008*ab*) that, respectively, together with Maskin monotonicity and strict monotonicity, ensure the implementability of many unanimous SCCs. We precise that there is no relationship between the I-weak no-veto power, weak no-veto power, and strict weak no-veto power conditions.

[2] A SCC F satisfies unanimity if for any $a \in A$ and any $R \in \mathscr{D}$, if for any $i \in N$, $L(a, R_i) = A$, then $a \in F(R)$.

the unknown sets are based on the property of richness for preference domain, which fails to check in private good economies, and hence these algorithms do not apply.

3 Main result

We now present a new necessary condition that a social choice correspondence must satisfy in order to be Nash implementable in the presence of at least one partially honest agent. We use this necessary condition, called *Partial-Equivalency*, to identify various social choice correspondences that cannot be implemented with partial honesty in a symmetric information setting, precisely because they fail to satisfy this necessary condition.

Definition 2 *(Partial-Equivalency)*
A SCC F satisfies Partial-Equivalency if for any $R, R' \in \mathscr{D}$ and $x \in F(R')$, if $[R \neq R'$ implies for all $b \in A \setminus \{x\}$, $xP'_i b$ for all $i \in N]$, then $x \in F(R')$.

This condition stipulates that if an alternative x is socially chosen in a preference profile R, and if x is the unique maximal element in R' when at least one agent changes his preference in groining from R to R', then x is socially chosen in R'.

To understand better this new necessary condition of Partial Equivalency, we consider the following example.

Example 1. Let $N = \{1, 2, 3\}$ and $A = \{a, b, c, d\}$. Let $R, R' \in \mathscr{D}$ be defined by:

R:

R_1	R_2	R_3
b	a	a
c	c,d	b,c
a, d	b	d

R':

R'_1	R'_2	R'_3
a	a	a
b,d	c,d	b,c
c	b	d

Let $F(R) = \{a, c\}$ and $F(R') = \{a\}$. In this example, we have $a \in F(R)$, and a is an unique maximal element for all players in R' when player 1 changes his preference in going from R to R', therefore Partial-Equivalency says that $a \in F(R')$.

Since the sufficient conditions provided by Dutta and Sen (2009,2012) and Doghmi and Ziad (2013) are weak, the gap between a full characterization and these conditions is very small. Thus, the new necessary condition of Partial-Equivalency is an extremely weak property.

***Observation* 1** *The properties of no-veto power and unanimity imply partial equivalency.*

Proof. Let $R, R' \in \mathscr{D}$, $x \in A$, and $x \in F(R)$. Assume that $[R \neq R'$ implies for all $b \in A \setminus \{x\}$, $xP'_i b$ for all $i \in N]$. This means that $L(x, R'_i) = A$ for all $i \in N$. By unanimity $x \in F(R')$, and hence partial equivalency is implied by no-veto power. Q.E.D

Now, we present our first result of this paper.

Theorem 1 *Let Assumption A hold. If a SCC F is partial honest implementable in Nash equilibria, then F satisfies Partial-Equivalency.*

Proof. Let F be a Partial honest Nash implementable SCC. Therefore, for all $R \in \mathscr{D}$, $F(R) = g(N(g, \succeq^R, S))$. Let $x \in F(R)$, by definition of implementability, $x \in F(R) = g(N(g, \succeq^R, S))$, means that there exists $s \in N(g, \succeq^R, S)$ such that $g(s) = x$. Suppose that F does not satisfy Partial-Equivalency. Therefore, if for any $R, R' \in \mathscr{D}$ and $x \in F(R')$, if
$[R \neq R'$ implies for all $b \in A \setminus \{x\}$, $xP'_i b$ for all $i \in N]$ (1), but $x \notin F(R')$. By implementability, $x \notin F(R') = g(N(g, \succeq^{R'}, S))$ implies that $s \notin N(g, \succeq^{R'}, S)$. Therefore, there exists a player k with a strategy $\widetilde{s}_k$ such that $(\widetilde{s}_k, s_{-k}) \succ_k^{R'} (s_k, s_{-k})$ (2). By Assumption A there is at least one partially honest player, denoted h. Thus, the deviator player k can be the agent h, i.e. $k = h$ or he is different to h, i.e., $k \neq h$.

α) If $k = h$, then, since s is not a Nash equilibrium at R', the partially honest agent h can deviate to the truthful announcement of R', i.e., for any $s_h \notin \tau_h(R')$, there exists $\widetilde{s}_h \in \tau_h(R')$ such that $(\widetilde{s}_h, s_{-h}) \succ_h^{R'} (s_h, s_{-h})$ (3). Assume that $g(\widetilde{s}_h, s_{-h}) = b$, $g(s_h, s_{-h}) = x$, it follows from (1) that $b = g(\widetilde{s}_h, s_{-h})R'_h x$ implies that $R = R'$. From (3), we obtain $(\widetilde{s}_h, s_{-h}) \succ_h^R (s_h, s_{-h})$. Therefore, $s \notin N(g, \succeq^R, S)$, a contradiction.

β) If $k \neq h$, i.e., the agent k is not partially honest, then by definition, the relation (2) is equivalent to $g(\widetilde{s}_k, s_{-k})P'_k g(s_k, s_{-k})$. Assume that $g(\widetilde{s}_k, s_{-k}) = b$, $g(s_k, s_{-k}) = x$, it follows from (1) that $b = g(\widetilde{s}_k, s_{-h})P'_k x$ implies that $R = R'$. From (2), we obtain $(\widetilde{s}_k, s_{-k}) \succ_k^R (s_k, s_{-k})$. Therefore, $s \notin N(g, \succeq^R, S)$, a contradiction. Q.E.D.

In the next section, we use the result of Theorem 1 to study the implementability of some non-monotonic correspondences which violate unanimity in the domain of private good economies with single-peaked preferences.

4 Applications to private good economies with single-peaked preferences

In the following subsections, we present the private good economies model with single-peaked preferences, introduce some well-known correspondences, and study the problem of their implementability.

4.1 The economic environment

There is an amount $\Omega \in \mathbb{R}_{++}$ of a certain infinitely divisible good that is to be allocated among a set $N = \{1, ..., n\}$ of n agents. The preference of each agent $i \in N$ is represented by a continuous and single-peaked preference relation R_i over $[0, \Omega]$ [3]: This means that there is a number $p(R_i)$ such that for all $x_i, y_i \in [0, \Omega]$: (i) if $y_i < x_i \leq p(R_i)$ or $p(R_i) \leq x_i < y_i$, then $x_i P_i y_i$. We call $p(R_i)$ the *peak* of R_i.

The class of all single-peaked preference relations is represented by $\Re_{sp_i} \subseteq \Re_i$. Let $\Re_{sp} = \Re_{sp_1} \times ... \times \Re_{sp_n}$ be the domain of single-peaked preferences. For $R \in \Re_{sp}$, let

[3] For all $x_i, y_i \in [0, \Omega]$, $x_i R_i y_i$ means that, for the agent i, to consume a share x_i is as good as to consume the quantity y_i. The asymmetrical and symmetrical parts of R_i are written respectively P_i and $\sim_i$.

$p(R) = (p(R_1), ..., p(R_n))$ be the profile of peaks (or of preferred consumptions). A single-peaked preference relation $R_i \in \mathscr{D}_{sp_i}$ is described by the function $r_i : [0, \Omega] \to [0, \Omega]$ which is defined as follows: $r_i(x_i)$ is the consumption of the agent i on the other side of the peak which is indifferent to x_i (if it exists), or else, it is 0 or Ω; i.e., if $x_i \leq p(R_i)$, then, $r_i(x_i) \geq p(R_i)$ and $x_i \sim_i r_i(x_i)$ if such a number exists or $r_i(x_i) = \Omega$ otherwise; if $x_i \geq p(R_i)$, then, $r_i(x_i) \leq p(R_i)$ and $x_i \sim_i r_i(x_i)$ if such a number exists or $r_i(x_i) = 0$ otherwise.

For $R \in \Re_{sp}$, a feasible allocation for the economy (R, Ω) is a vector $x \equiv (x_i)_{i \in N} \in \mathbb{R}^n_+$ such that $\sum_{i \in N} x_i = \Omega$ and X is the set of the feasible allocations. We note that the feasible allocations set is $X \subseteq [0, \Omega] \times ... \times [0, \Omega]$. Thus, $L(x, R_i) = X$ is equivalent to $L(x_i, R_i) = [0, \Omega]$. For the set $L(x, R_i) = X$, xR_iy for all $y \in X$ implies that $x_iR_iy_i$. Thus, the agents preferences are defined over individual consumption spaces, not over allocation space. Then the properties of implementation theory, presented in general setup in Section 2, become as follows. An SCC F is a multi-valued mapping from $\Re_{sp}$ into X. A SCC F satisfies Partial-Equivalency if for $R, R' \in \mathscr{D}$, $x \in F(R)$, if $R_i = R'_i$ and, $R_{-i} \neq R'_{-i}$ implies for all $y \in X \setminus \{x\}$, $x_jP'_jy_j$ for all $j \in N \setminus \{i\}$, then $x \in F(R')$. We note that the free disposability of the good is not assumed.

In the next subsection, we study the implementability of some examples of the solutions of the problem of fair division.

4.2 Examples of correspondences violate Maskin monotonicity, unanimity, and partial equivalency

Here we firstly provide some solutions of the problem of fair division, and we secondly inspect their implementability in the standards setting and in the partially honest environment.

4.2.1 Egalitarian-Equivalence correspondence from equal division(EE_{ed})

The egalitarian-equivalence correspondence from equal division is a solution adapted from a notion proposed by Pazner and Schmeidler (1978) for the fair division problem. An allocation x is said to be *egalitarian-equivalence correspondence from equal division* if there is a bundle $z = (\frac{\Omega}{n}, ..., \frac{\Omega}{n})$ so that for all $i \in N$, $x_i \sim_i \frac{\Omega}{n}$; such a vector z is called an egalitarian-reference-bundle.

Proposition 1 *In private good economies with single-peaked preferences, the EE_{ed} correspondence does not satisfy Maskin monotonicity.*

Proof. Let $R, R' \in \mathscr{D}_{sp}$, $N = \{1, 2, 3\}$, $x = (2, 1, 9) \in X$ with $\sum_{i=1}^3 x_i = \Omega = 12$. Let $p(R) = (2.75, 2.5, 6)$, and $p(R') = (2, 1, 9)$. Figure 1 illustrates such representations.

Note that $x \in EE_{ed}(R)$ and for all $i \in N$, $L(x_i, R_i) \subseteq L(x_i, R'_i)$. However, in profile R', $x \notin EE_{ed}(R')$.Q.E.D.

As an immediate corollary of Proposition 1 and Theorem 2 of Maskin (1999), we have the following result.

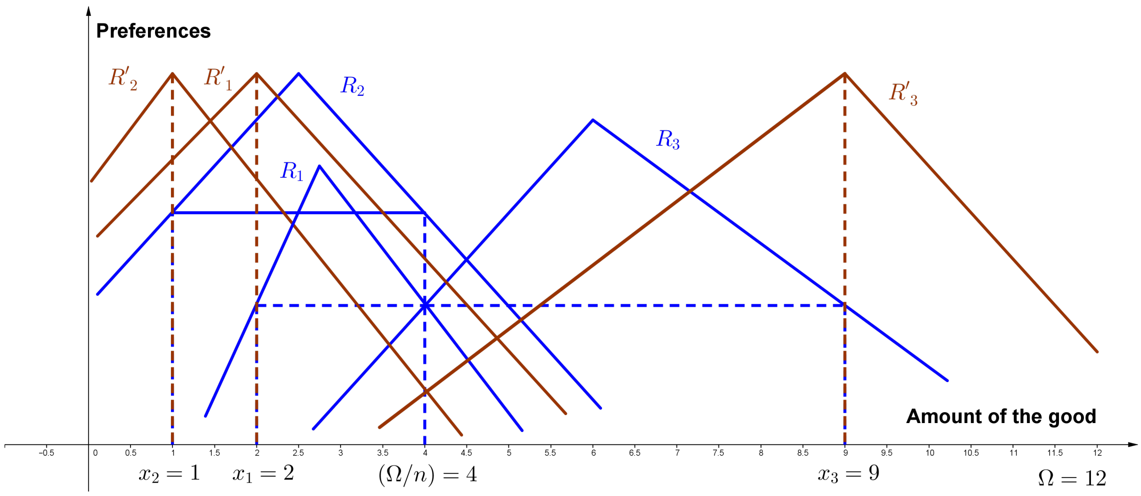

Figure 1: The EE_{ed} correspondence does not satisfy Maskin monotonicity, unanimity, or partial equivalency.

Corollary 1 *In private good economies, if the preferences are single-peaked, the EE_{ed} correspondence can not be implemented in Nash equilibria with standard agents.*

Proposition 2 *In private good economies with single-peaked preferences, the EE_{ed} correspondence does not satisfy unanimity.*

Proof. It follows from Figure 1 that, in profile R', $L(x_i, R'_i) = [0, \Omega]$ for all $i \in N$, but $x \notin EE_{ed}(R')$.Q.E.D.

From Proposition 2, we conclude that Theorem 1 of Doghmi and Ziad (2013) and the results of Lombardi and Yoshihara (2013) can not give us an answer on the implementability of the EE_{ed} correspondence. To examine its implementation, we use Theorem 1. We give the following result.

Proposition 3 *In private good economies with single-peaked preferences, the EE_{ed} correspondence does not satisfy Partial-Equivalency.*

Proof. It follows from Figure 1 that $x \in EE_{ed}(R)$, and $R \neq R'$ implies that x_i is the unique maximal element in R', but $x \notin EE_{ed}(R')$.Q.E.D.

Corollary 2 *In private good economies, if the preferences are single-peaked, the the EE_{ed} correspondence can not be implemented in Nash equilibria with partially honest agents.*

4.2.2 Pareto Indifferent (PI)

Two allocations $x, y \in X$ are *Pareto indifferent* under R if $x_i \sim_i y_i$ for all $i \in N$. This solution fail to satisfy Maskin monotonicity as shown in the following proposition.

Proposition 4 *In private good economies with single-peaked preferences, the PI correspondence does not satisfy Maskin monotonicity.*

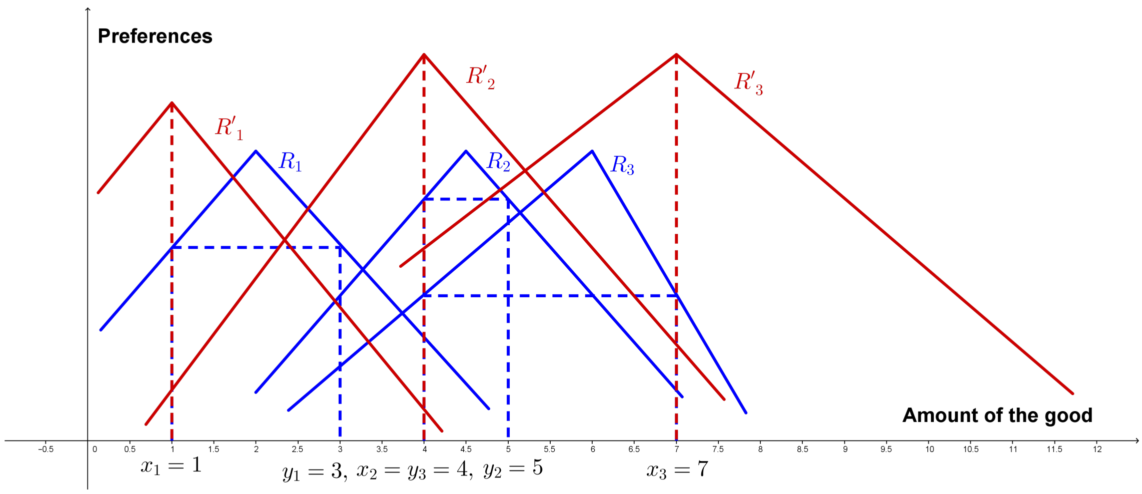

Figure 2: The PI correspondence does not satisfy Maskin monotonicity, unanimity, or partial equivalency.

Proof. Let $R, R' \in \mathscr{D}_{sp}$, $N = \{1, 2, 3\}$, $x = (1, 4, 7)$, $y = (3, 5, 4) \in X$ with $\sum_{i=1}^{3} x_i = \Omega = 12$. Let $p(R) = (2, 4.5, 6)$, and $p(R') = (1, 4, 7)$. Figure 2 illustrates such representations.

Note that $x \in PI(R)$ and for all $i \in N$, $L(x_i, R_i) \subseteq L(x_i, R'_i)$. However, in profile R', $x \notin PI(R')$.Q.E.D.

Corollary 3 *In private good economies, if the preferences are single-peaked, the PI correspondence can not be implemented in Nash equilibria with standard agents.*

Proposition 5 *In private good economies with single-peaked preferences, the PI correspondence does not satisfy unanimity.*

Proof. It follows from Figure 2 that $L(x_i, R') = [0, \Omega]$ for $i = 1, 2, 3$, but $x \notin PI(R')$.Q.E.D.

Proposition 6 *In private good economies with single-peaked preferences, the PI correspondence does not satisfy Partial-Equivalency.*

Proof. It follows from Figure 2 that $x \in PI(R)$, and $R \neq R'$ implies that x_i is the unique maximal element in R', but $x \notin PI(R')$.Q.E.D.

Corollary 4 *In private good economies, if the preferences are single-peaked, the PI correspondence can not be implemented in Nash equilibria with partially honest agents.*

A third non-monotonic and non-unanimous solution can be produced from the intersection of the EE_{ed} correspondence and the PI correspondence. Thus, from Propositions 1, 3, 4, and 6, we give the following result.

Corollary 5 *In private good economies, if the preferences are single-peaked, the $EE_{ed} \cap PI$ correspondence can not be implemented in Nash equilibria neither with standards agents nor with partially honest ones.*

Others examples of solutions can be constructed by intersection with the above correspondences and they can be studied in others specific domains restrictions like single-peaked preferences with worst indifferent allocations, and single-troughed preferences.

4.3 Partial-Equivalency and full characterization

In this subsection, we prove that Partial-Equivalency is not only necessary, but becomes also sufficient in private good economies with single-peaked preferences. To show this, we introduce the mild requirement of citizen sovereignty that is trivially checked for many solutions including those which violate the property of unanimity in this area..

Definition 3 *(Citizen sovereignty)*
A SCC F satisfies the property of citizen sovereignty if for each $x \in X$, there is a profile $R \in \mathscr{D}_{sp}$ such that $x \in F(R)$.

In the next proposition, we prove that in private good economies with single-peaked preferences, if the property of citizen sovereignty holds, any partially-equivalence SCC is unanimous.

Lemma 1 *In the private good economies with single-peaked preferences, if the property of citizen sovereignty holds, then any partially-equivalence SCC satisfies unanimity.*

Proof. Suppose not. Let $x \in X$ and $\widetilde{R} \in \mathscr{D}_{sp}$ be such that for any $i \in N$, $[0, \Omega] = L(x_i, \widetilde{R}_i)$, and $x \notin F(\widetilde{R})$. By the property of citizen sovereignty, for all $x \in X$, there is a profile $R \in \mathscr{D}_{sp}$ such that $x \in F(R)$. Therefore, $\widetilde{R} \neq R$. Since $L(x_i, \widetilde{R}_i) = [0, \Omega]$ for all $i \in N$ at $\widetilde{R}$, it follows from single-peakedness that for all $i \in N$, $x_i \widetilde{P}_i y_i$ for all $y \in X \setminus \{x\}$. By Partial-Equivalency, $x \in F(\widetilde{R})$, a contradiction. Q.E.D.

Through Observation 1 and Lemma 1 we complete the proof of the following proposition.

Proposition 7 *In the private good economies with single-peaked preferences, if the requirement of citizen sovereignty holds, an SCC satisfies unanimity if and only if it is partially-equivalence.*

Through Proposition 7 and Theorem 1 of Doghmi and Ziad (2013), we complete the proof of the second main Theorem of the paper.

Theorem 2 *: Let $n \geq 3$. In the private good economies with single-peaked preferences, if the requirement of citizen sovereignty holds, an SCC is partially honest Nash implementable if and only if it satisfies Partial-Equivalency.*

According to Diss, Doghmi and Tliti (2015), this result give an important connection with *strategy-proofness*, which is considered as a central property in implementation theory. It is a necessary condition for *dominant strategy implementation* in general environment, and it is also a necessary property for *Nash implementation* in private good economies with single-plateaued preferences as shown in Doghmi and Ziad (2015), and hence when preferences are single-peaked. Diss, Doghmi and Tliti (2015) proved that, under some requirements and when the welfare of a society is represented by a single-valued function, strategy-proofness is a necessary and sufficient condition for partially honest Nash implementation, standard Nash implementation, and dominant strategy implementation, and they concluded that these theories become equivalent.

5 Partially honest environment with incomplete information

To complement Dutta and Sen's result for sufficiency in incomplete information setting, we offer our second main result in this paper for necessity. For this, we reconsider Dutta and sen's model in this environment. Let j a particular partially honest agent with probability $\epsilon > 0$ and self-interested with probability $1-\epsilon$. The $N \setminus \{j\}$ agents are self-interested. As in section 2, we consider the mechanism $\Gamma = (S, g)$ where S is a product strategy set and $g : S \to A$. Also, we assume that $S_i = \mathscr{D} \times C_i$ where C_i denotes other components of agent i's strategy space. Let $R \in \mathscr{D}$. We assume that there is an individual j with two types: a truthful type denoted t and self-interested type denoted m. The $N \setminus \{j\}$ individuals have a single type m. The action set for individual i is S_i. As in Definition 1, individual j of type t has preferences over S in the preference profile R denoted by $\succeq^R$. All individuals of type m have preferences over lotteries with outcomes in A. Let l be an arbitrary individual of type m. Let the mapping $\upsilon : A \to \Re$ be an utility function which represents R_l. This function satisfies the following requirement. For all $x, y \in A$, $xPy \Leftrightarrow \upsilon(x) > \upsilon(x)$ and $x \sim y \Leftrightarrow \upsilon(x) = \upsilon(x)$. Let $p = \{p_x\}$, $x \in A$ be a lottery over elements of A with $p(x) \geq 0$ and $\sum_{x \in A} p_x = 1$. We say that for a individual l, a lottery p is at least as good as a lottery p' according to cardinal function υ, denoted by $pR_l^{\upsilon}p'$, if $\sum_{x \in A} \upsilon(x)p_x \geq \sum_{x \in A} \upsilon(x)p'_x$. Let υ_l be a cardinalization R_l for all $l \in N$. The individual j has two strategies $s_j^t, s_j^m \in S_j$. An individual $l \neq j$ has a strategy $s_l \in S_l$.

A strategy profile $((\overline{s}_j^t, \overline{s}_j^m), \overline{s}_{-j})$ is a Bayesian-Nash equilibrium (BNE) if

α) $\Gamma(\overline{s}_j^t, \overline{s}_{-j}) \succeq_j^R \Gamma(s_j, \overline{s}_{-j})$ for all $s_j \in S_j$;

β) $\Gamma(\overline{s}_j^m, \overline{s}_{-j}) R_j \Gamma(s_j, \overline{s}_{-j})$ for all $s_j \in S_j$;

γ) $\upsilon_l(\Gamma(\overline{s}_j^t, \overline{s}_l, \overline{s}_{-l,j}))\epsilon + \upsilon_l(\Gamma(\overline{s}_j^m, \overline{s}_l, \overline{s}_{-l,j}))(1-\epsilon) \geq \upsilon_i(\Gamma(\overline{s}_j^t, s_l, \overline{s}_{-l,j}))\epsilon + \upsilon_l(\Gamma(\overline{s}_j^m, s_l, \overline{s}_{-l,j}))(1-\epsilon)$ for all $s_l \in S_l$ and all $l \neq j$.

Let $R \in \mathscr{D}$. We define by the pair (Γ, R) a game of incomplete information where Γ is a ordinal mechanism as in Dutta and Sen (2009,2012). Thus, the set of equilibria does not depend on the chosen cardinalization. We say that Γ implements a SCC F if, for all $R \in \mathscr{D}$,

a) For all $a \in F(R)$, there exists a BNE of the game (Γ, R) denoted by $((\overline{s}_j^t, \overline{s}_j^m), \overline{s}_{-j})$ such that $g(\overline{s}_j^t, \overline{s}_{-j}) = g(\overline{s}_j^m, \overline{s}_{-j}) = a$.

b) Let $((\overline{s}_j^t, \overline{s}_j^m), \overline{s}_{-j})$ be an arbitrary BNE of (Γ, R). Then $g(\overline{s}_j^t, \overline{s}_{-j}), g(\overline{s}_j^m, \overline{s}_{-j}) \in F(R)$.

The next theorem shows that a SCC which is partial honesty implementable in Bayesian Nash equilibria must satisfy Partial-Equivalency.

Theorem 3 *If a SCC F is partial honest implementable in Bayesian Nash equilibria, then F satisfies Partial-Equivalency.*

Proof. Let F a SCC Partial honest implementable in Bayesian Nash equilibria. Therefore, for all $R \in \mathscr{D}$,

a) For all $x \in F(R)$, there exists a BNE of the game $\Gamma, R)$ denoted by $((\overline{s}^t_j, \overline{s}^m_j), \overline{s}_{-j})$ such that $g(\overline{s}^t_j, \overline{s}_{-j}) = g(\overline{s}^m_j, \overline{s}_{-j}) = x$.

b) Let $((\overline{s}^t_j, \overline{s}^m_j), \overline{s}_{-j})$ be an arbitrary BNE of (Γ, R). Then $g(\overline{s}^t_j, \overline{s}_{-j}), g(\overline{s}^m_j, \overline{s}_{-j}) \in F(R)$.

Suppose that F does not satisfy Partial-Equivalency. Therefore, for $R, R' \in \mathscr{D}$, $x \in F(R)$, we have $[R \neq R'$ implies for all $b \in A \setminus \{x\}$, $xP'_i b$ for all $i \in N]$ (1), but $x \notin F(R')$. By implementability, the strategy $((\overline{s}^t_j, \overline{s}^m_j), \overline{s}_{-j})$ is not a BNE of (Γ, R'). Therefore, there exists a player k with a strategy $s_k \in S_k$ such that (α') $\Gamma(s_k, \overline{s}_{-k}) \succ^{R'}_k \Gamma(\overline{s}^t_k, \overline{s}_{-k})$ for some $s_k \in S_k$ or (β') $\Gamma(s_k, \overline{s}_{-k}) P'_k \Gamma(\overline{s}^m_k, \overline{s}_{-k})$ for some $s_k \in S_k$ or (γ') For the cardinal function υ_l which represents R'_l, we have $\upsilon_l(\Gamma(\overline{s}^t_k, s_l, \overline{s}_{-l,k}))\epsilon + \upsilon_l(\Gamma(\overline{s}^m_k, s_l, \overline{s}_{-l,k}))(1-\epsilon) > \upsilon_l(\Gamma(\overline{s}^t_k, \overline{s}_l, \overline{s}_{-l,k}))\epsilon + \upsilon_l(\Gamma(\overline{s}^m_k, \overline{s}_l, \overline{s}_{-l,k}))(1-\epsilon)$ for some $s_l \in S_l$ and $l \neq k$. By Assumption A there is at least one partially honest player, denoted h. Thus, the deviator player k can be the agent h, i.e. $k = h$ or he is different to h, i.e., $k \neq h$.

*Case.*1) If $k = h$, then for possibility α' we have the following: since s is not a Bayesian Nash equilibrium at R', the partially honest agent h can deviate to the truthful announcement of R', i.e., for any $\overline{s}^t_h \notin \tau_h(R')$, there exists $s_h \in \tau_h(R')$ such that $\Gamma(s_h, \overline{s}_{-h}) \succ^{R'}_h \Gamma(\overline{s}^t_h, \overline{s}_{-h})$ (2). Assume that $g(s_h, \overline{s}_{-h}) = b$ and $g(s^t_h, \overline{s}_{-h}) = x$, then it follows from implication (1) that $b = g(s_h, \overline{s}_{-h}) R'_h g(s^t_h, \overline{s}_{-h}) = x$ implies that $R = R'$. From (2), we obtain $\Gamma(s_h, \overline{s}_{-h}) \succ^{R}_h \Gamma(\overline{s}^t_h, \overline{s}_{-h})$. For β', it follows from (1) that $\Gamma(s_h, \overline{s}_{-h}) P_h \Gamma(\overline{s}^m_h, \overline{s}_{-h})$ (3). For possibility γ', assume that $l \in N \setminus \{h\}$ is an arbitrary player of type m. Let υ_l be the cardinal function which represents R_l. Since $\epsilon > 0$, it follows from (2) and (3) that for some $s_h \in S_h$ and $l \neq h$, $\upsilon_l(\Gamma(\overline{s}^t_h, s_l, \overline{s}_{-l,h}))\epsilon + \upsilon_l(\Gamma(\overline{s}^m_h, s_l, \overline{s}_{-l,h}))(1-\epsilon) > \upsilon_l(\Gamma(\overline{s}^t_h, \overline{s}_l, \overline{s}_{-l,h}))\epsilon + \upsilon_l(\Gamma(\overline{s}^m_h, \overline{s}_l, \overline{s}_{-l,h}))(1-\epsilon)$ (4). From (2),(3), and (4), we conclude that the strategy $((\overline{s}^t_h, \overline{s}^m_h), \overline{s}_{-h})$ is not a BNE of (Γ, R), a contradiction.

*Case.*2) If $k \neq h$, i.e., the agent k is not partially honest, then by definition, the relations (α') and (β') are equivalent. Assume that $g(s_k, \overline{s}_{-k}) = b$ and $g(s^t_k, \overline{s}_{-k}) = x$, then from implication (1) it follows that $b = g(s_k, \overline{s}_{-k}) P'_k g(s^t_k, \overline{s}_{-k}) = x$ implies that $R = R'$. Thus, the the relations (α') and (β') respectively become as follows: $\Gamma(s_k, \overline{s}_{-k}) \succ^{R}_k \Gamma(\overline{s}^t_k, \overline{s}_{-k})$ for some $s_k \in S_k$ (5), $\Gamma(s_k, \overline{s}_{-k}) P_k \Gamma(\overline{s}^m_k, \overline{s}_{-k})$ for some $s_k \in S_k$ (6). For possibility γ', assume that $l \in N \setminus \{k\}$ is an arbitrary player of type m. Let υ_l be the cardinal function which represents R_l. Since $\epsilon > 0$, it follows from (5) and (6) that for some $s_k \in S_k$ and $l \neq k$, $\upsilon_l(\Gamma(\overline{s}^t_k, s_l, \overline{s}_{-l,k}))\epsilon + \upsilon_l(\Gamma(\overline{s}^m_k, s_l, \overline{s}_{-l,k}))(1-\epsilon) > \upsilon_l(\Gamma(\overline{s}^t_k, \overline{s}_l, \overline{s}_{-l,k}))\epsilon + \upsilon_l(\Gamma(\overline{s}^m_k, \overline{s}_l, \overline{s}_{-l,k}))(1-\epsilon)$ (7). From (5),(6), and (7), we conclude that the strategy $((\overline{s}^t_h, \overline{s}^m_h), \overline{s}_{-h})$ is not a BNE of (Γ, R), a contradiction. Q.E.D.

6 Concluding remarks

We have contributed to the growing literature on Nash implementation with partially honest agents by deriving a simple necessary condition, called *Partial-Equivalency*, that a social choice correspondence must satisfy in order to be Nash implementable in the presence of a partially honest agent. We have then used this necessary condition to identify various social choice correspondences that cannot be implemented in the presence of a partially honest agent were the results of Dutta and Sen (2009,2012) and Doghmi and Ziad (2013) do not give any answer.

We have given application in the domain of private good economies with single-peaked preferences. We have provided examples of non-monotonic solutions of the problem of fair division which violate unanimity. We have shown that they are not partially honest Nash implementable. We have have introduced the mild requirement of citizen sovereignty and we have proved that Partial-Equivalency becomes sufficient for partially honest implementation. We have also extended our result to the case where there exists a particular agent who is partially honest with a strictly positive probability.

We conclude that even if Dutta and Sen (2009,2012) have made a significant contribution that reveals the ramifications of partial honesty for implementation theory by dropping Maskin monotonicity, it remains again many very demanding social rules that can not be implemented in this domain of partial honesty. However, the honesty of players can be modeled in others ways and so it is therefore possible to capture the implementability of the family of choice rules considered in this paper and the implementability of others social choice rules that can be constructed by developing certain specific combinations. This is a fruitful area which we leave for future research.

References

[1] J.-P. Benoît and E.A. Ok. Maskin's theorem with limited veto power. *Games and Economic Behavior*, 55:331–339, 2006.

[2] M. Diss, A. Doghmi, and A. Tlidi. Strategy proofness and unanimity in private good economies with single-peaked preferences. *Working paper*, 2015.

[3] A. Doghmi. Nash implementation in an allocation problem with single-dipped preferences. *Games*, 4:38–49, 2013.

[4] A. Doghmi and A. Ziad. Reexamination of Maskin's theorem on Nash implementability. *Economics Letters*, 100:150–152, 2008a.

[5] A. Doghmi and A. Ziad. Nash implementation in exchange economies with single-peaked preferences. *Economics Letters*, 100:157–160, 2008b.

[6] A. Doghmi and A. Ziad. Nash implementation in private good economies with single-plateaued preferences. *Working paper*, 2012.

[7] A. Doghmi and A. Ziad. On partially honest Nash implementation in private good economies with restricted domains: A sufficient condition. *The B.E. Journal of Theoretical Economics*, 13:415–428, 2013.

[8] B. Dutta and A. Sen. Nash implementation with partially honest individuals. *Working Paper*, 2009. Published 2012 in *Games and Economic Behavior* 74, 154-169.

[9] R. Holden, N. Kartik, and O. Tercieux. Simple mechanisms and preferences for honesty. *Games and Economic Behavior*, 83:284–290, 2014.

[10] N. Kartik and O. Tercieux. Implementation with evidence. *Theoritical Economics*, 7:323–355, 2012.

[11] V. Korpela. Bayesian implementation with partially honest individuals. *Working paper, University of Turku*, 2012.

[12] M. Lombardi and N. Yoshihara. Partially-honest Nash implementation: A full characterization. *MPRA Working Paper No. 48234, ASBS, University of Glasgow, IER, Hitotsubashi University, available online at:http://mpra.ub.uni-muenchen.de/48234/*, 2013.

[13] E. Maskin. Nash equilibrium and welfare optimality. M.I.T. *mimeo*, 1977. Published 1999 in the *Review of Economic Studies* 66, 23-38.

[14] H. Matsushima. Role of honesty in full implementation. *Journal of Economic Theory*, 139:353–359, 2008 a.

[15] H. Matsushima. Behavioral aspects of implementation theory. *Economics Letters*, 100:161–164, 2008 b.

[16] J. Moore and R. Repullo. Nash implementation: a full characterization. *Econometrica*, 58:1083–1100, 1990.

[17] E. Pazner and D. Schmeidler. Egalitarian-equivalent allocations: a new concept of economic equity. *Quarterly Journal of Economics*, 92:671–687, 1978.

[18] T. Sjöström. On the necessary and sufficient conditions for Nash implementation. *Social Choice and Welfare*, 8:333–340, 1991.