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2015

Online at <https://mpra.ub.uni-muenchen.de/67334/>
MPRA Paper No. 67334, posted 20 Oct 2015 05:08 UTC

Non-manipulable rules for land rental problems*

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October 19, 2015

Abstract

We consider land rental problems where there are several communities that can act as lessors and a single tenant who does not necessarily need all the available land. A rule should determine which communities become lessors, how much land they rent and at which price. Our first result is a complete characterization of the family of rules that satisfy land monotonicity and non-manipulability under land reassignment. We also define two parametric subfamilies. The first one is characterized by adding a property of weighted standard for two-person. The second one is characterized by adding consistency and continuity.

Keywords: *land rental, non-manipulability, land reassignment, land monotonicity, consistency, standard for two-person.*

*Latest version at <http://vidalpuga.webs.uvigo.es/>. Alfredo Valencia-Toledo thanks the Ministry of Education of Peru for its financial support through the grant “Beca Presidente de la República del Programa Nacional de Becas y Crédito Educativo” (PRONABEC). Juan Vidal-Puga acknowledges financial support from the Spanish Ministerio de Ciencia e Innovación through grant ECO2011-23460 and from the Spanish Ministerio de Economía y Competitividad through grant 10PXIB362299PR.

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1 Introduction

The management of land and natural resources is one of the most critical challenges facing developing countries (Kaye and Yahya, 2012). In particular, natural resource exploitation is an industrial activity that has recently been generating conflicts between firms and indigenous communities in many countries in Latin America, Africa and Asia. Examples include Mexico (Tetreault, 2015), Guatemala (Walter and Urkidi, 2015), India (Sosa, 2011), Peru (Arellano-Yanguas, 2011), Sierra Leone (Akiwumi, 2014) and Indonesia (Welker, 2009). Indirectly, land tenure also has a major influence in labour policy (Dell, 2010). Another example is a restitution problem in Colombia where two agents have rights over the land (Jaramillo et al., 2014).

In these land conflicts, there exist rights over the land for each side. For the case of mining activities, article 10 of the United Nations Declaration on the Rights of Indigenous People defined Free Prior and Informed Consent (FPIC) as the principle that indigenous communities have the right to give or withhold its consent to proposed projects that may affect the land they customarily own, occupy or otherwise use (UN, 2007). On the other hand, the mining firm has an investment and a concession over those lands, or, even if a concession has not been granted yet, the firm may have a profit opportunity high enough to make it possible to compensate the land owners in a fair way (Helwege, 2015). In order to solve these land conflicts, it is fundamental for the planner (e.g. the government) to have all the relevant information about both sides.

In many situations, land identification and demarcation may be not clear, as in the case of customary land (Gildenhuys, 2005; Azima et al., 2015). This situation can lead to manipulation by merging or splitting of the communities, due to the fact that they may have incentives to strategically misrepresent their identity in order to influence the final outcome to their own advantage. The study of this kind of manipulation is common in the strategy-proofness literature in the context of cost sharing (Moulin and Shenker, 2001; Sprumont, 2005; Gómez-Rúa and Vidal-Puga, 2011; Ju, 2013; Massó

et al., 2015), resource allocation (Erlanson and Flores-Szwagrzak, 2015), job scheduling (Moulin, 2007, 2008), indivisible object allocation (Sun and Yang, 2003; Svensson, 2009; Morimoto and Serizawa, 2015), assigning problems (Kojima and Manea, 2010), and taxation problems (Ju and Moreno-Tertero, 2011), among others. Splitting and merging proofness have also been deeply studied in bankruptcy problems where an estate $E > 0$ should be divided among a set of claimants N with claims given by $c \in \mathbb{R}^N$. Several authors (O'Neill, 1982; Moulin, 1987; Chun, 1988; de Frutos, 1999; Ju, 2003; Moreno-Tertero, 2006, 2007; Ju et al., 2007) have showed that merging and splitting proofness in bankruptcy problems leads to a proportional share of the estate. See for example Thomson (2003, 2015a).

In this article, we assume that the government or planner seeks to assign a price and amount of land fairly, stably and efficiently, and at the same time, to guarantee non-manipulability by land reassignment. We model this situation as a land rental problem, where there is a single tenant who can be a mining firm, and several lessors who can be a group of communities. Each community has some available amount of land c_i with a reservation price r per unit, that for simplicity we consider equal for all of them. The mining firm needs to rent an optimal amount of land E , which is a completely divisible object¹². The mining firm assures an optimal profit K over E .

A rule determines, for each land rental problem, a quantity of land to be rented by each community and a price that the mining firm must pay as a way of compensation.

In order to study rules that guarantee non-manipulability, we propose a version of strategy-proofness such that communities should not find it profitable to re-assign the land among them. Also, we propose a version of land monotonicity that assures fairness, in the sense that an increase in the quan-

¹We can also consider other situations with similar characteristics, in the sense that there are several land communities or owners, and a single agent who is interested in their land. For example, construction of facilities (hotels, malls, convention centers, schools, resorts, etc.), urbanisation in populated areas, or settlement of governmental bases.

²We use the terms c and E because of their resemblance to bankruptcy problems.

tity of available land affects positively the final profits to both sides.

Our first result is a complete characterization of the family of rules that satisfy land reassignment and land monotonicity. By adding a property inspired by standard for two-person in Hart and Mas-Colell (1989), we characterize a parametric subfamily. Another property is consistency, that states that the rule should behave in a similar way independently of the number of agents involved. This is a classical property in cooperative games (see van den Brink et al. (2013) and Huettner (2015) for two recent applications), and it has also been studied in bankruptcy problems (see Thomson (2008, 2015b) and references herein) and cost sharing problems (see for example Albizuri and Zarzuelo (2007) and Koster (2012)). By adding consistency and continuity we characterize another parametric subfamily of rules. The intersection of both parametric subfamilies singles out two particular rules: one of them optimal for the tenant and the other optimal for the lessors.

We organize the paper as follows: In Section 2, we present the model. In Section 3, we study and characterize the family of rules that satisfy land reassignment and land monotonicity. In Section 4, we characterize the family of rules that also satisfy weighted standard for two-person. Finally, in Section 5, we characterize the subfamily of rules that satisfy land reassignment, land monotonicity, consistency and continuity.

2 The model

Let $\mathbb{N}_+ = \{1, 2, \dots\}$ be the set of potential lessors and let \mathcal{N} be the set of non-empty finite subsets of \mathbb{N}_+ . Let $N = \{1, 2, \dots, n\}$ be a generic set of lessors, and let S be a generic subset of N . Given $y \in \mathbb{R}^S$, we write $y(S) = \sum_{i \in S} y_i$. Given $x, y \in \mathbb{R}^S$, we write $x \leq y$ when $x_i \leq y_i$ for all $i \in S$. Moreover 0_S denotes the vector $(0, \dots, 0) \in \mathbb{R}^S$. We denote the set of nonnegative real numbers as \mathbb{R}_+ , and the set of positive real numbers as \mathbb{R}_{++} .

A *land rental problem* is a tuple (N_0, μ, c, r) where $N_0 = \{0\} \cup N$ is the

set of agents with 0 the unique tenant and N the set of lessors, $c \in \mathbb{R}_{++}^N$ is the vector whose coordinates represent the amount of available land for each lessor, $r \in \mathbb{R}_+$ is the reservation price per unit of land for lessors, and $\mu : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a function that assigns to each amount of land the tenant's profit when that amount is rented. Hence, the aggregate profit when the tenant rents l units of land is $\mu(l) + r(c(N) - l)$. We normalise $\mu(0) = 0$, and assume there exists a unique $E \in]0, c(N)]$ such that $\mu(E) + r(c(N) - E)$ is maximum³ on $[0, c(N)]$ ⁴. We then denote $K = \mu(E)$ as the optimal profit that the agents can obtain. This implies that $K > rE$, i.e. there exists benefit of cooperation.

Under these conditions, an efficient allocation implies that the amount of rented land is E and the profit of the tenant is K . Thus, only E and K are relevant. Furthermore, for convenience we use N instead of N_0 . Hence, we denote the land rental problem as (N, K, E, c, r) instead of (N_0, μ, c, r) . Let \mathcal{L} be the set of all land rental problems.

A feasible agreement is a pair $(x, p) \in \mathbb{R}_+^N \times \mathbb{R}_+$ with $x \leq c$, where x_i is the land rented by lessor $i \in N$, and p is the price per unit of land. The set of feasible agreements on a land rental problem L is denoted as A^L . Let $\mathcal{A} = \bigcup_{L \in \mathcal{L}} A^L$ be the set of all potential feasible agreements.

Given $(x, p) \in A^L$, the utility for tenant and each lessor $i \in N$ are $u_0(x, p) = \mu(x(N)) - px(N)$ and $u_i(x, p) = (p - r)x_i$, respectively.

We define a *rule* as a function $\psi : \mathcal{L} \rightarrow \mathcal{A}$ that assigns to each problem $L = (N, K, E, c, r) \in \mathcal{L}$ a pair $(x, p) = \psi(L) \in A^L$, satisfying:

- (i) $x(N) = E$,
- (ii) $p(N, \alpha K, \beta E, \beta c, \frac{\alpha}{\beta} r) = \frac{\alpha}{\beta} p(N, K, E, c, r)$ and $x(N, \alpha K, \beta E, \beta c, \frac{\alpha}{\beta} r) = \beta x(N, K, E, c, r)$ for all $\alpha, \beta > 0$,
- (iii) $r \leq p \leq \frac{K}{E}$.

³Since $rc(N)$ is constant, this condition is equivalent to $\mu(E) - rE$ be maximum.

⁴This condition holds, for example, when μ is increasing and strictly concave.

The first condition (*efficiency*) says that the amount of land rented is optimal. The second condition (*scale invariance*) says that the final price and the amount of land rented are independent of changes of scale. The third condition (*individual rationality*) says that the lessors get at least zero (this is implied by $r \leq p$), and under efficiency, the tenant also gets at least zero (this is implied by $p \leq \frac{K}{E}$). Under efficiency, the utility of the tenant can be rewritten as $u_0(x, p) = K - pE$.

There exist two special classes of rules: On the one hand, a rule is *optimal for tenant* when the price is given by $p = r$. In that case, x_i is irrelevant for each $i \in N$, because their payoffs are zero, and so the final payoff allocation is unique. On the other hand, a rule is *optimal for lessors* when the price is given by $p = \frac{K}{E}$. In the latter case, there are many possible payoff allocations when $E > c(N)$, all of them giving zero to the tenant.

3 Land reassignment and monotonicity

Since there may be no official registration and demarcation of the customary land, the lessors can reach an agreement of reallocating it in order to share extra benefits so created under a rule. For the planner it is not possible to see this customary land situation, and it may be hard to get the outcome that the rule is supposed to attain. In our context manipulation implies that the lessors will benefit by merging or splitting under reallocating their land. Our aim is to fully identify rules that are free from this concern. We formalise this property as follows.

Land Reassignment (LR) Given $(N, K, E, c, r), (N', K, E, c', r) \in \mathcal{L}$ such that $c_i = c'_i$ for all $i \in S = N \cap N'$ and $c(N \setminus S) = c'(N' \setminus S)$,

$$\sum_{i \in N \setminus S} u_i(\psi(N, K, E, c, r)) = \sum_{i \in N' \setminus S} u_i(\psi(N', K, E, c', r)).$$

If the right-hand side of expression is larger than the left-hand side and the problem is (N, K, E, c, r) , then lessors in $N \setminus S$ can gain by reallocating

their land so that the problem becomes (N', K, E, c', r) . Analogously, if the left-hand side of expression is larger than the right-hand side and the problem is (N', K, E, c', r) , then lessors in $N' \setminus S$ can gain by reallocating their land so that the problem becomes (N, K, E, c, r) . S is the set of lessors that remain unchanged ($S = \emptyset$ is also possible). This property avoids lessors to have incentives for merging or splitting by reallocating their land.

The following property says that an increase of the available land is (weakly) beneficial for everyone involved.

Land Monotonicity (LM) Given $(N, K, E, c, r), (N, K, E, c', r) \in \mathcal{L}$,

- (i) $c \leq c'$ implies $u_0(\psi(N, K, E, c, r)) \leq u_0(\psi(N, K, E, c', r))$, and
- (ii) for each $i \in N$, $c_i \leq c'_i$ and $c_j = c'_j$ for all $j \in N \setminus \{i\}$ imply $u_i(\psi(N, K, E, c, r)) \leq u_i(\psi(N, K, E, c', r))$.

Under this property, the tenant will be weakly better off when there are more available land. Furthermore, when only one lessor has more available land and the rest of lessors remain unchanged, this lessor will be weakly better off.

Let \mathcal{F} be the set of functions $f : [0, 1] \rightarrow [0, 1]$ with $f(t) \geq t$ for all $t \in [0, 1]$. Now, we consider the family of rules defined by $p = \frac{K}{E} f(\frac{rE}{K})$ for some $f \in \mathcal{F}$ and $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$. So, we obtain different rules with different functions $f \in \mathcal{F}$. These functions determine the price, whereas the among of land is always divided proportionally, in line with the known results on invariance under reassignment in cost and surplus sharing (cf. Theorem 1.1 in Moulin (2002)).

Figure 1 represents six examples of these functions.

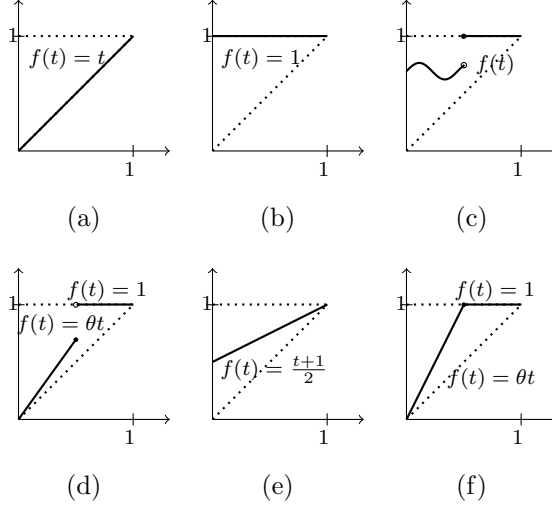


Figure 1: Examples of functions in \mathcal{F} that determine six different rules, including an optimal rule for the tenant (a) and an optimal rule for the lessors (b).

Theorem 3.1 *A rule ψ satisfies LR and LM if and only if there exists $f \in \mathcal{F}$ such that the price is given by $p = \frac{K}{E} f\left(\frac{rE}{K}\right)$ and, when $p \neq r$, the assigned amount of land is given by $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$.*

Proof. (\Leftarrow) Let ψ be a rule given by $p = \frac{K}{E} f\left(\frac{rE}{K}\right)$ for some $f \in \mathcal{F}$ and, when $p \neq r$, $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$. We will prove that ψ satisfies LR and LM. In order to prove that ψ satisfies LR, let $L = (N, K, E, c, r) \in \mathcal{L}$, $L' = (N', K, E, c', r) \in \mathcal{L}$ and $S = N \cap N'$ given as the definition of LR. Let $t = \frac{rE}{K} \in [0, 1[$. On the one hand, we have $\sum_{i \in N \setminus S} u_i(\psi(L)) = \sum_{i \in N \setminus S} \left(\frac{K}{E} f(t) - r\right) \frac{c_i E}{c(N)} = K(f(t) - t) \left(1 - \frac{c(S)}{c(N)}\right)$. Analogously, on the other hand, we have $\sum_{i \in N' \setminus S} u_i(\psi(L')) = K(f(t) - t) \left(1 - \frac{c'(S)}{c'(N')}\right)$. Since $c(N \setminus S) = c'(N' \setminus S)$ and $c_i = c'_i$ for all $i \in S$, we have that $c(S) = c'(S)$ and $c(N) = c'(N')$. Hence the last two expressions coincide. We now prove that ψ satisfies LM. Let L and $L' = (N, K, E, c', r) \in \mathcal{L}$ given as in the definition of LM. If $c \leq c'$, then, by efficiency, $u_0(\psi(L)) = K - \frac{K}{E} f\left(\frac{rE}{K}\right) E = u_0(\psi(L'))$, hence condition (i) holds. If $c_i \leq c'_i$ and $c_j = c'_j$ for all $j \in N \setminus \{i\}$ then

$u_i(\psi(L)) = \left(\frac{K}{E}f\left(\frac{rE}{K}\right) - r\right) \frac{c_i E}{c(N)} \leq \left(\frac{K}{E}f\left(\frac{rE}{K}\right) - r\right) \frac{c'_i E}{c'(N)} = u_i(\psi(L'))$ for all $i \in N$, hence condition (ii) also holds.

(\Rightarrow) Let ψ be a rule that satisfies LR and LM. For simplicity, we write (x, p) instead of $\psi(N, K, E, c, r)$, (x', p') instead of $\psi(N', K, E, c', r)$ and so on. Furthermore, we write u_i instead of $u_i(x, p)$, u'_i instead of $u_i(x', p')$ and so on. We proceed by series of claims.

Claim 3.1 *If $K = E = 1$ and $N = \{1\}$, then the price p does not depend on c .*

Proof. By LM, if $c_1 \leq c'_1$, then $u_0 \leq u'_0$. By efficiency, $u_0 \leq u'_0$ can be rewritten as $1 - p \leq 1 - p'$, hence $p \geq p'$ (the higher c_1 , the higher p). Analogously, $c_1 \leq c'_1$ implies $u_1 \leq u'_1$ and $p \leq p'$ (the higher c_1 , the lower p). Therefore, $p = p'$. \square

We define $f(t) = p(\{1\}, 1, 1, (1), t)$ for all $t \in [0, 1]$. By individual rationality, $t \leq p(\{1\}, 1, 1, (1), t) \leq 1$ for all $t \in [0, 1]$, so $f \in \mathcal{F}$.

Claim 3.2 *If $K = E = 1$, then $p = f(r)$.*

Proof. By LR, $u(N) = u_1(\psi(\{1\}, 1, 1, (c(N)), r))$. Under Claim 3.1 and efficiency, this is equal to $p(\{1\}, 1, 1, (1), r) - r$, hence $u(N) = f(r) - r$. Furthermore, by efficiency, $u(N) = \sum_{i \in N} (p - r)x_i = p - r$. Therefore, we have $p = f(r)$. \square

Claim 3.3 $p = \frac{K}{E}f\left(\frac{rE}{K}\right)$.

Proof. By scale invariance, $p = \frac{K}{E}p\left(N, 1, 1, \left(\frac{c}{E}\right), \frac{rE}{K}\right)$, and under Claim 3.2 we have that $p = \frac{K}{E}f\left(\frac{rE}{K}\right)$. \square

Therefore, the price is determined by Claim 3.3. Now we focus on the amount of land x .

Claim 3.4 *If $p \neq r$ and there exist $i, j \in N$ such that $c_i = c_j$, then $x_i = x_j$.*

Proof. Fix $\alpha \in \mathbb{N} \setminus N$. We define $(N^{i\alpha}, K, E, c^{i\alpha}, r) \in \mathcal{L}$, where $N^{i\alpha} = \{i, \alpha\}$, $c_i^{i\alpha} = c_i$ and $c_\alpha^{i\alpha} = c(N \setminus \{i\})$. Since $N \cap N^{i\alpha} = \{i\}$, then by LR, $u(N \setminus \{i\}) = u^{i\alpha}(N^{i\alpha} \setminus \{i\})$. Under Claim 3.3 and $p \neq r$, we obtain $x(N \setminus \{i\}) = x_\alpha^{i\alpha}$. Furthermore, by efficiency $x(N \setminus \{i\}) + x_i = E$ and $x_i^{i\alpha} + x_\alpha^{i\alpha} = E$. From these last three equalities we obtain that $x_i = x_i^{i\alpha}$. We define $(N^{j\alpha}, K, E, c^{j\alpha}, r) \in \mathcal{L}$, where $N^{j\alpha} = \{j, \alpha\}$, $c_j^{j\alpha} = c_i^{i\alpha} = c_i = c_j$ and $c_\alpha^{j\alpha} = c_\alpha^{i\alpha}$. Since $N^{i\alpha} \cap N^{j\alpha} = \{\alpha\}$ and $c_i^{i\alpha} = c_j^{j\alpha}$, by LR, $u_i(x^{i\alpha}, p^{i\alpha}) = u_j(x^{j\alpha}, p^{j\alpha})$. Under Claim 3.3 and $p \neq r$, we obtain $x_i^{i\alpha} = x_j^{j\alpha}$. Since $N^{j\alpha} \cap N = \{j\}$ and $c_\alpha^{j\alpha} = c(N \setminus \{j\})$, by LR, $u_\alpha(x^{j\alpha}, p^{j\alpha}) = u(N \setminus \{j\})$, and under Claim 3.3 and $p \neq r$, we obtain $x_\alpha^{j\alpha} = x(N \setminus \{j\})$. Furthermore, by efficiency we have $x_j^{j\alpha} + x_\alpha^{j\alpha} = E$ and $x_j + x(N \setminus \{j\}) = E$. So, from these last three equalities we obtain $x_j^{j\alpha} = x_j$. Then, from $x_i = x_i^{i\alpha}$, $x_i^{i\alpha} = x_j^{j\alpha}$ and $x_j^{j\alpha} = x_j$ we get that $x_i = x_j$. \square

Claim 3.5 *If $N = \{i, j\}$, $p \neq r$ and $c_i, c_j \in \mathbb{Q}$, then $x_i = \frac{c_i E}{c_i + c_j}$ and $x_j = \frac{c_j E}{c_i + c_j}$.*

Proof. Assume $c_i = \frac{a_i}{b_i}$ and $c_j = \frac{a_j}{b_j}$ where a and b are non-negative integers. Let $N^i, N^j \subset \mathbb{N} \setminus N$ with $N^i \cap N^j = \emptyset$, $|N^i| = a_i b_j$ and $|N^j| = a_j b_i$. We define $(N^{*i}, K, E, c^{*i}, r) \in \mathcal{L}$ with $N^{*i} = N^i \cup \{j\}$ and $c_k^{*i} = \frac{1}{b_i b_j}$ for all $k \in N^i$ and $c_j^{*i} = c_j$. Since $N \cap N^{*i} = \{j\}$ and $c^{*i}(N^{*i} \setminus \{j\}) = c_i$, by LR, $u_i = u^{*i}(N^{*i} \setminus \{j\})$. Under Claim 3.3, this is equivalent to write $(p - r)x_i = (p - r)x^{*i}(N^{*i} \setminus \{j\})$. Since $p \neq r$, $x_i = x^{*i}(N^{*i} \setminus \{j\})$. We now define $(N^{*ij}, K, E, c^{*ij}, r) \in \mathcal{L}$ with $N^{*ij} = N^i \cup N^j$ and $c_k^{*ij} = \frac{1}{b_i b_j}$ for all $k \in N^{*ij}$. Since $c^{*i}(N^{*i} \setminus \{j\}) = c^{*ij}(N^{*ij} \setminus N^j)$, $N^{*i} \cap N^{*ij} = N^i$ and $c^{*ij}(N^{*ij} \setminus N^i) = \sum_{l=1}^{a_j b_i} \frac{1}{b_i b_j} = c_j = c_j^{*i}$, by LR, $u_j^{*i} = u^{*ij}(N^{*ij} \setminus N^i)$. Under Claim 3.3, this is equivalent to write $(p - r)x_j^{*i} = (p - r)x^{*ij}(N^{*ij} \setminus N^i)$. Since $p \neq r$, $x_j^{*i} = x^{*ij}(N^{*ij} \setminus N^i) = x^{*ij}(N^j)$. On the one hand, by efficiency, $x_i + x_j = E$ and $x^{*i}(N^{*i} \setminus \{j\}) + x_j^{*i} = E$. Since $x_i = x^{*i}(N^{*i} \setminus \{j\})$ and $x_j^{*i} = x^{*ij}(N^j)$, we obtain $x_j = x^{*ij}(N^j)$. On the other hand, by efficiency, $x^{*i}(N^{*i} \setminus \{j\}) + x_j^{*i} = E$ and $x^{*ij}(N^i) + x^{*ij}(N^j) = E$. Since $x_i = x^{*i}(N^{*i} \setminus \{j\})$ and $x_j^{*i} = x^{*ij}(N^j)$, we obtain $x_i = x^{*ij}(N^i)$. We have $(N^{*ij}, K, E, c^{*ij}, r)$ with $N^{*ij} \cap N = \emptyset$ and $c^{*ij}(N^{*ij}) = c(N)$. By LR, $u^{*ij}(N^{*ij}) = u(N)$.

By $p \neq r$ and Claim 3.3, this is equivalent to write $x^{*ij}(N^{*ij}) = x(N)$. Under Claim 3.4 and efficiency, we obtain that $x_k^{*ij} = \frac{E}{|N^{*ij}|} = \frac{E}{a_i b_j + a_j b_i}$ for all $k \in N^{*ij}$. By efficiency and $x_j = x^{*ij}(N^j)$, we have that $x_i = E - x^{*ij}(N^j)$. Under Claim 3.4, this is equivalent to write $x_i = E - a_j b_i x_k^{*ij}$ for each $k \in N^{*ij}$. Since $x_k^{*ij} = \frac{E}{a_i b_j + a_j b_i}$ for all $k \in N^{*ij}$, we have $x_i = E - \frac{a_j b_i E}{a_i b_j + a_j b_i} = \frac{c_i E}{c_i + c_j}$. Analogously, $x_j = \frac{c_j E}{c_i + c_j}$. \square

Claim 3.6 *If $N = \{i, j\}$, $p \neq r$ and $c_j \in \mathbb{Q}$, then $x_i = \frac{c_i E}{c_i + c_j}$ and $x_j = \frac{c_j E}{c_i + c_j}$.*

Proof. Assume first $c_i + c_j = E$. Then, $\frac{c_i E}{c_i + c_j} = c_i$ and $\frac{c_j E}{c_i + c_j} = c_j$. By efficiency, $x_i = c_i$ and $x_j = c_j$. Therefore, $x_i = \frac{c_i E}{c_i + c_j}$ and $x_j = \frac{c_j E}{c_i + c_j}$. Assume now $c_i + c_j > E$. Let $\{c_i^s\}_{s=1}^\infty$ be a decreasing sequence of rational numbers that converges to c_i . For each s , we take $(N, K, E, c^s, r) \in \mathcal{L}$ with $c^s = (c_i^s, c_j)$. Under Claim 3.5, we have $x^s = (\frac{c_i^s E}{c_i^s + c_j}, \frac{c_j E}{c_i^s + c_j})$. By LM, $u_i(x, p) \leq u_i(x^s, p^s)$. Under Claim 3.3, this is equivalent to write $(p-r)x_i \leq (p-r)x_i^s$. Since $p \neq r$, this is equivalent to $x_i \leq x_i^s$. Under Claim 3.5, this is equivalent to write $x_i \leq \frac{c_i^s E}{c_i^s + c_j}$. Hence, $x_i \leq \frac{c_i E}{c_i + c_j}$. Let $\{\widehat{c}_i^s\}_{s=1}^\infty$ be an increasing sequence of positive rational numbers that converges to c_i and such that $\widehat{L}^s = (N, K, E, \widehat{c}^s, r) \in \mathcal{L}$, where $\widehat{c}^s = (\widehat{c}_i^s, c_j)$. We can find such a sequence because $c_i > 0$ and $c_i + c_j > E$. Under Claim 3.5, we have $\widehat{x}^s = (\frac{\widehat{c}_i^s E}{\widehat{c}_i^s + c_j}, \frac{c_j E}{\widehat{c}_i^s + c_j})$. By LM, $\widehat{u}_i^s \leq u_i$. Under Claim 3.3, this is equivalent to write $(p-r)\widehat{x}_i^s \leq (p-r)x_i$. Since $p \neq r$, this is equivalent to $\widehat{x}_i^s \leq x_i$. Under Claim 3.5, this is equivalent to write $\frac{\widehat{c}_i^s E}{\widehat{c}_i^s + c_j} \leq x_i$. Hence, $\frac{c_i E}{c_i + c_j} \leq x_i$. Since $x_i \leq \frac{c_i E}{c_i + c_j}$ and $\frac{c_i E}{c_i + c_j} \leq x_i$, we obtain $x_i = \frac{c_i E}{c_i + c_j}$. By efficiency, $x_j = E - x_i$. Since $x_i = \frac{c_i E}{c_i + c_j}$, we deduce $x_j = E - \frac{c_i E}{c_i + c_j} = \frac{c_j E}{c_i + c_j}$. \square

Claim 3.7 *If $N = \{i, j\}$ and $p \neq r$, then $x_i = \frac{c_i E}{c_i + c_j}$ and $x_j = \frac{c_j E}{c_i + c_j}$.*

Proof. Assume first $c_i + c_j = E$. Then, $\frac{c_i E}{c_i + c_j} = c_i$ and $\frac{c_j E}{c_i + c_j} = c_j$. By efficiency, $x_i = c_i$ and $x_j = c_j$. Therefore, $x_i = \frac{c_i E}{c_i + c_j}$ and $x_j = \frac{c_j E}{c_i + c_j}$. Assume now $c_i + c_j > E$. Let $\{c_j^s\}_{s=1}^\infty$ a decreasing sequence of rational numbers that converges to c_j . For each s , we take $(N, K, E, c^s, r) \in \mathcal{L}$ with $c^s = (c_i, c_j^s)$. Under Claim 3.6, we have $x^s = (x_i, \frac{c_j^s E}{c_i + c_j^s})$. By LM, $u_j(x, p) \leq u_j(x^s, p^s)$.

Under Claim 3.3, this is equivalent to write $(p - r)x_j \leq (p - r)x_j^s$. Since $p \neq r$, this is equivalent to $x_j \leq x_j^s = \frac{c_j^s E}{c_i + c_j^s}$. Hence, $x_j \leq \frac{c_j E}{c_i + c_j}$. Let $\{\widehat{c}_j^s\}_{s=1}^\infty$ be an increasing sequence of rational numbers that converges to c_j and such that $\widehat{L}^s = (N, K, E, \widehat{c}^s, r) \in \mathcal{L}$, where $\widehat{c}^s = (c_i, \widehat{c}_j^s)$. We can find such a sequence because $c_i > 0$ and $c_i + c_j > E$. Under Claim 3.6, we have $\widehat{x}^s = (x_i, \frac{\widehat{c}_j^s E}{c_i + \widehat{c}_j^s})$. By LM, $\widehat{u}_j^s \leq u_j$. Under Claim 3.3, this is equivalent to write $(p - r)\widehat{x}_j^s \leq (p - r)x_j$. Since $p \neq r$, this is equivalent to $\widehat{x}_j^s \leq x_j$, or $\frac{\widehat{c}_j^s E}{c_i + \widehat{c}_j^s} \leq x_j$. Hence, $\frac{c_j E}{c_i + c_j} \leq x_j$. Since $x_j \leq \frac{c_j E}{c_i + c_j}$ and $\frac{c_j E}{c_i + c_j} \leq x_j$, we obtain $x_j = \frac{c_j E}{c_i + c_j}$. By efficiency, $x_i = E - x_j$. Since $x_j = \frac{c_j E}{c_i + c_j}$, we deduce $x_i = E - \frac{c_j E}{c_i + c_j} = \frac{c_i E}{c_i + c_j}$. \square

Claim 3.8 *If $p \neq r$, then $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$.*

Proof. Let $i \in N$, $j \in N \setminus N$ and $(N^{ij}, K, E, c^{ij}, r) \in \mathcal{L}$ with $N^{ij} = \{i, j\}$, $c_i^{ij} = c_i$ and $c_j^{ij} = c(N \setminus \{i\})$. By efficiency, $x_i = E - x(N \setminus \{i\})$. By LR and $p \neq r$, we have $x(N \setminus \{i\}) = x_j^{ij}$, so that $x_i = E - x_j^{ij}$. Under Claim 3.7, $x_j^{ij} = \frac{c_j^{ij} E}{c_i^{ij} + c_j^{ij}}$. Hence, $x_i = E - \frac{c_j^{ij} E}{c_i^{ij} + c_j^{ij}} = E - \frac{c(N \setminus \{i\}) E}{c_i + c(N \setminus \{i\})} = \frac{c_i E}{c(N)}$. \square

Therefore, the amount of land is determined by Claim 3.8. \blacksquare

We denote ψ^f as the rule corresponding to $f \in \mathcal{F}$ that is given by $p = \frac{K}{E} f\left(\frac{rE}{K}\right)$, and $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$.

4 Weighted standard for two-person

We study a property that is inspired on the so called standard for two-person property by Hart and Mas-Colell (1989). This property follows a “divide the surplus equally” idea for two-person situations. In our context, the two-person case arises when $|N| = 1$, i.e. the only agents are the tenant and a single lessor. Standard for two-person says that both the tenant and the lessor obtain equal benefit. We formalize this property as follows. Let \mathcal{L}^2 be the set of land rental problems with a unique lessor.

Standard for 2-person (S2) Given $L = (\{1\}, K, E, c, r) \in \mathcal{L}^2$,

$$u_0(\psi(L)) = u_1(\psi(L)).$$

Next theorem characterizes the unique rule that satisfies LR, LM and S2. The function that determines this rule is represented in Figure 1(e).

Theorem 4.1 *A rule ψ satisfies LR, LM and S2 if and only if the price is given by $p = \frac{K+rE}{2E}$ and the amount of land is given by $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$.*

Proof. (\Leftarrow) Let ψ be a rule given by $p = \frac{K+rE}{2E}$ and $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$. It is straightforward to check that $\psi = \psi^f$ with $f(t) = \frac{1+t}{2}$ for all t and $p = \frac{K+rE}{2E}$. By Theorem 3.1, ψ satisfies LR and LM. So, we just need to prove that $u_0(\psi(\{1\}, K, E, c, r)) = u_1(\psi(\{1\}, K, E, c, r))$. The left side of the equality is equal to $K - \frac{K+rE}{2E}E = \frac{K-rE}{2}$. Analogously, the right side of the equality is equal to $\left(\frac{K+rE}{2E} - r\right)x_1$. By efficiency, $x_1 = E$, and hence we obtain $u_1(\psi(\{1\}, K, E, c, r)) = \frac{K-rE}{2}$. Therefore, the equality holds.

(\Rightarrow) Let ψ be a rule that satisfies LR, LM and S2. By Theorem 3.1 there exists $f \in \mathcal{F}$ such that $p = \frac{K}{E}f\left(\frac{rE}{K}\right)$ and, when, $p \neq r$, $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$. We need to prove that $\frac{K}{E}f\left(\frac{rE}{K}\right) = \frac{K+rE}{2E}$ or equivalently $f(t) = \frac{1+t}{2}$ for $t = \frac{rE}{K} \in [0, 1]$. By S2, we have $u_0(\psi(\{1\}, 1, 1, (1), t)) = u_1(\psi(\{1\}, 1, 1, (1), t))$. This is equivalent to $1 - f(t)x_1 = (f(t) - t)x_1$. By efficiency, $x_1 = 1$, which is equivalent to write $1 - f(t) = f(t) - t$. Hence, $f(t) = \frac{1+t}{2}$. Finally, since $K > rE$, we deduce $p \neq r$ so $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$. \blacksquare

Next, we generalize the standard for two-person concept in a nonsymmetric way. Notice that S2 determines the final payoffs for two-person problems, forcing both the tenant and the unique lessor to receive the same value. Since tenant and lessor are not symmetric, we can reasonably allow one side of the market to extract a higher value than the other. In our context, since the rules satisfy efficiency, it is enough to fix the relative payoff between both agents.

Weighted Standard for 2-person (WS2) There exists $\omega \in [0, 1]$ such that

$$(1 - \omega)u_0(\psi(L)) = \omega u_1(\psi(L))$$

for all $L = (\{1\}, K, E, c, r) \in \mathcal{L}^2$.

Next theorem characterizes the parametric subfamily of rules that satisfy LR, LM and WS2. We can see three examples of functions that determine these rules in Figure 1 (a), (b) and (e), respectively.

Theorem 4.2 *A rule ψ satisfies LR, LM and WS2 if and only if there exists $\omega \in [0, 1]$ such that the price is given by $p = \frac{K - \omega(K - rE)}{E}$ and, when $\omega < 1$, the quantity of land is given by $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$.*

Proof. (\Leftarrow) Fix $\omega \in [0, 1]$. Let ψ be a rule given by $p = \frac{K - \omega(K - rE)}{E}$ and, if $\omega < 1$, then $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$. By Theorem 3.1, ψ satisfies LR and LM for $f(t) = 1 - \omega(1 - t)$ and $p = \frac{K - \omega(K - rE)}{E}$. Fix $L = (\{1\}, K, E, c, r)$. We just need to prove that $(1 - \omega)u_0(\psi(L)) = \omega u_1(\psi(L))$. The left side of the equality is equal to $(1 - \omega) \left(K - \frac{K - \omega(K - rE)}{E} E \right) = (1 - \omega)\omega(K - rE)$. Analogously, the right side of the equality is equal to $\omega \left(\frac{K - \omega(K - rE)}{E} - r \right) x_1$. By efficiency, $x_1 = E$, and hence we obtain $\omega(1 - \omega)(K - rE)$. Therefore, equality holds.

(\Rightarrow) Let ψ be a rule that satisfies LR, LM and WS2. Let $\omega \in [0, 1]$. By Theorem 3.1, there exists $f \in \mathcal{F}$ such that $p = \frac{K}{E} f\left(\frac{rE}{K}\right)$ and, when $p \neq r$, $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$. It is clear that $\omega < 1$ implies $p \neq r$. To see why, notice that $p = r$ implies $u_1 = 0$, whereas $u_0 + u_1 = K - rE > 0$, so $u_0 > 0$, and by WS2, $(1 - \omega)u_0 = \omega u_1 = 0$, so $(1 - \omega)u_0 = 0$, which implies $\omega = 1$. We still need to prove that $\frac{K}{E} f\left(\frac{rE}{K}\right) = \frac{K - \omega(K - rE)}{E}$ or, equivalently, $f(t) = 1 - \omega(1 - t)$ for all $t \in [0, 1]$. Let $t = \frac{rE}{K} \in [0, 1]$. By WS2 we have $(1 - \omega)u_0(\psi(\{1\}, 1, 1, (1), t)) = \omega u_1(\psi(\{1\}, 1, 1, (1), t))$. This is equivalent to $(1 - \omega)(1 - f(t)x_1) = \omega(f(t) - t)x_1$, which by efficiency is equivalent to write $(1 - \omega)(1 - f(t)) = \omega(f(t) - t)$. Rearranging terms, we deduce $f(t) = 1 - \omega(1 - t)$. \blacksquare

Notice that, when $\omega = 1$, we obtain an optimal rule for the tenant, and when $\omega = 0$, we obtain an optimal rule for the lessors. Given $\omega \in [0, 1]$, we denote ψ^ω as the rule corresponding to the function ψ^f with $f(t) = 1 - \omega(1 - t)$ for all t and $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$. In particular, $\psi^{\frac{1}{2}}$ is the rule given in Theorem 4.1.

5 Consistency

Consistency is a well-known principle. Assume that there exists an agreement on what the right price and land share are, and that some lessors take this price and leave. The tenant and the rest of lessors can proceed in two ways: On the one hand, they can keep the previous price and land share. On the other hand, they can recompute the right price and land share following the same principle as before in the new reduced land renting problem. This new reduced land rental problem is defined as $L' = (N', K', E', c', r) \in \mathcal{L}$ given by $N' = N \setminus S$ where $S \subset N$ is the set of lessors that leave, $K' = K - px(S)$ is the new maximal profit of the tenant, $E' = E - x(S)$ is the amount of land that the tenant still needs in the new reduced land rental problem, $c' = c_{N \setminus S} \in \mathbb{R}_{++}^{N \setminus S}$ is the vector whose coordinates represent the amount of available land, and r is the reservation price, which is equal as in the original land rental problem. If this procedure always gives the same result for agents in $N_0 \setminus S$ as before, we say that ψ is consistent.

Consistency For all $(N, K, E, c, r) \in \mathcal{L}$ and $S \subset N$ such that $x(S) < E$,

$$u_i(\psi(N', K', E', c', r)) = u_i(\psi(N, K, E, c, r))$$

for all $i \in N_0 \setminus S$, where $N' = N \setminus S$, $K' = K - px(S)$, $E' = E - x(S)$ and $c'_i = c_i$ for all $i \in N'$.

Next proposition characterizes the second parametric subfamily of rules that satisfy LR, LM and consistency. We can see some examples of functions that determine these rules in Figure 1 (a), (b), (d) and (f), respectively.

Proposition 5.1 *A rule ψ satisfies LR, LM and consistency if and only if there exist $\alpha, \beta \in [0, 1]$ with $\alpha \leq \beta$ such that:*

a) *The price is given as follows:*

a.1) *If $r = 0$, then either $p = 0$ or $p = \frac{K}{E}$ for all $L \in \mathcal{L}$.*

a.2) *If $r > 0$ and $rE < \alpha K$, then $p = \frac{\beta}{\alpha}r$ for all $L \in \mathcal{L}$.*

a.3) If $r > 0$ and $rE = \alpha K$, then $p = \frac{\beta}{\alpha}r$ or $p = \frac{K}{E}$ for all $L \in \mathcal{L}$.

a.4) If $r > 0$ and $rE > \alpha K$, then $p = \frac{K}{E}$ for all $L \in \mathcal{L}$.

b) The amount of land when $p \neq r$ is given by $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$.

Proof. (\Leftarrow) Let $\alpha, \beta \in [0, 1]$ with $\alpha \leq \beta$ so that the price and the amount of land are given by a) and b), respectively. Let $f \in \mathcal{F}$ defined as follows: $f(0) \in \{0, 1\}$, $f(t) = \frac{\beta}{\alpha}t$ if $0 < t < \alpha$, $f(\alpha) \in \{\beta, 1\}$ if $\alpha > 0$, and $f(t) = 1$ if $t > \alpha$. Then, the price can be written as $p = \frac{K}{E}f\left(\frac{rE}{K}\right)$. Hence, by Theorem 3.1, ψ satisfies LR and LM. Let $L = (N, K, E, c, r)$ and $L' = (N', K', E', c', r)$ given as in the definition of consistency. We will prove that $u_i(x, p) = u_i(x', p')$ for all $i \in N_0 \setminus S$, where $(x, p) = \psi(L)$ and $(x', p') = \psi(L')$. Firstly, we prove that $p' = p$. We distinguish the following cases:

Case 1: $r = 0$ and $p = 0$. In this case, $f(0) = 0$. Hence, $p' = 0$ and $p = 0$.

Case 2: $r = 0$ and $p = \frac{K}{E}$. In this case, $f(0) = 1$. Hence, $p' = \frac{K'}{E'}$. Therefore,

$$p' = \frac{K'}{E'} = \frac{K - \frac{K}{E}x(S)}{E - x(S)} = \frac{K}{E} = p.$$

Case 3: $r > 0$, $rE < \alpha K$ and $p = \frac{\beta}{\alpha}r$. Under a.2), we know that $p' = \frac{\beta}{\alpha}r$ when $r > 0$ and $rE' < \alpha K'$. Since $r > 0$, it is enough to check that $rE' < \alpha K'$. Equivalently, $r(E - x(S)) < \alpha(K - \frac{\beta}{\alpha}rx(S))$. Since $rE < \alpha K$, it is enough to check that $rx(S) \geq \beta rx(S)$. This is trivially true when $rx(S) = 0$. Otherwise, it is equivalent to check that $\beta \leq 1$, which is true by definition.

Case 4: $r > 0$, $rE = \alpha K$ and $p = \frac{\beta}{\alpha}r$. In this case, $p = \frac{K}{E}\beta$, so $f\left(\frac{rE}{K}\right) = \beta$. Since $\frac{rE'}{K'} = \frac{r(E-x(S))}{K - \frac{K}{E}\beta x(S)} = \frac{rE(E-x(S))}{K(E-\beta x(S))}$ and $\beta \leq 1$, we have that $\frac{rE'}{K'} \leq \frac{rE(E-x(S))}{K(E-x(S))} = \frac{rE}{K} = \alpha$. Hence, $rE' \leq \alpha K'$. We will show that $f\left(\frac{rE'}{K'}\right) = \frac{\beta}{\alpha} \frac{rE'}{K'}$. We have two sub-cases: First, if $rE' < \alpha K'$, then it holds by a.2) and the fact that $p' = \frac{K'}{E'}f\left(\frac{rE'}{K'}\right)$. Second, if $rE' = \alpha K'$, then $f\left(\frac{rE'}{K'}\right) = f(\alpha) \stackrel{(rE=\alpha K)}{=} f\left(\frac{rE}{K}\right) = \beta$. Since $rE' = \alpha K'$, we obtain that $f\left(\frac{rE'}{K'}\right) = \frac{\beta}{\alpha} \frac{rE'}{K'}$. Hence, $p' = \frac{K'}{E'}f\left(\frac{rE'}{K'}\right) = \frac{K'}{E'} \frac{\beta}{\alpha} \frac{rE'}{K'} = \frac{\beta}{\alpha} r \stackrel{(rE=\alpha K)}{=} \frac{K}{E}\beta = p$.

Case 5: $r > 0$, $rE = \alpha K$, and $p = \frac{K}{E}$. Since $p = \frac{K}{E} f\left(\frac{rE}{K}\right)$ and $p = \frac{K}{E}$, we deduce $f\left(\frac{rE}{K}\right) = 1$. Moreover, $\frac{rE'}{K'} = \frac{r(E-x(S))}{K - \frac{K}{E}x(S)} = \frac{rE(E-x(S))}{K(E-x(S))} = \frac{rE}{K} = \alpha$. Hence, $f\left(\frac{rE'}{K'}\right) = f(\alpha)$. Since $rE = \alpha K$ and $f\left(\frac{rE}{K}\right) = 1$, we deduce $f\left(\frac{rE'}{K'}\right) = 1$. Hence, $p' = \frac{K'}{E'} f\left(\frac{rE'}{K'}\right) = \frac{K'}{E'} = \frac{K - \frac{K}{E}x(S)}{E - x(S)} = \frac{K}{E} = p$.

Case 6: $rE > \alpha K$. In this case, $p = \frac{K}{E}$. Under *a.4*), we know that $p' = \frac{K'}{E'}$ when $rE' > \alpha K'$. Since $\frac{K'}{E'} = \frac{K - \frac{K}{E}x(S)}{E - x(S)} = \frac{K}{E}$, it is enough to check that $rE' > \alpha K'$. This is equivalent to check that $r(E - x(S)) > \alpha\left(K - \frac{K}{E}x(S)\right)$. Equivalently, $r(E - x(S)) > \alpha K \left(\frac{E - x(S)}{E}\right)$. Since $E - x(S) > 0$, this is equivalent to $rE > \alpha K$, which is true in this case.

We check now that $u_i(\psi(L')) = u_i(\psi(L))$ for all $i \in N_0 \setminus S$. Assume first $i \in N \setminus S$. We need to prove that $(p - r)x'_i = (p - r)x_i$. This is trivially true when $p = r$. Hence, assume $p \neq r$. We need to prove $x'_i = x_i$. Since $c(N) = c(N \setminus S) + c(S)$, then $x'_i = \frac{c_i}{c(N \setminus S)}(E - x(S)) = \frac{c_i}{c(N \setminus S)}\left(E - \frac{c(S)E}{c(N)}\right) = \frac{c_i}{c(N \setminus S)}\left(\frac{c(N) - c(S)}{c(N)}\right)E = \frac{c_i}{c(N \setminus S)}\left(\frac{c(N \setminus S)}{c(N)}\right)E = \frac{c_i}{c(N)}E = x_i$. Assume now $i = 0$. We check that $u_0(\psi(L')) = u_0(\psi(L))$, or $K' - pE' = K - pE$. By definition, $K' - pE' = (K - px(S)) - p(E - x(S)) = K - pE$.

(\Rightarrow) Let ψ be a rule that satisfies LR, LM and consistency. Under LR and LM, by Theorem 3.1 there exists $f \in F$ such that $p = \frac{K}{E} f\left(\frac{rE}{K}\right)$ and, when $p \neq r$, $x_i = \frac{c_i}{c(N)}E$ for all $i \in N$.

Denote $L = (N, K, E, c, r)$ and let $S \subset N$ with $E < x(S)$ and $L' = (N', K', E', c', r)$ be defined as in the definition of consistency. Hence, we have $u_i(x, p) = u_i(x', p')$ for all $i \in N_0 \setminus S$, where $(x, p) = \psi(L)$ and $(x', p') = \psi(L')$. In particular, $u_0(x, p) = u_0(x', p')$. By definition, this is equivalent to $K' - p'E' = K - pE$, or $K - px(S) - p'(E - x(S)) = K - pE$. Since $E \neq x(S)$, we deduce $p' = p$.

We will prove the existence of α and β with $\alpha \leq \beta$, such that the price is given as in *a*). Or, equivalently,

$$\begin{aligned}
f(0) &\in \{0, 1\}, \\
f(t) &= \frac{\beta}{\alpha}t \text{ if } t \in]0, \alpha[, \\
f(\alpha) &\in \{\beta, 1\} \text{ when } \alpha > 0, \text{ and} \\
f(t) &= 1 \text{ if } t > \alpha.
\end{aligned} \tag{1}$$

If $f(t) = 1$ for all $t \in [0, 1]$, then $\alpha = 0$ and $\beta = 1$ satisfy (1). Hence, we can assume that there exists \hat{t} such that $f(\hat{t}) < 1$. Let $\alpha = \text{Sup}\{t : f(t) < 1\}$ and $\beta = \text{Sup}\{f(t) : f(t) < 1\}$. Then, $f(t) \geq t$ for all t implies $\alpha, \beta \in [0, 1]$ and $\alpha \leq \beta$.

For each $r \in [0, 1]$ and $\gamma \in]0, 1[$, assume $L = (\{1, 2\}, 1, 1, (\gamma, 1 - \gamma), r)$ and $S = \{2\}$. So, $K' = K - px_2 = 1 - f(r)(1 - \gamma)$ and $E' = E - x_2 = \gamma$. Hence, we have $L' = (\{1\}, 1 - f(r)(1 - \gamma), \gamma, (\gamma), r)$.

$$\text{So, } p = \frac{K}{E}f\left(\frac{rE}{K}\right) = f(r) \text{ and } p' = \frac{K'}{E'}f\left(\frac{rE'}{K'}\right) = \frac{1-f(r)(1-\gamma)}{\gamma}f\left(\frac{r\gamma}{1-f(r)(1-\gamma)}\right).$$

Since $p' = p$, from the last two expressions, we have

$$f\left(\frac{r\gamma}{1-f(r)(1-\gamma)}\right) = \frac{f(r)\gamma}{1-f(r)(1-\gamma)} \text{ for all } r \in [0, 1] \text{ and } \gamma \in]0, 1[. \tag{2}$$

In particular, for $r = 0$, we have $f(0) = \frac{f(0)\gamma}{1-f(0)(1-\gamma)}$ for all γ . If $f(0) \neq 0$, then $\gamma = 1 - f(0)(1 - \gamma)$ for all γ , or equivalently, $(1 - f(0))\gamma = 1 - f(0)$ for all γ , which implies that $f(0) = 1$. Hence $f(0) \in \{0, 1\}$. This is the first line of (1).

For $t > \alpha$, we have $f(t) = 1$. This is the fourth line of (1).

For each $r \in]0, 1[$, we define $F^r(\delta) = \frac{r\delta}{1-f(r)(1-\delta)} \in [0, r]$ for all $\delta \in]0, 1[$. If $f(r) < 1$, then F^r is a strictly increasing and continuous function, and its inverse is given by $G^r(t) = \frac{(1-f(r))t}{r-f(r)t} \in]0, 1[$ for all $t \in]0, r]$. Given $t \in]0, r]$ and $r \in]0, 1[$ such that $f(r) < 1$,

$$\begin{aligned}
f(t) &= f(F^r(G^r(t))) = f\left(\frac{rG^r(t)}{1-f(r)(1-G^r(t))}\right) \\
&\stackrel{(2)}{=} \frac{f(r)G^r(t)}{1-f(r)(1-G^r(t))} \\
&= \frac{f(r)\frac{(1-f(r))t}{r-f(r)t}}{1-f(r)\left(1-\frac{(1-f(r))t}{r-f(r)t}\right)} = \frac{f(r)}{r}t.
\end{aligned}$$

Assume $\alpha > 0$. Then we can fix $r \in]0, 1[$ such that $f(r) < 1$. Hence, $\frac{f(t)}{t} = \frac{f(r)}{r}$ for all $t \in]0, r]$. We will prove that $f(t) = \theta t$ for all $t \in]0, \alpha[$, where $\theta = \frac{f(r)}{r}$. For all $t \in]0, \alpha[$, there exists $r' > t$ such that $f(r') < 1$ and $\frac{f(t)}{t} = \frac{f(r')}{r'}$. If $t < r$, we can take $r' = r$, thus $\frac{f(t)}{t} = \theta$. If $t \geq r$, then $r' > r$, thus $\frac{f(r)}{r} = \frac{f(r')}{r'} = \theta$. Hence, $f(t) = \theta t$ for all $t \in]0, \alpha[$. We will prove that $\theta = \frac{\beta}{\alpha}$, or equivalently $r\beta = \alpha f(r)$. We have two cases:

Case I. If $f(\alpha) = 1$, then $\beta = \text{Sup}\{f(t) : t \in]0, \alpha]\} = \text{Sup}\{\theta t : t \in]0, \alpha]\} = \theta\alpha$. Hence, $\theta = \frac{\beta}{\alpha}$.

Case II. If $f(\alpha) < 1$, then $\frac{f(\alpha)}{\alpha} = \theta$, so that $f(t) = \theta t$ for all $t \in]0, \alpha]$ and $\beta = \text{Sup}\{f(t) : t \in]0, \alpha]\} = \text{Sup}\{\theta t : t \in]0, \alpha]\} = \theta\alpha$. Hence, $\theta = \frac{\beta}{\alpha}$.

Then, the second line of (1) is satisfied.

From Case I and Case II we can deduce $f(\alpha) \in \{\beta, 1\}$. This is the third line of (1). ■

Given $\alpha, \beta \in [0, 1]$ with $\alpha \leq \beta$, we define $\psi^{\alpha, \beta}$ as the rule corresponding to the function given in Proposition 5.1 with $f(0) = 0$, $f(\alpha) = \beta$ and such that the amount of land is given by $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$.

Next property says that small changes in the land rental problem should not cause large changes in the chosen allocation.

Continuity The price p and the amount of land x are continuous functions on \mathcal{L} .

The rules that satisfy LR, LM, consistency and continuity constitute a particular subfamily of rules from the one determined in Proposition 5.1 and

it is characterized in the next theorem. We can see examples of functions that determine these rules in Figure 1 (a), (b) and (f).

Theorem 5.1 *A rule ψ satisfies LR, LM, consistency and continuity if and only if there exists $\alpha \in [0, 1]$ such that:*

a)

$$p = \begin{cases} \frac{r}{\alpha} & \text{if } rE < \alpha K \\ \frac{K}{E} & \text{if } rE \geq \alpha K \end{cases}$$

and

b)

$$x_i = \frac{c_i E}{c(N)} \text{ for all } i \in N.$$

Proof. (\Leftarrow) Let $\alpha \in [0, 1]$ such that the price and amount of land are given by a) and b) respectively. Part a) can be written as $p = \frac{K}{E} f(\frac{rE}{K})$ with $f \in \mathcal{F}$ given as $f(t) = \frac{t}{\alpha}$ if $t < \alpha$ and $f(t) = 1$ if $t \geq \alpha$. Hence, by Proposition 5.1, ψ satisfies LR, LM and consistency. To prove it that satisfies continuity we still need to check that p is continuous at the points where $rE = \alpha K$. Equivalently, $\lim_{rE \rightarrow \alpha K^+} \frac{r}{\alpha} = \frac{K}{E}$, which holds trivially. Moreover, $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$ also determines a continuous function.

(\Rightarrow) Under LR, LM and consistency, by Proposition 5.1 there exist $\alpha, \beta \in [0, 1]$ with $\alpha \leq \beta$ such that the price is given as in part a) of Proposition 5.1, and the amount of land when $p \neq r$, is given by $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$. By adding continuity, we will prove that $p = \frac{r}{\alpha}$ if $rE \leq \alpha K$ and $p = \frac{K}{E}$ if $rE \geq \alpha K$. Moreover, $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$. In this sense, we have that p is a continuous functions in $]0, \alpha[\cup]\alpha, 1]$. We still need to prove the following cases:

i) If $r = 0$, by continuity, $p = \lim_{t \rightarrow 0} \frac{\beta}{\alpha} t = 0 = \frac{t}{\alpha}$.

ii) If $rE = \alpha K$, by continuity, $\frac{\beta}{\alpha} r = \frac{K}{E}$. Then, $\beta = \frac{K\alpha}{rE} = 1$.

We need to prove that $x_i = \frac{c_i E}{c(N)}$ for all $i \in N$ when $p = r$. Let $L^t = (N, K, E, c, r^t) \in \mathcal{L}$ with $\lim_{t \rightarrow \infty} r^t = 0$ and $r^t > 0$ for all t . Therefore, $x_i^t = \frac{c_i E}{c(N)}$ for all $t \in [0, 1]$. Then, under continuity of the land function, $x_i = \lim_{r \rightarrow 0} \frac{c_i E}{c(N)} = \frac{c_i E}{c(N)}$. ■

Notice that the functions provided in Theorem 5.1 are those $\psi^{\alpha, \beta}$ with $\beta = 1$. In particular, $\psi^{0,1} = \psi^0$ (rule ψ^ω when $\omega = 0$) is a rule optimal for the tenant and ψ^1 (rule ψ^ω when $\omega = 1$) is a rule optimal for the lessors.

These two rules are the only ones that belong to both parametric sub-families defined at Theorem 4.2 and Theorem 5.1, respectively. Both rules are characterized in the next proposition. We can see the functions that determine these rules in Figure 1 (a) and (b), respectively.

Proposition 5.2 *ψ^0 and ψ^1 are the only rules that satisfy LR, LM, WS2, consistency and continuity.*

Proof. It is straightforward to check that both ψ^0 and ψ^1 satisfy these properties. Let ψ be a rule that satisfies these properties. We will prove that ψ is either ψ^0 or ψ^1 . On the one hand, by Theorem 5.1, there exists $\alpha \in [0, 1]$ such that $p = \frac{K}{E}$ when $rE = \alpha K$. On the other hand, by Theorem 4.2, there exists $\omega \in [0, 1]$ such that $p = \frac{K - \omega(K - rE)}{E}$ for all r . Hence, given $r = \frac{\alpha K}{E}$, we have $\frac{K}{E} = \frac{K - \omega(K - rE)}{E}$, or $\omega(K - rE) = 0$. There are two possibilities: On the one hand, $\omega = 0$ which gives $\psi = \psi^0$. On the other hand, $K = rE$ and $rE = \alpha K$ imply $K = \alpha K$. Since $K > 0$, we deduce $\alpha = 1$ which gives $\psi = \psi^{1,1} = \psi^1$. ■

A summary of the results is presented in Table 1.

Property	ψ^f	ψ^ω	$\psi^{\alpha,1}$	$\psi^{\frac{1}{2}}$	ψ^0	ψ^1
Land Reassignment	Yes*	Yes*	Yes*	Yes*	Yes	Yes
Land Monotonicity	Yes*	Yes*	Yes*	Yes*	Yes	Yes
Standard for 2-person	-	-	No	Yes*	No	No
Weighted Standard for 2-person	-	Yes*	-	Yes	Yes	Yes
Consistency	-	-	Yes*	No	Yes	Yes
Continuity	-	Yes	Yes*	Yes	Yes	Yes

Table 1: Summary of the results. Symbol * means that this property, together with others in the same column, characterizes the family/rule.

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