On the Maximal Domain Theorem

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Abstract

The maximal domain theorem by Gul and Stacchetti (J. Econ. Theory **87** (1999), 95-124) implies that for markets with indivisible objects and sufficiently many agents, the set of gross substitutable preferences is a largest set for which the existence of a competitive equilibrium is guaranteed, and hence no relaxation of the gross substitutability can ensure the existence of a competitive equilibrium. However, we note that there is a flaw in their proof, and give an example to show that a claim used in the proof may fail to be true. We correct the proof and sharpen the result by showing that even there are only two agents in the market, if the preferences of one agent are not gross substitutable, then gross substitutable preferences can be found for another agent such that no competitive equilibrium exists. Moreover, we introduce the new notion of implicit gross substitutability, which is weaker than the gross substitutability condition and is still sufficient for the existence of a competitive equilibrium when the preferences of some agent are monotone.

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1 Introduction

An essential issue for markets with heterogeneous indivisible objects is under which conditions an efficient allocation of objects can be supported by a system of competitive prices as an equilibrium outcome.[1] A sufficient condition for the existence of a competitive equilibrium is the gross substitutability (GS) condition, which requires that objects are substitutes in the sense that the demand of each agent for an object does not decrease when prices of some other objects increase. Kelso and Crawford [7] introduce a price adjustment procedure and show that under gross substitutable preferences, such procedure will give rise to a competitive equilibrium.

Gul and Stacchetti [3] study markets with monotone preferences by adopting a less restrictive condition, the weak gross substitutability (WGS) condition, which requires that agents view objects as substitutes for each other when prices are non-negative. Based on the price adjustment procedure by Kelso and Crawford, they first note that under the monotonicity assumption, WGS preferences are sufficient for the existence of a competitive equilibrium. Then they prove that the WGS condition is also necessary in the maximal domain sense: for a market with sufficiently many agents, if the preferences of some agent violate the WGS condition, then WGS preferences can be found for other agents such that no competitive equilibrium exists.

Nevertheless, we note that there is a flaw in the proof of the maximal domain result by Gul and Stacchetti, and present an example to show that a claim used in their proof may fail to be true. To correct the proof, we give an equivalent characterization of the GS

[1] A sampling of relevant works includes Kelso and Crawford [7], Bikhchandani and Mame [2], Ma [8], Beviá et al. [1], Gul and Stacchetti [3, 4], Sun and Yang [10], and Teytelboym [11].

condition[2] and an alternative maximal domain result which shows that if the preferences of some agent fail the GS condition, we can construct GS preferences for another agent such that no competitive equilibrium exists in the two-agent market. This implies that even for markets with few agents, no relaxation of the GS condition (or the WGS condition together with the monotonicity assumption[3]) can guarantee the existence of a competitive equilibrium, improving upon the Gul-Stacchetti maximal domain theorem, but making it seem more difficult to give new existence results with conditions weaker then the gross substitutability.

One way to circumvent the above difficulty is to consider the markets in which not all agents have monotone preferences. It should be noted that, while monotonicity of preferences is a commonly used assumption in the literature, there are numerous economic situations in which monotonicity is not always satisfied.[4] For instance, an extra bed might be a burden for an agent with a small house. We introduce the new notion of implicit gross substitutability (IGS), which requires that allowing agents to dispose of undesirable objects for free will make objects become substitutes, and thus exhibits substitutability in an implicit way. We prove that the IGS condition is weaker than the WGS condition, and is still sufficient for the existence of a competitive equilibrium when there exists an agent with monotone preferences.

The rest of the paper is organized as follows. In Section 2, we recall the Gul-Stacchetti maximal domain theorem and give an example to show that there is a flaw in the proof. In Section 3, we give an alternative proof with a new characterization of the gross substitutability. Finally, we provide an existence result with the IGS condition in Section 4, and present two proofs in the Appendices.

[2]See Theorem 3 in Section 3.

[3]We prove that under monotonicity, GS and WGS are equivalent. (Corollary 6)

[4]See Manelli [9] and Hara [5, 6] for discussions on markets without the monotonicity assumption.

2 Gross substitutability as a maximal domain

Consider an economy with a finite set $N = \{1, \ldots, n\}$ of agents and a finite set $\Omega = \{a_1, \ldots, a_m\}$ of heterogeneous indivisible objects. Let $p = (p_a) \in \mathbb{R}^{|\Omega|}$ be a price vector, where p_a denotes the price of object $a \in \Omega$. Note that negative prices are allowed. For any bundle of objects $A \subseteq \Omega$, let $\chi_A \in \mathbb{R}^{|\Omega|}$ denote the characteristic price vector that has price 1 for objects $a \in A$ and price 0 for objects $a \notin A$. We assume that agents' net utility functions are quasilinear in prices in the sense that each agent i's utility of holding bundle $A \subseteq \Omega$ at price level p is

$$u_i(A, p) \equiv v_i(A) - p(A),$$

where $v_i : 2^\Omega \to \mathbb{R}$ is a valuation function satisfying $v_i(\emptyset) = 0$ and $p(A)$ is a shorthand for $\sum_{a \in A} p_a$. The valuation function v_i is called *monotone* if $v_i(B) \leq v_i(A)$ for $B \subseteq A \subseteq \Omega$. We also assume that agents are not subject to any budget constraints, and hence we can represent such an economy by $\mathcal{E} = \langle \Omega; (v_i, i \in N) \rangle$.

A *competitive equilibrium* for economy $\mathcal{E}$ is a pair $\langle p; \mathbf{X} \rangle$, where $\mathbf{X} = (X_1, \ldots, X_n)$ is a partition of objects among all agents and p is a price vector such that for all $i \in N$,

$$X_i \in D_{v_i}(p) \equiv \arg\max_{A \subseteq \Omega} u_i(A, p).$$

In that case, $\mathbf{X}$ is called an equilibrium allocation and p is called an equilibrium price vector. The possibility that $X_i = \emptyset$ for some agent i is allowed.

A crucial condition for the guaranteed existence of a competitive equilibrium is the *gross substitutability*. Formally, a valuation function $v_i : 2^\Omega \to \mathbb{R}$ is called *gross substi-*

tutable (GS) if for any vector $p \in \mathbb{R}^{|\Omega|}$, the following condition holds:

$$A \in D_{v_i}(p), p' \geq p \Rightarrow \exists B \in D_{v_i}(p') \text{ such that } \{a \in A : p_a = p'_a\} \subseteq B. \tag{1}$$

Moreover, we say that v_i is *weakly gross substitutable* (WGS) if condition (1) holds for all non-negative vectors $p \in \mathbb{R}_+^{|\Omega|}$. Note that WGS is strictly weaker than GS. Consider the function $v_i : 2^\Omega \to \mathbb{R}$ given by $\Omega = \{a, b, c\}$ and

$$v_i(A) = \begin{cases} 2, & \text{if A=\{a\}}, \\ 1, & \text{otherwise.} \end{cases}$$

It is not difficult to verify that v_i satisfies WGS, but violates GS.

Kelso and Crawford [7] introduce a price adjustment procedure and show that under gross substitutable preferences, such procedure will give rise to a competitive equilibrium. More precisely, a direct application of Theorem 2 of Kelso and Crawford [7] leads to the following result.

Theorem 1 (Kelso-Crawford) *For the economy $\mathcal{E} = \langle \Omega; (v_i, i \in N) \rangle$, there exists a competitive equilibrium if one of the following conditions holds:*

(a) *each agent's valuation function satisfies the GS condition.*

(b) *each agent's valuation function is monotone and satisfies the WGS condition.*

On the other hand, Theorem 2 of Gul and Stacchetti [3] shows that when there are sufficiently many agents and each agent's preferences are assumed to be monotone, the set of WGS preferences is a maximal domain for which the existence of a competitive equilibrium is guaranteed.

Theorem 2 (Gul-Stacchetti) *Let $v_1 : 2^\Omega \to \mathbb{R}$ be a monotone valuation function that violates the WGS condition. Then there exits an n-agent economy $\mathcal{E} = \langle \Omega; (v_1, \ldots, v_n) \rangle$ such that v_i is monotone and satisfis the WGS condition for $i = 2, \ldots, n$, but no competitive equilibrium exists in $\mathcal{E}$.*

To prove the above maximal domain theorem, Gul and Stacchetti [3, pp. 122-123] claim that if there exists a bundle $A \subseteq \Omega$ augmented with a vector $p \in \mathbb{R}^{|\Omega|}$ such that $|A \backslash B| > 1$ and $B \backslash A = \{b\}$, where B is an optimal solution for the problem

$$\begin{aligned} \arg\min \quad & |(A \backslash C) \cup (C \backslash A)| \\ s.t. \quad & v_1(C) - p(C) > v_1(A) - p(A), \end{aligned}$$

then no competitive equilibrium exists in the economy $\langle \Omega; (v_1, v_2, v_3, v_{a_1}, \ldots, v_{a_r}) \rangle$ given by $\Omega = A \cup B \cup \{a_1, \ldots, a_r\}$,

$$v_2(C) = \begin{cases} 0, & \text{if } C \cap (A \backslash B) = \emptyset, \\ \max\{p_a + v_1(\Omega) + 1 : a \in C \cap (A \backslash B)\}, & \text{otherwise,} \end{cases}$$

$$v_3(C) = \begin{cases} 0, & \text{if } C \cap [(A \backslash B) \cup \{b\}] = \emptyset, \\ \max\{p_a + v_1(\Omega) + 1 : a \in C \cap [(A \backslash B) \cup \{b\}]\}, & \text{otherwise,} \end{cases}$$

and,

$$v_{a_j}(C) = \begin{cases} v_1(\Omega) + 1, & \text{if } a_j \in C, \\ 0, & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, r$.

However, the following example shows that the claim is not correct.

Example 1 *Let $\Omega = \{a, b, c\}, A = \{a, c\}, B = \{b\}$, and let $p \in \mathbb{R}^{|\Omega|}$ be the vector such* *that* $p_a = p_c = 2$ *and* $p_b = 1$. *Consider the economy* $\mathcal{E} = \langle \Omega; (v_1, v_2, v_3) \rangle$ *given by*

$$v_1(C) = \begin{cases} 7, & \text{if } C = \{a,c\} \text{ or } \{a,b,c\}, \\ 5, & \text{if } C = \{b\}, \text{ or } \{a,b\} \text{ or } \{b,c\}, \\ 3, & \text{if } C = \{a\} \text{ or } \{c\}, \\ 0, & \text{if } C = \emptyset, \end{cases}$$

and

$$v_2(C) = \begin{cases} 10, & \text{if } C \cap A \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases} \qquad v_3(C) = \begin{cases} 10, & \text{if } C \cap A \neq \emptyset, \\ 9, & \text{if } C = \{b\}, \\ 0, & \text{if } C = \emptyset. \end{cases}$$

Clearly, the allocation $X_1 = \{b\}, X_2 = \{a\}, X_3 = \{c\}$ *can be supported by prices* $p_a = p_c = 2$ *and* $p_b = 1$ *as an equilibrium allocation.*

3 A correct proof of Theorem 2

Our approach relies on the notion of improvability, which requires that any suboptimal bundle $A \subseteq \Omega$ at price level $p \in \mathbb{R}^{|\Omega|}$ can be strictly improved by either removing an object from it, or adding a set of objects to it, or doing both. It should be noted that our improvability condition is similar to in spirit, but apparently weaker than the single improvement condition by Gul and Stacchetti [3]. Formally, a valuation function $v_i : 2^{\Omega} \to \mathbb{R}$ is said to be *improvable* (or *weakly improvable*) if for any price vector $p \in \mathbb{R}^{|\Omega|}$ (or non-negative price vector $p \in \mathbb{R}_+^{|\Omega|}$) and for any $A \in 2^{\Omega} \backslash D_{v_i}(p)$, there exists a bundle

$B \subseteq \Omega$ such that $|A \backslash B| \leq 1$ and $u_i(B,p) > u_i(A,p)$.

Theorem 3 *Consider the valuation function $v_i : 2^\Omega \to \mathbb{R}$.*

(a) *Assume that v_i is monotone. Then v_i satisfies the WGS condition if and only if it is weakly improvable.*

(b) *The valuation function v_i satisfies the GS condition if and only if it is improvable.*

Proof. See Appendix A. ■

The following result shows that even for markets with only two agents, the existence of a competitive equilibrium cannot be guaranteed by any relaxation of the GS condition.

Theorem 4 *Let $v_1 : 2^\Omega \to \mathbb{R}$ be a valuation function that violates the GS condition. Then there exists a GS valuation function v_2 such that no competitive equilibrium exists in the two-agent economy $\mathcal{E} = \langle \Omega; (v_1, v_2) \rangle$.*

Proof. Since v_1 violates the GS condition, the result of Theorem 3 (b) implies that v_1 is not improvable. Hence, there exit a vector $p^1 \in \mathbb{R}^{|\Omega|}$ and a bundle $A \notin D_{v_1}(p^1)$ such that the following condition holds:

$$C \subseteq \Omega \text{ and } u_1(C,p^1) > u_1(A,p^1) \Rightarrow |A \backslash C| \geq 2.$$

Let C^* be an optimal solution for the problem

$$\begin{array}{ll} \arg\min & |A \backslash C| \\ s.t. & u_1(C,p^1) > u_1(A,p^1) \end{array}$$

such that for any $C \subseteq \Omega$,

$$A \cap C = A \cap C^* \text{ and } u_1(C, p^1) > u_1(A, p^1) \Rightarrow |C \backslash A| \geq |C^* \backslash A.|$$

Consider the vector

$$p^2 = p^1 + \varepsilon \cdot \chi_{\Omega \backslash (A \cup C^*)} - \varepsilon \cdot \chi_{A \cap C^*} - \frac{u_1(C^*, p^1) - u_1(A, p^1)}{|A \backslash C^*|} \cdot \chi_{A \backslash C^*}.$$

Note that there exists some $\varepsilon > 0$ such that

$$A \in D_{v_1}(p^2) = \{C^*\} \cup \{C \subseteq \Omega : u_1(C, p^1) = u_1(A, p^1), A \subseteq C \subseteq A \cup C^*\}.$$

Let $M = \max\{|v_1(C)| : C \subseteq \Omega\}$ and let $\bar{p} \in \mathbb{R}^{|\Omega|}$ be the vector given by

$$\bar{p}_a = \begin{cases} p_a^2, & \text{if } a \in C^*, \\ p_a^2 - \delta, & \text{if } a \in A \backslash C^*, \\ M + 1, & \text{otherwise,} \end{cases} \tag{2}$$

where

$$\delta = \frac{1}{|A \backslash C^*|} \cdot \min\{u_1(A, p^2) - u_1(C, p^2) : C \notin D_{v_1}(p^2)\} > 0.$$

Let v_2 be the valuation function given by

$$v_2(C) = \bar{p}(C) + \begin{cases} M + 1, & \text{if } C \cap (A \backslash C^*) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Clearly, v_2 is gross substitutable since it is the sum of an additive function and a unit-demand function.[5]

We are going to prove that not competitive equilibrium exists in the economy $\mathcal{E} = \langle \Omega; (v_1, v_2) \rangle$. Suppose, to the contrary, that there exits an equilibrium $\langle p; (X_1, X_2) \rangle$ for $\mathcal{E}$. Since the allocation (X_1, X_2) must be efficient, we have $X_1 \subseteq A \cup C^*, X_2 \cap (A \backslash C^*) \neq \emptyset$, and hence $u_1(X_1, p^2) + \delta \cdot |A \backslash C^*| \leq u_1(C^*, p^2)$. Moreover, let $p' \in \mathbb{R}^{|\Omega|}$ be the vector given by

$$p'_a = \begin{cases} p^2_a, & \text{if } a \in C^*, \\ \\ p_a, & \text{otherwise,} \end{cases}$$

then $\langle p'; (X_1, X_2) \rangle$ is also a competitive equilibrium for $\mathcal{E}$. We consider two cases.

Case I. $X_1 \backslash C^* \neq \emptyset$. By (2) and (3), we have that $p'_a \geq p^2_a - \delta$ for each $a \in X_1 \backslash C^*$. This implies

$$\begin{aligned} u_1(X_1, p') &\leq u_1(X_1, p^2) + \delta \cdot |X_1 \backslash C^*| < u_1(X_1, p^2) + \delta \cdot |A \backslash C^*| \\ &\leq u_1(C^*, p^2) = u_1(C^*, p'). \end{aligned}$$

Since $X_1 \in D_{v_1}(p')$, this is impossible.

Case II. $X_1 \subseteq C^*$. Then we have $A \backslash C^* \subseteq X_2$. Since $|A \backslash C^| \geq 2$, it follows that $p'_a \leq p^2_a - \delta$ for each $a \in A \backslash C^*$, and hence

$$u_1(A, p') \geq u_1(A, p^2) + \delta \cdot |A \backslash C^*| > u_1(X_1, p^2) = u_1(X_1, p').$$

This is also impossible. ■

[5]A valuation function $v_i : 2^\Omega \to \mathbb{R}$ is *additive* if there exists a vector $p \in \mathbb{R}^{|\Omega|}$ such that $v_i(C) = p(C)$ for all $C \subseteq \Omega$. A monotone function v_i is *unit-demand* if $v_i(C) = max_{a \in C} v_i(\{a\})$ for all $C \subseteq \Omega$. One can easily check that the sum of an additive function and a unit-demand function is gross substitutable.

The following result improves upon the Gul-Stacchetti maximal domain theorem, and implies that even for markets with few agents, no relaxation of the weak gross substitutability, together with the monotonicity, can ensure the existence of a competitive equilibrium.

Theorem 5 *Assume that there are n agents and $n \geq 2$. Let $v_1 : 2^\Omega \to \mathbb{R}$ be a monotone valuation function that violates the WGS condition. Then there exists a set of monotone and WGS valuation functions, $\{v_2, \ldots, v_n\}$, such that no competitive equilibrium exists in the economy $\langle \Omega; (v_1, v_2, \ldots, v_n) \rangle$.*

Proof. Since v_1 violates the WGS condition and hence violates the GS condition, by Theorem 4, there exits a GS valuation function w_2 such that no competitive equilibrium exists in the economy $\langle \Omega; (v_1, w_2) \rangle$. Let $\hat{w}_2$ denote the valuation function given by

$$\hat{w}_2(A) = \max\{w_2(C) : C \subseteq A\} \text{ for } A \subseteq \Omega.$$

We first prove that $\hat{w}_2$ satisfies the GS condition.

Let w_3 be the valuation function given by $w_3(A) = 0$ for $A \subseteq \Omega$, and let w_4 be an arbitrary GS valuation function. By Theorem 1, we know that there exists a competitive equilibrium $\langle p; (X_2, X_3, X_4) \rangle$ for the economy $\langle \Omega; (w_2, w_3, w_4) \rangle$. For each bundle $A \subseteq \Omega$, let A' denote a subset of A such that $\hat{w}_2(A) = w_2(A') = \hat{w}_2(A')$. Then we have that for any $A \subseteq \Omega$,

$$\begin{aligned}
\hat{w}_2(X_2 \cup X_3) - p(X_2 \cup X_3) &\geq [w_2(X_2) - p(X_2)] + [w_3(X_3) - p(X_3)] \\
&\geq [w_2(A') - p(A')] + [w_3(A \backslash A') - p(A \backslash A')] \\
&= \hat{w}_2(A) - p(A).
\end{aligned}$$

This implies that $\langle p; (X_2 \cup X_3, X_4)\rangle$ is a competitive equilibrium for economy $\langle \Omega; (\hat{w}_2, w_4)\rangle$. Since w_4 is an arbitrary GS valuation function, the result of Theorem 4 implies that $\hat{w}_2$ satisfies the GS condition.

Consider the economy $\mathcal{E} = \langle \Omega; (v_1, \ldots, v_n)\rangle$, where $v_2 = \hat{w}_2$ and $v_i = w_3$ for $i \geq 3$. We are going to prove that no competitive equilibrium exists in $\mathcal{E}$. Suppose, to the contrary, that there is a competitive equilibrium $\langle q; (Y_1, \ldots, Y_n)\rangle$ for $\mathcal{E}$. Since each agent's valuation function is monotone, we have $q_a \geq 0$ for all $a \in \Omega$, and without loss of generality, we may assume that $Y_i = \emptyset$ for $i \geq 3$.

Let Y_2' be a subset of Y_2 such that $\hat{w}_2(Y_2) = w_2(Y_2') = \hat{w}_2(Y_2')$. Then for any $A \subseteq \Omega$, we have

$$\begin{aligned} w_2(A) - q(A) &\leq v_2(A) - q(A) \leq v_2(Y_2) - q(Y_2) = v_2(Y_2') - q(Y_2') - q(Y_2 \backslash Y_2') \\ &\leq v_2(Y_2') - q(Y_2') = w_2(Y_2') - q(Y_2'), \end{aligned}$$

which implies $Y_2' \in D_{w_2}(q)$ and $q_a = 0$ for all $a \in Y_2 \backslash Y_2'$. Since v_1 is monotone, it follows that $Y_1 \cup (Y_2 \backslash Y_2') \in D_{v_1}(q)$, contradicting to the fact that no competitive equilibrium exists in $\langle \Omega; (v_1, w_2)\rangle$. ■

Finally, we prove that under monotonicity, WGS and GS are equivalent. Based on this and Theorem 5, it can be shown that for markets without the monotonicity assumption, the set of GS preferences is a maximal domain for which the existence of a competitive equilibrium is guaranteed.

Theorem 6 *A monotone valuation function $v_i : 2^\Omega \to \mathbb{R}$ satisfies the GS condition if and only if it satisfies the WGS condition.*

Proof. Let $v_i : 2^\Omega \to \mathbb{R}$ be a monotone valuation function. By Theorem 3, it suffices to prove that v_i is improvable whenever it is weakly improvable. Assume that v_i is weakly improvable. Let $p \in \mathbb{R}^{|\Omega|}$ be a price vector such that $p_a < 0$ for some $a \in \Omega$ and choose an arbitrary bundle of objects $A \subseteq \Omega$ such that $A \notin D_{v_i}(p)$. Let $p^+ \in \mathbb{R}^{\Omega}_{+}$ denote the vector given by

$$p_a^+ = \begin{cases} p_a, & \text{if } p_a \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and let $\Omega' = \{a \in \Omega : p_a \neq p_a^+\}$. Since v_i is monotone, we have $C \cup \Omega' \in D_{v_i}(p^+)$ for all $C \in D_{v_i}(p^+)$, and hence

$$D_{v_i}(p) = \{C \cup \Omega' : C \in D_{v_i}(p^+)\} \subseteq D_{v_i}(p^+). \tag{4}$$

We consider two cases.

Case I. $A \cup \Omega' \in D_{v_i}(p^+)$. Let $B = A \cup \Omega'$. Since $A \notin D_{v_i}(p)$, by (4), it follows that A is a proper subset of B and $u_i(B, p) > u_i(A, p)$.

Case II. $A \cup \Omega' \notin D_{v_i}(p^+)$. Since v_i is weakly improvable, there exists $B' \subseteq \Omega$ such that $|(A \cup \Omega') \backslash B'| \leq 1$ and $u_i(B', p^+) > u_i(A \cup \Omega', p^+)$. Let $B = B' \cup \Omega'$. Since v_i is monotone, we have

$$u_i(B, p^+) \geq u_i(B', p^+) > u_i(A \cup \Omega', p^+) \geq u_i(A, p^+),$$

and hence $u_i(B, p) > u_i(A, p)$. ■

Corollary 1 *Assume that there are n agents and $n \geq 2$. Let $v_1 : 2^{\Omega} \to \mathbb{R}$ be a valuation function that violates the GS condition. Then there exists a set of GS valuation functions, $\{v_2, \ldots, v_n\}$, such that no competitive equilibrium exists in the economy $\langle \Omega; (v_1, v_2, \ldots, v_n) \rangle$.*

4 Implicit gross substitutability

The maximal domain results studied in Section 3 makes it seem difficult to establish existence results with relaxations of the WGS condition. To make a breakthrough, we first introduce the notion of implicit gross substitutability (IGS), which is inspired by the idea of "free disposal" condition, and generalizes the WGS condition. Then we prove that the IGS condition is sufficient for the existence of a competitive equilibrium when the preferences of some agent are known to be monotone.

Monotonicity of preferences is a commonly used assumption in the economic literature. This assumption can be justified by offering free disposal of unwanted objects. In that case, possessing more objects does not make any agent worse off, and each agent i's original valuation function v_i would thereby be replaced by its *monotone cover* $\hat{v}_i$, i.e., the valution function given by

$$\hat{v}_i(A) = \max\{v_i(C) : C \subseteq A\} \text{ for } A \subseteq \Omega.$$

A valuation function $v_i : 2^\Omega \to \mathbb{R}$ is called *implicitly gross substitutable* (IGS) if its monotone cover $\hat{v}_i$ is gross substitutable. Roughly speaking, the IGS condition requires that allowing agents to dispose of undesirable objects for free will make objects become substitutes for each other, and thus exhibits substitutability in an implicit way. The following result shows that IGS is weaker than WGS.

Theorem 7 *The monotone cover $\hat{v}_1$ of a WGS valuation function $v_1 : 2^\Omega \to \mathbb{R}$ satisfies the GS condition.*

Proof. Let v_1 be a WGS valuation function. Consider the price adjustment procedure of Kelso and Crawford [7] for the economy $\mathcal{E} = \langle \Omega; (v_1, v_2, v_3) \rangle$, where v_2 is the valuation function given by $v_2(A) = 0$ for all $A \subseteq \Omega$ and v_3 is an arbitrary GS valuation function. Since v_2 is monotone and each valuation function satisfies WGS, it follows that each object will receive at least one offer at the initial zero price vector $\mathbf{0} \in \mathbb{R}^{|\boldsymbol{\Omega}|}$ and the procedure will terminate at a competitive equilibrium $\langle p; (X_1, X_2, X_3) \rangle$ such that $p \in \mathbb{R}_+^{|\Omega|}$ and $p_a = 0$ for $a \in X_2$. For any bundle $A \subseteq \Omega$, let A' be a subset of A such that $\hat{v}_1(A) = v_1(A')$. Note that $v_2(X_2) - p(X_2) \geq 0$. This implies that for any $A \subseteq \Omega$, $\hat{v}_1(X_1 \cup X_2) - p(X_1 \cup X_2) \geq [v_1(X_1) - p(X_1)] + [v_2(X_2) - p(X_2)] \geq v_1(A') - p(A') = \hat{v}_1(A) - p(A) + p(A \backslash A') \geq \hat{v}_1(A) - p(A)$, and hence $\langle p; (X_1 \cup X_2, X_3) \rangle$ is a competitive equilibrium for the economy $\langle \Omega; (\hat{v}_1, v_2) \rangle$. Together with Theorem 4, it follows that $\hat{v}_1$ satisfies GS. ■

We conclude the paper with a new existence result, Theorem 9, in which we try to extend Theorem 1 with the notion of IGS condition. The result of Theorem 9 relies on a more general observation which shows that when there exists an agent with monotone preferences, the existence of a competitive equilibrium is irrelevant to whether agents are allowed to dispose of undesirable objects for free.

Theorem 8 *Let $\mathcal{E} = \langle \Omega; (v_i, i \in N) \rangle$ be an economy and denote $\hat{\mathcal{E}} \equiv \langle \Omega; (\hat{v}_i, i \in N) \rangle$. If v_1 is monotone, then the following results hold:*

(a) *Each equilibrium allocation for $\mathcal{E}$ is also an equilibrium allocation for $\hat{\mathcal{E}}$.*

(b) *Each equilibrium price vector for $\hat{\mathcal{E}}$ is also an equilibrium price vector for $\mathcal{E}$.*

(c) *$\mathcal{E}$ has a competitive equilibrium if and only if $\hat{\mathcal{E}}$ has a competitive equilibrium.*

Proof. See Appendix B. ■

Theorem 9 *For any economy $\mathcal{E} = \langle \Omega; (v_i, i \in N) \rangle$, there exists a competitive equilibrium if v_1 is monotone and each agent i's valuation function v_i satisfies IGS.*

Proof. Assume that v_1 is monotone and v_i satisfies IGS for $i = 1, \ldots, n$. This implies that $\hat{v}_i$ satisfies GS for $i = 1, \ldots, n$, and hence there exists a competitive equilibrium for the economy $\langle \Omega; (\hat{v}_1, \ldots, \hat{v}_n) \rangle$ by Theorem 1. Combining with Theorem 8, we obtain the desired result. ■

Appendix A. Proof of Theorem 3

The proof of Theorem 3 requires the following lemma.

Lemma 1 *Suppose that the valuation function $v_i : 2^\Omega \to \mathbb{R}$ is weakly improvable. Then for price vectors $p, p' \in \mathbb{R}^{|\Omega|}$ with $p' \geq p$ and for $A \in D_{v_i}(p) \backslash D_{v_i}(p')$, there exists $A^* \in \arg\min_{C \in D_{v_i}(p)}[p'(C) - p(C)]$ such that $\{a \in A : p'_a = p_a\} \subseteq A^*$.*

Proof. Let $C^* \in \arg\min_{C \in D_{v_i}(p)}[p'(C) - p(C)]$ and let $X = \{a \in A \backslash C^* : p'_a > p_a\}$. In case $X = \emptyset$, we have $A \in \arg\min_{C \in D_{v_i}(p)}[p'(C) - p(C)]$ and the proof is done. Assume that $X = \{a_1, \ldots, a_r\} \neq \emptyset$. Since v_i is weakly improvable and

$$\{A, C^*\} \subseteq D_{v_i}(p + \chi_{\Omega \backslash (A \cup C^*)}) = \{C \in D_{v_i}(p) : C \subseteq A \cup C^*\},$$

we can find a small positive number ε for which there exists

$$A_1 \in D_{v_i}(p + \chi_{\Omega \backslash (A \cup C^*)} + \varepsilon \chi_{\{a_1\}}) \subseteq D_{v_i}(p + \chi_{\Omega \backslash (A \cup C^*)})$$

such that $A \backslash \{a_1\} \subseteq A_1 \subseteq A \cup C^*$.

Inductively, we can construct a sequence of sets, $A_1, \ldots, A_r \in D_{v_i}(p + \chi_{\Omega \backslash (A \cup C^*)})$, such that $A \backslash \{a_1, \ldots, a_i\} \subseteq A_i \subseteq A \cup C^*$ for $i = 1, \ldots, r$. Since $A \backslash X \subseteq A_r$, it follows that $\{a \in A : p'_a = p_a\} \subseteq A_r$ and $\{a \in A_r \backslash C^* : p'_a > p_a\} = \emptyset$, and hence $A_r \in$

$\arg\min_{C \in D_{v_i}(p)}[p'(C) - p(C)]$. ■

We are now ready to prove Theorem 3.

(a) ($\Rightarrow$) Suppose that v_i is weakly gross substitutable, but there exists a bundle of objects $A \notin D_{v_i}(p)$ for some $p \in \mathbb{R}_+^{|\Omega|}$ such that $|A \backslash C| > 1$ for all $C \in \Gamma(A,p) \equiv \{C \subseteq \Omega : u_i(C,p) > u_i(A,p)\}$. Let $B \in \Gamma(A,p)$ be a bundle such that $|A \backslash B| \leq |A \backslash C|$ for all $C \in \Gamma(A,p)$. It follows that there exist two distinct objects $a, b \in A \backslash B$ and a price vector $p^1 = p + \varepsilon_1 \chi_{\Omega \backslash (A \cup B)}$ for some $\varepsilon_1 > 0$ such that $A \backslash C = A \backslash B$ for each bundle $C \in \Gamma(A, p^1)$

Let $B^* \in D_{v_i}(p^1)$. Since v_i is monotone and $A \cup B^* \notin \Gamma(A, p^1)$, we have

$$\begin{aligned} u_i(A,p^1) &\geq v_i(A \cup B^*) - p^1(A \cup B^*) \\ &\geq v_i(B^*) - p^1(B^*) - p^1(A \backslash B^*) \\ &= u_i(B^*, p^1) - p^1(A \backslash B^*), \end{aligned}$$

and hence $p^1(A \backslash B^*) \geq u_i(B^*, p^1) - u_i(A, p^1)$. Let $\lambda = [u_i(B^*, p^1) - u_i(A, p^1)]/p^1(A \backslash B^*)$ and let $p^2 \in \mathbb{R}_+^{|\Omega|}$ be the vector given by

$$p_a^2 = \begin{cases} p_a^1 - \lambda \cdot p_a^1, & \text{it } a \in A \backslash B^*, \\ p_a^1, & \text{otherwise.} \end{cases}$$

Clearly, $D_{v_i}(p^2) = D_{v_i}(p^1) \cup \{A\}$, Therefore, when the price increases from p^2 to $p^3 = p^2 + e^{\{a\}}$, no bundles in $D_{v_i}(p^3)$ would contain b, violating the WGS condition.

($\Leftarrow$) Let p and p' be two distinct nonnegative vectors in $\mathbb{R}_+^{|\Omega|}$ such that $p' \geq p$ and let A be a set of objects such that $A \in D_{v_i}(p) \backslash D_{v_i}(p')$.

Note that since $A \notin D_{v_i}(p')$, there exists a positive number $t_1 \in (0,1)$ such that $A_1 \in D_{v_i}(t_1 p' + (1-t_1) p^0)$ and $A_1 \notin D_{v_i}(tp' + (1-t) p^0)$ for $t > t_1$. Let $p^1 = t_1 p' + (1-t_1) p^0$.

By (a) again, there exists

$$A_2 \in \arg \min_{C \in D_{v_i}(p^1)} \left[p'(C) - p^1(C)\right]$$

such that $\{a \in A_1 : p'_a = p^1_a\} \subseteq A_2$. Since $\{a \in \Omega : p'_a = p^0_a\} = \{a \in \Omega : p'_a = p^1_a\}$, it follows that

$$\left\{a \in A_0 : p'_a = p^0_a\right\} \subseteq \left\{a \in A_1 : p'_a = p^1_a\right\} \subseteq A_2.$$

In case $A_2 \in D_{v_i}(p')$, the proof is done. Otherwise, there exists a positive number $t_2 \in (0,1)$ such that $A_2 \in D_{v_i}(t_2 p' + (1-t_2)p^1)$ and $A_2 \notin D_{v_i}(tp' + (1-t)p^1)$ for $t > t_2$. Let $p^2 = t_2 p' + (1-t_2)p^1$. Using (a), there exists

$$A_3 \in \arg \min_{C \in D_{v_i}(p^2)} \left[p'(C) - p^2(C)\right]$$

such that $\{a \in A_2 : p'_a = p^2_a\} \subseteq A_3$.

Since the number of sets of objects is finite, we may inductively construct a finite sequence of distinct price vectors $p' = p^r \geq p^{r-1} \geq \cdots \geq p^1 \geq p^0$ and a finite sequence of distinct sets of objects $A_0, A_1, \ldots, A_r$ such that $A_r \in D_{v_i}(p')$ and for $k = 1, \ldots, r$,

1. $A_k \in \arg\min_{C \in D_{v_i}\left(p^{k-1}\right)} \left[p'(C) - p^{k-1}(C)\right]$,

2. $\left\{a \in A_{k-1} : p'_a = p^{k-1}_a\right\} \subseteq A_k$, and

3. $\left\{a \in \Omega : p'_a = p^{k-1}_a\right\} \subseteq \left\{a \in \Omega : p'_a = p^k_a\right\}$.

This implies $\{a \in A : p'_a = p_a\} \subseteq A_r$, and hence completes the proof of (a).

(b) For any arbitrary vector $p \in \mathbb{R}^{|\Omega|}$, let $p' \in \mathbb{R}^{|\Omega|}$ be the vector given by

$$p'_a = \begin{cases} p_a, & \text{if } \forall A \subseteq \Omega \backslash \{a\}, p_a \leq v_i(A \cup \{a\}) - v_i(A), \\ \min_{A \subseteq \Omega \backslash \{a\}}[v_i(A \cup \{a\}) - v_i(A)], & \text{otherwise.} \end{cases}$$

Let v_i^p be the monotone valuation function given by $v_i^p(A) = v_i(A) - p'(A)$ for $A \subseteq \Omega$ and $\bar{p} \in \mathbb{R}_+^{|\Omega|}$ the non-negative vector such that $\bar{p}_a = p_a - p'_a$ for $a \in \Omega$. Since $v_i^p(A) - \bar{p}(A) = v_i(A) - p(A)$ for all $A \subseteq \Omega$, we note that

(i) v_i satisfies GS if and only if v_i^p satisfies WGS for all $p \in \mathbb{R}^{|\Omega|}$; and

(ii) v_i is improvable if and only if v_i^p is weakly improvable for all $p \in \mathbb{R}^{|\Omega|}$.

Putting (i), (ii) and (a) together yields the desired result.

Appendix B. Proof of Theorem 8

(a) Assume that $\langle p, \mathbf{X} \rangle$ is a competitive equilibrium for $\mathcal{E}$. We are going to prove that $\mathbf{X}$ is an equilibrium allocation for $\hat{\mathcal{E}}$. Let $p' \in \mathbb{R}_+^{|\Omega|}$ be the price vector given by

$$p'_a = \begin{cases} p_a, & \text{if } p_a \geq 0, \\ 0, & \text{if } p_a < 0. \end{cases}$$

We first prove that $\langle p', \mathbf{X} \rangle$ is a competitive equilibrium for $\mathcal{E}$. Let $\bar{A} = \{a \in \Omega : p_a < 0\}$. In case there exists $a \in \bar{A} \backslash X_1$, since v_1 is monotone, we have

$$v_1(X_1 \cup \{a\}) - p(X_1 \cup \{a\}) \geq v_1(X_1) - p(X_1) - p_a > v_1(X_1) - p(X_1),$$

violating the fact $X_1 \in D_{v_1}(p)$. This implies $\bar{A} \subseteq X_1$, and hence we have $X_i \in D_{v_i}(p')$ for $i \neq 1$ and for each bundle $A \in 2^{\Omega}$,

$$\begin{aligned} v_1(X_1) - p'(X_1) &= [v_1(X_1) - p(X_1)] + p(\bar{A}) \geq [v_1(A \cup \bar{A}) - p(A \cup \bar{A})] + p(\bar{A}) \\ &= v_1(A \cup \bar{A}) - p'(A \cup \bar{A}) \geq v_1(A) - p'(A). \end{aligned}$$

We next prove that $\hat{v}_i(X_i) = v_i(X_i)$ for all $i \in N$. In case there exists an agent $i \neq 1$ such that $\hat{v}_i(X_i) > v_i(X_i)$, there exists a proper subset B of X_i such that $\hat{v}_i(X_i) = v_i(B) = \hat{v}_i(B)$. Together with the fact $p_a \geq 0$ for all $a \in X_i$, we have $v_i(B) - p(B) > v_i(X_i) - p(B) \geq v_i(X_i) - p(X_i)$. Since $X_i \in D_{v_i}(p)$, this is impossible.

We are now ready to prove that $\langle p', \mathbf{X} \rangle$ is also a competitive equilibrium for $\hat{\mathcal{E}}$. In case there exists an agent $j \neq 1$ such that $\hat{v}_j(X_j) - p'(X_j) < \hat{v}_j(C) - p'(C)$ for some bundle $C \in 2^{\Omega}$. Since $X_j \in D_{v_j}(p')$ and $\hat{v}_j(X_j) = v_j(X_j)$, we have

$$v_j(C) - p'(C) \leq v_j(X_j) - p'(X_j) = \hat{v}_j(X_j) - p'(X_j) < \hat{v}_j(C) - p'(C).$$

This implies $v_j(C) < \hat{v}_j(C)$ and $\hat{v}_j(C) = v_j(C')$ for some proper subset C' of C, and hence

$$v_j(C') - p'(C') \geq \hat{v}_j(C) - p'(C) > v_j(X_j) - p'(X_j),$$

contradicting to the fact $X_j \in D_{v_j}(p')$.

(b) Assume that $\langle p, \mathbf{X} \rangle$ is a competitive equilibrium for $\hat{\mathcal{E}}$. Note that since all agents in $\hat{\mathcal{E}}$ have monotone preference, the equilibrium price vector p must be nonnegative. We are going to construct a competitive equilibrium $\langle p, \mathbf{Y} \rangle$ for $\mathcal{E}$ such that for $i \neq 1$, $Y_i \subseteq X_i$ and $\hat{v}_i(X_i) = v_i(Y_i) = \hat{v}_i(Y_i)$, and $Y_1 = [\cup_{i \neq 1}(X_i \backslash Y_i)] \cup X_1$.

For each $i \in \{2, \ldots, n\}$, we choose $Y_i \subseteq X_i$ such that $\hat{v}_i(X_i) = v_i(Y_i) = \hat{v}_i(Y_i)$. Since

$X_i \in D_{\hat{v}_i}(p)$, we have

$$\hat{v}_i(X_i) - p(X_i) \geq \hat{v}_i(Y_i) - p(Y_i) = \hat{v}_i(X_i) - p(Y_i) \geq \hat{v}_i(X_i) - p(X_i).$$

This implies $p_a = 0$ for $a \in X_i \backslash Y_i$, and for any subset A of Ω,

$$v_i(Y_i) - p(Y_i) = \hat{v}_i(X_i) - p(X_i) \geq \hat{v}_i(A) - p(A) \geq v_i(A) - p(A).$$

Let $Y_1 = [\cup_{i \neq 1}(X_i \backslash Y_i)] \cup X_1$. Since v_1 is monotone and $p_a = 0$ for all $a \in \cup_{i \neq 1}(X_i \backslash Y_i)$, it follows that for any subset A of Ω,

$$\begin{aligned} v_1(Y_1) - p(Y_1) &\geq v_1(X_1) - p(X_1) = \hat{v}_1(X_1) - p(X_1) \\ &\geq \hat{v}_1(A) - p(A) = v_1(A) - p(A), \end{aligned}$$

and the proof of (b) is done.

Finally, the result of (c) is an immediate consequence of the combination of (a) and (b).

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